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A geometric proof that $\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$ is not finitely presented

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We give a new proof of the theorem of Krstić–McCool from the title. Our proof has potential applications to the study of finiteness properties of other subgroups of SL_2 resulting from rings of functions on curves.

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1 Introduction

Our main result is a strengthening of the theorem of Krstić–McCool from the title.

Proposition A *The group* $\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$ *is not finitely presented, indeed it is not even of type* FP_2.

It will be clear from our proof that $\mathbb{Z}$ can be replaced in Proposition A with any ring of integers in an algebraic number field. Note that the theorem of Krstić–McCool [5] also allows for this replacement as well as for many other generalizations of the ring $\mathbb{Z}[t,t^{-1}]$, which include in particular any ring of the form $J[t,t^{-1}]$ where J is an integral domain.

Let us recall the definition of type FP_2.

Type FP_s A group Γ is of *type* FP_s if $\mathbb{Z}$, regarded as a $\mathbb{Z}\Gamma$–module via the trivial action, admits a partial projective resolution

$$P_s \to P_{s-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

by finitely generated $\mathbb{Z}\Gamma$–modules P_i.

Every group is of type FP_0. Type FP_1 is equivalent to the property of finite generation. Every finitely-presented group is of type FP_2, but Bestvina–Brady showed the converse does not hold in general [1, Example 6.3(3)].

Purpose In [4], we studied finiteness properties of subgroups of linear reductive groups arising from rings of functions on algebraic curves defined over finite fields.

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For example, we showed that $\mathrm{SL}_n(\mathbb{F}_q[t])$ is not of type FP_{n-1} and $\mathrm{SL}_n(\mathbb{F}_q[t,t^{-1}])$ is not of type $\mathrm{FP}_{2(n-1)}$ where $\mathbb{F}_q$ is a finite field.

We wrote this paper to show how the techniques in [4] might be applied to a more general class of groups.

In this paper we stripped down the general proof of the main result from [4] to the special case of showing that $\mathrm{SL}_2(\mathbb{F}_q[t,t^{-1}])$ is not of type FP_2, and then made some modest alterations until we arrived at the proof of Proposition A presented below.

It seems likely that more results along these lines can be proved, but it is not clear to us how much the results in [4] can be generalized. Below we phrase a question that seems a good place to start.

Rings of functions on curves Let C be an irreducible smooth projective curve defined over an algebraically closed field k. We let $k(C)$ be the field of rational functions defined on C, and we denote the set of nonzero elements of this field by $k(C)^*$.

For each point $x \in C$, there is a discrete valuation $v_x\colon k(C)^* \to \mathbb{Z}$ that assigns to any nonzero function f on C its vanishing order at x. Formally, we extend v_x to all of $k(C)$ by $v_x(0) := \infty$.

We let $S_1, S_2, \ldots, S_m \subseteq C$ be collections of pairwise disjoint finite nonempty sets of closed points in C. We call a ring $R \leq k(C)$ containing some nonconstant function and the constant function 1 an *m–place ring* if the following two conditions are satisfied:

(1) For all $f \in R$ and all $x \in C - \left(\bigcup_{i=1}^m S_i\right)$, we have $v_x(f) \geq 0$.

(2) If there is an i, an $x \in S_i$, and an $f \in R$ such that $v_x(f) < 0$, then $v_y(f) < 0$ for all $y \in S_i$.

For example, if $\mathbb{P}^1$ is the projective line, then $k(\mathbb{P}^1)$ is isomorphic to the field $k(t)$ of rational functions in one variable. Thus, if J is a subring of k, then $J[t] \leq k(\mathbb{P}^1)$ is a 1–place ring with $S_1 = \{\infty\}$, while $J[t,t^{-1}]$ is a 2–place ring with $S_1 = \{\infty\}$ and $S_2 = \{0\}$. For an example of a 1–place ring R that obeys condition 2 nontrivially, we can take $R = \mathbb{Z}\left[\frac{1}{t^2-2}\right] \leq \mathbb{C}(t)$ with $S_1 = \left\{\sqrt{2}, -\sqrt{2}\right\}$.

Note that the definition of an m–place ring is a generalization of the definition of a ring of S–integers of a global function field.

Finiteness properties of linear groups We ask the following question:

Question B *Is there an example of an m–place ring R such that $\mathrm{SL}_n(R)$ is of type $\mathrm{FP}_{m(n-1)}$?*

Specifically, is there an $n \geq 2$ such that $SL_n(\mathbb{Z}[t])$ is of type FP_{n-1} or such that $SL_n(\mathbb{Z}[t,t^{-1}])$ is of type $FP_{2(n-1)}$?

There seems to be no known example as above, though relatively few candidates have been examined for this property. Krstić–McCool [5; 6] proved that $SL_2(J[t,t^{-1}])$ and $SL_3(J[t])$ are not finitely presented for any integral domain J. In [4], we prove that there exist no examples when R is a ring of S–integers of a global function field. Examples of such rings include $\mathbb{F}_q[t]$ and $\mathbb{F}_q[t,t^{-1}]$.

We also know that there are no examples as asked for in Question B when $m = 1$ and $n = 2$. We give a proof of this fact in Section 4. This is an easy result, but as this general problem has not been studied extensively, it appears not to have been stated in this form in the literature.

About the proof Our proof of Proposition A is geometric in that it employs the action of $SL_2(\mathbb{Z}[t,t^{-1}])$ on a product of two Bruhat–Tits trees. It is essentially a special case of our proof that arithmetic subgroups of SL_n over global function fields are not of type FP_∞ [4]. The proof uses a result of K. Brown's which requires the action to have "nice" stabilizers. Unfortunately, the stabilizer types of $SL_2(R)$ are unknown to us for many of the more interesting 2–place rings R. This prevents us from applying our proof to groups other than $SL_2(\mathcal{O}[t,t^{-1}])$ where $\mathcal{O}$ is the ring of integers in an algebraic number field.

Other finiteness properties As an aside, we point out a few loosely related facts. In [6], Krstić–McCool showed that $SL_3(J[t])$ is not finitely presented for any integral domain J. Suslin proved in [9] that $SL_n(\mathbb{Z}[t])$ and $SL_n(\mathbb{Z}[t,t^{-1}])$ are finitely generated by elementary matrices when $n \geq 3$. It is not known whether $SL_2(\mathbb{Z}[t,t^{-1}])$ is also generated by elementary matrices. In fact, even finite generation is an open problem for this group.

Homology Our proof of Proposition A can be seen as a variant of Stuhler's proof [8] that $SL_2(\mathbb{F}_q[t,t^{-1}])$ is not of type FP_2. As Stuhler's proof establishes the stronger fact that the second homology $H_2(SL_2(\mathbb{F}_q[t,t^{-1}]);\mathbb{Z})$ is infinitely generated, it is natural to wonder if the proof of Proposition A below can be extended to show that $H_2(SL_2(\mathbb{Z}[t,t^{-1}]);\mathbb{Z})$ is infinitely generated.

Type F_s We will not use type F_s in this paper, but as it is related to type FP_s, we recall its definition here.

A group Γ is of *type* F_s if there exists an Eilenberg–Mac Lane complex $K(\Gamma,1)$ with finite s–skeleton. For $s \geq 2$, a group is of type F_s if and only if it is finitely presented and of type FP_s. In general, type F_s is stronger than type FP_s.

Outline of the paper In Section 2, we present the main body of the proof of Proposition A, leaving the verification that cell stabilizers are well-behaved for Section 3. In Section 4, we comment on Question B.

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2 The action on a product of trees

Let v_∞ be the degree valuation on $\mathbb{Q}(t)$ given by

$$v_\infty\left(\frac{r(t)}{s(t)}\right) = \deg(s(t)) - \deg(r(t)),$$

and let v_0 be the valuation at 0, that is, the valuation corresponding to the irreducible polynomial $t \in \mathbb{Q}[t]$. Thus

$$v_0\left(\frac{r(t)}{s(t)}t^n\right) = n$$

if t does not divide $r(t)$ nor $s(t)$.

Let T_∞ (resp. T_0) be the Bruhat–Tits tree associated to $\mathrm{SL}_2(\mathbb{Q}(t))$ with the valuation v_∞ (resp. v_0). We consider these trees as metric spaces by assigning a length of 1 to each edge. For a definition as well as for many of the facts we will use in this proof, we refer to Serre's book on trees [7].

Outline We put

$$X := T_\infty \times T_0,$$

and we let $\mathrm{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big)$ act diagonally on X.

We will begin by finding an $\mathrm{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big)$–invariant cocompact subspace $X_0 \subseteq X$. Then for each $n \in \mathbb{N}$, we will construct a 1–cycle γ_n in X_0 with the property that for any $\mathrm{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big)$–invariant cocompact subspace $Y \subseteq X$ containing X_0, there exists some $n \in \mathbb{N}$ such that γ_n represents a nontrivial element of the first homology group $\mathrm{H}_1(Y)$.

A direct application of K. Brown's filtration criterion then shows that $\mathrm{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big)$ is not of type FP_2 as long as the cell stabilizers of the $\mathrm{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big)$–action on X are not of type FP_2. We leave the verification of this last fact for Section 3.

Finding a cocompact subspace A crucial part of our construction will take place in a flat plane inside X, which we shall describe now.

Let $\mathcal{O}_\infty \leq \mathbb{Q}(t)$ be the valuation ring associated to v_∞, that is, the ring of all $f \in \mathbb{Q}(t)$ with $v_\infty(f) \geq 0$. Let $L_\infty \subseteq T_\infty$ be the unique bi-infinite geodesic stabilized by the diagonal subgroup of $\mathrm{SL}_2(\mathbb{Q}(t))$. We parameterize L_∞ by an isometry $l_\infty\colon \mathbb{R} \to L_\infty$ such that $l_\infty(0)$ is the unique vertex stabilized by $\mathrm{SL}_2(\mathcal{O}_\infty)$ and such that the end corresponding to the positive reals is fixed by all upper triangular matrices in $\mathrm{SL}_2(\mathbb{Q}(t))$. Analogously, we define $l_0\colon \mathbb{R} \to L_0$. The plane we shall consider is the product

$$L_\infty \times L_0.$$

We define a diagonal matrix $D \in \mathrm{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big)$ by:

$$D := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Note that for any $n \in \mathbb{Z}$, we have

$$D^n \cdot (l_\infty(0)\,, l_0(0)) = (l_\infty(2n)\,, l_0(-2n)).$$

Hence, if we denote by V the line in $L_\infty \times L_0$ of the form $\{(l_\infty(s)\,, l_0(-s))\}_{s\in\mathbb{R}}$, then V has a compact image under the quotient map

$$\pi\colon X \longrightarrow X/\,\mathrm{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big).$$

Note that

$$X_0 := \pi^{-1}(\pi(V)) \subseteq X$$

is an $\mathrm{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big)$–invariant cocompact subspace.

A family of loops in X_0 For any $n \in \mathbb{Z}$, we define the unipotent matrix $U_n \in \mathrm{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big)$ as

$$U_n = \begin{pmatrix} 1 & t^n \\ 0 & 1 \end{pmatrix}.$$

Note that U_n fixes a point of the form $(l_\infty(s)\,, l_0(s')) \in L_\infty \times L_0$ if and only if $s \geq n$ and $s' \geq -n$. Moreover, any points in the plane $L_\infty \times L_0$ that are not fixed by U_n are moved outside of $L_\infty \times L_0$.

For all $n \in \mathbb{N}$, we define the geodesic segment $\sigma_n \subseteq V$ to be the segment with endpoints $(l_\infty(-n)\,, l_0(n))$ and $(l_\infty(n)\,, l_0(-n))$. Note that U_n fixes the endpoint of σ_n given by $(l_\infty(n)\,, l_0(-n))$ whereas U_{-n} fixes its other endpoint $(l_\infty(-n)\,, l_0(n))$. Since U_n and U_{-n} commute, the union of geodesic segments

$$\gamma_n := \sigma_n \cup (U_n \cdot \sigma_n) \cup (U_{-n} \cdot \sigma_n) \cup (U_n U_{-n} \cdot \sigma_n)$$

is a loop. Note that $\gamma_n \subseteq X_0$.

How the loops can be filled It is easy to describe a filling disc for γ_n in X. Just let Δ_n be the closed triangle with geodesic sides and vertices at the endpoints of σ_n and at the point $(l_\infty(n), l_0(n))$, which is fixed by both U_n and U_{-n}. Then we define C_n to be the union of triangles

$$C_n := \Delta_n \cup (U_n \cdot \Delta_n) \cup (U_{-n} \cdot \Delta_n) \cup (U_n U_{-n} \cdot \Delta_n).$$

Since X is a 2–complex, it does not allow for simplicial 3–chains (using any appropriate simplicial decomposition of X). Since X is contractible, it follows that there are no nontrivial simplicial 2–cycles. Hence, there is a unique 2–chain bounding γ_n, and this consists of the simplices forming C_n. Since $(l_\infty(n), l_0(n)) \in C_n$, we have:

Lemma 2.1 *Each loop $\gamma_n \subseteq X_0$ represents a nontrivial class in the first homology group of $X - \{(l_\infty(n), l_0(n))\}$.*

Note how our proof relies on the commutator relations $U_n U_{-n} = U_{-n} U_n$ that were also essential in the argument of Krstić–McCool [5].

An unbounded sequence in the quotient We will need to know that the points $(l_\infty(n), l_0(n))$ move farther and farther away from X_0. We will use this to show that for any $\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$–invariant cocompact subspace $Y \subseteq X$ containing X_0, there exists some $n \in \mathbb{N}$ such that γ_n represents a nontrivial element of the first homology group $\mathrm{H}_1(Y)$.

Actually, it suffices to prove our claim for "half of the points":

Lemma 2.2 *The sequence $\{\pi((l_\infty(2n), l_0(2n)))\}_{n\in\mathbb{N}}$ is unbounded in the quotient space $X/\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$.*

Proof Note that $\mathrm{SL}_2(\mathbb{Q}(t)) \times \mathrm{SL}_2(\mathbb{Q}(t))$ acts on $T_\infty \times T_0$ componentwise and recall that the valuations v_∞ and v_0 define a metric on $\mathrm{SL}_2(\mathbb{Q}(t)) \times \mathrm{SL}_2(\mathbb{Q}(t))$ so that vertex stabilizers are bounded subgroups. Thus, to prove that a set of vertices in the quotient $X/\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$ is not bounded, it suffices to prove that it has an unbounded preimage under the canonical projection

$$\big(\mathrm{SL}_2(\mathbb{Q}(t)) \times \mathrm{SL}_2(\mathbb{Q}(t))\big)/\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}]) \longrightarrow X/\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$$

where $\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$ is embedded diagonally in $\mathrm{SL}_2(\mathbb{Q}(t)) \times \mathrm{SL}_2(\mathbb{Q}(t))$.

Put $A := (D, D^{-1}) \in \mathrm{SL}_2(\mathbb{Q}(t)) \times \mathrm{SL}_2(\mathbb{Q}(t))$, and observe that

$$A^n \cdot (l_\infty(0), l_0(0)) = (l_\infty(2n), l_\infty(2n)).$$

As we have argued, it suffices to prove that the sequence $\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])\, A^n$ is unbounded in $\mathrm{SL}_2(\mathbb{Q}(t)) \times \mathrm{SL}_2(\mathbb{Q}(t))$ modulo $\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$. So assume, for a contradiction, this sequence is bounded. By definition, this means that there is a global constant C satisfying the following condition:

> For any $n \in \mathbb{N}$, there is a matrix $M_n = \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$ such that the values of v_∞ of the coefficients of $M_n D^n$ are bounded from below by C and the values of v_0 of the coefficients of $M_n D^{-n}$ are also bounded from below by C.

Recall that $D = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ and that $v_\infty(t) = -1$ whereas $v_0(t) = 1$. Since

$$C \le v_\infty(a_n t^n) = v_\infty(a_n) + n v_\infty(t) = v_\infty(a_n) - n$$

and

$$C \le v_0(a_n t^{-n}) = v_0(a_n) - n v_0(t) = v_0(a_n) - n$$

we find that $v_\infty(a_n) \ge 1$ and $v_0(a_n) \ge 1$ whenever $n \ge 1 - C$, which implies $a_n = 0$. However, the same argument shows $c_n = 0$, for $n \ge 1 - C$. But then, $M_{1-C} \notin \mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$. □

Brown's criterion The following is an immediate consequence of [3, Theorem 2.2].

Lemma 2.3 *Suppose a group* Γ *acts by cell-permuting homeomorphisms on a contractible CW–complex* X *such that stabilizers of* d *–cells are of type* FP_{s+1-d}. *Assume that* X *admits a filtration*

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X = \bigcup_{j \in \mathbb{N}} X_j$$

by Γ*–invariant, cocompact subcomplexes* X_j. *Then* Γ *is not of type* FP_{s+1} *if each of the reduced homology homomorphisms*

$$\widetilde{\mathrm{H}}_s(X_0) \longrightarrow \widetilde{\mathrm{H}}_s(X_j)$$

is nontrivial.

In the following section, we will verify that cell stabilizers of the $\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$–action on X are of type FP_∞. Assuming this hypothesis for the moment, we can finish the proof of Proposition A as follows:

Proof of Proposition A The family of loops γ_n is contained within the cocompact subspace X_0, which is a subcomplex of (a suitable subdivision) of X. Since the quotient $X/\operatorname{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big)$ has countably many cells, we can extend X_0 to a filtration

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X$$

of X by $\operatorname{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big)$–invariant, cocompact subcomplexes X_j.

By Lemma 2.2, for each index j there is a natural number n such that

$$X_j \subseteq X - \{(l_\infty(n)\,,l_0(n))\}\,.$$

Therefore, by Lemma 2.1, γ_n represents a nontrivial class in $\widetilde{\mathrm{H}}_1\big(X_j\big)$, thus showing that

$$\widetilde{\mathrm{H}}_1(X_0) \longrightarrow \widetilde{\mathrm{H}}_1\big(X_j\big)$$

is nontrivial. By Lemma 2.3, $\operatorname{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big)$ is not of type FP_2. □

3 Finiteness properties of cell stabilizers

It remains to verify the hypothesis about cell stabilizers. Borel and Serre [2, 11.1] have shown that arithmetic groups are of type F_∞. Therefore, the following lemma proves what we need, and more:

Lemma 3.1 *The cell stabilizers of the $\operatorname{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big)$-action on X are arithmetic groups.*

This section is devoted entirely to the proof of this lemma.

Observation 3.2 *The set $\mathcal{B} := \{t^n \mid n \in \mathbb{Z}\}$ is a $\mathbb{Q}$–vector space basis for $\mathbb{Q}[t,t^{-1}]$ such that the subring $\mathbb{Z}[t,t^{-1}]$ consists precisely of those elements in $\mathbb{Q}[t,t^{-1}]$ whose coefficients with respect to $\mathcal{B}$ are all in $\mathbb{Z}$.*

Stabilizers of standard vertices We fix the following family of *standard vertices* in X. For $j \in \mathbb{N}$, put

$$\mathbf{x}_j := (l_\infty(j)\,,l_0(0)).$$

Recall that $\operatorname{SL}_2(\mathbb{Q}(t))$ acts on the tree T_∞. The vertex $l_\infty(j) \in T_\infty$ has the stabilizer

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}_2(\mathbb{Q}(t)) \;\middle|\; v_\infty(a)\,,v_\infty(d) \geq 0;\ v_\infty(b) \geq -j;\ v_\infty(c) \geq j \right\}.$$

Thus, the stabilizer

$$\operatorname{Stab}_{\mathbb{Q}[t,t^{-1}]}\big(\mathbf{x}_j\big)$$

of the vertex $\mathbf{x}_j$ under the diagonal $\mathrm{SL}_2\big(\mathbb{Q}[t,t^{-1}]\big)$-action on $X = T_\infty \times T_0$ is

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\Big(\mathbb{Q}[t,t^{-1}]\Big) \;\middle|\; \begin{array}{l} v_\infty(a)\,,v_\infty(d) \geq 0;\;\; v_\infty(b) \geq -j;\;\; v_\infty(c) \geq j \\ v_0(a)\,,v_0(b)\,,v_0(c)\,,v_0(d) \geq 0 \end{array} \right\},$$

which is an affine algebraic $\mathbb{Q}$–group: Because of the bounds on the valuations v_∞ and v_0, each matrix in $\mathrm{Stab}_{\mathbb{Q}[t,t^{-1}]}\big(\mathbf{x}_j\big)$ can be considered as a 4–tuple (a,b,c,d) in the finite dimensional $\mathbb{Q}$–vector space $V_0 \times V_j \times V_{\bar{j}} \times V_0$ where

$$V_j := \left\{ \sum_{i=0}^{j} q_i t^i \;\middle|\; q_i \in \mathbb{Q} \right\}, \qquad V_{\bar{j}} := \begin{cases} \mathbb{Q} & \text{for } j = 0 \\ \{0\} & \text{for } j > 0. \end{cases}$$

The requirement that the determinant be 1 translates into a system of algebraic equations defining an affine variety in $V_0 \times V_j \times V_{\bar{j}} \times V_0$. This variety is an affine $\mathbb{Q}$–group by means of matrix multiplication.

Note that the vector space V_j carries an integral structure: the lattice of integer points is $\big\{\sum_{i=0}^{j} q_i t^i \;\big|\; q_i \in \mathbb{Z}\big\}$. Thus, the stabilizer $\mathrm{Stab}_{\mathbb{Z}[t,t^{-1}]}\big(\mathbf{x}_j\big)$ of $\mathbf{x}_j$ in $\mathrm{SL}_2\big(\mathbb{Z}[t,t^{-1}]\big)$ is the arithmetic subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\Big(\mathbb{Z}[t,t^{-1}]\Big) \;\middle|\; \begin{array}{l} v_\infty(a)\,,v_\infty(d) \geq 0;\;\; v_\infty(b) \geq -j;\;\; v_\infty(c) \geq j \\ v_0(a)\,,v_0(b)\,,v_0(c)\,,v_0(d) \geq 0 \end{array} \right\}.$$

The idea of the proof is to push this result forward to other vertices.

Other vertices are translates We claim that every vertex $\mathbf{y} = (y_\infty, y_0) \in X$ can be written as $M \cdot \mathbf{x}_j$ for some $M \in \mathrm{Gl}_2\big(\mathbb{Q}[t,t^{-1}]\big)$ and some $j \in \mathbb{N} \cup \{0\}$.

To see this, we will use that the ray

$$F_0 := l_0(0) \text{ --- } l_0(1) \text{ --- } l_0(2) \text{ --- } l_0(3) \text{ --- } \cdots$$

is a fundamental domain for the action of $\mathrm{SL}_2\big(\mathbb{Q}[t^{-1}]\big)$ on T_0. This follows from the discussion in Serre [7, page 86f] and the fact that $t \mapsto t^{-1}$ induces a ring automorphism of $\mathbb{Q}[t,t^{-1}]$ that interchanges $\mathbb{Q}[t]$ and $\mathbb{Q}[t^{-1}]$.

The matrix $\left(\begin{smallmatrix} t^k & 0 \\ 0 & 1 \end{smallmatrix}\right)$ translates $l_0(k)$ to $l_0(0)$ as t is a uniformizing element for the valuation v_0. Thus, within two moves, we can adjust the second coordinate of $\mathbf{y}$ to $l_0(0)$.

Now, we consider $\mathbb{Q}[t]$. In this case, the discussion in Serre [7] applies directly: the ray

$$F_\infty := l_\infty(0) \text{ --- } l_\infty(1) \text{ --- } l_\infty(2) \text{ --- } l_\infty(3) \text{ --- } \cdots$$

is a fundamental domain in T_∞ for the action of $\mathrm{SL}_2(\mathbb{Q}[t])$. This allows us to adjust the first coordinate. Note that every matrix in $\mathrm{SL}_2(\mathbb{Q}[t])$ fixes the vertex $l_0(0) \in T_0$. Thus, we do not change the second coordinate during the third and final move.

We conclude:

Lemma 3.3 *Every vertex stabilizer in* $\mathrm{SL}_2\left(\mathbb{Z}[t,t^{-1}]\right)$ *is of the form*

$$M\ \mathrm{Stab}_{\mathbb{Q}[t,t^{-1}]}(\mathbf{x}_j)\ M^{-1} \cap \mathrm{SL}_2\left(\mathbb{Z}[t,t^{-1}]\right)$$

for some j *and some matrix* $M \in \mathrm{Gl}_2\left(\mathbb{Q}[t,t^{-1}]\right)$.

We also make the following:

Observation 3.4 *Since multiplication by* M *can lower valuations only by a bounded amount, we can find* $N \in \mathbb{N}$ *such that*

$$M\ \mathrm{Stab}_{\mathbb{Q}[t,t^{-1}]}(\mathbf{x}_j)\ M^{-1} \subseteq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\left(\mathbb{Q}[t,t^{-1}]\right) \;\middle|\; a,b,c,d \in W_N \right\}$$

where $W_N := \left\{ \sum_{i=-N}^{N} q_i t^i \;\middle|\; q_i \in \mathbb{Q} \right\}$.

Finite dimensional approximations We want to use Observation 3.4 and argue that $M\ \mathrm{Stab}_{\mathbb{Q}[t,t^{-1}]}(\mathbf{x}_j)\ M^{-1}$ is an affine $\mathbb{Q}$–group with arithmetic subgroup

$$M\ \mathrm{Stab}_{\mathbb{Q}[t,t^{-1}]}(\mathbf{x}_j)\ M^{-1} \cap \mathrm{SL}_2\left(\mathbb{Z}[t,t^{-1}]\right).$$

This is accomplished as follows.

Lemma 3.5 *Fix* $N \in \mathbb{N}$ *and let* $\mathbf{G}$ *be a* $\mathbb{Q}$*–subvariety of the affine* $\mathbb{Q}$*–variety* $\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2\left(\mathbb{Q}[t,t^{-1}]\right) \;\middle|\; a,b,c,d \in W_N \right\}$. *Assume that* $\mathbf{G}$ *is a* $\mathbb{Q}$*–group with respect to matrix multiplication. Then* $\mathbf{G} \cap \mathrm{SL}_2\left(\mathbb{Z}[t,t^{-1}]\right)$ *is an arithmetic subgroup of* $\mathbf{G}$.

Proof The integer points in W_N are $W_N \cap \mathbb{Z}[t,t^{-1}]$. Thus the integer points in $\mathbf{G}$ are $\mathbf{G} \cap \mathrm{SL}_2\left(\mathbb{Z}[t,t^{-1}]\right)$. □

We note that Lemma 3.5 and Observation 3.4 imply:

Corollary 3.6 *All vertex stabilizers in* $\mathrm{SL}_2\left(\mathbb{Z}[t,t^{-1}]\right)$ *are arithmetic groups.*

Extending the argument to cell stabilizers So far we have argued that vertex stabilizers are arithmetic. To extend this argument to stabilizers of cells of higher dimension, note that the action of $\mathrm{SL}_2\big(\mathbb{Q}[t,t^{-1}]\big)$ on X is type-preserving. Hence the stabilizer of a cell is the intersection of the stabilizers of its vertices. To recognize such a group as arithmetic using the above method, we just have to choose N large enough to accommodate for all the involved vertex stabilizers simultaneously. This concludes the proof of Lemma 3.1.

4 Comments on Question B

We shall begin with answering Question B when $m=1$ and $n=2$.

Proposition 4.1 *If R is a 1–place ring, then $\mathrm{SL}_2(R)$ is not finitely generated.*

Proof By our hypothesis on R, there is an algebraically closed field k, and an irreducible smooth projective curve C defined over k such that R is a subring of the field of rational functions $k(C)$.

Let $S_1 \subseteq C$ be the finite set of closed points given in the definition of R as a 1–place ring, and pick some $x \in S_1$. We let T be the Bruhat–Tits tree for $\mathrm{SL}_2(k(C))$ with the valuation v_x. We regard T as a metric space by assigning unit length to all edges.

Denote the geodesic in T corresponding to the diagonal subgroup of $\mathrm{SL}_2(k(C))$ by L, and parameterize L by an isometry $l\colon \mathbb{R} \to L$ such that the end of L corresponding to the positive reals is fixed by upper-triangular matrices.

It follows from the definition of a 1–place ring, that there exists an element $f \in R$ such that $v_x(f) < 0$. We use this element to define for each $n \in \mathbb{N}$ a matrix

$$U_n := \begin{pmatrix} 1 & f^n \\ 0 & 1 \end{pmatrix}$$

Note that for sufficiently large n, there is an $s_n > 0$ such that

$$U_n \cdot l([0,s_n]) \cap l([0,s_n]) = \{l(s_n)\}\,.$$

Note also that $s_n = -n v_x(f) + a$ for some $a \in \mathbb{R}$.

We claim that for any $r>0$, the r–metric neighborhood of the orbit $\mathrm{SL}_2(R)\cdot l(0) \subseteq T$ is not connected. Indeed, for large n, the unique path between $l(0)$ and $U_n \cdot l(0)$ contains $l(s_n)$, thus it suffices to show that $\mathrm{SL}_2(R)\cdot l(s_n)$ is an unbounded sequence in the quotient space $T/\mathrm{SL}_2(R)$.

Observe that for each $n \in \mathbb{N}$, the diagonal matrix

$$D_n := \begin{pmatrix} f^n & 0 \\ 0 & f^{-n} \end{pmatrix}$$

acts by translations on L and that $D_n \cdot l(0) = l(-2nv_x(f))$. Thus, to prove our claim it suffices to show that $\mathrm{SL}_2(R)\, D_n \cdot l(0)$ is an unbounded sequence in $T/\,\mathrm{SL}_2(R)$.

Since point stabilizers in $\mathrm{SL}_2(k(C))$ are bounded, we can further reformulate our task as showing the sequence $\mathrm{SL}_2(R)\, D_n$ is unbounded in $\mathrm{SL}_2(k(C))\,/\,\mathrm{SL}_2(R)$. For this, we will employ a proof by contradiction: Assuming that $\mathrm{SL}_2(R)\, D_n$ is bounded, there exist matrices

$$M_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \mathrm{SL}_2(R)$$

such that the image of the matrix entries of $M_n D_n$ under the valuation v_x are bounded from below by a constant C. In particular,

$$C \leq v_x\big(f^n a_n\big) = n v_x(f) + v_x(a_n)\,.$$

Since $v_x(f) < 0$, it follows that $v_x(a_n) > 0$ for all but finitely many n. Combining conditions (1) and (2) of the definition of a 1–place ring, $v_y(a_n) \geq 0$ for all $y \in C$. Therefore, a_n is a constant function on C. As $v_x(a_n) > 0$, we conclude that $a_n = 0$. Similarly, $c_n = 0$ for sufficiently large n which contradicts that M_n is invertible. We have completed our proof of the claim that for any $r > 0$, the r–metric neighborhood of the orbit $\mathrm{SL}_2(R) \cdot l(0) \subseteq T$ is not connected.

Proposition 4.1 now follows from an application of the following lemma. □

Lemma 4.2 *Suppose a finitely generated group Γ acts on a geodesic metric space X. Then, for any point $x \in X$, there is a number $r > 0$ such that the metric r–neighborhood of the orbit of $\Gamma \cdot x \subseteq X$ is connected.*

Proof Let $\{\xi_1, \xi_2, \ldots, \xi_s\}$ be a finite generating set for Γ. Choose r such that the ball $\mathrm{B}_r(x)$ contains all translates $\xi_i \cdot x$. Then $\Gamma \cdot \mathrm{B}_r(x) = \mathrm{Nbhd}_r(\Gamma \cdot x)$ is connected. □

The question of $\mathbf{FP_2}$ After modest adjustments, the proofs in Section 2 apply to $\mathrm{SL}_2(R)$ for many other 2–place rings R. Thus, the only obstruction to substituting one of these groups for $\mathrm{SL}_2\big(\mathbb{Z}[t, t^{-1}]\big)$ in the proof of Proposition A is proving results about finiteness properties of stabilizers as in Section 3.

Certainly there are more 2–place rings that produce stabilizers of type FP_2 than the rings $\mathcal{O}[t,t^{-1}]$ where $\mathcal{O}$ is a ring of integers in an algebraic number field, but this is not the case for all 2–place rings. For instance, this is clearly not the case for any uncountable ring R. For a countable example, consider $\mathbb{Z}[s,t,t^{-1}]$ as the 2–place ring contained in $\overline{\mathbb{C}(s)}(\mathbb{P}^1) \cong \overline{\mathbb{C}(s)}(t)$ where $\overline{\mathbb{C}(s)}$ is the algebraic closure of the field $\mathbb{C}(s)$ (we take $S_1 := \{0\}$ and $S_2 := \{\infty\}$). Then the stabilizer in $\mathrm{SL}_2\big(\mathbb{Z}[s,t,t^{-1}]\big)$ of the "standard vertex" $\mathbf{x}_0$ in the product of Bruhat–Tits trees corresponding to valuations at 0 and ∞ is equal to $\mathrm{SL}_2(\mathbb{Z}[s])$ and thus is not finitely generated by Proposition 4.1 since $\mathbb{Z}[s]$ is a 1–place ring.

The question of higher finiteness properties Note that the results of Section 3 can easily be extended to the groups $\mathrm{SL}_n(\mathbb{Z}[t])$ and $\mathrm{SL}_n\big(\mathbb{Z}[t,t^{-1}]\big)$. Thus, the complication in extending our proof of Proposition A to these groups lies in generalizing the material of Section 2.

Of course, for the general m–place ring R and for $n > 2$, most of the details of this paper cannot be easily extended to $\mathrm{SL}_n(R)$.

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