Cohomology of preimages with local coefficients

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Let M, N and $B \subset N$ be compact smooth manifolds of dimensions $n + k, n$ and ℓ, respectively. Given a map $f: M \to N$, we give homological conditions under which $g^{-1}(B)$ has nontrivial cohomology (with local coefficients) for any map g homotopic to f. We also show that a certain cohomology class in $H^j(N, N-B)$ is Poincaré dual (with local coefficients) under f^* to the image of a corresponding class in $H_{n+k-j}(f^{-1}(B))$ when f is transverse to B. This generalizes a similar formula of D Gottlieb in the case of simple coefficients.

55M20; 55S35

1 Preliminaries

In 1978, D Gottlieb [8] proved a beautiful formula which, in particular for fibrations, relates the homology fundamental class of the fiber with the cohomology fundamental class of the base. More precisely, let $f : M \to N$ be a smooth map where M and N are closed orientable smooth manifolds. Let $y \in N$ be a regular value for f and let $F = f^{-1}(y)$ and consider $i_1 \colon F \hookrightarrow M$. If $[z_F]$ and $[z_M]$ denote the homology fundamental classes of F and of M respectively, and μ_N denotes the cohomology fundamental class of N, then Gottlieb [8] proved that

$$(1\text{--}1) \qquad f^*(\mu_N) \frown [z_M] = i_{1_*}([z_F]).$$

According to A Dold [8], this fact "is a special case of a vague principle known to Hopf, which says that if a cycle and a cocycle are related by Poincaré duality in N, then $f^{-1}(\text{cycle})$ and $f^*(\text{cocycle})$ should be related by Poincaré duality in M." In fact, a similar relation was established by H Samelson in [11] as follows. Suppose $F \subset M$ and M are closed connected oriented manifolds of dimensions k and $n + k$ respectively, and ν and μ are the corresponding homology fundamental classes of F and of M. Then it was shown in Theorem B of [11] that

$$(1\text{--}2) \qquad i_{1_*}(\nu) = u_F \frown \tilde{\mu}$$

where $i_1 \colon F \hookrightarrow M$ is the inclusion, $u_F \in H^n(M, M - F)$ is the Thom class of F in M and $\widetilde{\mu} \in H_{n+k}(M, M - F)$ is the element determined by μ. The map $f \colon M \to N$ induces an isomorphism $H^k(N) \cong H^k(N, N - y_0) \to H^k(M, M - F)$ in the setting of [8], and one can deduce (1–1) from (1–2).

Gottlieb used the formula (1–1) to obtain results about the transfer [8] and about the trace of a group action [9]. A similar formula was also obtained for Poincaré duality groups by Daccach and Franco [2]. Local coefficients have played an important role in the study of nonorientable manifolds and in obstruction theory (see also Borsari and Gonçalves [1]). In particular, a certain type of Gottlieb's formula has been obtained and used in [7] in the study of the coincidence problem of maps between manifolds of different dimensions. Let $p, q \colon X \to Y$ be maps between two closed orientable manifolds such that p and q are transverse so that $C(p, q) = \{x \in X \mid p(x) = q(x)\}$ is a submanifold of dimension $m - n$ where $m = \dim X \geq \dim Y = n$. Under conjugation, $\mathbb{Z}[\pi]$ is a local system on $Y \times Y$ where $\pi = \pi_1(Y)$. The first author, Wong and Jezierski defined a twisted Thom class $\widetilde{\tau}_Y \in H^n(Y^\times; \mathbb{Z}[\pi])$ [7]. Here $Y^\times$ denotes the pair $(Y \times Y, Y \times Y - \Delta Y)$ where ΔY is the diagonal of Y in $Y \times Y$. The primary obstruction to deforming p and q to be coincidence free on the n–th skeleton of X is defined by

$$o_n(p, q) := [j(p \times q)d]^*(\widetilde{\tau}_N)$$

where $d \colon X \to X \times X$ is the diagonal map and $j \colon Y \times Y \hookrightarrow Y^\times$ is the inclusion. Then it was shown in [7] that $o_n(p, q)$ is Poincaré dual to the twisted fundamental class representing $C(p, q)$ with local coefficients π^* induced from $\mathbb{Z}[\pi]$, ie,

$$(1\text{–}3) \qquad\qquad o_n(p, q) \frown [z_X] = [z_{C(p,q)}^{\pi^*}] \quad \in H_{m-n}(X; \pi^*).$$

Furthermore, this formula has been employed successfully in Gonçalves and Wong [6] in the study of coincidences and of roots. While Gottlieb's formula (1–1) relates the cohomology fundamental class of the base N with the homology fundamental class of the fiber F, it does not determine whether F is empty or not and hence it does not give any information about the (homological) size of $F = f^{-1}(y)$.

To see how cohomology with local coefficients can yield more information, let us consider the following example.

Example 1.1 Let $\mathcal{H}$ be the Heisenberg 3–dimensional nilmanifold, T the 2–dimensional torus and

$$S^1 \xrightarrow{i} \mathcal{H} \xrightarrow{f} T$$

a fibration inducing the homomorphism in the fundamental group $f_\# \colon \pi_1(\mathcal{H}) \to \pi_1(T)$ which is the abelianization. To avoid technical difficulties in dimension 2, consider the

fibration $p: \mathcal{H} \times S^1 \to T \times S^1 = T^3$ where $p = f \times 1_{S^1}$ is the product map. The homomorphism $i_*: H_1(S^1) \to H_1(\mathcal{H} \times S^1)$ induced by the inclusion of the fiber is the zero homomorphism since $H_1(\mathcal{H} \times S^1) \to H_1(T^3)$ is an isomorphism. By Gottlieb's formula, we have

$$i_*(\mu_1') = p^*(z_2) \frown \mu_1$$

where μ_1 denotes the homology fundamental class of $\mathcal{H} \times S^1$, z_2 the cohomology fundamental class of T^3 and μ_1' the homology fundamental class of the circle fiber $S^1 = p^{-1}(y)$, $y \in T^3$. This shows that $p^*(z_2) = 0$.

On the other hand, using the local coefficient system $\mathbb{Z}[\pi_2]$ on $\mathcal{H} \times S^1$ induced by the map p, one can show that $H^3(\pi_1; \mathbb{Z}[\pi_2]) \cong \mathbb{Z} \neq 0$ and in addition that the inclusion $S^1 \hookrightarrow \mathcal{H} \times S^1$ induces an isomorphism on homology with local coefficients. Here, $\pi_1 = \pi_1(\mathcal{H} \times S^1)$ and $\pi_2 = \pi_1(T^3)$. Because the target is a group, a root problem can be regarded as a coincidence problem (see Proposition 3.3). Then we can apply (1–3) and see that the cohomology class representing the primary obstruction to deforming the map to be root free is Poincaré dual to the twisted fundamental class

$$[z_{p^{-1}(y)}^{\pi_2^*}] \in H_1(\pi_1; \mathbb{Z}[\pi_2]) \cong H^3(\pi_1; \mathbb{Z}[\pi_2]).$$

This class turns out to be nontrivial [6]. It follows that the obstruction to deforming p to $\hat{p}$ with $\hat{p}^{-1}(y) = \varnothing$ is nontrivial. Hence, F has nontrivial homology (with local coefficients) in dimension one.

This example uses two important facts: (i) the primary obstruction to deforming a map to be root free is Poincaré dual to the twisted fundamental class by means of a Gottlieb type formula, and (ii) the homomorphism induced by p is nontrivial when local coefficients are employed. Another example which illustrates the phenomenon above can be found in [7].

In studying injective points, A Wyler [14] proved an interesting result concerning the cohomological size of $f^{-1}(B)$. He showed that if R is a ring (with unity) and $f_{*n}: H_n(M; R) \to H_n(N; R)$ is surjective then the composite

$$\check{H}^{i+k}(f^{-1}(B); R) \to H_{n-i}(M, M - f^{-1}(B); R) \overset{f_*}{\to} H_{n-1}(N, N - B; R) \to H^i(B; R)$$

is also surjective for all i. This holds for all maps in the same homotopy class of f.

In topological coincidence theory, one studies the coincidence set

$$C(p, q) = \{x \in X \mid p(x) = q(x)\}$$

of two maps $p, q: X \to Y$. Let ΔY denote the diagonal of Y in $Y \times Y$. Then $C(p, q) = f^{-1}(B)$ with $M = X, N = Y \times Y$ and $B = \Delta Y$, and $f: M \to N$ is

given by $f = (p \times q) \circ d$ where $d \colon X \to X \times X$ is the diagonal. In [7], we studied the coincidence problem for transverse maps. Since $C(p', q')$ changes for $p' \sim p, q' \sim q$, it is natural to look for homological information about $C(p', q')$.

Motivated by this question in coincidence theory and by the work of Wyler [14], we explore the general problem of obtaining a (co)homological measure of the coincidence set $C(p, q)$ or the preimage set $f^{-1}(B)$ in general.

Our first objective of this paper is to give a general formula in the context of homology and cohomology with local coefficients which contains both Gottlieb's formula (1–1) and the formula (1–3) as special cases, thereby further exploring the so-called *vague principle* of H Hopf. Secondly, we give homological conditions on a map $f \colon M \to N \supset B$ such that $f^{-1}(B)$ has nontrivial cohomology with local coefficients.

This paper is organized as follows. In Section 2, we give a generalized Gottlieb's theorem (Theorem 2.4) with local coefficients. This formula reduces to Gottlieb's formula in the root case with trivial integral coefficients and orientable manifolds (Theorem 3.1) and it reduces to the formula (1–3) in the coincidence setting (Theorem 3.2). The formula in the root case does not seem to be a special case of that in the coincidence case unless further conditions are imposed (see Proposition 3.3). In Section 4, we generalize the work of A Wyler [14] in studying the nonvanishing of the induced homomorphism of a map with simple coefficients. In Section 5, we obtain the results of Section 4 for local coefficients.

2 Generalized Gottlieb's formula

Let M, N and $B \subset N$ be closed smooth compact manifolds of dimensions $k+n, n$ and ℓ, respectively. Let $f \colon M \to N$ be a map transverse to B and $i_1 \colon f^{-1}(B) \hookrightarrow M$ be the inclusion. Therefore, $F = f^{-1}(B)$ is a closed submanifold of dimension $k + \ell$. No orientability is assumed.

Recall that over a manifold N there is a local coefficient system $\mathcal{O}_N$ called *the orientation local coefficient system* which has $\mathbb{Z}$ as the typical group and the homomorphism

$w: \pi_1(N) \to \mathrm{Aut}(\mathbb{Z})$ is given by $w(\alpha) = \mathrm{sgn}_N(\alpha)$, the sign of α in N. Consider a local coefficient system R_N over N with typical group R. Then we have the local system over M induced by f, denoted by R_M, and define the local system $R'_M = R_M \otimes \mathcal{O}_M$. Denote by R'_F the local system over F induced by $i_1: F \hookrightarrow M$ from the system R'_M, ie $R'_F = i_1^* R'_M$.

We now define homomorphisms $\phi(j): H^j(N, N-B; R_N) \to H_{n+k-j}(F; R'_F)$ for $0 \le j \le n$ and then we prove the general Gottlieb's formula.

Definition 2.1 For $0 \le j \le n$, let $\phi(j)$ be the composite homomorphism

$$H^j(N, N-B; R_N) \xrightarrow{f^*} H^j(M, M-F; R_M) \xrightarrow{A^{-1}} H_{n+k-j}(F; R'_F)$$

where A is the duality isomorphism as in Theorem 6.4 of Spanier [12]

A closer look at Definition 2.1 suggests that $\phi(j)$ can be expressed using the homology transfer as in Dold [5, Definition 10.5, p.310], provided we use local coefficients. This alternative way to describe $\phi(j)$ may have interest in its own right.

First we recall (for maps between oriented manifolds) that Dold's homology transfer is the composite

$$(2\text{--}1) \quad H_{n-j}(U_2) \xrightarrow{(\frown[z_N])^{-1}} H^j(N, N-U_2) \xrightarrow{f^*} H^j(M, M-U_1) \xrightarrow{(\frown[z_M])} H_{n+k-j}(U_1)$$

where $U_2 \xhookrightarrow{i_2} N$ is an open set of N and $f^{-1}(U_2) = U_1$ By incorporating local coefficients without assuming orientability, we define a similar transfer $f_!$ to be the composite

$$(2\text{--}2) \quad H_{n-j}(U_2; i_2^* R'_N) \xrightarrow{D_N} H^j(N, N-U_2; R_N) \xrightarrow{f^*} H^j(M, M-U_1; R_M)$$
$$\xrightarrow{D_M^{-1}} H_{n+k-j}(U_1; i_1^* R'_M)$$

where D_N and D_M are the duality isomorphisms in N and in M, respectively, with local coefficients and $R'_N = R_N \otimes \mathcal{O}_N$.

To relate the definition of $\phi(j)$ to the Dold transfer we introduce $\phi'(j)$ as follows.

Definition 2.2 Let $U \subset N$ be an open ($\bar{U}$ be a closed) tubular neighborhood of B such that $f^{-1}(U) = U_1 \subset M$ is an open ($\bar{U}_1$ a closed) tubular neighborhood of $f^{-1}(B) = F$. Consider the composite

$$(2\text{--}3) \qquad\qquad \phi'(j) := r_{1_*} \circ f_! \circ (r_{2_*})^{-1} \circ D_N{}^{-1}$$

where $r_2: U_2 \to B$ and $r_1: U_1 \to F$ are the retractions.

It can be shown that $\phi(j) = \phi'(j)$.

Next we compute the groups $H^{n-\ell}(N, N-B; R_N)$ and $H_{k+\ell}(F; R'_F)$ for an arbitrary system R_N, where R'_F is obtained from R'_N as above. Let $i_2: B \to N$ be the inclusion. Write

$$R_1 = \{r \in R \mid \operatorname{sgn}_N(\theta) \operatorname{sgn}_B(\theta) i_\#(\theta) r = r\} \qquad \text{for all } \theta \in \pi_1(B),$$

$$R_2 = \{r \in R \mid \operatorname{sgn}_F(\alpha) \operatorname{sgn}_M(\alpha) f_\#(\alpha) r = r\} \qquad \text{for all } \alpha \in \pi_1(f^{-1}(B)).$$

Proposition 2.3 *The groups $H^{n-\ell}(N, N-B; R_N)$ and $H_{k+\ell}(F, R'_F)$ are isomorphic to R_1 and R_2 respectively and $R_1 \subset R_2$.*

Proof By duality with local coefficients [12], $H^{n-\ell}(N, N-B; R_N)$ is isomorphic to $H_\ell(B; R'_B)$ where $R'_B = i_2^* R'_i N$ is the restriction of $R'_N = R_N \otimes \mathcal{O}_N$ to B. Now we apply duality again in B and we obtain $H^0(B; R'_B \otimes \mathcal{O}_B)$ which is precisely the subgroup R_1. For the calculation of $H_{k+\ell}(F; R'_F)$, again we use duality. The local coefficient system $R'_F \otimes \mathcal{O}_F$ is exactly the one given by $\alpha * r = \operatorname{sgn}_F(\alpha) \operatorname{sgn}_M(\alpha) f_\#(\alpha) r$.

In order to show that $R_1 \subset R_2$, observe that for $\alpha \in \pi_1(f^{-1}(B))$ we have that $\operatorname{sgn}_F(\alpha) \operatorname{sgn}_M(\alpha) = \operatorname{sgn}_N(f_\#(\alpha)) \operatorname{sgn}_B(f_\#(\alpha))$. This follows since we have U_2 and U_1 tubular neighborhoods of B and of F respectively, where $U_1 = f^{-1}(U_2)$. So the two actions coincide on the subset $\pi_1(f^{-1}(B)) = \pi_1(F)$ and the result follows. $\quad\square$

Now, we prove the following general Gottlieb's formula.

Theorem 2.4 *Let $f: M \to N$ be transverse to the submanifold $B \subset N$. Let $\phi(j): H^j(N, N-B; R_N) \to H_{n+k-j}(F; R'_F)$ be the homomorphism defined as above. With respect to the local coefficient systems R_N, R_M, R'_M and R'_F over N, M, M and F, respectively, for every $r \in H^j(N, N-B; R_N)$, the element $f^* j_2^*(r)$ is the Poincaré dual of $i_{1*}\phi(j)(r)$, ie,*

(2–4) $$i_{1*}\phi(j)(r) = f^*(j_2^*(r)) \frown [z_M].$$

Furthermore, the homomorphism $\phi(n-\ell)$ corresponds to the inclusion after identifying $H^{n-\ell}(N, N-B; R_N)$ and $H_{k+\ell}(F; R'_F)$ with the subgroups R_1 and R_2 of R, respectively as in Proposition 2.3.

Proof Consider the following commutative diagram where $j_2: (N, \varnothing) \to (N, N-B)$ and $j_1: (M, \varnothing) \to (M, M-F)$ are inclusions.

(2–5)
$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} \\ (M, M-F) & \longrightarrow & (N, N-B) \end{array}$$

Together with duality isomorphisms, (2–5) induces the commutative diagram:

$$
\begin{array}{ccccc}
H^j(N,N-B;R_N) & \xrightarrow{\;f^*\;} & H^j(M,M-F;R_M) & \xrightarrow{\;A^{-1}\;} & H_{n+k-j}(F;R'_F) \\
\Big\downarrow{j_2^*} & & \Big\downarrow{j_1^*} & & \Big\downarrow{i_{1*}} \\
H^j(N;R_N) & \xrightarrow{\;f^*\;} & H^j(M;R_M) & \xrightarrow{\;\frown[z_M]\;} & H_{n+k-j}(M;R'_M)
\end{array}
$$

(2–6)

The composite homomorphism of the top row of (2–6) gives $\phi(j)$ by Definition 2.1. Thus, the commutativity of (2–6) implies (2–4). Now, for $j = n - \ell$, Proposition 2.3 completes the second assertion. $\qquad\square$

Remark In [3], R Dobreńko defined the primary obstruction $o_B(f)$ to deforming a map f (transverse to B) out of a subspace $B \subset N$. In view of Theorem 2.4, our general duality formula (2–4) shows that Dobreńko's obstruction class $o_B(f)$ is Poincaré dual to the image of the homology class representing $f^{-1}(B)$.

3 Roots and coincidences

In this section we consider in details two special cases of Theorem 2.4. For one of them, we take B to be a point. This is related to the study of roots. The other case, we take $p, q \colon X \to Y$ a pair of maps between two closed manifolds. Then we consider the map $f \colon M \to N$ where $f = (p \times q) \circ d$, $M = X$, $N = Y \times Y$ and $B = \Delta Y$. It turns out that $f^{-1}(B) = C(p,q) = \{x \in X \mid p(x) = q(x)\}$ is the coincidence set of the two maps $p, q \colon X \to Y$. This latter case corresponds to the study of coincidences of a pair of maps.

Suppose B is a point and R_N is an arbitrary local coefficient system over N with typical group R. We obtain:

Theorem 3.1 *In the case $B = y_0$ is a point, the local coefficient system R'_F is $R \otimes \mathcal{O}_F$ and we have the following:*

(1) $H^j(N, N - y_0; R_N) = 0$ *for $j \neq n$ and $H^n(N, N - y_0; R_N) = R$.*

(2) $H_k(F; R \otimes \mathcal{O}_F) = R$.

(3) *The homomorphism $\phi(n) \colon H^n(N, N - y_0; R_N) \to H_k(F; R'_F)$ under the identification above is the identity. Therefore, if y_0 lies in the interior of an n–simplex σ_0 and c_n denotes the elementary cocycle representing the cohomology fundamental class so that $c_n(\sigma_0) = r \in R$, then the pullback of $[c_n]$ is dual to the image of the homology class $[z_F] \otimes r$.*

(4) When $R_N = \mathbb{Z}$ is the trivial local coefficient system and the manifolds are closed and orientable, we obtain Gottlieb's formula (1–1).

Proof (1) By duality, $H^j(N, N - y_0; R_N) \cong H_{n-j}(y_0; R_{\{y_0\}}) = 0$ for $j \neq n$. On the other hand, $H_0(y_0; R_{\{y_0\}}) = R$.

(2) The system $R'_F = \mathcal{O}_F \otimes R$ and $k = \dim F$ since $\ell = 0$ so the assertion follows.

(3) This follows from Theorem 2.4.

(4) When $R_N = \mathbb{Z}$ and $\ell = 0$, $\phi(n)$ is the identity from (3) and j_2 induces an isomorphism identifying $H^n(N)$ with $H^n(N, N - y_0)$. Now (2–4) reduces to Gottlieb's formula (1–1). $\square$

For the second case we consider $f \colon M \to N$ where f is the map $(p \times q) \circ d$ from $M = X$ to $N = Y \times Y$, $B = \Delta Y$, and $F = (p \times q)^{-1}(\Delta Y)$ which is the same as $C(p, q)$. If $\dim Y = n$ then we have $\dim N = 2n$, $\dim M = k + 2n$ and $\ell = n$. Denote by $\psi \colon \pi_1(N \times N) \to \mathrm{Aut}(R)$ the representation corresponding to the given local coefficient system. We obtain:

Theorem 3.2 *In the case $N = Y \times Y$ and $B = \Delta Y$, the local coefficient systems $R_{\Delta Y}(= R_B)$ and $R_{C(p,q)}(= R_F)$ are given by the equations $(\alpha, \alpha) * r = \psi(\alpha, \alpha)$ and $\beta * r = \psi(p_\#(\beta), q_\#(\beta))$, respectively. We have the following:*

(1) $H^j(Y^\times; R_{Y \times Y}) = 0$ *for $j < n = \dim Y$, and $H^n(Y^\times; R_{Y \times Y})$ consists of the elements of R that are fixed by the action of $\mathrm{sgn}_Y(\alpha)\psi(\alpha, \alpha)$ for all $\alpha \in \pi_1(Y)$. In particular, when $R_{Y \times Y}$ is given by $\pi_n(Y, Y - y_0)$, the group $H^n(Y^\times; R_{Y \times Y})$ is isomorphic to the sum $\sum \mathbb{Z}$ indexed by the center of the group $\pi = \pi_1(Y)$.*

(2) $H_{k+n}(C(p, q); R'_{C(p,q)})$ *consists of the elements of R that are fixed by the action of $\psi(p_\#\beta, p_\#(\beta))$ for all $\beta \in \pi_1(C(p, q))$. In particular, when $R_{Y \times Y}$ is given by $\pi_n(Y, Y - y_0)$, the group $H_{k+n}(C(p, q); R'_{C(p,q)})$ is isomorphic to the sum $\sum \mathbb{Z}$ indexed by the centralizer of the subgroup $p_\#(\pi_1(C(p, q)))$ in $\pi_1(Y)$.*

(3) *The homomorphism*

$$\phi(n) \colon H^n(Y^\times; R_{Y \times Y}) \to H_{k+n}(C(p, q); R'_{C(p,q)})$$

under the identification above is the inclusion. Thus, for σ_0 an n–simplex transversal to the diagonal and c_n an elementary cocycle with $c_n(\sigma_0) = r \in R$, the pullback of $[c_n]$ is dual to the image of $[z_F] \otimes m$ where $[z_F]$ is the fundamental homology class of F.

(4) When $R_N = \pi_n(Y, Y - y_0)$ and the manifolds are closed and orientable, we obtain the formula (1–3).

Proof (1) By duality, $H^j(Y^\times; R_{Y \times Y}) \cong H_{2n-j}(\Delta Y; R'_{\Delta Y})$. Since $\dim \Delta Y = n$, it follows that $H^j(Y^\times; R_{\Delta Y}) = 0$ for $j < n$. When $j = n$, by duality again, we have $H_n(\Delta Y; R'_{\Delta Y}) \cong H^0(\Delta Y; R_{\Delta Y})$ which is isomorphic to the submodule of $R_{\Delta Y}$ that is fixed by the action of ψ. This submodule in the case where $R_{Y \times Y} = \pi_n(Y, Y - y_0)$ is easily seen to be isomorphic to $\sum_\alpha \mathbb{Z}$ where α varies over the center of $\pi_1(Y)$.

(2) Note that $F = C(p, q)$ is a submanifold of dimension $k + n$. By duality, $H_{k+n}(F; R'_F) \cong H^0(F; R'_F \otimes \mathcal{O}_F)$. A similar argument as in (1) gives the result.

(3) This follows from Theorem 2.4.

(4) The formula (2–4) reduces to (1–3) when $R_N = \pi_n(Y, Y - y_0)$. $\qquad\square$

Remark The special cases above with local coefficients given by $\pi_n(Y, Y - y_0)$ are related to the study of obstructions in the root problem and in the general coincidence problem. While the root problem can sometimes be regarded as a special case of the coincidence problem, the formula (2–4) in the root case as in Theorem 3.1 does not seem to follow from that in the general coincidence case as in Theorem 3.2.

If the space Y is a topological group, then by means of the map $h: Y \times Y \to Y$ defined by $h(x, y) = xy^{-1}$, we obtain the formula for the root case from the general coincidence case as follows.

Proposition 3.3 *Given a topological group Y, a local coefficient system R_Y and a map $f: X \to Y$ which is transverse to the neutral element $e \in Y$, the general Gottlieb's formula for roots follows from that for the coincidence case.*

Proof Consider the local coefficient system $R_{Y \times Y}$ over $Y \times Y$ induced by h from the system R_Y and the map $f \times \overline{e}: X \to Y \times Y$ where $\overline{e}$ denotes the constant map at e. It follows that $F = f^{-1}(1) = (f \times \overline{e})^{-1}(Y \times Y)$. Now, for a given class $x \in H^j(Y, Y - e; R_Y)$, we apply the formula (1–3) to $h^*(x) \in H^n(Y^\times; R_{Y \times Y})$. Then, we obtain

$$i_{1*}\phi(j)(h^*(x)) = (f \times \overline{e})^*(j_2^*(h^*(x))) \frown [z_X].$$

But $(f \times \overline{e})^*(j^*(h^*(x)) \frown [z_X] = f^*(j^*(x)) \frown [z_X]$ since $f = h \circ (f \times \overline{e})$ and it is straightforward to see that

$$i_{1*}\phi(j)(h^*(x)) = j_{2*}\phi(j)((x)).\qquad\square$$

Remark The hypothesis that Y be a group in Proposition 3.3 can be slightly relaxed by requiring Y to possess a multiplication with inverse. For example, Proposition 3.3 holds if $Y = S^7$, the unit Cayley numbers.

4 (Co)homological estimates for preimages – simple coefficients

Let R be a ring with unity. Suppose M and N are R–orientable manifolds of dimension m and n respectively, and $B \subset N$ is an R–orientable submanifold of dimension k. We study the relationship between $H^{*+m-n+k}(g^{-1}(B); R)$ and $H^*(B; R)$ (similarly in homology between $H_{*+m+n-k}(g^{-1}(B); R)$ and $H_*(B; R)$) for trivial coefficient systems, where g is an arbitrary map belonging to $[f]$, the homotopy class of a map $f \colon M \to N$. In particular, this will give lower bounds for the minimal dimension (cohomological, homological) of $g^{-1}(B)$ among all possible maps $g \in [f]$.

4.1 Bounds for the cohomology of $f^{-1}(B)$

Here, we extend the results of [14]. Cech cohomology will be used. Since B is a k–dimensional submanifold, we identify the Cech cohomology $\check{H}^*(B; R)$ with the singular cohomology $H^*(B; R)$ of B.

Theorem 4.1 Let R_0 be the image of the homomorphism f_{*n} from $H_n(M; R)$ to $H_n(N; R)$. Let i_0 be the first integer (if it exists) such that the induced homomorphism in homology $f_{*i} \colon H_i(M; R) \to H_i(N, N-B; R)$ is nonzero. Then

(a) If $i_0 < n$ then $\check{H}^{m-i_0}(g^{-1}(B); R)) \neq 0$ for any map g homotopic to f.

(b) If f_{*n} is nontrivial then for $n - k \leq j \leq n$ the image of the homomorphism $f_{*j} \colon H_j(M, M-g^{-1}(B); R) \to H_j(N, N-B; R)$ contains $R_0 \cdot H_j(N, N-B; R)$ and the image of the composite

$$\check{H}^{m-j}(g^{-1}(B); R) \to H_j(M, M - g^{-1}(B); R)$$
$$\to H_j(N, N-B; R) \to H^{n-j}(B; R)$$

contains $R_0 \cdot H^{n-j}(B; R)$.

In particular, $\check{H}^{m-n}(g^{-1}(B); R)$ and $\check{H}^{m-n+k}(g^{-1}(B); R)$ are nontrivial.

Proof (a) First we have a commutative diagram

$$
\begin{array}{ccc}
H_i(M;R) & \xrightarrow{\ f_*\ } & H_i(N;R) \\
\ \ \downarrow{\scriptstyle j_{1*}} & & \ \ \downarrow{\scriptstyle j_{2*}} \\
H_i(M, M-g^{-1}(B);R) & \xrightarrow{\ f_*\ } & H_i(N, N-B;R)
\end{array}
$$

(4–1)

where j_1, j_2 are inclusions.

Since the composite $H_{i_0}(M;R) \to H_{i_0}(M, M-g^{-1}(B);R) \to H_{i_0}(N, N-B;R)$ is nonzero, it follows that $H_{i_0}(M, M-g^{-1}(B);R)$ is nontrivial. Now applying Poincaré Duality as in [5] using the fact that $g^{-1}(B)$ is compact, it follows that $\check{H}^{m-i_0}(g^{-1}(B);R) \neq 0$.

(b) Given an element $\alpha \in H_j(N, N-B;R)$ and $r_0 \in R_0$, the element $r_0 \cdot \alpha$ is the cap product of some class $\beta \in H^{n-j}(B;R)$ with the multiple $r_0 \cdot \mu_N$ of the homology fundamental class $\mu_N \in H_n(N;R)$. Using the naturality of the cap product [5, p.239] and the fact that $r_0 \cdot \mu_N$ is in the image of f_{*_n}, we have

$$
r_0 \cdot \alpha = \beta \frown r_0 \cdot \mu_N = \beta \frown (f_*(\psi)) = f_*((f^*\beta) \frown \psi).
$$

For $j = n - k$, we have $H_{n-k}(N, N-B;R)$ isomorphic to $H^k(B;R)$ which is isomorphic to R. For $j = n$, we have $H_n(N, N-B;R)$ isomorphic to $H^0(B;R)$ which is isomorphic to R. In either case, $R_0 \cdot H^{n-j}(B;R)$ is nontrivial. $\qquad\square$

Remark In Theorem 4.1, for any closed connected B, the image of

$$
f_{*_i}\colon H_i(M, M-g^{-1}(B);R) \to H_i(N, N-B;R)
$$

is equal to $R_0 \cdot H_i(N, N-B;R)$. Furthermore, part (b) is similar to the result of A Wyler [14] in that we obtain nontriviality of f_{*_i} for all i given the nontriviality of f_{*_n}.

4.2 Bounds for the homology of $f^{-1}(B)$

Next, we estimate the homology of $g^{-1}(B)$ for any g in the homotopy class $[f]$ of f. We use Steenrod homology, singular cohomology and Alexander–Spanier cohomology, denoted by H_*, H^* and $\bar{H}^*$ respectively. Our proof uses the same ingredients as in the last subsection.

Theorem 4.2 *Let i_0 be the first integer (if it exists) such that the induced homomorphism in cohomology $f^{*^{i_0}}\colon H^{i_0}(N, N-B;R) \to H^{i_0}(M;R)$ is a nonzero homomorphism. Then*

(a) If $i_0 < n$ then $H_{m-i_0}(g^{-1}(B); R) \neq 0$ for any map g homotopic to f.

(b) If $f^{*^n}: H^n(N; R) \to H^n(M; R)$ is nontrivial then $f^{*^{n-k}}$ is also different from zero and $H_{m-n+k}(g^{-1}(B); R) \neq 0$.

Proof Note that $H^i(N, N-B; R)$ is zero for $i < n - k$ and for $i = n - k$ it is isomorphic to $H_k(B; R) \approx R$.

(a) First we have a commutative diagram where the cohomology here is Alexander–Spanier cohomology.

$$(4\text{--}2) \qquad \begin{array}{ccc} \bar{H}^i(N, N-B; R) & \xrightarrow{f^*} & \bar{H}^i(M, M - g^{-1}(B); R) \\ {\scriptstyle j_2^*}\downarrow & & {\scriptstyle j_1^*}\downarrow \\ \bar{H}^i(N; R) & \xrightarrow{f^*} & \bar{H}^i(M; R) \end{array}$$

Since M, N are compact and B is a submanifold, we can identify $\bar{H}^i(N, N-B; R)$ and $H^i(M; R)$, the Alexander–Spanier cohomology, with their respective singular cohomology groups. Since the composite

$$\bar{H}^{i_0}(N, N-B; R) \to \bar{H}^{i_0}(M, M - g^{-1}(B); R) \to \bar{H}^{i_0}(M; R)$$

is nonzero, it follows that $\bar{H}^{i_0}(M, M - g^{-1}(B); R)$ is nontrivial. Now applying Alexander–Spanier duality [10, Theorem 11.15] and using the fact that $g^{-1}(B)$ is compact, $H_{m-i_0}(g^{-1}(B); R) \neq 0$ and part (a) follows.

(b) Using duality, the hypothesis on f^{*^n} implies that the homomorphism f_{*_n} from $H_n(M; R)$ to $H_n(N; R)$ is nontrivial. Again, we let $R_0 = f_{*_n}(H_n(M; R))$ be the subring of R. Therefore, $r_0 \cdot \mu_N$ is in the image of f_{*_n} for any $r_0 \in R_0$. As in the previous case, we use again the formula from [5, p.239] for the naturality as follows. Given an element $x' \in H^i(N, N-B; R)$, suppose that there exists $r_0 \in R_0$ such that $r_0 \cdot x' \neq 0$. Thus

$$r_0 \cdot x' \frown \mu_N = x' \frown r_0 \cdot \mu_N = x' \frown f_{*n}(\beta) = f_*((f^*(x')) \frown \psi),$$

which then implies that $f^{*^i}(x') \neq 0$. It follows that $f^{*^{n-k}}$ is nontrivial since we have $H^{n-k}(N, N-B; R) \approx R$. □

One natural way to use the results of the two subsections above is to find suitable coefficients R such that our hypotheses hold. It may happen that for some ring the hypotheses do not hold but do hold for another ring. In the next section, we study the same situations as in this section when local coefficients are used.

Remark Similar results can be obtained for B a closed connected subset of N, using the same techniques above. However, one must use a suitable (co)homology in order to apply duality [10, Theorems 11.11 and 11.15] and replace $k = \dim B$ by the (co)homology dimension of the set B.

4.3 Roots and coincidences

We focus on two special cases, namely (i) when $B = \{b\}$ is a point (root case) and (ii) when $N = Y \times Y$, $B = \Delta Y$ and $f = p \times q$ where $p, q \colon M \to Y$ with $\dim N = 2n$, $\dim Y = n$ and $m \geq n$.

In situation (i), the homomorphism

$$f_{*i} \colon H_i(M; R) \to H_i(N, N - b; R)$$

is always zero except possibly for $i = n$. Thus, Theorem 4.1 reduces to asserting that $H^{m-n}(g^{-1}(b); R) \neq 0$. Similarly, Theorem 4.2 reduces to asserting that $\check{H}_{m-n}(g^{-1}(b); R) \neq 0$.

In fact, a similar phenomenon occurs in the coincidence case (ii) when simple coefficients are used.

Let $\tau_Y \in H^n(Y^\times; R)$ be the Thom class of the normal bundle of the diagonal ΔY in $Y \times Y$ where $Y^\times$ denotes the pair $(Y \times Y, Y \times Y - \Delta Y)$. Then the Thom Isomorphism Theorem asserts that every class $\beta \in H^{n+i}(Y^\times; R)$ is given by $\beta = \tau_Y \smile p_1^*(\alpha)$ for some $\alpha \in H^i(Y; R)$ where $p_1 \colon Y \times Y \to Y$ denotes the projection on the first factor. Now, suppose the homomorphism

$$H^n(Y^\times; R) \xrightarrow{f^*} H^n(M; R)$$

is trivial then for any $z \in H^{n+i}(Y^\times; R)$, $z = \tau_Y \smile p_1^*(z')$ for some $z' \in H^i(Y; R)$. Now, $f^*(z) = f^*(\tau_Y) \smile f^*(z') = 0$. For dimension reasons, $H^i(Y^\times; R) = 0$ for $i < n$. Hence, the first dimension i at which $f^i \colon H^*(Y^\times; R) \to H^*(M; R)$ can possibly be nonzero is $* = n$.

Remark The occurrence of the first nontrivial homomorphism f^i at $i = n$ does not always hold true for arbitrary B (other than the root or the coincidence cases). See for instance Example 5.7 even when one uses trivial coefficients.

5 (Co)homological estimates for preimages – local coefficients

In this section, we no longer require M, N and B to be orientable for we will use local coefficients.

Recall that in Example 1.1, the homomorphism f^* is zero with trivial coefficients (integral coefficients) but nonzero for some local coefficients. In fact, using the universal coefficient theorem, the homomorphism f^* is zero for any abelian group G as simple coefficients. Although our results are stated in terms of arbitrary local coefficient systems, certain coefficient systems play an important role since they are closely related with the primary obstruction to deforming a map out of a subspace [3].

Let us recall from Dobreńko [4] a few useful results. We have that the pair $(N, N-B)$ is $(n-k-1)$–connected. The local system over N given by the group $\pi_{n-k}(N, N-B)$ is computed as follows.

Theorem 5.1 [4] *Let $i\colon B \to N$ be the inclusion and $\pi_1^0(B, b)$ be the subgroup of the elements of $\pi_1(B, b)$ which preserve the orientation of B. Then*

(1) *The groups $\pi_j(N, N-B)$ vanish for $0 \le j < \dim N - \dim B = n - k$.*

(2) *If $\mathrm{Ker}(i_\#) \subset \pi_1^0(B, b)$ then $\pi_{n-k}(N, N-B)$ is isomorphic to a sum of $\mathbb{Z}$ indexed in $W(b)$, where $W(b) = \pi_1(N, b)/\mathrm{Im}(i_\#))$ (left cosets).*

(3) *Otherwise, $\pi_{n-k}(N, N-B)$ is isomorphic to a sum of $\mathbb{Z}_2$ indexed in $W(b)$, where $W(b) = \pi_1(N, b)/\mathrm{Im}(i_\#)$.*

(4) *The action of $\pi_1(N-B)$ on $\pi_{n-k}(N, N-B)$ is given by*

$$\beta[\alpha] = \mathrm{sgn}_N(\beta)[\beta * \alpha]$$

for $\beta \in \pi_1(Y) = \pi_1(Y - B)$ and $\alpha \in \pi_{n-k}(N, N-B)$.

From the above result we can easily compute the first homology and cohomology of the pair $(N, N-B)$.

Theorem 5.2 [4] *Using the local coefficient system $V_1 := \pi_{n-k}(N, N-B)$ given by Theorem 5.1,*

(1) $H^{n-k}(N, N-B; \pi_{n-k}(N, N-B))$ *is equal to the direct sum of R (for R either $\mathbb{Z}$ or $\mathbb{Z}_2$) indexed over the set of elements of $\pi_1(N, b)/\mathrm{Im}(i_\#)$ fixed by all elements of $\pi_1(B)$;*

(2) $H_{n-k}(N, N-B; \pi_{n-k}(Y, Y-B))$ *is equal to the quotient of* $\pi_{n-k}(N, N-B)$ *by the action of* $\pi_1(B)$ *(induced from the action of* $\pi_1(N)$*).*

The next two results are the local coefficients analogs of Theorem 4.1 and Theorem 4.2.

5.1 Bounds for the cohomology of $f^{-1}(B)$

Theorem 5.3 *Let* R_N *be a local coefficient system over* N*. Then* $H_i(N, N-B; R_N)$ *is zero for* $i < n-k$ *and it is isomorphic to* $H_0(B; R_N \otimes \mathcal{O}_N \otimes \mathcal{O}_B) \approx R/R_0$ *for* $i = n - k$ *where* R_0 *denotes the image of the action. Denote by* S_*^0 *the image of the homomorphism* $f_*\colon H_n(M; f^*(R'_N)) \to H_n(N; R'_N) = S^0$ *where the ring* $S^0 = H^0(N; R_n) \cong H_n(N; R'_N)$ *is the subring of elements of* R_N *that are fixed by the action, and* $R'_N = R_N \otimes \mathcal{O}_N$ *is the induced local system corresponding to the orientation system* $\mathcal{O}_N$*. Let* i_0 *be the first integer (if it exists) such that the induced homomorphism in homology* $f_*\colon H_*(M; f^*R_N) \to H_*(N, N-B; R_N)$ *is nonzero. Then*

(a) *If* $i_0 < n$ *then we have that* $\check{H}^{m-i_0}(g^{-1}(B); f^*R_N) \neq 0$ *for any map* g *homotopic to* f*.*

(b) *If* $i_0 = n$ *then for* $n - k \leq j \leq n$ *the image of*

$$f_{*j}\colon H_j(M, M - g^{-1}(B); f^*R_N) \to H_j(N, N-B; R_N)$$

contains $S_*^0 \cdot H_j(N, N-B; R_N)$ *and the image of the composite*

$$\check{H}^{m-j}(g^{-1}(B); f^*(R'_N)) \to H_j(M, M - g^{-1}(B); f^*R_N)$$
$$\to H_j(N, N-B; R_N) \to H^{n-j}(B; \iota^*(R'_N))$$

contains $S_*^0 \cdot H^{n-j}(B; \iota^*(R'_N))$ *where* $\iota\colon B \to N$ *is the inclusion.*

In particular, $\check{H}^{m-n}(g^{-1}(B); f^*R_N)$ and $\check{H}^{m-n+k}(g^{-1}(B); f^*R_N) \neq 0$ are nontrivial.

Proof The proof is similar to the proof of Theorem 4.1 where we use duality with local coefficients as in [12]. $\qquad\square$

5.2 Bounds for the homology of $f^{-1}(B)$

As in the trivial local coefficient case, H_* means Steenrod homology. We remark that for a compact space the homology $\bar{H}_*$ used in [12] coincide with the Steenrod homology.

Theorem 5.4 *Let R_N be a local coefficient system over N. For $i < n-k$, we know that $H^i(N, N-B; R_N)$ is zero. For $i = n-k$, it is isomorphic to $H^0(B; i_2^*(R_N)) \approx R^0$ where R^0 denotes the set of elements of R that are fixed by the action. Let i_0 be the first integer (if it exists) such that the induced homomorphism in cohomology $f^*: H^i(N, N-B; R_N) \to H^i(M; f^*R_N)$ is nonzero. Here, f^*R_N denotes the local system over M by pulling back R_N by f. Then*

(a) *If $i_0 < n$ then $H_{m-i_0}(g^{-1}(B); f^*R_N) \neq 0$ for any map g homotopic to f.*

(b) *If $i_0 = n$ then $f^{*^{n-k}}$ is also nonzero and $H_{m-n+k}(g^{-1}(B); f^*R_N) \neq 0$.*

Proof The proof is similar to the proof of Theorem 4.2 where we use duality with local coefficients as in [12]. □

Recall from [3, Theorem 2.7, p.15], there is a universal element, which is the primary obstruction to deforming the identity map 1_N off the subspace B. We shall call this element the *Thom class* of B in N with coefficients in R_U, denoted by τ_B. It was shown that for any map $f: (M, M - f^{-1}(B)) \to (N, N-B)$, the primary obstruction $o^{n-k}{}_B(f)$ to deforming f off the subspace B on the $(n-k)$–th skeleton of M is the pullback $f^*(\tau_B)$ of the Thom class where $k = \dim B$ and $n = \dim N$.

The next result shows the relevancy of the local coefficient R_U. Since the pair $(N, N-B)$ is $(n-k-1)$–connected, the first possible dimension for which the homomorphism $f^*: H^*(N, N-B; R_U) \to H^*(M; f^*(R_U))$ is nontrivial is $n-k$. Thus, for any local system R_N, we call the homomorphism $f^*: H^{n-k}(N, N-B; R_N) \to H^{n-k}(M; f^*(R_N))$ the *primary* homomorphism.

Proposition 5.5 *If the pullback of the Thom class in $H^{n-k}(N, N-B; R_U)$ by the map f is trivial, then the primary homomorphism $f^*: H^{n-k}(N, N-B; R_N) \to H^{n-k}(M; f^*(R_N))$ is trivial for any local coefficient system R_N. In particular the primary homomorphism with local coefficient R_N is trivial.*

Proof From the hypothesis, it follows that the primary obstruction to deforming f into $N-B$ through the $(n-k)$–th skeleton is zero. So, up to homotopy, the restriction $f|_{M^{n-k}}: M^{n-k} \to N$ of f to the $(n-k)$–th skeleton factors through $N-B$.

Note that the homomorphism $i^*\colon H^{n-k}(M; i^*R_N) \to H^{n-k}(M^{n-k}; R_N)$ in cohomology induced by the inclusion $i\colon M^{n-k} \to M$ is injective for any local system of coefficients R_N. This follows from the cellular definition of cohomology with local coefficients as presented by Whitehead [13]. For the cellular chains $C^j(\ ; R_N)$ for the two complexes M and M^{n-k} are isomorphic for $j \leq n-k$, and $C^j(M^{n-k}; R_N)$ is zero for $j > n-k$. So $H^{n-k}(M^{n-k}; R_N)$ is a quotient of $C^{n-k}(M^{n-k}; R_N)$ by the subgroup Im δ^{n-k-1} and $H^{n-k}(M; R_N)$ is the quotient of the $(n-k)$–cocycles contained in $C^{n-k}(M^{n-k}; R_N)$ by the same subgroup Im δ^{n-k-1}. Hence the induced homomorphism by the composite map $M^{n-k} \to (N-B) \to N \to (N, N-B)$ is the zero homomorphism for any local coefficients because of the long exact sequence of the pair $(N, N-B)$ in cohomology. But this homomorphism is the composite of the primary homomorphism followed by i^*. Since i^* is injective, the first assertion follows. The last part is clear from the first part. $\qquad\square$

Example 5.6 The primary homomorphism with local coefficient in general gives more information than the primary homomorphism with trivial coefficients (with $\mathbb{Z}$ coefficients for example). If we take the fiber map $p\colon N_3 \to T^2$ from the three-dimensional Heisenberg manifold into the two-dimensional torus, this map has the property that the pullback of the fundamental class of T^2 with trivial $\mathbb{Z}$ coefficients is zero. However, the pullback of the twisted Thom class with local coefficients as given by Dobreńko [4] is nonzero. See Gonçalves and Wong [6] for more details about this and other examples.

Example 5.7 Let $n > 2$, $M = N = S^{2n}$ and $B = S^n$. If f is the identity then the primary homomorphism is trivial because $H^n(S^{2n}; R_N) = 0$ for any local coefficient system R_N. But the $2n$–th homomorphism is clearly the identity.

Remark In view of the proof of Proposition 5.5, if f is deformable into $N-B$ off the $(n-k+j)$–th skeleton of M then the homomorphism

$$f^*\colon H^{n-k+j}(N, N-B; R_N) \to H^{n-k+j}(M; f^*(R_N))$$

is trivial for any R_N. This implies that the nonvanishing of the $(n-k+j)$–th homomorphism implies that there exist higher obstructions to deformation. In contrast to defining secondary or higher obstruction (sets), these higher homomorphisms are easily defined and therefore can be used as sufficient conditions for the existence of preimages.

Remark For the coincidence case with simple coefficients in Section 4.3, if the homomorphism $f^{*n}\colon H^n(Y^\times; R) \to H^n(M; R)$ is trivial then all the homomorphisms f^{*k}

are trivial for $k > n$. We do not know if the same phenomenon would hold true when local coefficients are used. In particular, we do not know whether the higher homomorphisms f^{*k} would vanish if $f^{*n}: H^n(Y^\times; \pi_n(N, N-b)) \to H^n(M; \pi_n(N, N-b))$ is trivial.

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