

CAT(0) groups with specified boundary

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We specify exactly which groups can act geometrically on CAT(0) spaces whose visual boundary is homeomorphic to either a circle or a suspension of a Cantor set.

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1 Introduction

Suppose X is a proper CAT(0) space. We would like to understand what can be said about the geometry of X only knowing the homeomorphism type of the visual boundary. For example, if you specify that $\partial_\infty X$ is homeomorphic to the circle, then what does this say about X ? X could be the Euclidean plane, but X could also be \mathbb{H}^2 or a Euclidean cone with cone angle greater than 2π . Such a space can be obtained by gluing five (or more) quarter planes together along the boundary rays. If one further specifies that X admits a geometric group action, then one can rule out this last possibility, but the other two possibilities remain.

There are two topologies on the boundary ∂X that are used in this paper. In general, these boundaries are very different topological objects. For example, the *visual* topology on the boundary of $X = \mathbb{H}^2$ gives $\partial_\infty X \cong S^1$ while the Tits topology gives $\partial_{\text{Tits}} X$ an uncountable discrete space. There is however, always a continuous map from $\partial_{\text{Tits}} X$ to $\partial_\infty X$ given by the identity map (Bridson and Haefliger [3]). If you know the homeomorphism type of the visual topology and some of the structure of the Tits boundary, then you can obtain information about X . However, it can be difficult to extract any information about the Tits topology only given information about the visual topology. This article will explore the situation when the visual boundary is either a circle or a suspension of a Cantor set. The results themselves are not very surprising, but the proof techniques are quite interesting and can hopefully be used to analyze other boundaries.

The first result concerning the circle follows from the Flat Plane Theorem (Theorem 2.14 here) and work of Gromov, Casson–Jungreis, Tukia, and Gabai. This result is interesting in its own right but is also used as a crucial step in the proof of the more

difficult Theorem 3.7. Both results use the same method of proof so it is worth reading the easier one first.

Recall that a CAT(0) space X has *local extendability of geodesics* if every geodesic segment in X can be extended to a geodesic line in X .

Theorem 3.5 *Suppose X is a CAT(0) space with local extendability of geodesics and G is a group acting geometrically on X . Suppose the visual boundary of X is homeomorphic to a circle. Then exactly one of the following is true.*

- (1) X is isometric to a Euclidean plane and G is a Bieberbach group (virtually $\mathbb{Z} \oplus \mathbb{Z}$).
- (2) X is quasi-isometric to \mathbb{H}^2 and G is topologically conjugate to a fuchsian group.

The statement of the second result is along the same lines only we assume $\partial_\infty X$ is homeomorphic to ΣC where C denotes a Cantor set, however this does not follow from the work cited above. The proof of this theorem requires a careful analysis of the interplay between the visual and Tits topologies and the group action.

Theorem 3.7 *Suppose G acts geometrically on a CAT(0) space X and suppose $\partial_\infty X$ is homeomorphic to ΣC where C denotes the Cantor set. Then, the following are true.*

- (1) $\partial_{\text{Tits}} X$ is isometric to the suspension of an uncountable discrete space.
- (2) There exists a closed, convex, G -invariant subset X' of X which splits as $Y \times \mathbb{R}$. If X has local extendability of geodesics, then $X = Y \times \mathbb{R}$.
- (3) The subspace Y is CAT(0) and has $\partial_\infty Y$ homeomorphic to a Cantor set.

And finally, we obtain information about the group G in the previous theorem using an algebraic result proven by M. Bridson and relayed to the author here. The result of Bridson is Theorem 4.1 here.

Theorem 4.4 *Suppose G acts geometrically on a CAT(0) space X with $\partial_\infty X$ homeomorphic to the suspension of a Cantor set. Then G contains a subgroup G_0 of finite index which is isomorphic to $F \times \mathbb{Z}$ where F is a nonabelian free group.*

As a final remark, we mention that a much more difficult question would be to classify all CAT(0) groups with Sierpinski carpet boundary. An example of a CAT(0) group with this boundary is the fundamental group of the figure eight knot complement (Ruane [12]). This group has $\mathbb{Z} \oplus \mathbb{Z}$ subgroups so is not word hyperbolic. However, if

M is a compact hyperbolic three manifold with nonempty totally geodesic boundary then $G = \pi_1(M)$ is a word hyperbolic group with Sierpinski carpet boundary. The following conjecture concerning word hyperbolic groups with Sierpinski boundary was posed by Kapovich and Kleiner [10].

Conjecture [10] Let G be a word hyperbolic group with Sierpinski carpet boundary. Then G acts discretely, cocompactly, and isometrically on a convex subset of \mathbb{H}^3 with nonempty totally geodesic boundary.

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2 CAT(0) preliminaries

In this section, we assume X is a proper, complete, geodesic metric space. The CAT(0) inequality is a curvature condition on X first introduced by Alexandrov in [1]. The idea is to compare geodesic triangles in X to triangles of the same size in the Euclidean plane and require those in X to be at least as thin as the corresponding triangle in \mathbb{E}^2 . The formal definition is given below. Examples of CAT(0) spaces include \mathbb{E}^n and the universal covers of compact Riemannian manifolds of nonpositive curvature.

Definition 2.1 (CAT(0)) Let (X, d) be a proper complete geodesic metric space. If Δabc is a geodesic triangle in X , then we consider $\Delta \bar{a}\bar{b}\bar{c}$ in \mathbb{E}^2 , a triangle with the same side lengths, and call this a *comparison triangle*. Then we say X satisfies the CAT(0) inequality if given Δabc in X , then for any comparison triangle and any two points p, q on Δabc , the corresponding points \bar{p}, \bar{q} on the comparison triangle satisfy

$$d(p, q) \leq d(\bar{p}, \bar{q})$$

Definition 2.2 (Asymptotic) Let X be a metric space. Two geodesic rays $c, c': [0, \infty) \rightarrow X$ are called *asymptotic* if there exists a constant K such that $d(c(t), c'(t)) \leq K$ for all $t \geq 0$.

Let (X, d) be a CAT(0) space. First, define the boundary, ∂X as a point set as follows:

Definition 2.3 (Boundary) The *boundary* of X , denoted ∂X , is the set of equivalence classes of geodesic rays where two rays are equivalent if and only if they are asymptotic. The union $X \cup \partial X$ is denoted by \bar{X} .

Remark 2.4 Fix $x_0 \in X$ and consider the set $\partial_{x_0} X$ of geodesic rays in X that begin at x_0 . Fixing a basepoint essentially picks out a unique representative of an equivalence class of geodesic rays. More precisely, by [3, Proposition 8.2], given a geodesic ray $c: [0, \infty) \rightarrow X$ with $c(0) = x_0$ and any other $x \in X$, there is a unique geodesic ray $c': [0, \infty) \rightarrow X$ issuing from x and is asymptotic to c . Thus one can identify ∂X with $\partial_{x_0} X$ for a given basepoint. We describe a topology on $\partial_{x_0} X$ below and it follows that the obvious bijection $\partial_{x_0} X \rightarrow \partial_{x_1} X$ for any other $x_1 \in X$ is a homeomorphism. Thus we often write ∂X even though we may be considering a fixed basepoint.

There is a natural neighborhood basis for a point in \bar{X} . Let c be a geodesic ray emanating from x_0 and $r > 0$, $\epsilon > 0$. Also, let $S(x_0, r)$ denote the sphere of radius r centered at x_0 with $p_r: X \rightarrow S(x_0, r)$ denoting projection. Define

$$U(c, r, \epsilon) = \{x \in \bar{X} \mid d(x, x_0) > r, d(p_r(x), c(r)) < \epsilon\}$$

This consists of all points in \bar{X} such that when projected back to $S(x_0, r)$, this projection is not more than ϵ away from the intersection of that sphere with c . These sets along with the metric balls about x_0 form a basis for the *cone topology* on \bar{X} . The set ∂X with the subspace topology is often called the *visual boundary*. As one expects, the visual boundary of \mathbb{E}^n is S^{n-1} as is the visual boundary of \mathbb{H}^n . Thus the visual boundary does not capture the difference between these two CAT(0) spaces. Thus we need the Tits topology to distinguish these types of spaces. The notation ∂X is used to denote the visual topology and $\partial_{\text{Tits}} X$ to denote the Tits boundary.

We first develop a general technique for measuring angles between points in ∂X . We assume the reader has some knowledge of how to measure angles in a metric space, but we add the necessary definitions for completeness. Alexandrov used the method of comparison triangles to define the notion of angle between two geodesics leaving a point x_0 in a metric space X , [3]. We recall that definition here.

Definition 2.5 (Angles) Let $c: [0, a] \rightarrow X$ and $c': [0, a'] \rightarrow X$ be two geodesics with $c(0) = c'(0) = x_0$. Given $t \in [0, a]$, $t' \in [0, a']$, and let $\alpha_{c, c'}^{t, t'}$ denote the angle in a comparison triangle in Euclidean space at the vertex corresponding to x_0 . The (upper) angle between c, c' at x_0 is defined to be the following number:

$$\angle_{c, c'} := \limsup_{t, t' \rightarrow 0} \alpha_{c, c'}^{t, t'}$$

Note: The lim sup is used because the limit may not always exist, but in CAT(0) spaces, the limit does exist and instead of calling it an “upper” angle, we call it the angle. For a proof of this, see [3].

Definition 2.6 (Angle metric) Let X be a CAT(0) space. Given $x \in X$ and $u, v \in \partial X$, we denote by $\angle_x(u, v)$ the angle between the unique geodesic rays which issue from x ending at u and v respectively. Then we define the angle between u and v to be

$$\angle(u, v) = \sup_{x \in X} \angle_x(u, v)$$

One can verify this angle satisfies the triangle inequality and so $\angle(u, u')$ defines a metric on ∂X called the *angular metric*.

Example 2.7 Consider the CAT(0) space $X = \mathbb{H}^2$. We know ∂X can be identified with the unit circle in the disk model. For any two points p, q on the boundary circle, there is a geodesic line in \mathbb{H}^2 joining p and q . Clearly the supremum is attained by taking any other point on the line between p and q so $\angle(p, q) = \pi$. This means the angle metric on $\partial(\mathbb{H}^2)$ is discrete.

Definition 2.8 (Tits metric) The *Tits metric* on ∂X , denoted Td , is the *length metric* associated to the angular metric. Thus for $v, w \in \partial X$, $Td(v, w)$ is defined to be the infimum of the lengths of rectifiable curves in the angular metric between v and w . If there are no rectifiable curves joining them, then $Td(v, w) = \infty$.

From the example above, we can conclude that for any two points $u, v \in \partial\mathbb{H}^2$, $Td(u, v) = \infty$. The basic facts needed here concerning the Tits metric can be found in Ballmann–Gromov–Schroeder [2] for the manifold setting and in [3] for the CAT(0) setting. A CAT(1) space is defined similarly to a CAT(0) space except the comparison triangles live in the standard S^2 rather than \mathbb{E}^2 and the inequality must hold for all triangles with perimeter less than 2π . Note, that this perimeter restriction guarantees the existence of a comparison triangle in S^2 .

Remark 2.9 The main facts concerning $\partial_{\text{Tits}} X$ needed here are as follows.

- (1) The Tits boundary of a flat plane in X is a circle that is isometrically embedded in $\partial_{\text{Tits}} X$
- (2) If $u, v \in \partial X$ have $Td(u, v) = \infty$, then there is a geodesic line in X joining u and v and this line does not bound a flat half plane.
- (3) $\partial_{\text{Tits}} X$ is a CAT(1) space. In particular, if $p, q \in \partial_{\text{Tits}} X$ have $Td(p, q) < \pi$, then there is a unique geodesic between them in $\partial_{\text{Tits}} X$ and $Td(p, q) = \angle(p, q)$.

Remark 2.10 It follows that if X and Y are two CAT(0) spaces, then with the product metric, $X \times Y$ is also CAT(0) and $\partial_\infty(X \times Y) \cong \partial_\infty X \star \partial_\infty Y$ where \star denotes the spherical join. The Tits topology is isometric to the join of the Tits topologies of the factors. In this paper the case in which one of the factors is \mathbb{R} is used so we point out the following: $\partial(Y \times \mathbb{R}) \equiv \Sigma(\partial Y)$ where Σ denotes suspension.

We will also need the following facts concerning lines and planes in a proper CAT(0) space X . Again, proofs can be found in [3].

Definition 2.11 (Parallel) Suppose $c, c': \mathbb{R} \rightarrow X$ are geodesic lines in X . Then we say c, c' are *parallel* in X if there exists a constant $K \geq 0$ with $d(c(t), c'(t)) \leq K$ for all $t \in \mathbb{R}$.

Theorem 2.12 Suppose $c: \mathbb{R} \rightarrow X$ is a geodesic line. Let $\mathcal{P}(c)$ denote the set of all geodesic lines in X that are parallel to c . Then $\mathcal{P}(c)$ is a closed convex subset of X which splits isometrically as $\mathcal{P}(c) \equiv Y \times \mathbb{R}$ where the \mathbb{R} factor is determined by c . Y is also a closed, convex subset of X .

The next result is analogous to the previous result but for any closed convex subset C of X instead of a line. This is often referred to the Sandwich Lemma, [3, page 182]. Write $d_C(x) = \inf\{d(x, c) \mid c \in C\}$ to denote the distance from a point $x \in X$ to the set C .

Theorem 2.13 (Sandwich Lemma) Let C_1, C_2 be closed convex subspaces of a CAT(0) space X . If the restriction of d_{C_1} to C_2 is constant and equal to some number a , and the restriction of d_{C_2} to C_1 is constant, then the convex hull of $C_1 \cup C_2$ is isometric to $C_1 \times [0, a]$.

We now state the version of the Flat Plane Theorem used in this paper. This version is stated as [3, Theorem 3.1, page 459].

Theorem 2.14 (Flat Plane Theorem) If a group Γ acts properly and cocompactly by isometries on a CAT(0) space X , then Γ is word hyperbolic if and only if X does not contain an isometrically embedded copy of the Euclidean plane.

Remark 2.15 In the case that Γ is word hyperbolic, we have Γ is quasi-isometric to X as metric spaces. This forces X to be δ -hyperbolic as well since hyperbolicity is preserved under quasi-isometries.

We will also need the following result concerning free abelian central subgroups. This can be found as [3, Theorem 6.12, page 234].

Theorem 2.16 *Let X be a CAT(0) space and let Γ be a finitely generated group acting by isometries on X . Suppose Γ contains a central subgroup $A \cong \mathbb{Z}^n$ that acts faithfully by hyperbolic isometries (apart from the identity), then there exists a subgroup of finite index $H \subset \Gamma$ which contains A as a direct factor.*

3 Geometric results

One of the important facts used for the proofs in this section is that the identity map $\partial X \rightarrow \partial X$ gives a continuous function $\partial_{\text{Tits}} X \rightarrow \partial_{\infty} X$ between these two topological spaces. Intuitively, if $p, q \in \partial X$ have small angle in the sense of Definition 2.6, then if we take rays from a fixed basepoint x_0 to p and q , these rays must follow travel for a long time. This is not necessarily true in reverse as can be seen in \mathbb{H}^2 since $\angle(p, q) = \pi$ for all $p \neq q \in \partial \mathbb{H}^2$. The next lemma follows easily from point-set topology, however the proof given here uses this interaction between the two topologies.

Lemma 3.1 *Suppose X is a CAT(0) space and F is an n -flat in X . Then the identity map from $\partial_{\text{Tits}} X$ to $\partial_{\infty} X$ restricted to the boundary of F is a homeomorphism. In particular, the image of $\partial_{\text{Tits}} F$ in $\partial_{\infty} X$ is a simple closed, curve.*

Proof Clearly the restricted map is a continuous bijection from the Tits circle to the visual circle. We show the inverse map is continuous. Pick a basepoint $x_0 \in F$. Given a point $p \in \partial_{\text{Tits}} F$ and an $\alpha > 0$ we can find r and ϵ so that the set $U(p, r, \epsilon) \cap \partial_{\infty} F$ consists of points whose angle with p is less than $\alpha/2$. Indeed, use the law of cosines which holds in a CAT(0) space. \square

We will also need the following lemmas for both theorems. A (complete) CAT(0) space X has *local extendability of geodesics* if every geodesic segment in X can be extended to a geodesic line in X (compare [3, Definition II.5.7]). It is not true in general that a complete CAT(0) space has this property, however it is known that if X is proper and the full isometry group of X has a compact fundamental domain, then X is *almost extendable*. This means there exists a global constant $r \geq 0$ such that for any pair $a, b \in X$ there exists a geodesic ray beginning at a and passing within r of b . See Geoghegan and Ontaneda [6] for details.

The assumption of local extendability of geodesics is necessary for making certain conclusions about the space X , however for results about $\partial_{\infty} X$ only it is often unnecessary to have this extra assumption. Instead, the notion of a quasi-dense set is sufficient for results about $\partial_{\infty} X$. A subset M of X is called *quasi-dense* if there exists a constant $K > 0$ so that each point of X is within K of some point of M .

Notice that if a group Γ acts cocompactly, then the orbit of a point is a quasi-dense subset.

Lemma 3.2 *If $M \subset X$ is closed, convex, and quasi-dense, then $\partial X = \partial M$.*

Proof If $p \in \partial X$, then let $c: [0, \infty) \rightarrow X$ be a geodesic ray from x_0 whose endpoint in \bar{X} is p . For each $n \in \mathbb{N}$, choose $m_n \in M$ with $d(m_n, c(n)) < K$. The geodesics $[x_0, m_n]$ are all contained in M since M is convex. The sequence $[x_0, m_n]$ converges to a geodesic ray c' in M with endpoint in ∂M . The endpoint of c' must be p since c' is asymptotic to c by construction, thus $p \in \partial M$ and we are done. \square

Corollary 3.3 *Suppose the group G acts geometrically on a CAT(0) space X and suppose Z is a closed, convex, G -invariant subset of X . Then $\partial Z = \partial X$. Furthermore, if X has local extendability of geodesics, then $Z = X$.*

Proof Let $z_0 \in Z$. Then the orbit $G \cdot z_0 \subset Z$ is a quasi-dense subset of X which forces Z to be quasi-dense. The corollary now follows from the previous lemma. \square

Lemma 3.4 *Suppose X is a CAT(0) space with 1-dimensional visual boundary $\partial_\infty X$. Suppose F is a 2-flat in X and let $\mathcal{P}(F)$ be the set of all flats in X that are parallel to F . Then $\mathcal{P}(F)$ is isometric to a product $F \times Y$ with Y compact.*

Proof By the Sandwich Lemma (Theorem 2.13 here), the set $\mathcal{P}(F)$ of all flats in X parallel to F isometrically splits as $F \times Y$ for some closed, convex subset Y of X . We claim that Y must be compact. Indeed, otherwise, $\partial Y \neq \emptyset$ and $\partial_{\text{Tits}}(F \times Y)$ is isometric to the spherical join of $\partial_{\text{Tits}} F$ and $\partial_{\text{Tits}} Y$. If $Y \neq \emptyset$, this would force $\partial_{\text{Tits}} X$ to be 2-dimensional since $\partial_{\text{Tits}} F$ is a circle. This would force the visual topology to contain a 2-dimensional subset which gives a contradiction. \square

The following result will provide one of the crucial steps in Theorem 3.7.

Theorem 3.5 *Suppose X is a CAT(0) space with local extendability of geodesics and G is a group acting geometrically on X . Suppose the visual boundary of X is homeomorphic to a circle. Then exactly one of the following is true.*

- (1) X is isometric to a Euclidean plane and G is a Bieberbach group (virtually $\mathbb{Z} \times \mathbb{Z}$).
- (2) X is quasi-isometric to \mathbb{H}^2 and G is topologically conjugate to a fuchsian group.

Proof We know by the Flat Plane Theorem that X is either δ -hyperbolic or X contains a 2-flat. Suppose X contains a 2-flat – ie, an isometric embedding of $f: \mathbb{E}^2 \rightarrow X$. Denote by F , the image of f . Let $g \in G$ and consider the flat $g \cdot F$. Each of F and $g \cdot F$ contribute a circle to the Tits boundary of X . There are three possibilities – the circles are the same, they intersect but are unequal, or they are disjoint. We claim the last two cannot happen. Indeed, if the two circles are disjoint, then they map homeomorphically to disjoint circles in the visual topology. This obviously cannot happen since $\partial_\infty X$ is homeomorphic to a single circle. If the two circles intersect but are not the same, then again, this would happen in the visual topology which is absurd. Thus F and $g \cdot F$ have the same boundary circle. This implies that $g \cdot F$ is parallel to F . Consider the set $\mathcal{P}(F)$ of all flats in X that are parallel to F . By Lemma 3.4, $\mathcal{P}(F) = F \times Y$ where Y is compact. Since the chosen g was arbitrary, $F \times Y$ is a closed, convex, G -invariant subset of X . By Corollary 3.3, we have $X = F \times Y$. With the assumption of local extendability, Y must be a single point and we are in the first case of the theorem.

Suppose X is δ -negatively curved. Then the group G is word hyperbolic and ∂G (the Gromov boundary of the group) is the same as $\partial_\infty X$, [3]. We also know G acts as a convergence group on $\partial_\infty X$ (Gromov [9]). But now the work of Casson–Jungreis [4], Tukia [16], Gabai [5] implies that G is topologically conjugate to a fuchsian group. This also gives G quasi-isometric to \mathbb{H}^2 which, in turn, gives X is quasi-isometric to \mathbb{H}^2 as needed for case two of the theorem. \square

Remark 3.6 Note that in case two of the theorem, we do not get that X is isometric to the hyperbolic plane. The following example was explained to me by Phil Bowers. If you take the constant seven degree triangulation of the plane, make all edge lengths one, and glue in equilateral triangles, you will obtain a CAT(0) space X which is quasi-isometric to \mathbb{H}^2 but not isometric to it. Indeed, there are flat triangles in X . Furthermore, the (2,3,7) triangle group acts in the obvious way as a geometric group action on X . Thus we leave the statement as it is in the theorem and cannot hope for more.

Theorem 3.7 *Suppose G acts geometrically on a CAT(0) space X and suppose $\partial_\infty X$ is homeomorphic to ΣC where C denotes the Cantor set. Then, the following are true.*

- (1) $\partial_{\text{Tits}} X$ is isometric to the suspension of an uncountable discrete space.
- (2) There exists a closed, convex, G invariant subset X' of X which splits as $Y \times \mathbb{R}$. If X has local extendability of geodesics, then $X = Y \times \mathbb{R}$.
- (3) The subspace Y is CAT(0) and has $\partial_\infty Y$ homeomorphic to a Cantor set.

Proof By the Flat Plane Theorem, we know X is either δ -negatively curved or else X contains a flat plane. If X negatively curved, then G is word hyperbolic. Since $\partial_\infty X$ is connected, G is one-ended. But $\partial_\infty X$ is a suspension of a Cantor set and is thus not locally connected. This contradicts the work of Swarup [14] on boundaries of one-ended hyperbolic groups.

Thus X must contain a flat plane, F . Since F is a closed, convex subset of X , we know $\partial_\infty F \cong S^1$ embeds in $\partial_\infty X$. Also $\partial_{\text{Tits}} F \equiv S^1$ embeds in $\partial_{\text{Tits}} X$. (\cong means “homeomorphic to” while \equiv means “isometric to”).

Note that any circle in $\partial_\infty X$ must go through the suspension points of $\partial_\infty X$. Let’s call these suspension points N and S .

We claim $Td(N, S) = \pi$. We prove this fact in a subsequent lemma.

Thus there is a geodesic line $c: \mathbb{R} \rightarrow X$ with image in F , with endpoints N and S in $\partial_\infty X$. Consider $\mathcal{P}(c)$ in X , the set of all geodesic lines in X that are parallel to the line c .

We know by Theorem 2.12 that $\mathcal{P}(c) \equiv Y \times \mathbb{R}$ is a closed convex subset of X . Furthermore, $\partial_{\text{Tits}} \mathcal{P}(c) \equiv \Sigma(\partial_{\text{Tits}} Y)$. The suspension points of this are N and S , the suspension points of $\partial_\infty X$.

Given $g \in G$, we have $g \cdot F$ is another flat in X with $\partial_{\text{Tits}}(g \cdot F)$ an S^1 in $\partial_{\text{Tits}} X$. There are three possible configurations.

- (1) $\partial_{\text{Tits}} F$ and $\partial_{\text{Tits}}(g \cdot F)$ are disjoint.
- (2) $\partial_{\text{Tits}} F = \partial_{\text{Tits}}(g \cdot F)$.
- (3) $\partial_{\text{Tits}} F \cap \partial_{\text{Tits}}(g \cdot F) \neq \emptyset$ but are unequal.

If the two circles are disjoint, then they would map to disjoint circles in $\partial_\infty X$ under the identity map but this is impossible since any pair of circles in $\partial_\infty X$ intersect. Thus case (1) cannot happen.

If $\partial_{\text{Tits}} F = \partial_{\text{Tits}}(g \cdot F)$, then the flats F and $g \cdot F$ are parallel in X . We claim there must be some $g \in G$ for which this does not happen. If this were true for all elements $g \in G$, then consider the set $\mathcal{P}(F) \equiv Y \times F$ of all flats in X that are parallel to F . Y must be compact by Lemma 3.4. Now the group G leaves $\mathcal{P}(F)$ invariant and thus acts geometrically on $\mathcal{P}(F)$. But $\partial_\infty \mathcal{P}(F)$ is a circle so G and X fall into the setting of Theorem 3.5.

Thus there exists $g \in G$ as in the case (3) above. In this case, the circles $\partial_{\text{Tits}} F$ and $\partial_{\text{Tits}}(g \cdot F)$ must both contain N and S since any two distinct circles in $\partial_\infty X$ share the points N and S . Thus the line $g \cdot c$ in $g \cdot F$ is parallel to the line c in F .

Since the action of G must take flats in X to flats in X , we clearly have that G leaves $\mathcal{P}(c)$ invariant. Thus $\mathcal{P}(c)$ is a closed, convex, G -invariant subset of X . By Corollary 3.3 $\mathcal{P}(c)$ is quasi-dense in X and $\partial X = \partial \mathcal{P}(c) = \Sigma \partial Y$. If X has local extendability of geodesics, we indeed have $\mathcal{P}(c) = X$ by the same lemma. We have now shown the second item in the theorem is true. We also know that $\partial_\infty X = \partial_\infty \mathcal{P}(c) \cong \Sigma(\partial_\infty Y)$ where N and S are the suspension points.

Let $z \neq w \in \partial Y$. Since $\partial_{\text{Tits}} X$ is isometric to the suspension of ∂Y , we clearly have $Td(z, w) \leq \pi$ because we can construct a Tits path from z to w by combining the segments $[z, N]$ and $[N, w]$ (or S instead of N). Each of these segments is length $\pi/2$ and thus we have a path from z to w of length π . We claim that $Td(z, w) = \pi$.

If there were a shorter Tits path between them, then this path would not pass through N or S . This path would also give a path in the visual topology after mapping via the identity map between the two topologies. Since $\partial_\infty X$ is the suspension of a Cantor set and $z, w \notin \{N, S\}$, the only paths between them go through N or S unless they are on the same suspension line. But this is clearly not the case for two points in ∂Y . Thus the Tits metric restricted to ∂Y is discrete and item (1) of the theorem is complete.

Since $\partial_\infty X \cong \Sigma(\partial_\infty Y)$, each point $p \in \partial X$ which is neither N or S lies on a unique arc between N and S in $\partial_\infty X$. The assumption that $\partial_\infty X$ is also homeomorphic to ΣC (with suspension points N and S) implies $\partial_\infty Y$ is homeomorphic to C . Thus all three items in the theorem are true. \square

The proof of Lemma 3.13 below requires several facts about the action of a hyperbolic isometry on the boundary. If G is a group acting geometrically on a CAT(0) space X and $g \in G$ of infinite order, then g acts on by translation on a geodesic line in X with two endpoints in $\partial_\infty X$. This g also acts on $\partial_\infty X$ as a homeomorphism with a special fixed point set which we discuss below. Every $g \in G$ will fix the suspension points of $\partial_\infty X$ simply because of topology. The key observation is that if two infinite order elements (which do not have a common power) fix the same suspension line between the suspension points of $\partial_\infty X$, then one of them must have the suspension points as its endpoints in which case the lemma will be true.

Let g be an isometry of a CAT(0) space X . Recall that $\text{Min}(g) = \{x \in X \mid d(x, g \cdot x) = |g|\}$ where $|g| = \inf\{d(x, g \cdot x) \mid x \in X\}$. We say g is *hyperbolic* if $|g| > 0$ and $\text{Min}(g) \neq \emptyset$. In this case, we have the following facts whose proofs can be found in Ruane [11], Theorem 1.2.3 and Theorem 3.2.

Theorem 3.8 *Suppose g is a hyperbolic isometry of a CAT(0) space X . Then*

- (1) *There exists a geodesic line A_g called an axis on which g acts via translation by $|g|$. The set $\text{Min}(g)$ consists of all points in X which lie on such an axis.*

- (2) $\text{Min}(g)$ is isometric to $Y \times \mathbb{R}$ where Y is a closed, convex subset of X .
- (3) $\partial_\infty \text{Min}(g)$ is homeomorphic to $\Sigma(\partial_\infty Y)$ and $\partial_{\text{Tits}} \text{Min}(g)$ is isometric to $\Sigma(\partial_{\text{Tits}} Y_g)$ where the suspension points are the endpoints of A_g (and hence any g -axis)
- (4) Suppose $g \in G$ where G acts geometrically on X . Then the centralizer $C(g)$ in G acts geometrically on $\text{Min}(g)$ and $C(g)/\langle g \rangle$ acts geometrically on Y (via the projected action).

If $g \in G$ where G acts geometrically on a CAT(0) space X , then any infinite order element is a hyperbolic isometry. This is the contents of [3, Proposition 6.10(2), page 233].

Any isometry g of X extends to a homeomorphism of $\partial_\infty X$ and an isometry of $\partial_{\text{Tits}} X$. We denote both of these maps by \bar{g} even though this is a slight abuse of notation. It will be clear which one of these we are talking about in context. Recall we use ∂X to denote the point set of the boundary, $\partial_\infty X$ to denote the visual boundary and $\partial_{\text{Tits}} X$ to denote the Tits boundary. A proof of the following can be found as [11, Theorem 3.2] and also in [15].

Theorem 3.9 $\text{Fix}(\bar{g}) = \partial_\infty \text{Min}(g)$.

Finally, we will need two key results from [15] that make our proof work. The first is Theorem 11 from that paper.

Theorem 3.10 *If an infinite group G acts geometrically on a CAT(0) space X , then G contains an element of infinite order.*

A subgroup H of a group G acting geometrically on a CAT(0) space X is called *convex* if there exists a closed, convex H -invariant subset A of X on which H acts geometrically. In this case the limit set ΛH is the same as $\partial_\infty A$. The following is really the key result we use in the proof. The proof of this fact from Swenson uses the full strength of a geometric group action on a CAT(0) space. In particular, the cocompactness of the actions are essential. This is [15, Theorem 16].

Theorem 3.11 *Suppose H and K are convex subgroups of a group G acting geometrically on a CAT(0) space X . Then*

- (1) $H \cap K$ is convex.
- (2) $\Lambda(H \cap K) = \Lambda H \cap \Lambda K$.

Finally, we point out an obvious corollary to the Flat Plane Theorem which will be used in the proof of Lemma 3.13.

Corollary 3.12 *Suppose G acts geometrically on the CAT(0) space X . Then $\partial_{\text{Tits}}X$ cannot be isometric to a segment of length less than or equal to π .*

Proof By Theorem 2.14, either X is δ -hyperbolic or X contains a flat plane. If X were δ -hyperbolic, $\partial_{\text{Tits}}X$ would be discrete and thus would not contain a segment. Thus X contains a flat plane which implies $\partial_{\text{Tits}}X$ contains at least a circle. \square

We are now ready to prove the Lemma.

Lemma 3.13 *The suspension points N and S in Theorem 3.7 have $Td(N, S) = \pi$.*

Proof We know $Td(N, S) \leq \pi$ since N and S lie in the boundary of a common flat. Suppose $Td(N, S) = \alpha < \pi$. Using Remark 2.9 item (3), there is a unique segment in $\partial_{\text{Tits}}X$ between N and S of length α . Since the entire group G must fix N and S , G must fix this segment in $\partial_{\text{Tits}}X$. Thus either N and S are each fixed by all of G or they are interchanged. In the second case, we can pass to a subgroup of index 2 in G which fixes both N and S . Thus we can assume each $g \in G$ fixes both N and S .

Using Theorem 3.10, we know there exists $g \in G$ of infinite order. Consider $\text{Min}(g) = Y_g \times \mathbb{R}$. By Theorem 3.8, this a closed, convex (and thus CAT(0)) subset of X on which the subgroup $C(g)$ acts geometrically with $C(g)/\langle g \rangle$ acting geometrically on Y_g . Let $\{g^{\pm\infty}\}$ denote the endpoints in $\partial_\infty X$ of an axis for g . From Theorem 3.9, we know $\partial_\infty \text{Min}(g)$ is exactly the fixed point set of \bar{g} viewed as a homeomorphism acting on $\partial_\infty X$. We also know $\partial_\infty \text{Min}(g) = \Sigma \partial_\infty Y_g$ and $\partial_{\text{Tits}} \text{Min}(g) = \Sigma(\partial_{\text{Tits}} Y_g)$ where $\{g^{\pm\infty}\}$ are the suspension points.

Since $[N, S]$ is fixed by \bar{g} , we must have $[N, S] \subset \partial_{\text{Tits}} \text{Min}(g)$. In particular, this implies $\partial Y_g \neq \emptyset$ just as a set.

Since $C(g)/\langle g \rangle$ acts geometrically on Y_g and $\partial Y_g \neq \emptyset$, Y_g has 1, 2, or infinitely many ends by a theorem of Hopf which can be found on [3, page 146]. Since $\partial_\infty X$ is 1-dimensional, we must have $\partial_\infty Y_g$ 0-dimensional, thus $\partial_\infty Y_g$ has 1, 2, or infinitely many points.

If Y_g has one end, then we would have $\text{Min}(g)$ is isometric to a half plane. This is impossible by Corollary 3.12.

If Y_g has 2 ends $\partial_\infty \text{Min}(g)$ a circle. From Theorem 3.5, we must have $\text{Min}(g)$ isometric to $\mathbb{E}^2 \times Z$ where Z is compact since $\partial_{\text{Tits}} \text{Min}(g)$ contains $[N, S]$ (ie, is not discrete). Thus $\partial_{\text{Tits}} \text{Min}(g)$ is isometric to a circle as well.

If each $h \in G$ were to leave $\text{Min}(g)$ invariant, we would have $\partial_\infty \text{Min}(g) = \partial_\infty X$ by Corollary 3.3. But then $\partial_\infty \text{Min}(g)$ is not a circle as assumed for this case. Thus let $g' = hgh^{-1}$ an element of infinite order in G with $\text{Min}(g') \neq \text{Min}(g)$. As for g , we can deduce that $\partial_{\text{Tits}} \text{Min}(g')$ is a circle containing $[N, S]$. In particular $\partial_{\text{Tits}} \text{Min}(g) \cap \partial_{\text{Tits}} \text{Min}(g')$ contains $[N, S]$. Since these are distinct circles, they can intersect in at most a segment of length π .

Using Theorem 3.11, we have that $K = C(g) \cap C(g')$ is a convex subgroup acting on a CAT(0) subset A of X . We also know $\partial_{\text{Tits}} A = \partial_{\text{Tits}} \text{Min}(g) \cap \partial_{\text{Tits}} \text{Min}(g')$ is a segment of length less than or equal to π that contains $[N, S]$. Since K acts geometrically on A , K must contain an element k of infinite order with $C_K(k)$ acting geometrically on $\text{Min}(k)$. But now $\text{Min}(k) = Y_k \times \mathbb{R}$ must have $\partial_{\text{Tits}} \text{Min}(k)$ is a segment of length less than or equal to π . Again using Corollary 3.12, this is impossible.

Thus Y_g has infinitely many ends and $\{g^{\pm\infty}\} = \{N, S\}$. Indeed, N and S are the only points in $\partial_\infty X$ which could possibly be the suspension points for $\partial_\infty \text{Min}(g)$ since all other points of $\partial_\infty X$ are points of non-local connectivity. But then N and S are the end points of a geodesic line which gives $Td(N, S) = \pi$ as needed. \square

4 Group theory consequences

The following theorem is due to M Bridson.

Theorem 4.1 *Suppose X is a proper CAT(0) space and $\text{Isom}(X)$ has discrete orbits. If a finitely generated group G acts properly discontinuously by semi-simple isometries on $X \times \mathbb{R}$, then there exists a subgroup G_0 , finite index in G , with $G_0 \cong K \times \mathbb{Z}$ (or just K) where K acts properly on X .*

Proof Any $g \in G$ acts on $X \times \mathbb{R}$ via $g = (g_X, t_g)$ where $g_X \in \text{Isom}(X)$ and $t_g \in \text{Isom}(\mathbb{R})$. Consider the action of G on the X factor given by $g \cdot x = g_X \cdot x$. If this action is proper, then we are done by taking $G_0 = G$. If this action is not proper, then there is an infinite stabilizer $\text{Stab}(x)$ (since $\text{Isom}(X)$ has discrete orbits).

This group $\text{Stab}(x)$ must act properly on $\{x\} \times \mathbb{R}$ since G acts properly on $X \times \mathbb{R}$. Since $\text{Stab}(x)$ acts properly on $L := \{x\} \times \mathbb{R}$ and is infinite, it contains a cyclic subgroup $\langle \gamma \rangle$ of finite index where γ acts as a translation.

Let $G^+ = \langle g_1, \dots, g_l \rangle$ be the subgroup of index at most 2 in G whose action on the second factor of $X \times \mathbb{R}$ preserves the orientation.

For each g_i and all $p, q \in L$ we have $d(g_i(p), p) = d(g_i(q), q)$ since L must go to a line parallel to L under the action of any isometry. In particular, letting $p_n = \gamma^n(x)$, for all $n \in \mathbb{Z}$ we have:

$$d(\gamma^{-n}g_i\gamma^n(x), x) = d(g_i(p_n), p_n) = d(g_i(x), x).$$

Since the action of G is proper, the set $\{\gamma^{-n}g_i\gamma^n \mid n \in \mathbb{Z}\}$ must be finite. Hence, for $i = 1, \dots, l$ there exists $r_i \neq 0$ such that $\gamma^{-r_i}g_i\gamma^{r_i} = g_i$. Thus if we let $R = r_1 \dots r_l$, then γ^R commutes with each of the generators of G^+ . We can now apply Theorem 2.16 to split a subgroup of finite index in G^+ as $K \times \langle \gamma^R \rangle$, where K acts trivially on the the second factor of $X \times \mathbb{R}$ and hence properly on the first factor. \square

Corollary 4.2 *If G acts geometrically on a CAT(0) space $X \times \mathbb{R}$ where $\text{Isom}(X)$ has discrete orbits, then G contains a subgroup of finite index G_0 of the form $K \times \mathbb{Z}$ where K acts geometrically on X .*

Proof As above, consider the induced action of G on the X factor. If this action is proper, then G would act geometrically on X as well as on $X \times \mathbb{R}$. Indeed, if the G translates of a compact set C in $X \times \mathbb{R}$ cover $X \times \mathbb{R}$, then the G translates of the projection of C onto X will cover X under the induced G -action on X . If G acts geometrically on X , then we could build a geometric action of $G \times \mathbb{Z}$ on $X \times \mathbb{R}$ using the product action. This would force G to be quasi-isometric to $G \times \mathbb{Z}$. This cannot happen because these two groups have different cohomological properties. See the next lemma for details.

Now by the previous theorem G contains a subgroup of finite index of the form $K \times \mathbb{Z}$ with K acting properly on X . It follows that K must also act cocompactly on X since G acts cocompactly on $X \times \mathbb{R}$. \square

Lemma 4.3 *Suppose G acts geometrically on a CAT(0) space X . Then G and $G \times \mathbb{Z}$ cannot be quasi-isometric.*

Proof Since G acts geometrically on X , we also know that $G \times \mathbb{Z}$ acts geometrically on $X \times \mathbb{R}$ just by considering the obvious product action. We can apply [15, Theorem 12] to conclude that $\partial_\infty X$ has finite Lebesgue covering dimension, call that dimension d . Using this d , a consequence of the main theorem of [6] is that since G acts geometrically on X , $H^{d+1}(G, \mathbb{Z}G) \neq 0$, while $H^n(G, \mathbb{Z}G) = 0$ for all $n > d + 1$. For all k we know $H^{k+1}(G \times \mathbb{Z}, \mathbb{Z}(G \times \mathbb{Z})) \cong H^k(G, \mathbb{Z}G)$ for all k . Thus $H^{d+2}(G \times \mathbb{Z}, \mathbb{Z}(G \times \mathbb{Z}))$ is non-zero while $H^{d+2}(G, \mathbb{Z}G) = 0$. Finally, since $H^*(G, \mathbb{Z}G)$ is a q.-i. invariant, [7], G and $G \times \mathbb{Z}$ cannot be quasi-isometric. \square

Theorem 4.4 *Suppose G acts geometrically on a CAT(0) space X with $\partial_\infty X$ homeomorphic to the suspension of a Cantor set. Then G contains a subgroup G_0 of finite index which is isomorphic to $F \times \mathbb{Z}$ where F is a nonabelian free group.*

Proof From Theorem 3.7, we know X splits isometrically as $Y \times \mathbb{R}$ for some closed convex subset Y of X with $\partial_\infty Y$ homeomorphic to a Cantor set and $\partial_{\text{Tits}} Y$ discrete. By Corollary 4.2, we know that up to finite index G splits as $K \times \mathbb{Z}$ where K acts geometrically on Y . Thus Y is a proper, cocompact CAT(0) space so we can apply the Flat Plane Theorem to Y . If there were a flat plane in Y , then the Tits metric on Y would not be discrete, and thus Y must be δ -negatively curved. Since K acts geometrically on Y , K is a δ -hyperbolic group.

But now K is a δ -hyperbolic group acting geometrically on a space Y with Cantor set boundary. Thus K is virtually free by work of Stallings, Gromov, Ghys and de la Harpe [13; 9; 8]. \square

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