Rigidification of algebras over multi-sorted theories

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We define the notion of a multi-sorted algebraic theory, which is a generalization of an algebraic theory in which the objects are of different "sorts." We prove a rigidification result for simplicial algebras over these theories, showing that there is a Quillen equivalence between a model category structure on the category of strict algebras over a multi-sorted theory and an appropriate model category structure on the category of functors from a multi-sorted theory to the category of simplicial sets. In the latter model structure, the fibrant objects are homotopy algebras over that theory. Our two main examples of strict algebras are operads in the category of simplicial sets and simplicial categories with a given set of objects.

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1 Introduction

Algebraic theories are useful in studying many standard algebraic objects, such as monoids, abelian groups, and commutative rings. An algebraic theory provides a functorial means of describing particular algebraic objects without specifying generating sets for the operations to which the objects are subject, or for the relations between these operations (Lawvere [12]). Given a category $\mathcal{C}$ of algebraic objects, the associated algebraic theory $\mathcal{T}_{\mathcal{C}}$ (if it exists) is a small category with products satisfying the property that specifying an object of $\mathcal{C}$ is equivalent to giving a product-preserving functor $\mathcal{T}_{\mathcal{C}} \to \mathcal{S}ets$.

Consider a category $\mathcal{C}$ with an associated algebraic theory $\mathcal{T}$. If a functor from $\mathcal{T}$ to the category of simplicial sets preserves products, then it is essentially a simplicial object in $\mathcal{C}$ and is thus a combinatorial model for a topological object in $\mathcal{C}$, such as a topological group when $\mathcal{C}$ is the category of groups. We call such a functor a *strict* $\mathcal{T}$*–algebra* (Definition 2.3). If the functor preserves products up to homotopy, we call it a *homotopy* $\mathcal{T}$*–algebra* (Definition 2.4). A homotopy $\mathcal{T}$–algebra can be viewed as a simplicial set with the appropriate algebraic structure "up to homotopy," in a higher-order sense. Using an appropriate notion of weak equivalence on homotopy $\mathcal{T}$–algebras (Badzioch [2, 5.6]), the following result relates strict and homotopy $\mathcal{T}$–algebras:

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Theorem 1.1 (Badzioch [2, 1.4]) *Let $\mathcal{T}$ be an algebraic theory. Any homotopy $\mathcal{T}$–algebra is weakly equivalent as a homotopy $\mathcal{T}$–algebra to a strict $\mathcal{T}$–algebra.*

As a motivation for the work in this paper, consider the category of monoids. There is an associated algebraic theory $\mathcal{T}_M$, and thus a simplicial monoid can be specified by a $\mathcal{T}_M$–algebra. However, the notion of simplicial monoid can be generalized to that of a simplicial category, by which we mean a category enriched over simplicial sets, since a simplicial monoid is a simplicial category with one object. We would like to have a generalization of Badzioch's theorem which applies to simplicial categories. From the point of view of algebraic structure, the main difference between a simplicial monoid and a simplicial category with more than one object is that in the latter case the description of the algebraic structure is more complicated, in that two morphisms can be combined by the composition operation only if they satisfy certain compatibility conditions on the domain and range. Therefore, we would like to describe a more general notion of theory which is capable of describing algebraic structures in which the elements have various sorts or types, and in which the operations which can be used to combine a collection of elements depend on these sorts.

There is in fact such a "multi-sorted" theory, $\mathcal{T}_{\mathcal{O}\mathcal{C}at}$, such that a product-preserving functor $\mathcal{T}_{\mathcal{O}\mathcal{C}at} \to \mathcal{S}ets$ is essentially a category with object set $\mathcal{O}$ (Example 3.5). A simplicial category, analogously, can be viewed as a product-preserving functor $\mathcal{T}_{\mathcal{O}\mathcal{C}at} \to \mathcal{SS}ets$.

A simpler example of an algebraic structure which requires the use of a multi-sorted theory, which we will describe in more detail in Example 3.2, is that of a group acting on a set. There are two sorts of elements, namely, the elements of the group and the elements of the set. Two elements of the group can be combined via multiplication, or an element can be inverted. An element of the group and an element of the set can be combined via the group action. However, the elements of the set cannot be combined with one another in any nontrivial way, so the operations which we allow depend on the sort of element involved. The example of a module over a ring is constructed similarly in Example 3.3.

Another application of the notion of a multi-sorted theory gives a convenient description of an operad. In Example 3.4, we characterize the theory $\mathcal{T}_{\text{operad}}$ of operads. An operad in the category of sets is then a product-preserving functor from $\mathcal{T}_{\text{operad}}$ to the category of sets. Thus, we can describe an operad of sets as a diagram of sets given by this multi-sorted theory. We can similarly describe operads of spaces.

A multi-sorted theory $\mathcal{T}$ is a category with products, so we can define strict and homotopy $\mathcal{T}$–algebras as before (see Definitions 3.6 and 3.7). Using a definition of weak equivalence for homotopy $\mathcal{T}$–algebras (Proposition 4.11), the main result which we prove for multi-sorted theories is the following generalization of Theorem 1.1:

Theorem 1.2 *Let $\mathcal{T}$ be a multi-sorted algebraic theory. Any homotopy $\mathcal{T}$–algebra is weakly equivalent as a homotopy $\mathcal{T}$–algebra to a strict $\mathcal{T}$–algebra.*

As Badzioch does, we actually prove a stronger statement in terms of a Quillen equivalence of model category structures (Theorem 5.1).

Using our example of the theory $\mathcal{T}_{\text{operad}}$ of operads, an operad in the category of simplicial sets is a strict $\mathcal{T}_{\text{operad}}$–algebra. A homotopy operad, or sequence of simplicial sets with the structure of an operad only up to homotopy, is then a homotopy $\mathcal{T}_{\text{operad}}$–algebra and can be rigidified to a strict operad using this theorem.

Returning to the example of simplicial categories, let $\mathcal{O}$ be a set and $\mathcal{SC}_{\mathcal{O}}$ the category of simplicial categories with object set $\mathcal{O}$ in which the morphisms are the identity on the objects. In [3], we use Theorem 1.2 to prove a relationship between $\mathcal{SC}_{\mathcal{O}}$ and the category of Segal categories with the same set $\mathcal{O}$ in dimension zero. In [4], we use the ideas of this proof to prove an analogous relationship between the category of all small simplicial categories and the category of all Segal categories.

Throughout this paper, we frequently work in the category of simplicial sets, $\mathcal{SS}ets$. Recall that a simplicial set is a functor ${}^{op} \to \mathcal{S}ets$, where Δ denotes the *cosimplicial category* whose objects are the finite ordered sets $[n] = (0, \dots, n)$ and whose morphisms are the order-preserving maps. The *simplicial category* ^{op} is then the opposite of this category. Some examples of simplicial sets are, for each $n \geq 0$, the n–simplex $\Delta[n]$, its boundary $\dot{\Delta}[n]$, and, for any $0 \leq k \leq n$, the simplicial set $V[n,k]$, which is $\dot{\Delta}[n]$ with the kth face removed. More information about simplicial sets can be found in Goerss and Jardine [8, I.1].

In this paper, we begin by recalling the definition of an algebraic theory and stating some of its basic properties. Using this definition as a model, we then define a multi-sorted theory. We should note here that this notion is not a new one; similar definitions are given by Adámek and Rosický [1, 3.14] and by Boardman and Vogt [5, 2.3]. (The still more general definition of a finite limit theory is used by Johnson and Walters [11], and Rosický proves a similar result to Theorem 1.2 for limit theories [17].) Because our perspective is slightly different, however, we will give a precise definition followed by some examples. Given a multi-sorted theory $\mathcal{T}$, we define strict and homotopy $\mathcal{T}$–algebras over a multi-sorted theory $\mathcal{T}$ and show that the existence of a model category structure on the category of all $\mathcal{T}$–algebras. We also show the existence of a model category structure on the category of all functors $\mathcal{T} \to \mathcal{SS}ets$ in which the fibrant objects are the homotopy $\mathcal{T}$–algebras. We then show that there is a Quillen equivalence between these two model categories.

We note that the key here is the fact that we are considering functors which preserve *categorical* products. An interesting question which we hope to address further in future work is whether such a rigidification holds in a category such as chain complexes where the “product” we are interested in, the tensor product, is not a categorical product.

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2 A summary of algebraic theories

We first recall the definition of an ordinary algebraic theory. More details about algebraic theories can be found in Borceux [6, Chapter 3].

Definition 2.1 An *algebraic theory* $\mathcal{T}$ is a small category with finite products and which has as objects T_n for $n \geq 0$ together with, for each n, an isomorphism $T_n \cong (T_1)^n$. In particular, T_0 is the terminal object in $\mathcal{T}$.

We can use theories to describe certain algebraic categories, namely those which are determined by sets with n–ary operations for each $n \geq 2$. To do so, we need to use the notion of adjoint pairs of functors. Recall that a pair of functors

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D} : R$$

is *adjoint* (where F is the left adjoint and R is the right adjoint) if there is a natural isomorphism

$$\varphi_{X,Y}\colon \mathrm{Hom}_{\mathcal{D}}(FX, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, RY)$$

for all objects X in $\mathcal{C}$ and Y in $\mathcal{D}$. The adjoint pair is sometimes written as the triple (F, R, φ) (Mac Lane [13, IV.1]).

Now, consider a category $\mathcal{C}$ such that there exists a forgetful functor

$$\Phi\colon \mathcal{C} \to \mathcal{S}ets$$

taking an object of $\mathcal{C}$ to its underlying set, and its left adjoint (a free functor)

$$L\colon \mathcal{S}ets \to \mathcal{C}.$$

In other words, $\mathcal{C}$ is required to have free objects. If the category $\mathcal{C}$ and the adjoint pair (Φ, L) satisfy some additional technical conditions (see [6, 3.9.1] for details), we will call $\mathcal{C}$ an *algebraic category*.

Given an object X of an algebraic category $\mathcal{C}$, we have a natural map

$$\varepsilon_X \colon L\Phi(X) \to X$$

and given a set A, we have another map

$$\eta_A \colon A \to \Phi L(A).$$

In order to discuss a theory over the algebraic category $\mathcal{C}$, consider a set A together with a map $m_A \colon \Phi L(A) \to A$ satisfying two conditions: the composite map

$$A \xrightarrow{\eta_A} \Phi L(A) \xrightarrow{m_A} A$$

is the identity map on A, and the diagram

$$(\Phi L)^2 A \overset{\Phi L(m_A)}{\underset{\Phi \varepsilon_{LA}}{\rightrightarrows}} \Phi L(A) \xrightarrow{m_A} A$$

is a coequalizer. These maps define an algebraic structure on the set A, specifically the structure possessed by the objects of $\mathcal{C}$ [12]. (Note that via this structure ΦL defines a monad on the category of sets [13, VI.1].)

For example, if $\mathcal{C} = \mathcal{G}$, the category of groups, Φ is the forgetful functor taking a group to its underlying set, and L is the free group functor taking a set to the free group on that set, then these two conditions are precisely the ones defining a group structure on the set A.

We would like to discuss the algebraic theory $\mathcal{T}$ corresponding to $\mathcal{C}$ to simplify this way of talking about algebraic structure. Let X be an object of $\mathcal{C}$. We consider natural transformations of functors $\mathcal{C} \to \mathcal{S}ets$

$$\underbrace{\Phi(-) \times \cdots \times \Phi(-)}_{n} \to \Phi(-).$$

Using the adjointness of Φ and L, we have that

$$\Phi(X) \cong \mathrm{Hom}_{\mathcal{S}ets}(\{1\}, \Phi(X)) \cong \mathrm{Hom}_{\mathcal{C}}(L\{1\}, X)$$

where $\{1\}$ denotes the set with one object, and we can think of $L\{1\}$ as the free object in $\mathcal{C}$ on one generator, since L is the free functor. Hence, we have

$$\begin{aligned}\Phi(X)^n &= \mathrm{Hom}_{\mathcal{S}ets}(\{1\}, \Phi(X))^n \\ &= \mathrm{Hom}_{\mathcal{S}ets}(\coprod_n \{1\}, \Phi(X)) \\ &= \mathrm{Hom}_{\mathcal{S}ets}(\{1, \dots, n\}, \Phi(X)) \\ &= \mathrm{Hom}_{\mathcal{C}}(L\{1, \dots, n\}, X).\end{aligned}$$

Now, by Yoneda's Lemma we have a bijection between the set of natural maps $\Phi(X)^n \to \Phi(X)$ and the set $\mathrm{Hom}_{\mathcal{C}}(L\{1\}, L\{1, \dots, n\})$. The objects

$$L\{\phi\} = T_0, L\{1\} = T_1, \dots, L\{1, \dots, n\} = T_n, \dots$$

are the objects of the algebraic theory $\mathcal{T}$ corresponding to $\mathcal{C}$. The morphisms are the opposites of the ones in $\mathcal{C}$ between these objects. More precisely stated, $\mathcal{T}$ is the opposite of the full subcategory of representatives of isomorphism classes of finitely generated free objects of $\mathcal{C}$.

Given an object X of $\mathcal{C}$, define a functor $H_X\colon \mathcal{T} \to \mathcal{S}ets$ such that

$$H_X(L\{1, \dots, n\}) = \mathrm{Hom}_{\mathcal{C}}(L\{1, \dots, n\}, X) = \Phi(X)^n.$$

Now, the algebraic category $\mathcal{C}$ is equivalent to the category of the functors H_X, namely, the full subcategory of the category of functors $A\colon \mathcal{T} \to \mathcal{S}ets$ whose objects preserve products, or those for which the canonical map $A(T_n) \to A(T_1)^n$ induced by the n projection maps is an isomorphism of sets for all $n \geq 0$ [12].

Example 2.2 Let $\mathcal{G}$ denote the category of groups. Consider the full subcategory of $\mathcal{G}$ whose objects T_n are the free groups on n generators for $n \geq 0$ (where T_0 is the trivial group). The opposite of this category is $\mathcal{T}_{\mathcal{G}}$, the theory of groups. It can be shown that the category of product-preserving functors $\mathcal{T}_{\mathcal{G}} \to \mathcal{S}ets$ is equivalent to the category $\mathcal{G}$.

Product-preserving functors from the theory $\mathcal{T}$ to $\mathcal{S}ets$ are called *algebras* over $\mathcal{T}$. We would also like to consider functors from an algebraic theory to the category $\mathcal{SS}ets$ of simplicial sets. To do so, we must first define a simplicial algebra over a theory $\mathcal{T}$. For simplicity, we will also use the term "algebra" to refer to these simplicial algebras.

Definition 2.3 [2, 1.1] Given an algebraic theory $\mathcal{T}$, a *(strict simplicial) $\mathcal{T}$–algebra* A is a product-preserving functor $A\colon \mathcal{T} \to \mathcal{SS}ets$. Namely, the canonical map

$$A(T_n) \to A(T_1)^n,$$

induced by the n projection maps $T_n \to T_1$, is an isomorphism of simplicial sets. In particular, $A(T_0)$ is the one-point space $\Delta[0]$.

The category of all $\mathcal{T}$–algebras will be denoted $\mathcal{A}lg^{\mathcal{T}}$. Similarly, we have the notion of a homotopy algebra, for which we only require products to be preserved up to homotopy:

Definition 2.4 [2, 1.2] Given an algebraic theory $\mathcal{T}$, a *homotopy $\mathcal{T}$–algebra* is a functor $X\colon \mathcal{T} \to \mathcal{SS}ets$ which preserves products up to homotopy, ie, for each n the canonical map

$$X(T_n) \to X(T_1)^n$$

is a weak equivalence of simplicial sets. In particular, we assume that $X(T_0)$ is weakly equivalent to $\Delta[0]$.

There exists a forgetful functor, or evaluation map,

$$U_{\mathcal{T}}\colon \mathcal{A}lg^{\mathcal{T}} \to \mathcal{SS}ets$$

such that $U_{\mathcal{T}}(A) = A(T_1)$. This functor has a left adjoint, the free $\mathcal{T}$–algebra functor

$$F_{\mathcal{T}}\colon \mathcal{SS}ets \to \mathcal{A}lg^{\mathcal{T}}$$

where, if Y is any simplicial set,

$$F_{\mathcal{T}}(Y)(T_1) = \coprod_{n \geq 0} \mathrm{Hom}_{\mathcal{T}}(T_n, T_1) \times Y^n / \sim$$

where the identifications come from the structure of the algebraic theory [2, 2.1]. More specifically, if $\mathcal{T}_0$ denotes the initial theory (given by representatives of isomorphism classes of finite sets), this free functor is given by a coend

$$F_{\mathcal{T}}(Y)(T_1) = \int^{\mathcal{T}_0} \mathrm{Hom}_{\mathcal{T}}(T_n, T_1) \times Y^n$$

as given by Schwede in [18, 2.3].

3 Multi-sorted algebraic theories

We now generalize the definition of an algebraic theory to that of a multi-sorted theory.

Definition 3.1 Given a set S, an *S–sorted algebraic theory* (or *multi-sorted theory*) $\mathcal{T}$ is a small category with objects $T_{\underline{\alpha}^n}$ where $\underline{\alpha}^n = <\alpha_1, \ldots, \alpha_n>$ for $\alpha_i \in S$ and $n \geq 0$ varying, and such that each $T_{\underline{\alpha}^n}$ is equipped with an isomorphism

$$T_{\underline{\alpha}^n} \cong \prod_{i=1}^{n} T_{\alpha_i}.$$

For a particular $\underline{\alpha}^n$, the entries α_i can repeat, but they are not ordered. In other words, $\underline{\alpha}^n$ is a an n–element subset with multiplicities. There exists a terminal object T_0 (corresponding to the empty subset of S).

Notation Lower-case Greek letters (with or without subscripts), say α or α_i, will be used to denote objects of S, whereas underlined ones, say $\underline{\alpha}^n$ or simply $\underline{\alpha}$, will denote an n–element subset of objects of S (with multiplicities) for $n \geq 1$.

Notice that a theory with a single sort is a theory in the sense of the previous section.

We would like to speak of multi-sorted theories corresponding to categories which are analogous to the algebraic categories which we had in the ordinary case. However, because we have several objects (or "sorts") T_α where we only had the object T_1 in an ordinary theory, we have many pairs of adjoint functors, one for each sort.

Let $\mathcal{C}$ a category with coproducts such that given any element $\beta \in S$, we have a forgetful functor

$$\Phi_\beta \colon \mathcal{C} \to \mathcal{S}ets$$

and its left adjoint, the free functor

$$L_\beta \colon \mathcal{S}ets \to \mathcal{C}.$$

We would like the category $\mathcal{C}$ and these adjoint pairs to satisfy the following analogous conditions to those of [6, 3.9.1]:

(1) The category $\mathcal{C}$ has coequalizers and kernel pairs (ie, pullbacks of diagrams $X \to Y \leftarrow X$).

(2) Each Φ_β reflects isomorphisms and preserves regular epimorphisms (ie, those that are coequalizers).

(3) For all $\beta \in S$, the composite functor $\Phi_\beta L_\beta$ preserves filtered colimits.

These conditions make $\mathcal{C}$ a kind of generalized algebraic category.

Now, for each object X in $\mathcal{C}$ and element $\beta \in S$, we have a map

$$\varepsilon_{X,\beta}\colon L_\beta \Phi_\beta(X) \to X$$

and, for each set A a map

$$\eta_{A,\beta}\colon A \to \Phi_\beta L_\beta(A).$$

As before, in order to make sense of the notion of theory, we consider a set A together with, for each $\beta \in S$, a map

$$m_{A,\beta}\colon \Phi_\beta L_\beta(A) \to A$$

satisfying two conditions: the composite map

$$A \xrightarrow{\eta_{A,\beta}} \Phi_\beta L_\beta(A) \xrightarrow{m_{A,\beta}} A$$

is the identity map on A, and the diagram

$$(\Phi_\beta L_\beta)^2 A \underset{\Phi_\beta \varepsilon_{L_\beta A,\beta}}{\overset{\Phi_\beta L_\beta(m_{A,\beta})}{\rightrightarrows}} \Phi_\beta L_\beta(A) \xrightarrow{m_{A,\beta}} A$$

is a coequalizer. These maps define a "multi-sorted algebraic structure" on $\mathcal{C}$. In particular, we have a notion of composition for certain elements of $\mathcal{C}$ depending on their sorts. Given this structure, we can now construct the S–sorted theory corresponding to the category $\mathcal{C}$.

Given $\alpha_i, \beta \in S$, we consider natural transformations of functors $\mathcal{C} \to \mathcal{S}ets$

$$\Phi_{\alpha_1}(-) \times \cdots \times \Phi_{\alpha_n}(-) \to \Phi_\beta(-).$$

As before, we can apply these functors to an object X of $\mathcal{C}$ and rewrite to obtain a map

$$\mathrm{Hom}_{\mathcal{S}ets}(\{1\}, \Phi_{\alpha_1}(X)) \times \cdots \times \mathrm{Hom}_{\mathcal{S}ets}(\{1\}, \Phi_{\alpha_n}(X)) \to \mathrm{Hom}_{\mathcal{S}ets}(\{1\}, \Phi_\beta(X))$$

which, by adjointness, is equivalent to

$$\mathrm{Hom}_{\mathcal{C}}(L_{\alpha_1}\{1\}, X) \times \cdots \times \mathrm{Hom}_{\mathcal{C}}(L_{\alpha_n}\{1\}, X) \to \mathrm{Hom}_{\mathcal{C}}(L_\beta\{1\}, X).$$

Since $\mathcal{C}$ has coproducts, we can rewrite this map as

$$\mathrm{Hom}_{\mathcal{C}}(L_{\alpha_1}\{1\} \amalg \cdots \amalg L_{\alpha_n}\{1\}, X) \to \mathrm{Hom}_{\mathcal{C}}(L_\beta\{1\}, X).$$

Then, by Yoneda's Lemma, there is a bijection between the set of natural transformations

$$\Phi_{\alpha_1}(-) \times \cdots \times \Phi_{\alpha_n}(-) \to \Phi_\beta(-)$$

and the set

$$\mathrm{Hom}_{\mathcal{C}}(L_\beta\{1\}, \coprod_{k=1}^{n} L_{\alpha_k}\{1\}).$$

The objects of the theory $\mathcal{T}$ corresponding to $\mathcal{C}$ are given by finite coproducts of "free" objects $L_{\alpha_k}\{1\}$ of $\mathcal{C}$ for all choices of α_k, and the morphisms are the opposites of those of $\mathcal{C}$. Let X be an object of $\mathcal{C}$ and $(\alpha_1, \ldots, \alpha_n) \in S^n$ an n–tuple of elements in S. We define the functor $H_{X,\alpha_1,\ldots,\alpha_n}\colon \mathcal{T} \to \mathcal{S}ets$ such that

$$H_{X,\alpha_1,\ldots,\alpha_n}(\coprod_{k=1}^{n} L_{\alpha_k}\{1\}) = \mathrm{Hom}_{\mathcal{C}}(\coprod_{k=1}^{n} L_{\alpha_k}\{1\}, X) = \Phi_{\alpha_1}(X) \times \cdots \times \Phi_{\alpha_n}(X).$$

(Note that we still write the "coproduct" to denote an object of $\mathcal{T}$ to be consistent with previous notation, even though in $\mathcal{T}$ it is actually a product.) The category $\mathcal{C}$ is equivalent to the category of all such functors if it satisfies the conditions given above.

We now consider some examples.

Example 3.2 Consider pairs (G, X), where G is a group and X is a set. We can obtain two different 2–sorted theories from these pairs, one corresponding to the category of unstructured pairs, and the other corresponding to the category of pairs (G, X) with a given action of the group G on the set X.

In each case, we have two forgetful functors and their respective left adjoints. We begin with the category of unstructured pairs, which we denote $\mathcal{P}$. The objects are the pairs (G, X) and the morphisms $(G, X) \to (H, Y)$ consist of pairs (φ, f) where $\varphi\colon G \to H$ is a group homomorphism and $f\colon X \to Y$ is a map of sets. For each sort $i = 1, 2$ we have a forgetful map

$$\Phi_i\colon \mathcal{P} \to \mathcal{S}ets$$

and its left adjoint

$$L_i\colon \mathcal{S}ets \to \mathcal{P}.$$

When $i = 1$, we have, for any group G and set X,

$$\Phi_1(G, X) = G$$

(where on the right-hand side G denotes the underlying set of the group G) and for any set S

$$L_1(S) = (F_S, \phi)$$

where F_S denotes the free group on the set S.

Similarly, when $i = 2$, we define

$$\Phi_2(G, X) = X$$

and

$$L_2(S) = (e, S)$$

where e denotes the trivial group.

In order to determine the objects of our theory, consider functors

$$F_{i,j} \colon \mathcal{P} \to \mathcal{S}ets$$

such that $F_{i,j}(G, X) = G^i \times X^j$. In other words,

$$F_{i,j}(G, X) = \mathrm{Hom}_{\mathcal{P}}(L_1\{1, \ldots, i\} \amalg L_2\{1, \ldots, j\}, (G, X))$$

where $\{1, \cdots, i\}$ denotes the set with i elements and similarly for $\{1, \ldots, j\}$. The objects of the theory will be representatives of the isomorphism classes of the $L_1\{1, \ldots, i\}$ $\amalg L_2\{1, \ldots, j\}$ for all choices of i and j. This coproduct in $\mathcal{P}$ is defined to be the coproduct of each element in the pairs. Thus we have

$$(G, X) \amalg (G', X') = (G * G', X \amalg X')$$

where $G * G'$ denotes the free product of groups. So, our corresponding theory is the opposite of the full subcategory of $\mathcal{P}$ whose objects are of the form $L_1\{1, \ldots, i\} \amalg L_2\{1, \ldots, j\}$.

When we equip each pair (G, X) with an action of G on X to obtain another category which we denote $\mathcal{PA}$, the process is identical until we have to specify the coproduct, since in this case we need to take the group actions into account. We then have the coproduct in $\mathcal{PA}$

$$(G, X) \amalg (G', X') = (H, (H \times_G X) \amalg (H \times_{G'} X'))$$

where $H = G * G'$ and we have defined

$$H \times_G X = \{(h, x) | h \in H, x \in X\}/\sim$$

when $(hg, x) \sim (h, gx)$ for any $g \in G$. We can now take the opposite of a full subcategory of $\mathcal{PA}$ as above to obtain the corresponding theory. In particular, the objects of the theory look like

$$L_1\{1, \ldots, i\} \amalg L_2\{1, \ldots, j\} = (F_i, F_i \times \{1, \ldots, j\}),$$

where F_i denotes the free group on i generators.

Example 3.3 A very similar example is the case of a commutative ring R and an R–module A. Again, we have two different 2–sorted theories: one where we have a ring R and regard A merely as an abelian group, and the other where we consider the R–module structure on A.

As before, we begin with $\mathcal{PR}$, the category of pairs with no additional structure. We have the forgetful map

$$\Phi_1\colon \mathcal{PR} \to \mathcal{S}ets$$

where $\Phi_1(R, A) = R$ for any ring R and abelian group A, where on the right side R is the underlying set of the ring R. Its left adjoint is the functor

$$L_1\colon \mathcal{S}ets \to \mathcal{PR}$$

where for any set S, $L_1(S) = (\mathbb{Z}[S], e)$, where $\mathbb{Z}[S]$ is the free commutative ring on the set S and e denotes the trivial (abelian) group. Then we have the map

$$\Phi_2\colon \mathcal{PR} \to \mathcal{S}ets$$

such that $\Phi_2(R, A) = A$, where again on the right hand side A is the underlying set of the abelian group A. Its left adjoint is the map

$$L_2\colon \mathcal{S}ets \to \mathcal{PR}$$

where $L_2(S) = (\mathbb{Z}, FA_S)$ where FA_S denotes the free abelian group on the set S.

To know what the objects of this 2–sorted theory are, we need to know what the coproduct is. We have that

$$(R, A) \amalg (R', A') = (R \otimes_{\mathbb{Z}} R', A \oplus A'),$$

and from there we can obtain a theory as in the previous example.

Now consider the category $\mathcal{PM}$ whose objects are pairs (R, A) where R is a ring and A is a module over A. If A and A' are modules over R and R', respectively, we have a coproduct similar to that in the group action example. So, we say that

$$(R, A) \amalg (R', A') = (R \otimes_{\mathbb{Z}} R', (R' \otimes_{\mathbb{Z}} A) \oplus (R \otimes_{\mathbb{Z}} A'))$$

and construct the corresponding theory as before.

Example 3.4 Another example of a multi-sorted theory is the $\mathbb{N}$–sorted theory of symmetric operads. Recall that an operad in the category of sets is a sequence of sets $\{P(k)\}_{k \geq 0}$, a unit element $1 \in P(1)$, each with a right action of the symmetric group Σ_k, and operations

$$P(k) \times P(j_1) \times \cdots \times P(j_k) \to P(j_1 + \cdots + j_k)$$

satisfying associativity, unit, and equivariance conditions [14, II.1.4].

There is a notion of a free operad on n generators at levels $m_1, \dots, m_n$ (Markl, Shnider and Stasheff [14, Section II.1.9], Rezk [16, 2.3.6]). Specifically, such a free operad has, for each $1 \leq i \leq n$, a generator in $P(m_i)$. Note that the values of m_i can repeat. For example, one can think of the free operad on n generators, each at level 1, as the free monoid on n generators.

In the category of operads, consider the full subcategory of isomorphism classes of free operads. Each object in this category, then, can be described as the free operad on n generators at levels $m_1, \dots, m_n$ for some $n \geq 0$ and $m_1, \dots, m_n$. The opposite of this category is the theory of operads. Using the notation we have set up for multi-sorted theories, we have that T_α for $\alpha \in \mathbb{N}$ is just the free operad on one generator at level α and for $\underline{\alpha}^n = <\alpha_1, \dots, \alpha_n>$, we have that $T_{\underline{\alpha}^n}$ is the free operad on n generators at levels $\alpha_1, \dots \alpha_n$.

There is also a notion of non-symmetric (or non–Σ) operads, where we no longer have an action of the symmetric group or an equivariance condition [14, II.1.14]. We can define the theory of non–Σ operads analogously, taking the opposite of the full subcategory of isomorphism classes of free non–Σ operads in the category of all non–Σ operads.

Example 3.5 Consider the category $\mathcal{O}\mathcal{C}at$ whose objects are the categories with a fixed object set $\mathcal{O}$ and whose morphisms are the functors which are the identity map on the objects. There is a theory $\mathcal{T}_{\mathcal{O}\mathcal{C}at}$ associated to this category. The objects of the theory are isomorphism classes of categories which are freely generated by directed graphs with vertices corresponding to the elements of the set $\mathcal{O}$. This theory will be sorted by pairs of elements in $\mathcal{O}$, corresponding to the morphisms with source the first element and target the second. In other words, this theory is $(\mathcal{O} \times \mathcal{O})$–sorted.

In particular, consider $\alpha = (x, y) \in \mathcal{O} \times \mathcal{O}$. Then, if $x \neq y$, T_α is the category with object set $\mathcal{O}$ and one nonidentity morphism with source x and target y. If $x = y$, then T_α is the category freely generated by one morphism from x to itself and no other nonidentity morphisms.

In general, if $\underline{\alpha} = <\alpha_1, \dots, \alpha_n>$, then $T_{\underline{\alpha}}$ is the category with object set $\mathcal{O}$ and morphisms freely generated by the morphisms given for each α_i as in the previous case.

Consider the forgetful functor $\Phi_\alpha \colon \mathcal{O}\mathcal{C}at \to \mathcal{S}ets$ where, for any object X in $\mathcal{O}\mathcal{C}at$ and $\alpha = (x, y)$,

$$\Phi_\alpha(X) = \mathrm{Hom}_X(x, y).$$

Its left adjoint then is the free functor L_α defined by, for a set A,

$$L_\alpha(A) = \begin{cases} C \text{ with } \mathrm{Hom}_C(x,y) = A & \text{if } x \neq y \\ C \text{ with } \mathrm{Hom}_C(x,y) = F_A & \text{if } x = y \end{cases}$$

where F_A is the free monoid generated by the set A and where in each case there are no other nonidentity morphisms in the category C.

As with ordinary algebraic theories, we can define strict and homotopy $\mathcal{T}$–algebras for a multi-sorted theory $\mathcal{T}$.

Definition 3.6 Given an S–sorted theory $\mathcal{T}$, a *(strict simplicial) $\mathcal{T}$–algebra* is a product-preserving functor $A\colon \mathcal{T} \to \mathcal{SS}ets$. Here, product-preserving means that the canonical map

$$A(T_{\underline{\alpha}^n}) \to \prod_{i=1}^{n} A(T_{\alpha_i}),$$

induced by the projections $T_{\underline{\alpha}^n} \to T_{\alpha_i}$ for all $1 \leq i \leq n$, is an isomorphism of simplicial sets.

As before, we will denote the category of strict $\mathcal{T}$–algebras by $\mathcal{A}lg^{\mathcal{T}}$.

Definition 3.7 Given an S–sorted theory $\mathcal{T}$, a *homotopy $\mathcal{T}$–algebra* is a functor $X\colon \mathcal{T} \to \mathcal{SS}ets$ which preserves products up to homotopy, ie, for all $\alpha \in S^n$, the canonical map

$$X(T_{\underline{\alpha}^n}) \to \prod_{i=1}^{n} X(T_{\alpha_i})$$

induced by the projection maps $T_{\underline{\alpha}^n} \to T_{\alpha_i}$ (for each $1 \leq i \leq n$) is a weak equivalence of simplicial sets.

We would like to prove a rigidification result similar to Theorem 1.1 above. We begin by finding model category structures for $\mathcal{T}$–algebras and homotopy $\mathcal{T}$–algebras. We then find a Quillen equivalence between these model category structures $\mathcal{T}$–algebras for any multi-sorted theory $\mathcal{T}$.

4 Model category structures

In this section, we define, given a multi-sorted theory $\mathcal{T}$, model category structures on the category of diagrams $\mathcal{T} \to \mathcal{SS}ets$ and on the category of $\mathcal{T}$–algebras. We begin with a review of model category structures.

Recall that a model category structure on a category $\mathcal{C}$ is a choice of three distinguished classes of morphisms: fibrations, cofibrations, and weak equivalences. A (co)fibration which is also a weak equivalence will be called an *acyclic (co)fibration*. With this choice of three classes of morphisms, $\mathcal{C}$ is required to satisfy the following five axioms (Dwyer and Spaliński [7, 3.3]).

(MC1) $\mathcal{C}$ has all small limits and colimits.

(MC2) If f and g are maps in $\mathcal{C}$ such that their composite gf exists, then if two of f, g, and gf are weak equivalences, then so is the third.

(MC3) If a map f is a retract of g and g is a fibration, cofibration, or weak equivalence, then so is f.

(MC4) If $i\colon A \to B$ is a cofibration and $p\colon X \to Y$ is a fibration, then a dotted arrow lift exists in any solid arrow diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

if either

(i) p is a weak equivalence, or
(ii) i is a weak equivalence.

(In this case we say that i has the *left lifting property* with respect to p and that p has the *right lifting property* with respect to i.)

(MC5) Any map f can be factored two ways:

(i) $f = pi$ where i is a cofibration and p is an acyclic cofibration, and
(ii) $f = qj$ where j is an acyclic cofibration and p is a fibration.

An object X in $\mathcal{C}$ is *fibrant* if the unique map $X \to *$ from X to the terminal object is a fibration. Dually, X is *cofibrant* if the unique map $\phi \to X$ from the initial object to X is a cofibration. The factorization axiom MC5 guarantees that each object X has a weakly equivalent fibrant replacement $\widehat{X}$ and a weakly equivalent cofibrant replacement $\widetilde{X}$. These replacements are not necessarily unique, but they can be chosen to be functorial in the cases we will use Hovey [10, 1.1.3].

The model category structures which we will discuss are all cofibrantly generated. In a cofibrantly generated model category, there are two sets of morphisms, one of generating cofibrations and one of generating acyclic cofibrations, such that a map is a fibration if and only if it has the right lifting property with respect to the generating acyclic cofibrations, and a map is an acyclic fibration if and only if it has the right lifting property with respect to the generating cofibrations (Hirschhorn [9, 11.1.2]). To describe such model categories, we make the following definition.

We are now able to state the theorem, due to Kan, that we will use to prove our model category structures in this paper.

Theorem 4.1 [9, 11.3.2] *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. Let $\mathcal{N}$ be a category that satisfies axiom MC1 such that there exists a pair of adjoint functors*

$$F\colon \mathcal{M} \rightleftarrows \mathcal{N} :U.$$

If $FI = \{Fu \mid u \in I\}$ and $FJ = \{Fv \mid v \in J\}$, and if

(1) *each of the sets FI and FJ permits the small object argument [9, 10.5.15], and*
(2) *U takes (possibly transfinite) colimits of pushouts along maps in FJ to weak equivalences,*

then there is a cofibrantly generated model category structure on $\mathcal{N}$ for which FI is a set of generating cofibrations and FJ is a set of generating acyclic cofibrations, and the weak equivalences are the maps that U sends to weak equivalences in $\mathcal{M}$.

We will refer to the standard model category structure on the category $\mathcal{SS}ets$ of simplicial sets. In this case, a weak equivalence is a map of simplicial sets $f\colon X \to Y$ such that the induced map $|f|\colon |X| \to |Y|$ is a weak homotopy equivalence of topological spaces. The cofibrations are monomorphisms, and the fibrations are the maps with the right lifting property with respect to the acyclic cofibrations [8, I.11.3]. This model category structure is cofibrantly generated; a set of generating cofibrations is $I = \{\dot{\Delta}[n] \to \Delta[n] \mid n \geq 0\}$, and a set of generating acyclic cofibrations is $J = \{V[n,k] \to \Delta[n] \mid n \geq 1, 0 \leq k \leq n\}$.

We will also need the notion of a simplicial model category $\mathcal{M}$. For any objects X and Y in a simplicial category $\mathcal{M}$, the *function complex* is the simplicial set $\operatorname{Map}(X,Y)$.

Definition 4.2 [9, 9.1.6] A *simplicial model category* $\mathcal{M}$ is a model category $\mathcal{M}$ that is also a simplicial category such that the following two axioms hold:

(SM6) For every two objects X and Y of $\mathcal{M}$ and every simplicial set K, there are objects $X \otimes K$ and Y^K in $\mathcal{M}$ such that there are isomorphisms of simplicial sets

$$\operatorname{Map}(X \otimes K, Y) \cong \operatorname{Map}(K, \operatorname{Map}(X,Y)) \cong \operatorname{Map}(X, Y^K)$$

that are natural in X, Y, and K.

(SM7) If $i\colon A \to B$ is a cofibration in $\mathcal{M}$ and $p\colon X \to Y$ is a fibration in $\mathcal{M}$, then the map of simplicial sets

$$i^* \times p_*\colon \mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$

is a fibration which is an acyclic fibration if either i or p is a weak equivalence.

It is important to note that a function complex in a simplicial model category is only homotopy invariant in the case that X is cofibrant and Y is fibrant. For the general case, we have the following definition:

Definition 4.3 [9, 17.3.1] A *homotopy function complex* $\mathrm{Map}^h(X, Y)$ in a simplicial model category $\mathcal{M}$ is the simplicial set $\mathrm{Map}(\widetilde{X}, \widehat{Y})$ where $\widetilde{X}$ is a cofibrant replacement of X in $\mathcal{M}$ and $\widehat{Y}$ is a fibrant replacement for Y.

Several of the model category structures that we use are obtained by localizing a given model category structure with respect to a map or a set of maps. Suppose that $P = \{f\colon A \to B\}$ is a set of maps with respect to which we would like to localize a model category $\mathcal{M}$.

Definition 4.4 A *P–local* object W is a fibrant object of $\mathcal{M}$ such that for any $f\colon A \to B$ in P, the induced map on homotopy function complexes

$$f^*\colon \mathrm{Map}^h(B, W) \to \mathrm{Map}^h(A, W)$$

is a weak equivalence of simplicial sets. A map $g\colon X \to Y$ in $\mathcal{M}$ is then a *P–local equivalence* if for every local object W, the induced map on homotopy function complexes

$$g^*\colon \mathrm{Map}^h(Y, W) \to \mathrm{Map}^h(X, W)$$

is a weak equivalence of simplicial sets.

Given a multi-sorted theory $\mathcal{T}$, let $\mathcal{SS}ets^{\mathcal{T}}$ denote the category of functors $\mathcal{T} \to \mathcal{SS}ets$. Note that the category $\mathcal{A}lg^{\mathcal{T}}$ of strict $\mathcal{T}$–algebras is a full subcategory of $\mathcal{SS}ets^{\mathcal{T}}$.

The category $\mathcal{SS}ets^{\mathcal{T}}$ is an example of a category of diagrams. In general, given any small category $\mathcal{D}$, there is a category $\mathcal{SS}ets^{\mathcal{D}}$ of $\mathcal{D}$–diagrams in $\mathcal{SS}ets$, or functors $\mathcal{D} \to \mathcal{SS}ets$. We can obtain two model category structures on $\mathcal{SS}ets^{\mathcal{D}}$ by the following results.

Theorem 4.5 [8, IX 1.4] *Given the category $\mathcal{SS}ets^{\mathcal{D}}$ of $\mathcal{D}$–diagrams of simplicial sets, there is a simplicial model category structure $\mathcal{SS}ets^{\mathcal{D}}_f$ in which the weak equivalences and fibrations are objectwise and in which the cofibrations are the maps which have the left lifting property with respect to the maps which are both fibrations and weak equivalences.*

Theorem 4.6 [8, VIII 2.4] *There is a simplicial model category $\mathcal{SS}ets_c^{\mathcal{D}}$ in which the weak equivalences and the cofibrations are objectwise and in which the fibrations are the maps which have the right lifting property with respect to the maps which are cofibrations and weak equivalences.*

We now return to the situation where our small category is a multi-sorted theory $\mathcal{T}$. We would like to have an evaluation map and its left adjoint as in the ordinary case (see the end of section 2 above), but here we will have one for each $\alpha \in S$. These evaluation maps look like

$$U_\alpha \colon \mathcal{A}lg^{\mathcal{T}} \to \mathcal{SS}ets$$

such that

$$U_\alpha(A) = A(T_\alpha)$$

for any $\mathcal{T}$–algebra A.

Each functor U_α has a left adjoint, the free functor

$$F_\alpha \colon \mathcal{SS}ets \to \mathcal{A}lg^{\mathcal{T}}$$

such that, given a simplicial set Y and object $T_{\underline{\beta}}$ in $\mathcal{T}$,

$$F_\alpha(Y)(T_{\underline{\beta}}) = \coprod_{n \geq 0} (\mathrm{Hom}_{\mathcal{T}}(T_{\alpha,\dots,\alpha}, T_{\underline{\beta}}) \times Y^n)/\sim .$$

As before, this free functor can be defined precisely as a coend over the initial (single-sorted) theory, regarded as the subcategory of the initial S–sorted theory whose objects are $(T_\alpha)^n$ for $n \geq 0$,

$$F_\alpha(Y)(T_{\underline{\beta}}) = \int^{\mathcal{T}_0} \mathrm{Hom}_{\mathcal{T}}((T_\alpha)^n, T_{\underline{\beta}}) \times Y^n.$$

Given a theory $\mathcal{T}$ (regular or multi-sorted), define a weak equivalence in the category $\mathcal{A}lg^{\mathcal{T}}$ of $\mathcal{T}$–algebras to be a map which induces a weak equivalence of simplicial sets after applying the evaluation functor U_α for each sort α. Similarly, define a fibration of $\mathcal{T}$–algebras to be a map f such that $U_\alpha(f)$ is a fibration of simplicial sets for all α. Then define a cofibration to be a map with the left lifting property with respect to the maps which are fibrations and weak equivalences.

The following theorem is a generalization of a result by Quillen [15, II.4].

Theorem 4.7 *Let $\mathcal{T}$ be an S–sorted theory. There is a cofibrantly generated model category structure on $\mathcal{A}lg^{\mathcal{T}}$ with the weak equivalences, fibrations, and cofibrations as defined above.*

Proof We use a slightly generalized version of Theorem 4.1 with the adjoint pairs $F_\alpha\colon \mathcal{S}\mathcal{S}ets \leftrightarrows \mathcal{A}lg^{\mathcal{T}} \colon U_\alpha$ for all $\alpha \in S$ and using the cofibrantly generated model structure on $\mathcal{S}\mathcal{S}ets$ as given above. The existence of limits and colimits follows just as they do in the case where $\mathcal{T}$ is an ordinary theory [15, II.4]. Thus, verifying conditions (1) and (2) will result in a model structure on $\mathcal{A}lg^{\mathcal{T}}$ for which the set $FI = \{F_\alpha \dot{\Delta}[n] \to F_\alpha \Delta[n] \mid \alpha \in S, n \geq 0\}$ is a set of generating cofibrations and $FJ = \{F_\alpha V[n,k] \to F_\alpha \Delta[n] \mid \alpha \in S, n \geq 1, 0 \leq k \leq n\}$ is a set of generating acyclic cofibrations.

We first show that FI and FJ satisfy the small object argument. Consider some $\mathcal{T}$–algebra A, which can be written as a directed colimit $\mathrm{colim}_m(A_m)$ and can therefore be computed objectwise. Thus, we can show that $F_\alpha \dot{\Delta}[n]$ is small:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}lg^{\mathcal{T}}}(F_\alpha \dot{\Delta}[n], \mathrm{colim}_m(A_m)) &= \mathrm{Hom}_{\mathcal{S}\mathcal{S}ets}(\dot{\Delta}[n], U_\alpha \mathrm{colim}_m(A_m)) \\
&= \mathrm{Hom}_{\mathcal{S}\mathcal{S}ets}(\dot{\Delta}[n], \mathrm{colim}_m(U_\alpha A_m)) \\
&= \mathrm{colim}_m \mathrm{Hom}_{\mathcal{S}\mathcal{S}ets}(\dot{\Delta}[n], U_\alpha A_m) \\
&= \mathrm{colim}_m \mathrm{Hom}_{\mathcal{A}lg^{\mathcal{T}}}(F_\alpha \dot{\Delta}[n], A_m).
\end{aligned}$$

The object $V[n,k]$ can be shown to be small analogously, so we have proved statement (1).

To prove statement (2), we need to show that taking a pushout along a map in FJ results in a map which is a weak equivalence in $\mathcal{A}lg^{\mathcal{T}}$. Note that since weak equivalences are taken levelwise, a (transfinite) directed colimit of weak equivalences is still a weak equivalence, so checking a single pushout suffices.

Consider a map $F_\alpha V[n,k] \to F_\alpha \Delta[n]$ in FJ and a map $F_\alpha V[n,k] \to A$ for some object A of $\mathcal{A}lg^{\mathcal{T}}$. We then take the pushout B in the following diagram:

$$\begin{tikzcd}
F_\alpha V[n,k] \arrow[r] \arrow[d] & A \arrow[d] \\
F_\alpha \Delta[n] \arrow[r] & B
\end{tikzcd} .$$

Suppose that $X \to Y$ is a map in $\mathcal{A}lg^{\mathcal{T}}$ with the right lifting property with respect to the maps in FJ. Note by adjointness that it is just a levelwise fibration of simplicial sets. Then in the diagram

$$\begin{tikzcd}
F_\alpha V[n,k] \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & X \arrow[d] \\
F_\alpha \Delta[n] \arrow[r] & B \arrow[r] & Y
\end{tikzcd}$$

a lift $F_\alpha\Delta[n]\to X$ exists, which implies by universality that there is also a lift $B\to X$. Now consider the diagram

$$\begin{tikzcd} V[n,k] \arrow[r] \arrow[d] & U_\alpha A \arrow[r] \arrow[d] & U_\alpha X \arrow[d] \\ \Delta[n] \arrow[r] & U_\alpha B \arrow[r] & U_\alpha Y \end{tikzcd}$$

where the left-hand square is given by adjointness and the right-hand square by applying U_α to the right-hand square of the previous diagram. Then there exists a lift $U_\alpha B\to U_\alpha X$. Since any fibration of simplicial sets occurs as $U_\alpha X\to U_\alpha Y$ for some $X\to Y$ in $\mathcal{A}lg^{\mathcal{T}}$, the map $U_\alpha A\to U_\alpha B$ is an acyclic cofibration of simplicial sets and in particular a weak equivalence. □

We now need a model category structure on the category of homotopy $\mathcal{T}$–algebras. However, the category of homotopy $\mathcal{T}$–algebras does not have all small limits and colimits (axiom MC1). Thus, we instead define a model category structure on all diagrams $\mathcal{T}\to\mathcal{SS}ets$ in such a way that the fibrant objects are homotopy $\mathcal{T}$–algebras.

The following theorem holds for model categories $\mathcal{M}$ which are left proper and cellular. We will not define these conditions here, but refer the reader to [9, 13.1.1, 12.1.1] for more details. It can be shown that $\mathcal{SS}ets^{\mathcal{T}}$ satisfies both these conditions [9, 13.1.14, 12.5.1].

Theorem 4.8 [9, 4.1.1] *Let $\mathcal{M}$ be a left proper cellular model category and P a set of morphisms of $\mathcal{M}$. There is a model category structure $\mathcal{L}_P\mathcal{M}$ on the underlying category of $\mathcal{M}$ such that:*

1. *The weak equivalences are the P–local equivalences.*
2. *The cofibrations are precisely the cofibrations of $\mathcal{M}$.*
3. *The fibrations are the maps which have the right lifting property with respect to the maps which are both cofibrations and P–local equivalences.*
4. *The fibrant objects are the P–local objects.*

To localize the model structure $\mathcal{SS}ets^{\mathcal{T}}_f$, we first need an appropriate map. To do so for ordinary algebraic theories, Badzioch uses free diagrams which are corepresented by the objects T_n of the theory $\mathcal{T}$ [2, 2.9]. In particular the n projection maps $T_n\to T_1$ induce maps

$$\coprod_{i=1}^{n}\mathrm{Hom}_{\mathcal{T}}(T_1,-)\to\mathrm{Hom}_{\mathcal{T}}(T_n,-).$$

He defines his localization with respect to the set of these maps. We would like to define similar free diagrams in a multi-sorted theory.

For each $\underline{\alpha}^n = <\alpha_1, \dots, \alpha_n>$ and $1 \leq i \leq n$, there exists a projection map $T_{\underline{\alpha}^n} \to T_{\alpha_i}$ inducing a map

$$\mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -) \to \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}^n}, -).$$

Taking the coproduct of all such maps results in a map

$$p_{\underline{\alpha}^n} \colon \coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -) \to \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}^n}, -).$$

These maps are the ones which we will use to localize $\mathcal{S}Sets^{\mathcal{T}}$. We define P to be the set of all such maps $p_{\underline{\alpha}^n}$ for each $\underline{\alpha}^n$ and $n \geq 0$.

Proposition 4.9 *There is a model category structure $\mathcal{LS}Sets^{\mathcal{T}}$ on the category $\mathcal{S}Sets^{\mathcal{T}}$ with weak equivalences the P–local equivalences, cofibrations as in $\mathcal{S}Sets^{\mathcal{T}}_f$, and fibrations the maps which have the right lifting property with respect to the maps which are cofibrations and weak equivalences.*

Proof This proposition is a special case of Theorem 4.8. □

The following propositions are proved by Badzioch for ordinary theories. His proofs can be generalized to apply to multi-sorted theories as well.

Proposition 4.10 [2, 5.5] *An object Z of $\mathcal{LS}Sets^{\mathcal{T}}$ is fibrant if and only if it is a homotopy $\mathcal{T}$–algebra which is fibrant as an object of $\mathcal{S}Sets^{\mathcal{T}}_f$.*

Proposition 4.11 [2, 5.6] *If Z and X' are homotopy $\mathcal{T}$–algebras in $\mathcal{S}Sets^{\mathcal{T}}$ and there is a P–local weak equivalence $f\colon Z \to X'$, then f is also a weak equivalence in $\mathcal{S}Sets^{\mathcal{T}}_f$, ie, an objectwise weak equivalence.*

Proposition 4.12 [2, 5.8] *A map $f\colon X \to X'$ is a P–local equivalence if and only if for any $\mathcal{T}$–algebra Y which is fibrant in $\mathcal{S}Sets^{\mathcal{T}}_c$, the induced map of function complexes*

$$f^*\colon \mathrm{Map}(X', Y) \to \mathrm{Map}(X, Y)$$

is a weak equivalence of simplicial sets.

These results can actually be stated in more generality; they are really just statements about the fibrant objects in a localized model category structure (see chapter 3 of [9] for more details).

Hence, we can consider the category $\mathcal{LS}Sets^{\mathcal{T}}$ to be our homotopy $\mathcal{T}$–algebra model category structure.

5 Rigidification of algebras over multi-sorted theories

We are now able to prove the following statement, which is a stronger version of Theorem 1.2:

Theorem 5.1 *There is a Quillen equivalence of model categories between $\mathcal{A}lg^{\mathcal{T}}$ and $\mathcal{LSS}ets^{\mathcal{T}}$.*

We begin with the necessary definitions.

Definition 5.2 [10, 1.3.1] If $\mathcal{C}$ and $\mathcal{D}$ are model categories, then the adjoint pair (F, R, φ) is a *Quillen pair* if one of the following equivalent statements is true:

(1) F preserves cofibrations and acyclic cofibrations.

(2) R preserves fibrations and acyclic fibrations.

The following theorem is useful for showing that we have a Quillen pair of localized model category structures.

Theorem 5.3 [9, 3.3.20] *Let $\mathcal{C}$ and $\mathcal{D}$ be left proper, cellular model categories and let (F, R, ψ) be a Quillen pair between them. Let S be a set of maps in $\mathcal{C}$ and $L_S\mathcal{C}$ the localization of $\mathcal{C}$ with respect to S. Then if $\mathbf{L}FS$ is the set in $\mathcal{D}$ obtained by applying the left derived functor of F to the set S [9, 8.5.11], then (F, R, ψ) is also a Quillen pair between the model categories $L_S\mathcal{C}$ and $L_{\mathbf{L}FS}D$.*

Definition 5.4 [10, 1.3.12] A Quillen pair is a *Quillen equivalence* if for all cofibrant X in $\mathcal{C}$ and fibrant Y in $\mathcal{D}$, a map $f\colon FX \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if the map $\varphi f\colon X \to RY$ is a weak equivalence in $\mathcal{C}$.

We need to find an adjoint pair of functors between $\mathcal{A}lg^{\mathcal{T}}$ and $\mathcal{LSS}ets^{\mathcal{T}}$ and prove that it is a Quillen equivalence. Let

$$J_{\mathcal{T}}\colon \mathcal{A}lg^{\mathcal{T}} \to \mathcal{SS}ets^{\mathcal{T}}$$

be the inclusion functor. We need to show we have an adjoint functor taking an arbitrary diagram in $\mathcal{SS}ets^{\mathcal{T}}$ to a $\mathcal{T}$–algebra. We first make the following definition.

Definition 5.5 Let $\mathcal{D}$ be a small category and $\mathcal{SS}ets^{\mathcal{D}}$ the category of functors $\mathcal{D} \to \mathcal{SS}ets$. Let P be a set of morphisms in $\mathcal{SS}ets^{\mathcal{D}}$. An object Y in $\mathcal{SS}ets^{\mathcal{D}}$ is *strictly P–local* if for every morphism $f\colon A \to B$ in P, the induced map on function complexes

$$f^*\colon \mathrm{Map}(B,Y) \to \mathrm{Map}(A,Y)$$

is an isomorphism of simplicial sets. A map $g\colon C \to D$ in $\mathcal{SS}ets^{\mathcal{D}}$ is a *strict P–local equivalence* if for every strictly P–local object Y in $\mathcal{SS}ets^{\mathcal{D}}$, the induced map

$$g^*\colon \mathrm{Map}(D,Y) \to \mathrm{Map}(C,Y)$$

is an isomorphism of simplicial sets.

Now, given a category of $\mathcal{D}$–diagrams in $\mathcal{SS}ets$ and the full subcategory of strictly P–local diagrams for some set P of maps, we have the following result. (Adámek and Rosický also prove this fact [1, 1.38], using slightly different terminology.)

Lemma 5.6 *Consider two categories, the category of all diagrams $X\colon \mathcal{D} \to \mathcal{SS}ets$ and the category of strictly local diagrams with respect to the set of maps $P = \{f\colon A \to B\}$. Then the forgetful functor from the category of strictly local diagrams to the category of all diagrams has a left adjoint.*

Proof Without loss of generality, assume that we have just one map f in P; otherwise replace f by $\coprod_\alpha f_\alpha$. Given an arbitrary diagram X, we would like to construct a strictly local diagram from X. So, suppose that X is not strictly local, ie, the map

$$f^*\colon \mathrm{Map}(B,X) \to \mathrm{Map}(A,X)$$

is not an isomorphism. To ensure that f^* is surjective, we obtain an object X' as the pushout in the following diagram:

$$\begin{array}{ccc} \coprod_{n\geq 0}\coprod_{A\times\Delta[n]\to X} A\times\Delta[n] & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod_{n\geq 0}\coprod_{A\times\Delta[n]\to X} B\times\Delta[n] & \longrightarrow & X' \end{array}$$

where each coproduct is taken over all maps $A\times\Delta[n] \to X$ for each $n \geq 0$. Then, to ensure that f is injective, we obtain X'' by taking the pushout

$$\begin{array}{ccc} \coprod_{n\geq 0}\coprod (B\coprod_A B)\times\Delta[n] & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \coprod_{n\geq 0}\coprod B\times\Delta[n] & \longrightarrow & X'' \end{array}$$

again where the second coproduct is over all maps $(B \coprod_A B) \times \Delta[n] \to X'$, and where the map

$$B \coprod_A B \to B$$

is the fold map.

In the construction of X', for any strictly local object Y we obtain a pullback diagram

$$\begin{tikzcd} \mathrm{Map}(X', Y) \arrow[r] \arrow[d, "\cong"] & \mathrm{Map}(\coprod B, Y) \arrow[d, "\cong"] \\ \mathrm{Map}(X, Y) \arrow[r] & \mathrm{Map}(\coprod A, Y) \end{tikzcd}$$

showing that the map $X \to X'$ is a strict local equivalence since $f\colon A \to B$ is.

In the construction of X'', we obtain a similar diagram, but it takes more work to show that the map $X' \to X''$ is a strict local equivalence. We first obtain the pullback diagram

$$\begin{tikzcd} \mathrm{Map}(X'', Y) \arrow[r] \arrow[d] & \mathrm{Map}(\coprod B, Y) \arrow[d] \\ \mathrm{Map}(X', Y) \arrow[r] & \mathrm{Map}(\coprod (B \coprod_A B), Y) \end{tikzcd}$$

Since it is a pullback diagram then suffices to show that the right hand vertical arrow is an isomorphism.

Recall that the object $B \coprod_A B$ is defined as the pushout in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ B \arrow[r] & B \coprod_A B \end{tikzcd}$$

which enables us to look at the pullback diagram

$$\begin{tikzcd} \mathrm{Map}(B \coprod_A B, Y) \arrow[r] \arrow[d] & \mathrm{Map}(B, Y) \arrow[d, "\cong"] \\ \mathrm{Map}(B, Y) \arrow[r] & \mathrm{Map}(A, Y). \end{tikzcd}$$

Hence the map

$$B \to B \coprod_A B$$

is a strict local equivalence. But, this map fits into a composite

$$\begin{tikzcd} B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B \coprod_A B \arrow[r] & B \end{tikzcd}$$

Since the identity map is a strict local equivalence, it follows that the map

$$B \coprod_A B \to B$$

is a strict local equivalence, since it can be shown that the strictly local equivalences satisfy model category axiom MC2.

Therefore, we obtain a composite map $X \to X''$ which is a strict local equivalence. However, we still do not know that the map

$$\mathrm{Map}(B, X'') \to \mathrm{Map}(A, X'')$$

is an isomorphism. So, we repeat this process, taking a (possibly transfinite) colimit to obtain a strictly local object $\widetilde{X}$ such that there is a local equivalence $X \to \widetilde{X}$.

It suffices to show that the functor which takes a diagram X to the local diagram $\widetilde{X}$ is left adjoint to the forgetful functor. So if J is the forgetful functor from the category of strictly local diagrams to the category of all diagrams and K is the functor we have just defined, we claim that

$$\mathrm{Map}(X, JY) \cong \mathrm{Map}(KX, Y)$$

for any diagram X and strictly local diagram Y. But, proving this statement is equivalent to showing that

$$\mathrm{Map}(X, Y) \cong \mathrm{Map}(\widetilde{X}, Y)$$

which was shown above for each step, and it still holds for the colimit. In particular, the map $X \to \widetilde{X} = KX$ and the identity $Y = JY$ induce natural isomorphisms

$$\mathrm{Map}(KX, Y) \to \mathrm{Map}(X, Y) \to \mathrm{Map}(X, JY),$$

and the restriction of this composite to the 0–simplices of each object,

$$\mathrm{Hom}(KX, Y) \to \mathrm{Hom}(X, JY)$$

is exactly the isomorphism we need to show that K is left adjoint to J. □

To apply this lemma to our situation, we first need to verify that $\mathcal{A}lg^{\mathcal{T}}$ is precisely the category of strictly local diagrams in $\mathcal{SS}ets^{\mathcal{T}}$ with respect to the set of maps

P, defined in the last section, to obtain the model category structure for homotopy $\mathcal{T}$–algebras.

To do so, we will need the following homotopical version of the Yoneda Lemma.

Lemma 5.7 *Let $T_{\underline{\alpha}}$ be an object in a multi-sorted algebraic theory $\mathcal{T}$ and A a strict $\mathcal{T}$–algebra. There is an isomorphism of simplicial sets*

$$\mathrm{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A) \cong A(T_{\underline{\alpha}}).$$

Proof Since A is a simplicial set-valued functor, we can regard it as a simplicial diagram of set-valued functors

$$A(-)_0 \Leftarrow A(-)_1 \Lleftarrow A(-)_2 \cdots$$

which further induces a simplicial diagram

$$\mathrm{Hom}_{\mathcal{SS}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A(-)_0) \Leftarrow \mathrm{Hom}_{\mathcal{SS}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A(-)_1) \Lleftarrow \cdots .$$

Using the classical Yoneda Lemma [9, 11.5.8], we have a natural isomorphism at each level

$$\mathrm{Hom}_{\mathcal{S}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A(-)_n) \cong A(T_{\underline{\alpha}})_n,$$

where $\mathcal{S}ets^{\mathcal{T}}$ denotes the category of functors $\mathcal{T} \to \mathcal{S}ets$.

Now, regarding sets as constant simplicial sets as necessary, notice that there are natural isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{S}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A(-)_n) \\
&\qquad \cong \mathrm{Hom}_{\mathcal{SS}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), \mathrm{Hom}_{\mathcal{SS}ets^{\mathcal{T}}}(\Delta[n], A(-))) \\
&\qquad \cong \mathrm{Hom}_{\mathcal{SS}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -) \times \Delta[n], A(-)) \\
&\qquad \cong \mathrm{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A)_n.
\end{aligned}$$

Since all the simplicial maps above are natural, we obtain a natural simplicial functor

$$\mathrm{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A) \to A(T_{\underline{\alpha}})$$

which is an isomorphism. □

Using this lemma, we are able to prove the following.

Lemma 5.8 *A diagram $A\colon \mathcal{T} \to \mathcal{SS}ets$ is a strict $\mathcal{T}$–algebra if and only if A is strictly local with respect to the maps*

$$p_{\underline{\alpha}^n}\colon \coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -) \to \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -).$$

Proof A diagram A is a strict $\mathcal{T}$–algebra if and only if for each $\underline{\alpha}^n = <\alpha_1, \dots, \alpha_n>$ there is a natural isomorphism

$$\prod_{i=1}^{n} A(T_{\alpha_i}) \cong A(T_{\underline{\alpha}})$$

induced by the projection maps in $\mathcal{T}$. Using Lemma 5.7, this statement is equivalent to having an isomorphism

$$\begin{aligned} \mathrm{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A) &\cong \prod_{i=1}^{n} \mathrm{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), A) \\ &\cong \mathrm{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), A) \end{aligned}.$$

Since all the isomorphisms in sight are induced by projections, it follows that this statement is equivalent to having A strictly local with respect to all the maps $p_{\underline{\alpha}^n}$. $\square$

In particular, a map $f\colon X \to X'$ is a strict P–local equivalence if and only if for every A in $\mathcal{A}lg^{\mathcal{T}}$ (regarded as an object in $\mathcal{SS}ets^{\mathcal{T}}$ via the map $J_{\mathcal{T}}$) the induced map

$$\mathrm{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(X', A) \to \mathrm{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(X, A)$$

is an isomorphism of simplicial sets.

Applying Lemma 5.6 to the functor $J_{\mathcal{T}}$, we obtain its left adjoint functor

$$K_{\mathcal{T}}\colon \mathcal{SS}ets^{\mathcal{T}} \to \mathcal{A}lg^{\mathcal{T}}.$$

Proposition 5.9 *The adjoint pair of functors*

$$K_{\mathcal{T}}\colon \mathcal{SS}ets^{\mathcal{T}} \rightleftarrows \mathcal{A}lg^{\mathcal{T}}\colon J_{\mathcal{T}}.$$

is a Quillen pair.

Proof Using Lemma 5.8, we can regard $\mathcal{A}lg^{\mathcal{T}}$ as a subcategory of $\mathcal{SS}ets^{\mathcal{T}}$ via the map $J_{\mathcal{T}}$. Since in both cases, the fibrations and weak equivalences are defined objectwise, $J_{\mathcal{T}}$ preserves fibrations and acyclic fibrations. $\square$

Lemma 5.10 *Each map $K_{\mathcal{T}}(p_{\underline{\alpha}^n})$ is a weak equivalence in $\mathcal{A}lg^{\mathcal{T}}$.*

Proof First, we note that the functor $\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -)$ is a strict $\mathcal{T}$–algebra and that $J_{\mathcal{T}} A = A$ for any strict $\mathcal{T}$–algebra A, again regarding $J_{\mathcal{T}}$ as an inclusion functor. Then, for each map $p_{\underline{\alpha}^n}$ we have the following composite isomorphism:

$$\begin{aligned}
\mathrm{Map}_{\mathcal{A}lg^{\mathcal{T}}}(K_{\mathcal{T}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -)), A) &\cong \mathrm{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A) \\
&\cong A(T_{\underline{\alpha}}) \\
&\cong \prod_{i=1}^{n} A(T_{\alpha_i}) \\
&\cong \prod_{i=1}^{n} \mathrm{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), A) \\
&\cong \mathrm{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), A) \\
&\cong \mathrm{Map}_{\mathcal{A}lg^{\mathcal{T}}}(K_{\mathcal{T}}(\coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -)), A).
\end{aligned}$$

Since all the isomorphisms are naturally induced by the map $p_{\underline{\alpha}^n}$ and adjoints, it follows that $K_{\mathcal{T}}$ is a strict local equivalence, or a weak equivalence in $\mathcal{A}lg^{\mathcal{T}}$. □

Now, we need to show that the same adjoint pair is still a Quillen pair when we replace the model structure $\mathcal{SS}ets^{\mathcal{T}}$ with the model structure $\mathcal{LSS}ets^{\mathcal{T}}$.

Proposition 5.11 *The adjoint pair*

$$K_{\mathcal{T}} \colon \mathcal{LSS}ets^{\mathcal{T}} \rightleftarrows \mathcal{A}lg^{\mathcal{T}} \colon J_{\mathcal{T}}$$

is a Quillen pair.

Proof Consider again the set of maps

$$P = \{p_{\underline{\alpha}^n} \colon \coprod_i \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -) \to \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -)\}.$$

Notice in particular that the objects involved in these maps are free diagrams and therefore cofibrant in $\mathcal{SS}ets_f^{\mathcal{T}}$. The model category structure $\mathcal{LSS}ets^{\mathcal{T}}$ is obtained by localizing with respect to these maps. Then using Lemma 5.10, we have that each map $K_{\mathcal{T}}(p_{\underline{\alpha}^n})$ is a weak equivalence in $\mathcal{A}lg^{\mathcal{T}}$. Hence, it follows from Theorem 5.3 that the pair of adjoints forms a Quillen pair even after the localization on $\mathcal{SS}ets_f^{\mathcal{T}}$. □

Before stating the main theorem, that the above Quillen pair is actually a Quillen equivalence, we first need a lemma. Badzioch's proof [2, 6.5] for ordinary theories generalizes for our case of multi-sorted theories, but we give a slightly different proof here.

Lemma 5.12 *If X is cofibrant in $\mathcal{LSS}ets^{\mathcal{T}}$, then the unit map $\eta\colon X \to K_{\mathcal{T}}X = J_{\mathcal{T}}K_{\mathcal{T}}X$ is a weak equivalence in $\mathcal{LSS}ets^{\mathcal{T}}$.*

Proof Case 1: The cofibrant object X is a free diagram, so it can be written as

$$\coprod_{\underline{\alpha}} \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}^n}, -).$$

The proof for such an object is then similar to the argument in the proof of Lemma 5.10.

Case 2: Let X be any cofibrant diagram. Then $X \simeq \mathrm{hocolim}_{\Delta^{op}} X_i$ where each X_i is a free diagram. Using Proposition 4.12, it then suffices to show that $\mathrm{Map}(K_{\mathcal{T}}X, Y) \simeq \mathrm{Map}(X, Y)$ for any $\mathcal{T}$–algebra Y which is fibrant in $\mathcal{SS}ets^{\mathcal{T}}_{cof}$. Using case 1, we have the following:

$$\begin{aligned}
\mathrm{Map}(X, Y) &\simeq \mathrm{Map}(\mathrm{hocolim}_{\Delta^{op}} X_i, Y)\\
&\simeq \mathrm{holim}\,\mathrm{Map}(X_i, Y)\\
&\simeq \mathrm{holim}\,\mathrm{Map}(K_{\mathcal{T}}X_i, Y)\\
&\simeq \mathrm{Map}(\mathrm{hocolim}_{\Delta^{op}} K_{\mathcal{T}}X_i, Y)\\
&\simeq \mathrm{Map}(K_{\mathcal{T}}X, Y).
\end{aligned}$$

Notice in particular that this weak equivalence is induced by the map η. The lemma follows. □

Now, the proof of the main theorem follows from this lemma exactly as it does for ordinary theories in [2, 6.4].

Theorem 5.13 *The Quillen pair of functors*

$$K_{\mathcal{T}}\colon \mathcal{LSS}ets^{\mathcal{T}} \rightleftarrows \mathcal{A}lg^{\mathcal{T}} \colon J_{\mathcal{T}}.$$

is a Quillen equivalence.

Proof Let X be a cofibrant object in $\mathcal{LSS}ets^{\mathcal{T}}$, A a fibrant object in $\mathcal{A}lg^{\mathcal{T}}$, and $f\colon X \to A = J_{\mathcal{T}}A$ a map in $\mathcal{LSS}ets^{\mathcal{T}}$. We need to show that f is a P–local equivalence if and only if its adjoint map $g\colon K_{\mathcal{T}}X \to A$ is a weak equivalence in $\mathcal{A}lg^{\mathcal{T}}$. There is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "f"'] & K_{\mathcal{T}}X \arrow[d, "g"] \\ J_{\mathcal{T}}A \arrow[r, "="] & A \end{tikzcd}$$

First assume that f is a P–local equivalence. Then g must also be a P–local equivalence since η is, by the previous lemma. However, g is a map in $\mathcal{A}lg^{\mathcal{T}}$, and so it is an objectwise weak equivalence, or a weak equivalence in $\mathcal{A}lg^{\mathcal{T}}$.

Conversely, suppose that g is a weak equivalence in $\mathcal{A}lg^{\mathcal{T}}$. Then it is a P–local equivalence. Hence, $f = g \circ \eta$ is also a P–local equivalence. □

Hence, we have a Quillen equivalence of model categories between strict $\mathcal{T}$–algebras and homotopy $\mathcal{T}$–algebras.

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