The C–polynomial of a knot

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In an earlier paper the first author defined a non-commutative A–polynomial for knots in 3–space, using the colored Jones function. The idea is that the colored Jones function of a knot satisfies a non-trivial linear q–difference equation. Said differently, the colored Jones function of a knot is annihilated by a non-zero ideal of the Weyl algebra which is generalted (after localization) by the non-commutative A–polynomial of a knot.

In that paper, it was conjectured that this polynomial (which has to do with representations of the quantum group $U_q(\mathfrak{sl}_2)$) specializes at $q=1$ to the better known A–polynomial of a knot, which has to do with genuine $\mathrm{SL}_2(\mathbb{C})$ representations of the knot complement.

Computing the non-commutative A–polynomial of a knot is a difficult task which so far has been achieved for the two simplest knots. In the present paper, we introduce the C–polynomial of a knot, along with its non-commutative version, and give an explicit computation for all twist knots. In a forthcoming paper, we will use this information to compute the non-commutative A–polynomial of twist knots. Finally, we formulate a number of conjectures relating the A, the C–polynomial and the Alexander polynomial, all confirmed for the class of twist knots.

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1 Introduction

1.1 The non-commutative A–polynomial of a knot

In [6] the first author defined a non-commutative A–polynomial for knots in 3–space, using the colored Jones function. The idea is that the colored Jones function of a knot satisfies a non-trivial linear q–difference equation. Said differently, the colored Jones function of a knot is annihilated by a non-zero ideal of the Weyl algebra. By localizing, the Weyl algebra becomes a principal ideal domain, so that there is a single polynomial generator, the non-commutative A–polynomial of a knot.

In [6], it was conjectured that this polynomial (which has to do with representations of the quantum group $U_q(\mathfrak{sl}_2)$) specializes at $q=1$ to the better known A–polynomial of

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a knot, which has to do with genuine $\mathrm{SL}_2(\mathbb{C})$ representations of the knot complement, Cooper–Culler–Gillet–Long–Shalen [4].

Computing the A–polynomial of a knot is a difficult task. For knots with a small (about 10) number of crossings, or with a small (about 7) number of ideal tetrahedra, a numerical method was developed by Culler, see [5]. For an alternative method that involves elimination, see Boyd [3]. For 2–bridge knots, simpler elimination methods are known. All methods exhibit that the complexity of the A–polynomial (both with respect to the degrees of the monomials appearing, and with respect to their coefficients) is exponential in the number of crossings.

1.2 Can we compute the non-commutative A–polynomial?

At a first glance, it is not obvious that one can compute the non-commutative A–polynomial of a knot. Let us explain a theoretical algorithm for computation. Given a planar projection of a knot with c crossings, there is an explicit c–dimensional multisum formula for the colored Jones polynomial, where the summand is q–proper hypergeometric, see Garoufalidis and Lê [8, Section 3]. This has been implemented in Bar-Natan's `KnotAtlas` as a way of computing the colored Jones function of a knot, see [1].

Given as input a multisum formula for the colored Jones polynomial, the general theory of Zeilberger–Wilf computes a linear q–difference equation by solving a system of linear equations; see [24]. If one is lucky (and for general multisums unlucky cases are known to exist) the linear q–difference equation is of minimal order, thus computing the non-commutative A–polynomial. Even if one is unlucky, there are costly factorization algorithms that in theory will compute a minimal order q–difference equation; see Petkovšek, Wilf and Zeilberger [21].

Using a computer implementation of the WZ method (Paule and Riese [18; 19; 20]), enabled the first author to give an explicit formula for the non-commutative A–polynomial of the two simplest knots: 3_1 and 4_1; see [6].

The main drawback of this implementation is that it works well when the number of summation variables is 1, but it becomes costly when the number of summation variables increases.

For 2–bridge knots, an alternative geometric method has been developed by Le that uses special properties of the Kauffman bracket skein module, Lê [16]. Unfortunately, this method cannot be extended to the case of non-2–bridge knots. In addition, the method is too costly to compute the non-commutative A–polynomial of the 5_2 knot.

Thus, two questions arise:

Question 1 How can we reduce the number of summation variables in the WZ method?

Question 2 How can we compute the non-commutative A–polynomial of the 5_2 and the 6_1 knots?

1.3 The cyclotomic function of a knot

To answer Question 1, we should look for efficient multisum formulas for the colored Jones function of a knot. Thinking geometrically, it would be better to use a single variable for a whole sequence of twists between two strands, rather than use one variable for each crossing.

As it turns out, Habiro [10] introduced such formulas for the colored Jones function of a knot. It is a good moment to review the colored Jones function, and Habiro's formulas.

A knot K in 3–space is a smoothly embedded circle, considered up to 1–parameter ambient motions of 3–space that avoid self-intersections. The *colored Jones function* J_K of a knot K is a sequence of Laurent polynomials with integer coefficients:

$$J_K \colon \mathbb{N} \longrightarrow \mathbb{Z}[q^{\pm}].$$

Technically, $J_K(n)$ is a quantum group invariant of the knot colored by the n–dimensional irreducible representation of $\mathfrak{sl}_2$, normalized to 1 for the unknot; see Turaev [23]. When $n = 2$, $J_K(2)$ is the celebrated *Jones polynomial* of a knot, introduced in [13]. One may think informally that the colored Jones function of a knot encodes the Jones polynomial of a knot and its parallels.

In [10], Habiro introduced a key repackaging of the colored Jones function J_K, namely the so-called *cyclotomic function*

$$\hat{J}_K \colon \mathbb{N} \longrightarrow \mathbb{Z}[q^{\pm}],$$

As the notation indicates, $\hat{J}_K$ is in a sense a *linear transformation* of J_K. More precisely, we have for every $n \geq 1$:

$$J_K(n) = \sum_{k=0}^{\infty} C(n,k)\hat{J}_K(k) \tag{1}$$

where

$$C(n,k) = \{n-k\}\dots\{n-1\}\{n+1\}\dots\{n+k\} \tag{2}$$

and

$$\{a\} = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}}.$$

Notice that for every fixed n, the summation in Equation (1) is finite, since $C(n,k) = 0$ for $k \geq n$.

Habiro used *an integrality property* of the cyclotomic function (namely, the fact that $\hat{J}_K(n) \in \mathbb{Z}[q^{\pm}]$ for all n) in order to show that the Ohtsuki series of an integer homology sphere determines its Witten–Reshetikhin–Turaev invariants, [10]. The same integrality property was used by Thang Lê and the first author to settle the *Volume Conjecture* to all orders, for small complex angles; see [7].

For our purposes, it is important that:

(a) The transformation $J_K \longrightarrow \hat{J}_K$ can be inverted to define $\hat{J}_K$ in terms of J_K; see for example [8, Section 4].

(b) $\hat{J}_K$ satisfies a linear q–difference equation; see [8].

(c) The cyclotomic function of twist knots has a single-sum formula; see Equation (3) below.

We will use (c) to compute a minimal q–difference equation for the cyclotomic function of twist knots. Since J_K determines and is determined by $\hat{J}_K$, in principle our results determine the non-commutative A–polynomial of twist knots. This motivates the results of our paper. En route, we will introduce the C–polynomial of a knot and its non-commutative cousin.

Due to its length, the computation of the non-commutative A–polynomial of twist knots will be postponed to a subsequent publication; see [9].

1.4 What is a q–holonomic function and a q–difference equation?

Since we will be dealing with q–difference equations all along this paper, let us review some general facts about the combinatorics and geometry of q–difference equations.

There are two synonymous terms to q–difference equations: namely *recursion relations*, and *operators*. We will adopt the *operator point of view* when dealing with recursion relations, in accordance to basic principles of physics and discrete math. An excellent reference is [21]. Likewise, there is a synonymous term to a solution of a q–difference linear equation: namely, a q–holonomic function.

For us, a *(discrete) function* f is a map:

$$f \colon \mathbb{N} \longrightarrow \mathbb{Q}(q)$$

with values in the field of rational functions in q. Consider two operators E and Q that act on the set of discrete functions by

$$(Ef)(n) = f(n+1), \qquad (Qf)(n) = q^n f(n).$$

It is easy to see that the operators E and Q satisfy $EQ = qQE$, and that E and Q generate a non-commutative *Weyl algebra*

$$\mathcal{A} = \mathbb{Q}(q)\langle Q, E\rangle/(EQ - qQE).$$

If $P = \sum_{j=0}^{d} a_j(Q,q)E^j$ is an element of $\mathcal{A}$, then the equation $Pf = 0$ is equivalent to the *linear q–difference equation*:

$$\sum_{j=0}^{d} a_j(q^n,q) f(n+j) = 0$$

for all natural numbers n. Given a a discrete function f as above, one may consider the set

$$I_f = \{P \in \mathcal{A} | Pf = 0\}$$

of all linear q–difference equations that f satisfies. It is easy to see that I_f is a left ideal in $\mathcal{A}$. The following is a key definition:

Definition 1.1 We say that f is q–holonomic iff $I_f \neq 0$.

In other words, f is q–holonomic iff it is a solution of a linear q–difference equation. Unfortunately, the Weyl algebra $\mathcal{A}$ is not a principal (left)-ideal domain. However, it becomes one after a suitable localization:

$$\mathcal{A}_{\mathrm{loc}} = \mathbb{Q}(q,Q)\langle E\rangle/\left(E\alpha(Q,q) - \alpha(qQ,q)E \,|\, \alpha(Q,q) \in \mathbb{Q}(Q,q)\right).$$

Moreover, the localized algebra still acts on discrete functions f. Thus, given a q–holonomic function f, one may define its *characteristic polynomial* $P_f \in \mathcal{A}_{\mathrm{loc}}$, which is a generator of the ideal I_f over $\mathcal{A}_{\mathrm{loc}}$. If we want to stress the dependence of an operator P on E, Q and q, we will often write $P = P(E,Q,q)$.

There are three ways to view a q–holonomic function f:

- The *D–module* $M_f = \mathcal{A}_{\mathrm{loc}}/(P_f)$ and some of its elementary invariants: its *rank*, and its *characteristic curve*

  $$\mathrm{Ch}_f = \{(E,Q) \in \mathbb{C}^2 | P_f(E,Q,1) = 0\}.$$

  The former is the E–degree of P_f and the latter is a Lagrangian complex curve in $\mathbb{C}^2$.

- The *quantization* point of view, where we think of the operator $P_f(E,Q,q)$ as a q–deformation of the polynomial $P_f(E,Q,1)$. The zeros of the latter polynomial define the characteristic curve, which is supposed to be a classical object.
- The *multi-graded* point of view. We may think of $P_f(E,Q,q)$ as a polynomial in three variables E, Q and q with integer coefficients. Then, $P_f(E,Q,q)$ and $P_f(E,Q,1)$ are, respectively, tri and bi-graded versions of $P_f(E,1,1)$.

Of course, $P_f(E,Q,q)$ is determined entirely by f.

As an example, consider the colored Jones function J_K of a knot K, and let $\mathrm{AJ}_K = \mathrm{AJ}_K(E,Q,q)$ denote its characteristic polynomial, which here and below we will call the *non-commutative A–polynomial* of the knot. The first author conjectured in [6] that the evaluation of the non-commutative A–polynomial at $q=1$ coincides with the *A–polynomial* of a knot, whose zeros parametrize the $\mathrm{SL}_2(\mathbb{C})$ character variety of the knot complement, restricted to a boundary torus. For a definition of the A–polynomial, see [4].

1.5 The non-commutative C–polynomial of twist knots

We now have all the ingredients to define the non-commutative C–polynomial of a knot.

Definition 1.2 Given a knot K, let $C_K(E,Q,q)$ denote the characteristic polynomial of its cyclotomic function $\widehat{J}_K$ and let $C_K(E,Q)$ denote $C_K(E,Q,1)$. We will call $C_K(E,Q,q)$ (resp. $C_K(E,Q)$) the *non-commutative C–polynomial* (resp. the *C–polynomial) of K*.

The reader should not confuse our C–polynomial with the *cusp polynomial* of a knot, due to X Zhang [26].

Consider the family of *twist knots* K_p for integer p, shown in Figure 1. The planar projection of K_p has $2|p|+2$ crossings, $2|p|$ of which come from the full twists, and 2 come from the negative *clasp*.

For small p, these knots may be identified with ones from Rolfsen's table (see [22]) as follows:

$$K_1 = 3_1, \quad K_2 = 5_2, \quad K_3 = 7_2, \quad K_4 = 9_2$$
$$K_{-1} = 4_1, \quad K_{-2} = 6_1, \quad K_{-3} = 8_1, \quad K_{-4} = 10_1.$$

Let $\widehat{J}_p(n)$ denote the cyclotomic function of K_p. Using Masbaum [17, Theorem 5.1] (compare also with [6, Section 3]), it follows that:

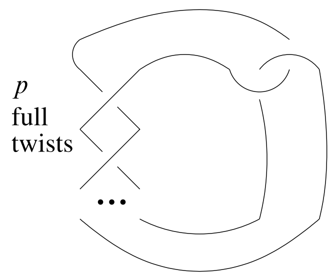

Figure 1: The twist knot K_p, for integers p

$$(3) \quad \hat{J}_p(n) = \sum_{k=0}^{\infty} q^{n(n+3)/2+pk(k+1)+k(k-1)/2}(-1)^{n+k+1}\frac{(q^{2k+1}-1)(q;q)_n}{(q;q)_{n+k+1}(q;q)_{n-k}},$$

where the *quantum factorial* and *quantum binomial coefficients* are defined by:

$$(A;q)_n = \begin{cases} (1-A)\cdots(1-Aq^{n-1}) & \text{if } n>0; \\ 1 & \text{if } n=0; \\ \frac{1}{(1-Aq^{-1})\cdots(1-Aq^{n})} & \text{if } n<0, \end{cases}$$

$$\binom{m}{n}_q = \begin{cases} \frac{(q^{m-n+1};q)_n}{(q;q)_n} & \text{if } n \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

We warn that we are using the unbalanced quantum factorials (common in discrete math) and not the balanced ones (common in the representation theory of quantum groups).

Equation (3) is the promised answer to Question 1 for the cyclotomic function of twist knots. For every fixed p, the summand in (3) is *q–proper hypergeometric* in the variables n, k. Notice that the summand is *not* q–hypergeometric in all three variables n, k, p.

Our first result is an explicit formula for the non-commutative C–polynomial of twist knots.

Definition 1.3 (a) For $p \in \mathbb{Z}$, let us define $C_p(E,Q,q) \in \mathcal{A}_{\text{loc}}$ by:

$$(4) \qquad C_p(E,Q,q) = E^{|p|} + \sum_{i=0}^{|p|-1} a_p(Q,i)E^i,$$

where
(5)

$$a_p(q^n,i)=\begin{cases} q^{(p-i)(n+p+1)}\frac{(q;q)_{n+p-1}}{(q;q)_{n+i}}\Big(\sum_{j=0}^{i} q^{(2n+p+i+1)j}\binom{p-j}{p-i}_q\binom{p-i+j-1}{j}_q \\ \qquad -\sum_{j=0}^{i-1} q^{(2n+p+i+1)j+n+p}\binom{p-j-1}{p-i}_q\binom{p-i+j-1}{j}_q\Big) & \text{if } p>0; \\ 0 & \text{if } p=0; \\ q^{(p-i+1)(n-p)}\frac{(q;q)_{n-p-1}}{(q;q)_{n+i}}\Big(-\sum_{j=0}^{i} q^{(2n-p+i)j}\binom{-p-j-1}{i-j}_q\binom{-p-i+j}{j}_q \\ \qquad +\sum_{j=0}^{i-1} q^{(2n-p+i)j+n-p}\binom{-p-j-2}{i-j-1}_q\binom{-p-i+j}{j}_q\Big) & \text{if } p<0. \end{cases}$$

In particular, $C_p(E,Q,q)$ is monic with respect to E with coefficients in $\mathbb{Z}[Q^{\pm},q^{\pm}]$.

(b) For $p\in\mathbb{Z}$, let us define $C_p(E,Q)\in\mathbb{Z}[E,Q^{\pm}]$ by:

$$C_p(E,Q)=E^{|p|}+\sum_{i=0}^{|p|-1} b_p(Q,i)E^i, \tag{6}$$

where
(7)

$$b_p(Q,i)=\begin{cases} Q^{p-i}(1-Q)^{p-i-1}\Big(\sum_{j=0}^{i} Q^{2j}\binom{p-j}{p-i}\binom{p-i+j-1}{j} \\ \qquad -\sum_{j=0}^{i-1} Q^{2j+1}\binom{p-j-1}{p-i}\binom{p-i+j-1}{j}\Big) & \text{if } p>0; \\ 0 & \text{if } p=0; \\ Q^{-p-i+1}(1-Q)^{-p-i-1}\Big(-\sum_{j=0}^{i} Q^{2j}\binom{-p-j-1}{i-j}\binom{-p-i+j}{j} \\ \qquad +\sum_{j=0}^{i-1} Q^{2j+1}\binom{-p-j-2}{i-j-1}\binom{-p-i+j}{j}\Big) & \text{if } p<0. \end{cases}$$

Notice that Equation (7) uniquely determines $a_p(Q,i)$. A direct definition of $a_p(Q,i)$ would be cumbersome, since there is no nice formula for $\binom{k}{l}_q$ as a rational function in q and $Q=q^n$ when k and l are linear forms on n.

Theorem 1 *For every $p\in\mathbb{Z}$, $C_p(E,Q,q)$ is the non-commutative C–polynomial of the twist knot K_p.*

An immediate corollary is:

Corollary 1.4 *For every $p\in\mathbb{Z}$, $C_p(E,Q)$ is the C–polynomial of the twist knot K_p.*

Our next result gives a 3–term recursion relation (with respect to p) for the C–polynomial of twist knots.

Theorem 2 (a) The C–polynomial of twist knots satisfies the 3–term relation:

$$C_{p+2}(E,Q) = (Q - Q^2 + E + Q^2 E)\, C_{p+1}(E,Q) - Q^2 E^2\, C_p(E,Q) \tag{8}$$

for all $p \geq 0$, with initial conditions:

$$C_0(E,Q) = 1, \qquad C_1(E,Q) = Q + E.$$

Likewise, for $p \leq 0$ it satisfies a 3–term recursion relation:

$$C_{p-2}(E,Q) = (Q - Q^2 + E + Q^2 E) Q^{-2}\, C_{p-1}(E,Q) - E^2 Q^{-2}\, C_p(E,Q), \tag{9}$$

with intial conditions

$$C_0(E,Q) = 1, \qquad C_{-1}(E,Q) = -1 + E.$$

(b) Moreover,

$$C_p^{\mathrm{op}}(M - 2 + M^{-1}, 1) = \Delta_p(M) \tag{10}$$

for all p.

Here, $\Delta_K(t) \in \mathbb{Z}[t^{\pm}]$ denotes the *Alexander polynomial* of a knot, normalized by $\Delta_K(t) = \Delta_K(t^{-1})$, and $\Delta_{\mathrm{unknot}}(t) = 1$; see [22]. Moreover, $\Delta_p(t)$ denotes the Alexander polynomial of the twist knot K_p.

In addition, if $P = P(E,Q)$ is a polynomial, then P^{op}

$$P^{\mathrm{op}}(E,Q) = E^{\deg_E P} P(E^{-1}, Q) \tag{11}$$

is essentially P, with its E–powers reversed.

1.6 Relation between the A–polynomial and the C–polynomial of twist knots

The next theorem relates the C–polynomial of twist knots to the better-known A–polynomial of [4]. In order to formulate our next theorem, we need to define a rational map of degree 2

$$\phi\colon \mathbb{Q}(E,Q) \longrightarrow \mathbb{Q}(L,M) \tag{12}$$

by

$$\phi(E) = \frac{L(M^2-1)^2}{M(L+M)(1+LM)}, \qquad \phi(Q) = \frac{1+LM}{L+M}.$$

For a motivation of this rational map, see Section 5.1.

Let $A_p(L,M)$ denote the A–polynomial of the twist knot K_p. The later has been computed by Hoste–Shanahan in [12, Theorem 1] (where is was denoted by $A_{J(2,2p)}$). It is known that the A–polynomial of a knot in S^3 has even powers in M^2.

Theorem 3 (a) For every $p \in \mathbb{Z}$ we have:

$$\phi C_p^{\mathrm{op}}(E,Q) = A_p(L,M^{1/2}) \cdot \begin{cases} \frac{(1+LM)^p}{M^p(L+M)^{3p-1}} & \text{if } p \geq 0; \\ \frac{1}{(M(L+M)(1+LM))^{|p|}} & \text{if } p < 0. \end{cases} \tag{13}$$

(b) For every p, C_p is an irreducible polynomial over $\mathbb{Q}[E,Q]$.

We can phrase the above theorem geometrically, as follows. The rational map ϕ gives a rational map $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ where the domain has coordinates (E,Q) and the range has coordinates (L,M). Then, we can restrict the above map to the affine curves defined by C_p^{op} and A_p.

Corollary 1.5 *For all twist knots K, the map ϕ of* (12) *induces a Zariski dense map of degree* 2*:*

$$\phi\colon \{(L,M) \in \mathbb{C}^2 \mid A_K(L,M) = 0\} \longrightarrow \{(E,Q) \in \mathbb{C}^2 \mid C_K^{\mathrm{op}}(E,Q) = 0\} \tag{14}$$

Thus, one can associate two *plane curves* to a knot, namely the A–curve, and the C–curve, which, in the case of twist knots, are related by the map ϕ above. Thus one may consider their *degrees* and their *genus*, discussed at length in Kirwan [14]. The genus has the advantage of being a *birational invariant*.

Rather than diverge to a lengthy algebraic geometry discussion, outside the scope of the present paper, we state our next result here, and postpone its proof in a subsequent publication.

Theorem 4 *For every $p \in \mathbb{Z}$, the genus of the $C_p(E,Q)$ polynomial is zero.*

The proof uses the *Noether formula* for the genus of a plane curve (see [14, Theorem 7.37]):

$$\operatorname{genus}(C) = \frac{(d-1)(d-2)}{2} - \sum_P \delta(P) \tag{15}$$

where d is the degree and the (finite) sum is over the delta invariants of the singular points of C. Since $\delta(P) > 0$ at the singular points of C, and the left hand side is nonnegative, if one finds enough singular points P' such that the contribution makes the right hand side vanish, then if follows that these are all the singular points of C

and moreover, the genus of C is zero. In our case, $d = 3|p|-2$, the singular points P' are

$$[0,0,1], \quad [1,0,0], \quad [0,1,1], \quad [0,1,0].$$

in homogeneous coordinates, and their delta invariants are given by:

$$\begin{aligned} \delta([0,0,1]) &= |p|(|p|-1) & \delta([1,0,0]) &= (2|p|-3)(|p|-1) \\ \delta([0,1,1]) &= |p|-1 & \delta([0,1,0]) &= |p|-1. \end{aligned}$$

As a comparison, a Maple computation confirms that for $|p| \leq 30$ we have:

$$\text{genus}(A_p(L, M^{1/2})) = \begin{cases} 2p-2 & \text{if } p > 0; \\ 2|p|-1 & \text{if } p < 0. \end{cases} \tag{16}$$

Unfortunately, the above method does not prove that the genus of the A_p polynomial is given by (16) for all p, since it is hard to prove that the *only* singular points of A_p for all p are the ones suggested by Maple.

We thank N. Dunfield suggestions and for pointing the curious fact about the genus of the C_p polynomials.

1.7 Plan of the proof

As is obvious from a brief look, the paper tries to bring together two largely disjoint areas: Quantum Topology and the Discrete Mathematics. Thus, the proofs require some knowledge of both areas. We have tried to separate the arguments in different sections, for different audiences.

In Section 2, we show that the sequence of polynomials $C_p(E, Q)$ from Equation (6) satisfy the 3–term recursion relations (8) and (9). Combining this result with a 3–term recursion relation for the A–polynomial of twist knots (due to Hoste–Shanahan), together with a matching of initial conditions allows us to prove Equation (13). A side-bonus of Equation (13) and of work of Hoste–Shanahan (that uses ideas from hyperbolic geometry) is that the non-commutative polynomials of Equation (4) are irreducible– a property that the WZ algorithms cannot guarantee in general.

In Section 3, we give a crash course on the WZ algorithm that computes recursion relations of sums of hypergeometric functions. The ideas are beautiful and use elementary linear algebra. Using an explicit formula for the cyclotomic function of twist knots (given in terms of a single sum of a q–hypergeometric function), in Section 4 we apply the WZ algorithm to confirm that the cyclotomic function of twist knots satisfies the q–difference equation

$$C_p(E, Q)\hat{J}_p(n) = 0$$

for all $n \in \mathbb{N}$. This, together with the irreducibility of $C_p(E, Q, q)$ obtained above, conclude the proof of Theorem 1.

In Section 5 we present some open questions (all confirmed for twist knots) about the structure of the C–polynomial and the A–polynomial of knots.

Finally, in the Appendix we give a table of the non-commutative C–polynomial of twist knots with at most 3 crossings.

1.8 Acknowledgements

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2 Proof of Theorems 2 and 3

2.1 Proof of Theorem 2

Proof (of Theorem 2) Consider the family of polynomials $C_p(E, Q)$ given by (6). In this section we will show that this family of polynomials satisfies the recursions stated in Theorem 2. Together with Theorem 1 (to be shown later), it will conclude the proof of Theorem 2.

For convenience, we will convert the recursions in (8) and (9) in backward shifts. That is, we define

$$\begin{aligned} D_p(E, Q) &= C_p^{\mathrm{op}}(E, Q) \\ &= C_p(E^{-1}, Q) E^{|p|}. \end{aligned}$$

Then we need to prove that

$$D_p(E, Q) = (QE - Q^2(E-1) + 1) D_{p-1}(E, Q) - Q^2 D_{p-2}(E, Q). \tag{17}$$

It is clear from Equation (31) that

$$D_p(E, Q) = 1 + \sum_{i=1}^{|p|} b'_p(Q, i) E^i \tag{18}$$

and

$$b'_p(Q,i) = Q^i(1-Q)^{i-1}$$
$$\cdot\begin{cases} \left(\sum_{j=0}^{p-i} Q^{2j}\binom{p-j}{i}\binom{i+j-1}{j} - \sum_{j=0}^{p-i-1} Q^{2j+1}\binom{p-j-1}{i}\binom{i+j-1}{j}\right) & \text{if } p>0; \\ 0 & \text{if } p=0; \\ Q^{2p+1}\left(-\sum_{j=0}^{-p-i} Q^{2j}\binom{-p-j-1}{i-1}\binom{i+j}{j} + \sum_{j=0}^{-p-i-1} Q^{2j+1}\binom{-p-j-2}{i-1}\binom{i+j}{j}\right) & \\ & \text{if } p<0. \end{cases}$$

When $p>0$, let

$$s_p^{(1)}(E,Q,i,j) = Q^i(1-Q)^{i-1}Q^{2j}\binom{p-j}{p-i-j}\binom{i+j-1}{j}E^i,$$
$$s_p^{(2)}(E,Q,i,j) = Q^i(1-Q)^{i-1}Q^{2j+1}\binom{p-j-1}{p-i-j-1}\binom{i+j-1}{j}E^i,$$
$$D_p^{(1)}(E,Q) = \sum_{i=1}^{p}\sum_{j=0}^{p-i} s_p^{(1)}(E,Q),$$
$$D_p^{(2)}(E,Q) = \sum_{i=1}^{p}\sum_{j=0}^{p-i-1} s_p^{(2)}(E,Q),$$

and we have

$$D_p(E,Q) = 1 + D_p^{(1)}(E,Q) - D_p^{(2)}(E,Q). \tag{19}$$

Using the same method mentioned before, it is easy to check that both $s_p^{(1)}(E,Q,i,j)$ and $s_p^{(2)}(E,Q,i,j)$ satisfy the *same* recursion

$$\begin{aligned} &-Q^2 s_{p-2}^{(l)}(E,Q,i,j-1) - E(-1+Q)Q s_{p-1}^{(l)}(E,Q,i-1,j) \\ &+Q^2 s_{p-1}^{(l)}(E,Q,i,j-1) + s_{p-1}^{(l)}(E,Q,i,j) - s_p^{(l)}(E,Q,i,j) = 0, \end{aligned}$$

for $l=1,2$.

Summing the above recursion over $i\ge 1$ and $j\ge 0$, and noticing that $t_p^{(1)}(E,Q,i,j) = 0$ when $i>p$, $j<0$, or $j>p-i$, we obtain

$$\begin{aligned} &-Q^2 D_{p-2}^{(1)}(E,Q) - E(-1+Q)QD_{p-1}^{(1)}(E,Q) + Q^2 D_{p-1}^{(1)}(E,Q) \\ &\qquad\qquad + D_{p-1}^{(1)}(E,Q) - D_p^{(1)}(E,Q) \\ &= E(-1+Q)Q s_{p-1}^{(1)}(E,Q,0,0) \\ &= -EQ. \end{aligned} \tag{20}$$

Similarly

$$(21)\qquad -Q^2D^{(2)}_{p-2}(E,Q)-E(-1+Q)QD^{(2)}_{p-1}(E,Q)+Q^2D^{(2)}_{p-1}(E,Q) + D^{(2)}_{p-1}(E,Q)-D^{(2)}_{p}(E,Q)$$
$$= E(-1+Q)Qs^{(2)}_{p-1}(E,Q,0,0)$$
$$= -EQ^2.$$

Now Equation (17) follows immediately from Equations (19), (20) and (21), which proves the theorem for $p>0$.

For the case of $p<0$, it is interesting that the backward-shifting 3–term recursion is the same as that when $p>0$. To prove it, we only need to define, like before,

$$s^{(1)}_p(E,Q,i,j)=Q^{2p+1+i}(1-Q)^{i-1}Q^{2j}\binom{p-j-1}{p-i-j}\binom{i+j}{j},$$
$$s^{(2)}_p(E,Q,i,j)=Q^{2p+1+i}(1-Q)^{i-1}Q^{2j+1}\binom{-p-j-2}{p-i-j-1}\binom{i+j}{j},$$

and realize that both of them satisfy the same recursion

$$-Q^2s^{(l)}_{p-2}(E,Q,i,j)-E(-1+Q)Qs^{(l)}_{p-1}(E,Q,i-1,j)$$
$$+Q^2s^{(l)}_{p-1}(E,Q,i,j-1)+s^{(l)}_{p-1}(E,Q,i,j)-s^{(l)}_{p}(E,Q,i,j-1)=0,$$

for $l=1,2$. This finishes the proof of part (a) of Theorem 2.

It remains to prove Equation (10). The recursion relation for C_p from part (a) together with the initial conditions imply that

$$C^{\mathrm{op}}_p(E,1)=1+pE.$$

Since $\Delta_{K_p}(M)=1+p(M+M^{-1}-2)$, this concludes Equation (10) and Theorem 2. □

2.2 Proof of Theorem 3

Consider the family of polynomials $C_p(E,Q)$ given by Equation (6). In Section 2.1 we showed that $C_p(E,Q)$ satisfy the 3–term recursion relations (8) and (9).

We will show that their evaluation $\phi C^{\mathrm{op}}(E,Q)$ satisfies Equation (13). In [12, Theorem 1], Hoste–Shanahan give a 3–term recursion relation for the A–polynomial of twist knots:

$$A_p(L,M)=x(L,M)A_{p-\mathrm{sgn}(p)}(L,M)-y(L,M)A_{p-2\mathrm{sgn}(p)}(L,M),$$

where

$$x(L,M) = -L+L^2+2LM^2+M^4+2LM^4+L^2M^4+2LM^6+M^8-LM^8,$$
$$y(L,M) = M^4(L+M^2)^4.$$

Theorem 2 gives a 3–term relation for the C–polynomial of twist knots. Assume, for simplicity, that $p>0$. Then, Theorem 2 implies that $C_p^{\mathrm{op}}(E,Q)$ satisfies a 3–term relation:

$$C_{p+2}^{\mathrm{op}}(E,Q) = (EQ - EQ^2 + 1 + Q^2)\, C_{p+1}^{\mathrm{op}}(E,Q) - Q^2\, C_p^{\mathrm{op}}(E,Q).$$

Using the rational map ϕ of Equation (12), a computation shows that

$$\phi(EQ - EQ^2 + 1 + Q^2) = \frac{1+LM}{M(L+M)^2} x(L, M^{1/2}), \qquad \phi(Q) = \frac{(1+LM)^2}{(L+M)^2}.$$

Thus, it follows that both sides of (13) satisfy the same 3–term recursion relation for $p>0$. Moreover, an explicit computation shows that (13) is verified for $p=1,2$. The result follows for $p>0$, and similarly for $p<0$.

For part (b), Hoste–Shanahan prove that the A–polynomial of twist knots is irreducible; see [11]. This, together with Equation (13) implies that any nontrivial factor of $C_p^{\mathrm{op}}(E,Q)$ must satisfy the property that its image under ϕ is a monomial in $1+LM$ or $L+M$. This implies that any nontrivial factor of $C_p^{\mathrm{op}}(E,Q)$ will be of the form $(Q\pm 1)^2 + QE$. If $C_p^{\mathrm{op}}(E,Q)$ had any such factor, then evaluating at $Q=\mp 1$, it follows that E divides $C_p(E,\mp 1)$. This is a contradiction, by the explicit formula of Corollary 1.4. □

3 A crash course on the WZ algorithm and Creative Telescoping

In this section we review briefly some key ideas of Zeilberger on recursion relations of combinatorial sums. An excellent reference is [21], which we urge the reader for references of the results in this section.

A *term* is $F(n,k)$ called *hypergeometric* if both $\frac{F(n+1,k)}{F(n,k)}$ and $\frac{F(n,k+1)}{F(n,k)}$ are rational functions over n and k. In other words,

(22) $$\frac{F(n+1,k)}{F(n,k)} \in \mathbb{Q}(n,k), \qquad \frac{F(n,k+1)}{F(n,k)} \in \mathbb{Q}(n,k).$$

Examples of hypergeometric terms are $F(n,k) = (an+bk+c)!$ (for integers a,b,c), and ratios of products of such. The latter are actually called *proper hypergeometric*. A key problem is to construct recursion relations for sums of the form

$$S(n) = \sum_k F(n,k), \tag{23}$$

where $F(n,k)$ is a proper hypergeometric term. The summation can be defined to be over all integers, even though in the cases that we consider in the paper, the summand vanishes for negative integers. Due to the telescoping nature of Sister Celine's method, we may allow for a definite or indefinite summation range. Sister Celine proved the following:

Theorem 5 *Given a proper hypergeometric term $F(n,k)$, there exist an integer I and a set of functions $a_i(n) \in \mathbb{Q}(n)$, $0 \le i \le I$, such that*

$$\sum_{i=0}^{I} a_i(n) F(n+i,k) = 0. \tag{24}$$

The important part of the above theorem is that the functions $a_i(n)$ are independent of k. Therefore if we take the sum over k on both sides, we get

$$\sum_{i=0}^{I} a_i(n) \sum_k F(n+i,k) = 0. \tag{25}$$

In other words, we have:

$$\sum_{i=0}^{I} a_i(n) S(n+i) = 0. \tag{26}$$

So, Equation (24) produces a recursion relation. How can we find functions $a_i(n)$ that satisfy Equation (24)? The idea is simple: divide Equation (24) by $F(n,k)$, and use (22) to convert the divided equation into an equation over the field $\mathbb{Q}(n,k)$. Moreover, $a_i(n)$ appear linearly. Clearing denominators, we arrive at an equation (linear with respect to $a_i(n)$) over $\mathbb{Q}(n)[k]$. Thus, the coefficients of every power of k must vanish, and this gives a linear system of equations over $\mathbb{Q}(n)$ with unknowns $a_i(n)$. If there are more unknowns than equations, one is guaranteed to find a nonzero solution. By a counting argument, one may see that if we choose I high enough (this depends on the complexity of the term $F(n,k)$), then we have more equations than unknowns.

We should mention that although it can be numerically challenging to find $a_i(n)$ that satisfy Equation (24), it is routine to check the equation once $a_i(n)$ are given. Indeed, one only need to divide the equation by $F(n,k)$, and then check that a function in $\mathbb{Q}(n,k)$ is identically zero. The latter is computationally trivial.

This algorithm produces a recursion relation for $S(n)$. However, it is known that the algorithm does not always yield a recursion relation of the smallest order.

Applying Gosper's algorithm, Wilf and Zeilberger invented another algorithm, the WZ algorithm. Instead of looking for 0 on the right-hand side of Equation (24), they instead looked for a function $G(n,k)$ such that

$$\sum_{i=0}^{N} a_i(n)F(n+i,k) = G(n,k+1) - G(n,k). \tag{27}$$

Summing over k, and using telescoping cancellation of the terms in the right hand side, we get a recursion relation for $S(n)$. How to find the $a_i(n)$ and $G(n,k)$ that satisfy (27)? The idea is to look for a *rational function* $\mathrm{Cert}(n,k)$ (the so-called *certificate* of (27)) such that

$$G(n,k) = \mathrm{Cert}(n,k)F(n,k).$$

Dividing out (27) by $F(n,k)$ as before, one reduces this to a problem of linear algebra. Just as before, given $a_i(n)$ and $\mathrm{Cert}(n,k)$, it is routine to check whether (27) holds.

Now, let us rephrase the above equations using operators. Let us define two operators N and K that act on a function $F(n,k)$ by:

$$(NF)(n,k) = F(n+1,k), \qquad (KF)(n,k) = F(n,k+1).$$

Then, we can rewrite Equation (27) as

$$\left(\sum_{i=0}^{I} a_i(n)N^i\right) F(n,k) = (K-1)G(n,k) = (K-1)\mathrm{Cert}(n,k)F(n,k). \tag{28}$$

Here, we think of n and k as operators acting on functions $F(n,k)$ by multiplication by n and k respectively. In other words,

$$(nF)(n,k) = nF(n,k), \qquad (kF)(n,k) = kF(n,k).$$

Beware that the operators N and n do not commute. Instead, we have:

$$Nn = (n+1)N,$$

and similarly for k and K.

Implementation of the algorithms are available in various platforms, such as, Maple and Mathematica. See, for example, [25] and [19].

Let us mention one more point regarding Creative Telescoping, namely the issue of dealing with boundary terms. In the applications below, one considers not quite the unrestricted sums of Equation (23), but rather restricted ones of the form:

$$S'(n) = \sum_{k=0}^{\infty} F(n,k), \tag{29}$$

where $F(n,k)$ is a proper hypergeometric term. When we apply the Creating Telescoping summation, we are left with some boundary terms $R(n) \in \mathbb{Q}(n)$. In that case, Equation (26) becomes:

$$\left(\sum_{i=0}^{I} a_i(n) N^i\right) S'(n) = R(n).$$

This is an inhomogeneous equation of order I which we can convert into a homogeneous recursion of order $I+1$ by following trick: apply the operator

$$(N-1)\frac{1}{R(n)}$$

on both sides of the recursion, we get

$$\left(\frac{1}{R(n+1)}N - \frac{1}{R(n)}\right)\left(\sum_{i=0}^{I} a_i(n) N^i\right) S'(n) = 0,$$

i.e. $$\left(\frac{a_I(n+1)}{R(n+1)}N^{I+1} + \sum_{i=1}^{I}\left(\frac{a_{i-1}(n+1)}{R(n+1)} - \frac{a_i(n)}{R(n)}\right)N^i - \frac{a_0(n)}{R(n)}\right) S'(n) = 0.$$

One final comment before we embark in the proof of the stated recursion relations. In Quantum Topology we are using q–factorials rather than factorials. The previous results translate without conceptual difficulty to the q–world, although the computer implementation costs more, in time. A *term* is $F(n,k)$ called *q–hypergeometric* if

$$\frac{F(n+1,k)}{F(n,k)}, \frac{F(n,k+1)}{F(n,k)} \in \mathbb{Q}(q, q^n, q^k).$$

Examples of q–hypergeometric terms are the quantum factorials of linear forms in n, k, and ratios of products of quantum factorials and q raised to quadratic functions of n and k. The latter are called *q–proper hypergeometric*.

Sister Celine's algorithm and the WZ algorithm work equally well in the q–case. In either algorithms, we can (roughly speaking) replace n and k with q^n and q^k respectively, and the rest of the original proofs still apply naturally. The implementations of the q–case include [18], [15] and [25].

4 The non-commutative C–polynomial of twist knots

4.1 Proof of Theorem 1

First, let us make a remark for the trivial twist knot K_0.

Remark 4.1 The colored Jones function of the trivial knot is $J_0(n) = 1$ for all $n \geq 1$. Consequently, the cyclotomic function of the trivial knot is $\hat{J}_0(n) = \delta_{n,0}$ (that is, 1 when $n = 0$ and 0 otherwise). The non-commutative C–polynomial of the trivial knot is $C_0(E, Q, q) = 1$. The A–polynomial of the trivial knot is $A_0(L, M) = 1$ and the Alexander polynomial of the trivial knot is $\Delta_0(M) = 1$. This confirms all our theorems for $p = 0$.

Proof (of Theorem 1) First we will prove that $\hat{J}_p(n)$ satisfies the recursion relation:

$$C_p(E, Q, q)\hat{J}_p = 0, \tag{30}$$

where $C_p(E, Q, q)$ is given by Equation (4). We begin with rewriting the above equation as a recursion in backward shifts:

$$\hat{J}_p(n) + \sum_{i=1}^{|p|} a'_p(n,i)\hat{J}_p(n-i) = 0, \tag{31}$$

where

$$a'_p(n,i) = \begin{cases} q^{i(n+1)}\frac{(q;q)_{n-1}}{(q;q)_{n-i}}\Big(\sum_{j=0}^{p-i} q^{(2n-i+1)j}\binom{p-j}{i}_q\binom{i+j-1}{j}_q \\ \qquad -\sum_{j=0}^{p-i-1} q^{(2n-i+1)j+n}\binom{p-j-1}{i}_q\binom{i+j-1}{j}_q\Big) & \text{if } p > 0; \\ 0 & \text{if } p = 0; \\ q^{(2p+i+1)n}\frac{(q;q)_{n-1}}{(q;q)_{n-i}}\Big(-\sum_{j=0}^{-p-i} q^{(2n-i)j}\binom{-p-j-1}{i-1}_q\binom{i+j}{j}_q \\ \qquad +\sum_{j=0}^{-p-i-1} q^{(2n-i)j+n}\binom{-p-j-2}{i-1}_q\binom{i+j}{j}_q\Big) & \text{if } p < 0. \end{cases}$$

When $p > 0$, we define a number of functions for the purpose of convenience:

$$s_p(n,k) = \frac{q^{n(n+3)/2+pk(k+1)+k(k-1)/2}(-1)^{n+k+1}(q^{2k+1}-1)(q;q)_n}{(q;q)_{n+k+1}(q;q)_{n-k}},$$

$$t_p^{(1)}(n,k,i,j) = (-1)^i q^{-i(2n-i+3)/2+i(n+1)+(2n-i+1)j}$$
$$\frac{(q;q)_{n-1}(q;q)_{n+k+1}(q;q)_{n-k}}{(q;q)_n(q;q)_{n+k-i+1}(q;q)_{n-i-k}}\binom{p-j}{p-i-j}_q\binom{i+j-1}{j}_q,$$

$$t_p^{(2)}(n,k,i,j) = (-1)^i q^{-i(2n-i+3)/2+i(n+1)+(2n-i+1)j} \cdot \frac{(q;q)_{n-1}(q;q)_{n+k+1}(q;q)_{n-k}}{(q;q)_n(q;q)_{n+k-i+1}(q;q)_{n-i-k}} \binom{p-j-1}{p-i-j-1}_q \binom{i+j-1}{j}_q,$$

$$r_p(n,k) = \sum_{i=1}^{p} \frac{a'_p(n,i)s_p(n-i,k)}{s_p(n,k)},$$

$$\mathrm{Cert}_p(n,k) = \frac{q^{pk+pn+p}(q^{k+1}-1)(q^n-q^k)}{(q^{2k+1}-1)(q^n-1)},$$

$$D_p(n,k) = \mathrm{Cert}_p(n,k) - \mathrm{Cert}_p(n,k-1)\frac{s_p(n,k-1)}{s_p(n,k)} - 1.$$

It is clear that

$$\sum_{k\geq 0} s_p(n,k) = \hat{J}_p(n).$$

Since

$$t_p^{(h)}(n,k,i,j) = 0 \quad \text{if } j > p-i-h+1 \text{ or } i > p, \quad \text{when } h = 1,2,$$

and

$$\frac{s_p(n-i,k)}{s_p(n,k)} = (-1)^{-i} q^{-i(2n-i+3)/2} \frac{(q;q)_{-i+n}(q;q)_{-k+n}(q;q)_{1+k+n}}{(q;q)_n(q;q)_{-i-k+n}(q;q)_{1-i+k+n}},$$

we obtain

$$\begin{aligned}\sum_{j\geq 0} t_p^{(1)}(n,k,i,j) - \sum_{j\geq 0} t_p^{(2)}(n,k,i,j) &= \sum_{j=0}^{p-i} t_p^{(1)}(n,k,i,j) - \sum_{j=0}^{p-i-1} t_p^{(2)}(n,k,i,j) \\ &= a'_p(n,i)\frac{s_p(n-i,k)}{s_p(n,k)},\end{aligned}$$

and therefore

$$\begin{aligned}\sum_{i\geq 1}\sum_{j\geq 0}\left(t_p^{(1)}(n,k,i,j) - t_p^{(2)}(n,k,i,j)\right) &= \sum_{i=1}^{p} a'_p(n,i)\frac{s_p(n-i,k)}{s_p(n,k)} \\ &= r_p(n,k).\end{aligned}$$

We are going to show that

$$1 + r_p(n,k) = \mathrm{Cert}_p(n,k) - \mathrm{Cert}_p(n,k-1)\frac{s_p(n,k-1)}{s_p(n,k)}. \tag{32}$$

If (32) is true, we can multiply both sides by $s_p(n,k)$ and obtain

$$s_p(n,k)+\sum_{i=1}^{p} a'_p(n,i)s_p(n-i,k)=\mathrm{Cert}_p(n,k)s_p(n,k)-\mathrm{Cert}_p(n,k-1)s_p(n,k-1).$$

Summing over $k\geq 0$, and using telescoping summation of the right hand side, and the boundary condition $\mathrm{Cert}_p(n,-1)=0$, completes the proof of (31). Notice incidentally that $\mathrm{Cert}_p(n,k)$ is the corresponding certificate of (4) in the WZ algorithm.

A recursion for both of the functions $t_p^{(h)}(n,k,i,j)$, $h=1,2$, is

$$\begin{aligned}
&-q^p(q^k-q^n)(-q+q^n)(-1+q^{1+k+n})t_{p-1}^{(h)}(-1+n,k,-1+i,j)\\
&+q^{2+k+2n}(-1+q^n)t_{p-2}^{(h)}(n,k,i,-1+j)\\
&-q^{2+k+2n}(-1+q^n)t_{p-1}^{(h)}(n,k,i,-1+j)\\
&-q^{2+k}(-1+q^n)t_{p-1}^{(h)}(n,k,i,j)+q^{2+k}(-1+q^n)t_p^{(h)}(n,k,i,j)=0.
\end{aligned}$$

This can be checked by dividing the equation by $t_p^{(h)}(n,k,i,j)$ and then both sides are rational functions in q,q^n,q^p,q^k; the identity can then be checked easily.

Summing over $i\geq 1$ and $j\geq 0$, and noticing that $t_p^{(h)}(n,k,i,-1)=0$, we get

$$\begin{aligned}
&-q^p(q^k-q^n)(-q+q^n)(-1+q^{1+k+n})r_{p-1}(n-1,k)\\
&+q^{2+k+2n}(-1+q^n)r_{p-2}(n,k)-q^{2+k+2n}(-1+q^n)r_{p-1}(n,k))\\
&-q^{2+k}(-1+q^n)r_{p-1}(n,k)+q^{2+k}(-1+q^n)r_p(n,k)\\
&=q^p(q^k-q^n)(-q+q^n)(-1+q^{1+k+n})\left(t_{p-1}^{(1)}(n,k,0,0)-t_{p-1}^{(2)}(n,k,0,0)\right)\\
&=q^p(-q+q^n)(-q^k+q^n)(-1+q^{1+k+n}).
\end{aligned}\tag{33}$$

What is left to prove now is that $D_p(n,k)$ satisfies the same recursion as in (33), and (32) is true for all n when $p=1$ and 2. Checking the former assertion is simple arithmetic since $D_p(n,k)$ is a rational function, while the latter can be proved by checking (31) directly for $p=1$ and 2. For any specific p, $s_p(n,k)$ is hypergeometric, so this can be done using any of the software packages developed for the WZ algorithm; see for example [18].

When $p<0$, we can define

$$t_p^{(1)}(n,k,i,j) = (-1)^{-i} q^{-i(2n-i+3)/2+(2p+i+1)n+(2n-i)j}$$
$$\frac{(q;q)_{n-1}(q;q)_{-k+n}(q;q)_{1+k+n}}{(q;q)_n(q;q)_{-i-k+n}(q;q)_{1-i+k+n}} \binom{-p-j-1}{-p-i-j}_q \binom{i+j}{j}_q,$$
$$t_p^{(2)}(n,k,i,j) = (-1)^{-i} q^{-i(2n-i+3)/2+(2p+i+1)n+(2n-i)j}$$
$$\frac{(q;q)_{n-1}(q;q)_{-k+n}(q;q)_{1+k+n}}{(q;q)_n(q;q)_{-i-k+n}(q;q)_{1-i+k+n}} \binom{-p-j-2}{-p-i-j-1}_q \binom{i+j}{j}_q,$$

and follow the same steps as above, where we only need to mention that both of the functions satisfy the following recursion

$$\begin{aligned}
&-q^p(q^k-q^n)(-q+q^n)(-1+q^{1+k+n})t_{p-1}(n-1,k,i-1,j)\\
&+q^{2+k+2n}(-1+q^n)t_{p-2}(n,k,i,j)-q^{2+k+2n}(-1+q^n)t_{p-1}(n,k,i,j-1)\\
&-q^{2+k}(-1+q^n)t_{p-1}(n,k,i,j)+q^{2+k}(-1+q^n)t_p(n,k,i,j-1)=0.
\end{aligned}$$

So far, we have shown that $\widehat{J}_p(n)$ is annihilated by an explicit operator $C_p(E,Q,q)$:

$$C_p(E,Q,q)\widehat{J}_p = 0.$$

If we prove that the above recursion has minimal E-degree, it will follow that $C_p(E,Q,q)$ is indeed the non-commutative C–polynomial of the twist knot K_p. Since $C_p(E,Q,q)$ is monic in E, minimality will follows from the fact that the polynomial $C_p(E,Q,1)$ is irreducible over $\mathbb{Q}[E,Q]$. This in turn follows from part (b) of Theorem 3 and by the fact that the A–polynomial of twist knots is irreducible (see [11]). This concludes the proof of Theorem 1. □

5 Odds and ends

5.1 Motivation for the rational map ϕ

In this section we give some motivation for the strange-looking rational map ϕ. We warn the reader that this section is heuristic, and not rigorous. However, it provides a good motivation.

Let us fix a sequence:

$$f\colon \mathbb{N} \longrightarrow \mathbb{Q}(q)$$

and let

$$g = \hat{f}\colon \mathbb{N} \longrightarrow \mathbb{Q}(q)$$

be defined by:

$$g(n) = \sum_{k=0}^{\infty} C(n,k) g(k),$$

where $C(n,k)$ are as in Equation (2). Let us suppose that $f(k)$ is annihilated by an operator

$$P_f(E_k, Q_k, q) = \sum_{j=0}^{d} a_j(q, q^j) E_k^j.$$

The question is to find (at least heuristically) an operator $P_g(E, Q, q)$ that annihilates $g(n)$. To achieve this, we will work in the Weyl algebra $\mathcal{W}$ generated by the operators E, Q, E_k and Q_k with the usual commutation relations.

Since $C(n,k)$ is closed form, a calculation shows that:

$$\begin{aligned}
\frac{C(n+1,k)}{C(n,k)} &= \frac{(1-q^{-n})(1-q^{1+k+n})}{(1-q^{k-n})(1-q^{1+n})} \\
\frac{C(n,k+1)}{C(n,k)} &= -q^{-1-k}(1-q^{1+k-n})(1-q^{1+k+n}).
\end{aligned}$$

In other words, $C(n,k)$ is annihilated by the left ideal in $\mathcal{W}$ generated by the operators P_1 and P_2 where

$$\begin{aligned}
P_2 &= (1-Q^{-1})(1-qQQ_k) - (1-Q_kQ^{-1})(1-qQ)E \\
P_1 &= -q^{-1}Q_k^{-1}(1-qQ_kQ^{-1})(1-qQ_kQ) - E_k
\end{aligned}$$

Lemma 5.1 *$g(n)$ is annihilated by the operators P_1 and P where*

$$P = \sum_{j=0}^{d} a_j(q, q^k) \frac{C(n,k+d)}{C(n,k+j)} E_k^j.$$

Proof It is easy to see that $g(n)$ is annihilated by P_1. Moreover,

$$\begin{aligned}
0 &= \sum_{j=0}^{d} a_j(q,q^k) f(k+j) \\
&= \frac{1}{C(n,k+d)} \sum_{j=0}^{d} a_j(q,q^k) \frac{C(n,k+d)}{C(n,k+j)} C(n,k+j) f(k+j)
\end{aligned}$$

$$= \frac{1}{C(n,k+d)} P g(n).$$

Thus, P annihilates $g(n)$. □

According to Sister Celine's algorithm, we want to eliminate Q_k (thus obtaining k–free operators), and then set $E_k = 1$. This will produce an operator in E, Q and q that annihilates $g(n)$. Finally, after setting $q = 1$, we will get a polynomial which contains the characteristic polynomial of $g(n)$.

Now, here comes the heuristic: let us commute the evaluation at $q = 1$ from last to first, and denote it by ϵ. Let us define two rational functions $R_1, R_2 \in \mathbb{Q}(Q, Q_k)$ by:

$$\begin{aligned} R_1(Q, Q_k) &= \frac{(1-Q^{-1})(1-Q_k Q)}{(1-Q_k Q^{-1})(1-Q)} \\ R_2(Q, Q_k) &= -Q_k^{-1}(1-Q_k Q^{-1})(1-Q_k Q). \end{aligned}$$

Observe that

$$\epsilon \frac{C(n,k+d)}{C(n,k+j)} = R_2(Q, Q_k)^{d-j}.$$

Thus, by the above calculation,

$$\begin{aligned} \epsilon P(E, Q, E_k, Q_k) &= R_2^d \epsilon P(R_2^{-1} E_k, Q_k) \\ \epsilon P_1 &= R_1(Q, Q_k) - E. \end{aligned}$$

Now, we want to eliminate Q_k and then set $E_k = 1$. The relation $\epsilon P_1 = 0$ is *linear* in Q_k. Solving, we obtain that:

$$Q_k = \frac{1+QE}{Q+E}.$$

Substituting this into $R_2^d \epsilon P_f(R_2^{-1} E_k, Q_k)$ and setting $E_k = 1$, we obtain:

$$\begin{aligned} R_2^d \epsilon P_f(R_2^{-1} E_k, Q_k) &= R_3^d \epsilon P(R_3, (1+QE)/(Q+E), 1) \\ &= \epsilon P_f^{\mathrm{op}}(R_3, (1+QE)/(Q+E), 1), \end{aligned}$$

where

$$R_3(E, Q) = \frac{E(Q^2-1)^2}{Q(E+Q)(1+EQ)}.$$

In other words, after we rename (E, Q) to (L, M), we have:

$$\begin{aligned} \epsilon P(L, M) &= \epsilon P_f^{\mathrm{op}}(R_3, (1+QE)/(Q+E), 1) \\ &= \phi \epsilon P^{\mathrm{op}}(E, Q, 1). \end{aligned}$$

In other words, we expect the characteristic polynomials $P_g(L,M)$ and $P_f(E,Q)$ of g and f to be related by:

$$P_g(L,M) \equiv_{L,M} \phi\, P_f^{\mathrm{op}}(E,Q), \tag{34}$$

where $\equiv_{L,M}$ means equality, up to multiplication by monomials in L, M, $L+M$, $1+LM$, $M-1$ and $M+1$. This is exactly how we came up with the strange looking rational map ϕ, and with Theorem 3.

In general, Equation (34) does not take into account repeated factors in $P_g(L,M)$ and $P_f^{\mathrm{op}}(E,Q)$. Let us make this more precise. Given $G \in \mathbb{Q}[L,M]$, let us factor $G = u\prod_i G_i^{n_i}$ where u is a unit, and G_i are irreducible and $n_i \in \mathbb{Z}$. This is possible, since $\mathbb{Q}[L,M]$ is a unique factorization domain. Now, let us define

$$\mathrm{rad}(G) = \prod_i G_i^{\mathrm{sgn}(n_i)}$$

to be the square-free part of G.

Then, the equation

$$\mathrm{rad}(G(L,M)) \equiv \mathrm{rad}(\phi\, F(E,Q)), \tag{35}$$

implies that F determines G up to multiplication by suitable monomials, and up to repeated factors. We may also invert the above equation, keeping in mind that the map ϕ is 2-to-1.

Let us end this heuristic section with a lemma that sheds some light into a possible relation between the A and the C–polynomials of a knot.

Let $\equiv'$ denote equality of rational functions in E,Q modulo multiplication by monomials in $E, Q, 1+2Q+Q^2+QS$ and $1-2Q+Q^2+QS$. These are precisely the monomials that map under ϕ to monomials in $L, M, L+M, 1+LM, M-1$ and $M+1$.

Lemma 5.2 *If F and G satisfy* (35)*, then*

$$\begin{aligned}&\mathrm{rad}(F(E,Q))\\ &\quad\equiv' \mathrm{rad}\left(\mathrm{Res}_M\left(G\left(\frac{MQ-1}{M-Q},M\right), EMQ-M^2Q-Q+Q^2M+M\right)\right)\end{aligned}$$

where Res_M *denotes the* resultant *with respect to* M*.*

Proof There is a geometric proof, which translates Equation (35) into the statement that ϕ induces a Zariski dense rational map of degree 2:

$$\{(L,M)\in\mathbb{C}^2 \mid G(L,M)=0\} \longrightarrow \{(E,Q)\in\mathbb{C}^2 \mid F(E,Q)=0\}.$$

Geometrically, it is clear that the domain determines the range and vice-versa.

There is an alternative algebraic proof. Let us try to invert the rational map ϕ. In other words, consider the system of equations

$$\begin{aligned} E &= \frac{L(M^2-1)^2}{M(L+M)(1+LM)}; \\ Q &= \frac{1+LM}{L+M}, \end{aligned}$$

with E, Q known and L, M unknown. Solving the last with respect to L gives:

$$L = \frac{MQ-1}{M-Q}.$$

Substituting into the first equation gives:

$$E = M + M^{-1} - Q - Q^{-1}.$$

Generically, this has two solutions in Q. Nevertheless, the above equation is equivalent to

$$EMQ - M^2Q - Q + Q^2M + M = 0.$$

So, we can take resultant to eliminate M:

$$\mathrm{Res}_M\left(G\left(\frac{MQ-1}{M-Q}, M\right), EMQ - M^2Q - Q + Q^2M + M\right).$$

The result follows. $\square$

5.2 Questions

In this section we formulate several questions regarding the structure and significance of the (non-commutative) C–polynomial of a knot.

Our first question may be thought of as a refined integrality property for the cyclotomic function of a knot.

Question 3 For which knots K, is the non-commutative C–polynomial monic in E with coefficients in $\mathbb{Z}[Q^{\pm}, q^{\pm}]$?

Motivated by Corollary 1.5, it is tempting to formulate the following

Question 4 For which knots K, does the map ϕ of Equation (12) give a Zariski dense rational map of degree 2:

$$\phi\colon \{(L,M)\in\mathbb{C}^2 \mid A_K(L,M^{1/2})=0\} \longrightarrow \{(E,Q)\in\mathbb{C}^2 \mid C_K^{\mathrm{op}}(E,Q)=0\}. \tag{36}$$

In view of Lemma 5.2, (36) is equivalent to

$$\mathrm{rad}(\phi C_K^{\mathrm{op}}(E,Q)) \equiv \mathrm{rad}(A(L,M^{1/2})). \tag{37}$$

Question 5 Is the genus of the C–polynomial $C_K(E,Q)$ of a knot always zero?

Remark 5.3 It seems that Questions 3 and 5 cannot be positive the same time for the 2–bridge knot $K_{13/5}$ and for the simplest hyperbolic non-2–bridge knot $m082$.

Theorem 6 *If $C_K(E,1)\neq 0$, then the Alexander polynomial $\Delta_K(M)$ divides*

$$C_K^{\mathrm{op}}(M-2+M^{-1},1).$$

The above theorem provides a nontrivial consistency check of Question 4. Indeed, in [4, Section 6] Cooper et al. prove that $A_K(1,M^{1/2})$ is divisible by the Alexander polynomial $\Delta_K(M)$ at least when the latter has unequal complex roots. On the other hand

$$\phi(E)|_{L=1} = M-2+M^{-1}, \qquad \phi(Q)|_{L=1}=1.$$

Thus,

$$\phi C_K^{\mathrm{op}}(E,Q)|_{L=1} = C_K^{\mathrm{op}}(M-2+M^{-1},1).$$

Thus, if a knot satisfies Question 4, then $\Delta_K(M)$ divides $C_K^{\mathrm{op}}(M-2+M^{-1},1)$. This is precisely Theorem 6.

Question 6 Does the C–polynomial of a knot have a classical geometric definition?

In other words, we are asking for a *geometric meaning* of the rational map ϕ of Equation (12).

Question 7 Is there any relation between the bi-graded knot invariant $C_K(E,Q,1)$ and some version of Knot Floer Homology? Theorem 6 states that under mild hypothesis, $C_K(M-2+M^{-1},1,1)$ is divisible by the Alexander polynomial of K.

5.3 Proof of Theorem 6

Proof The proof utilizes the algebra of generating functions and the fact that the generating function of the cyclotomic function of a knot (evaluated at 1) is given by the inverse Alexander polynomial. For a reference of the latter statement, see [7].

Now, let us give the details of the proof. We start from the recursion relation of the cyclotomic function:

$$C_K(E, Q, q)\hat{J}_K = 0.$$

Let us evaluate at $q = 1$, and set

$$C_K(E, 1, 1) = \sum_{j=0}^{d} a_j E^j, \qquad I_K(n) = \hat{J}_K(n)|_{q=1}.$$

Then, we have for all n:

$$\sum_{j=0}^{d} a_j I_K(n + j) = 0. \tag{38}$$

Let us use the generating function:

$$F_K(z) = \sum_{n=0}^{\infty} I_K(n) z^n.$$

Equation (38) implies that

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}\sum_{j=0}^{d} a_j I_K(n + j) z^n \\ &= \sum_{j=0}^{d} a_j z^{-j} \sum_{n=0}^{\infty} I_K(n + j) z^{n+j} \\ &= \sum_{j=0}^{d} a_j z^{-j} F_K(z) + \quad \text{terms}(z) \\ &= z^{-d} C_K^{\mathrm{op}}(z, 1, 1) F_K(z) + \quad \text{terms}(z), \end{aligned}$$

where $\text{terms}(z)$ is a Laurent polynomial in z. Thus, assuming that $C^{\mathrm{op}}(z, 1, 1) \neq 0$, it follows that

$$F_K(z) = -\frac{\text{terms}(z)}{C_K^{\mathrm{op}}(z, 1, 1)}.$$

The Melvin–Morton–Rozansky Conjecture (proven by Bar-Natan and the first author in [2]), together with the cyclotomic expansion of the colored Jones function implies that

$$F_K(M-2+M^{-1}) = \frac{1}{\Delta_K(M)}.$$

For example, see [7, Lemma 2.1]. Thus,

$$-\frac{\mathrm{terms}(M-2+M^{-1})}{C_K^{\mathrm{op}}(M-2+M^{-1},1,1)} = \frac{1}{\Delta_K(M)}.$$

The result follows. □

Appendix A A table of non-commutative C–polynomials

We finish with a table of the non-commutative C–polynomial of twist knots K_p for $p=-3,\dots,3$, taken from Theorem 1. In each matrix, the upper left entry indicates the $C_p(E,Q,q)$ polynomial and the entries in the E^i–row and Q^j–column indicate the coefficient of $Q^j E^i$ in $C_p(E,Q,q)$. For example, $C_1(E,Q) = E + q^2 Q$.

$$\left(\begin{array}{c|cc} C_1 & Q^0 & Q^1 \\ \hline E^0 & 0 & q^2 \\ E^1 & 1 & 0 \end{array}\right) \qquad \left(\begin{array}{c|cccc} C_2 & Q^0 & Q^1 & Q^2 & Q^3 \\ \hline E^0 & 0 & 0 & q^6 & -q^7 \\ E^1 & 0 & q^3+q^4 & -q^5 & q^7 \\ E^2 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{c|cccccc} C_2 & Q^0 & Q^1 & Q^2 & Q^3 & Q^4 & Q^5 \\ \hline E^0 & 0 & 0 & 0 & q^{12} & -q^{13}-q^{14} & q^{15} \\ E^1 & 0 & 0 & q^8+q^9+q^{10} & -q^{10}-2q^{11}-q^{12} & 2q^{13}+q^{14} & -q^{15}-q^{16} \\ E^2 & 0 & q^4+q^5+q^6 & -q^7-q^8 & q^{10}+q^{11} & -q^{13} & q^{16} \\ E^3 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{c|c} C_{-1} & Q^1 \\ \hline E^0 & -1 \\ E^1 & 1 \end{array}\right) \qquad \left(\begin{array}{c|ccc} C_{-2} & Q^{-2} & Q^{-1} & Q^0 \\ \hline E^0 & 0 & -q^{-2} & q^{-1} \\ E^1 & -q^{-4} & q^{-2} & -q^{-1}-1 \\ E^2 & 1 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{c|ccccc} C_{-3} & Q^{-4} & Q^{-3} & Q^{-2} & Q^{-1} & Q^0 \\ \hline E^0 & 0 & 0 & -q^{-6} & q^{-5}+q^{-4} & -q^{-3} \\ E^1 & 0 & -q^{-9}-q^{-8} & q^{-7}+2q^{-6} & -q^{-5}-2q^{-4}-q^{-3} & q^{-3}+q^{-2}+q^{-1} \\ E^2 & -q^{-12} & q^{-9} & -q^{-7}-q^{-6} & q^{-4}+q^{-3} & -q^{-2}-q^{-1}-1 \\ E^3 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

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