A categorification for the Tutte polynomial

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For each graph, we construct a bigraded chain complex whose graded Euler characteristic is a version of the Tutte polynomial. This work is motivated by earlier work of Khovanov, Helme-Guizon and Rong, and others.

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1 Introduction

In [5], Khovanov introduced a graded homology theory for classical links and showed it yields the Jones polynomial by taking the graded Euler characteristic. This construction has sparked a good deal of interests in recent years. In [3], Helme-Guizon and Rong constructed a graded homology theory for graphs. The graded Euler characteristic of the homology groups is the chromatic polynomial of the graph.

It is natural to ask if similar constructions can be made for other graph polynomials, especially the Tutte polynomial, which is universal among graph invariants satisfying the deletion-contraction rule. In this paper, we give such constructions for the Tutte polynomial. More precisely, for each graph G we define bigraded homology groups whose Euler characteristic is a variant of the Tutte polynomial. Our construction is different from the one by Khovanov and Rozansky for categorification for the Homflypt polynomial [7; 8].

Our construction starts with rewriting the Tutt polynomial using a state sum which is more amenable for a chain complex set up. This is done in Section 2. The chain groups will be built on two basic algebraic objects: a bigraded algebra A, and a bigraded $\mathbb{Z}$–module B. The differential will depend on the multiplication on A and a sequence of graded homomorphisms $f_k\colon B^{\otimes k} \to B^{\otimes k+1}$ $(k = 0, 1, 2, \cdots)$. In Section 3 we construct our homology groups using an obvious choice of A, B and f_k. A more general construction is shown in Section 4. These homology groups satisfy a long exact sequence which we explain in Section 5. In Section 6, we discuss additional properties, including a functorial property. Some computational examples are given in Section 7.

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2 The Tutte polynomial

We recall some basic properties of the Tutte polynomial. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Given an edge $e \in E(G)$, let $G-e$ denote the graph obtained from G by deleting the edge e, let G/e denote the graph obtained by contracting e to a vertex. Recall that e is called a loop if e joins a vertex to itself, e is called an isthmus if deleting e from G increases the number of components of the graph. The Tutte polynomial of G, denoted by $T(G;x,y)$, is uniquely defined by the following axioms:

- $T(G;x,y) = T(G-e;x,y) + T(G/e;x,y)$, if e is not a loop or isthmus.
- $T(G;x,y) = xT(G-e;x,y)$, if e is an isthmus.
- $T(G;x,y) = yT(G/e;x,y)$, if e is a loop.
- $T(G;x,y) = 1$ if G has no edges.

It is obvious that $T(G;x,y)$ is a 2–variable polynomial in x and y. Furthermore, it has a closed form described below. First we introduce some notations. Let $s \subset E(G)$. The rank of s, denoted by $r(s)$, is defined by $r(s) = |V(G)| - k(s)$ where $k(s)$ is the number of connected components of the graph $[G:s]$ having vertex set $V(G)$ and edge set s. We have the following well-known state sum formula (see, eg Welsh [10]):

$$T(G;x,y) = \sum_{s \subset E(G)} (x-1)^{r(E)-r(s)}(y-1)^{|s|-r(s)}$$

For our construction, we need to rewrite $T(G;x,y)$ in a new form that is easier to work with. Since $x-1 = -(1-x)$, $y-1 = -(1-y)$, we have

$$\begin{aligned} T(G;x,y) &= \sum_{s \subset E(G)} (-1)^{r(E)-r(s)}(1-x)^{r(E)-r(s)}(-1)^{|s|-r(s)}(1-y)^{|s|-r(s)} \\ &= (-1)^{r(E)} \sum_{s \subset E(G)} (-1)^{|s|}(1-x)^{r(E)-r(s)}(1-y)^{|s|-r(s)}. \end{aligned}$$

But $r(E)-r(s) = |V(G)|-k(E)-(|V(G)|-k(s)) = -k(E)+k(s)$. Note that $k(s)$ is the number of components of the graph $[G:s]$ which is the the 0^{th} Betti number of the (underlying space of the) graph $[G:s]$. We denote $k(s)$ by $b_0(s)$, and hence $r(E)-r(s) = -k(E)+b_0(s)$. For $|s|-r(s)$, we have $|s|-r(s) = |s|-(|V(G)|-k(s)) = -(|V(G)|-|s|)+k(s)$. But $|V(G)|-|s|$ is the Euler characteristic of the graph $[G:s]$. Let $b_1(s)$ denote the first betti number of $[G:s]$. We have $|V(G)|-|s| = \chi([G:s]) = b_0(s)-b_1(s)$, and $k(s) = b_0(s)$. It follows that $|s|-r(s) = b_1(s)$.

Therefore

$$
\begin{aligned}
T(G;x,y) &= (-1)^{r(E)} \sum_{s\subset E(G)} (-1)^{|s|}(1-x)^{-b_0(E)+b_0(s)}(1-y)^{b_1(s)} \\
&= (-1)^{r(E)}(1-x)^{-b_0(E)} \sum_{s\subset E(G)} (-1)^{|s|}(1-x)^{b_0(s)}(1-y)^{b_1(s)}.
\end{aligned}
$$

We have proved:

Proposition 2.1 *$T(G;x,y) = (-1)^{r(E)}(1-x)^{-b_0(E)}\widehat{T}(G;-x,-y)$, where $\widehat{T}(G;x,y) = \sum_{s\subset E(G)}(-1)^{|s|}(1+x)^{b_0(s)}(1+y)^{b_1(s)}$, and $b_i(s) = b_i([G:s])$ is the i^{th} Betti number of $[G:s]$ ($i=0,1$).*

It is the 2–variable polynomial $\widehat{T}(G;x,y)$ that we will categorify. This does recover the Tutte polynomial because of the following lemma.

Lemma 2.2 *The Tutte polynomial $T(G)$ is determined by $\widehat{T}(G)$.*

Proof We make the change of variables $u=1+x$, $v=1+y$ which turns $\widehat{T}(G;x,y)$ into $\sum_{s\subset E(G)}(-1)^{|s|}u^{b_0(s)}v^{b_1(s)}$, which we denote by $\widetilde{T}(G;u,v)$. Each subset $s\subset E(G)$ yields a term in $\widetilde{T}(G;u,v)$. Define the complexity of each such term to be $comp(s) = (-b_0(s), b_1(s))$ with the dictionary order. By Lemma 2.3 below, the complexity goes up when adding an edge to s. It follows that when $s=\varnothing$, we have the minimum term which is $u^{|V(G)|}$, and when $s=E(G)$, we have the maximum term which is $(-1)^{|E|}u^{b_0(E)}v^{b_1(E)}$. Therefore, we can recover $b_0(E)$, and $r(E)$ (which is $|V(G)|-b_0(E)$) from $\widehat{T}(G;x,y)$. By Proposition 2.1, $T(G;x,y)$ can be recovered from $\widehat{T}(G;x,y)$. □

Lemma 2.3 *Let s be a subset of $E(G)$, and e be an edge not in s. Then one of the following two cases occurs.*

(i) *If e joins two components of $[G:s]$, then $b_0(s\cup\{e\}) = b_0(s)-1$ and $b_1(s) = b_1(s\cup\{e\})$.*

(ii) *If e connects a component of $[G:s]$ to itself, then $b_0(s) = b_0(s\cup\{e\})$ and $b_1(s\cup\{e\}) = b_1(s)+1$.*

This lemma follows from standard algebraic topology and its proof is omitted.

3 The chain complex

3.1 Algebra background

Let A be a commutative algebra over a commutative ring. For our purpose, the ring will be $\mathbb{Z}$. Recall that A is called a graded algebra if it can be written as a direct sum $A = \oplus_{i\in\mathbb{Z}} A_i$ where each A_i is closed under addition, and $A_i A_j \subset A_{i+j}$ (ie, $a_i a_j \in A_{i+j}$ for all $a_i \in A_i, a_j \in A_j$). The elements in A_i are called homogeneous elements of degree i. Thus the condition $A_i A_j \subset A_{i+j}$ is equivalent to the condition that the degree is additive under multiplication.

The same definition can be made for modules over a ring by simply dropping the additivity condition of degree. Thus a $\mathbb{Z}$–module M is a graded module if we write it as a direct sum of submodules $M = \oplus_{i\in\mathbb{Z}} M_i$ where elements of M_i are called homogeneous elements of degree i. By definition, a graded algebra is automatically a graded module over the same ring.

Let $M = \oplus_{i\in\mathbb{Z}} M_i$ be a graded $\mathbb{Z}$–module. The *graded dimension* of M is the power series

$$q\dim M := \sum_i q^i \cdot \dim_{\mathbb{Q}}(M_i \otimes \mathbb{Q})$$

The same definition holds for a graded algebra.

An obvious generalization for graded algebras (modules) can be made by allowing the grading index i lying in an arbitrary abelian group. In this paper, we are interested in the case when the group is $\mathbb{Z}\oplus\mathbb{Z}$. Such an algebra (resp. module) will be called a $\mathbb{Z}\oplus\mathbb{Z}$ –graded, or a bigraded algebra (resp. module). By definition, a bigraded algebra is an algebra A with a decomposition $A = \oplus_{(i,j)\in\mathbb{Z}\oplus\mathbb{Z}} A_{i,j}$ where each $A_{i,j}$ is closed under addition and $A_{i_1,j_1}A_{i_2,j_2} \subset A_{i_1+i_2,j_1+j_2}$ for all $(i_1,j_1),(i_2,j_2) \in \mathbb{Z}\oplus\mathbb{Z}$. A bigraded module is a module M with a decomposition $M = \oplus_{(i,j)\in\mathbb{Z}\oplus\mathbb{Z}} M_{i,j}$ where elements of $M_{i,j}$ are called homogeneous elements with degree (i,j). The graded dimension of M is the 2–variable power series

$$q\dim M := \sum_{i,j} x^i y^j \cdot \dim_{\mathbb{Q}}(M_{i,j} \otimes \mathbb{Q})$$

3.2 The construction – a specific one

Let $A = \mathbb{Z}[x]/(x^2), B = \mathbb{Z}[y]/(y^2)$ where $\deg x = (1,0), \deg y = (0,1)$. Then A and B become bigraded algebras with $q\dim A = 1+x, q\dim B = 1+y$. Note that $A^{\otimes m}\otimes B^{\otimes n}$ is a bigraded $\mathbb{Z}$–module whose graded dimension is $q\dim A^{\otimes m}\otimes B^{\otimes n} =$

$(1+x)^m(1+y)^n$. We are not going to use the algebra structure on B, but its graded module structure will be needed.

Note that the letter x has two different meanings. First, it is an element in the algebra A. Second, it is the variable in the power series $q\dim A$. The same is true for the letter y. We believe such abuse of notation is convenient and will not lead to confusion.

Now, let G be a graph with $|E(G)|=n$. We fix an ordering on $E(G)$ and denote the edges by $e_1,\cdots,e_n$. Consider the n–dimensional cube $\{0,1\}^E=\{0,1\}^n$. Each vertex α of this cube corresponds to a subset $s=s_\alpha$ of E, where $e_i\in s_\alpha$ if and only if $\alpha_i=1$. The height $|\alpha|$ of α, is defined by $|\alpha|=\sum\alpha_i$, which is also equal to the number of edges in s_α.

For each vertex α of the cube, we associate the graded $\mathbb{Z}$–module $C^\alpha(G)$ as follows. Consider $[G:s]$, the graph with vertex set $V(G)$ and edge set s. We assign a copy of A to each component of $[G:s]$ and then take tensor product over the components. Let $A^\alpha(G)$ be the resulting graded $\mathbb{Z}$–module, with the induced grading from A. Therefore, $A^\alpha(G)\cong A^{\otimes k}$ where $k=b_0([G:s])$ is the number of components of $[G:s]$. Next, let $B^\alpha(G)=B^{\otimes l}$ where $l=b_1([G:s])$ is the first Betti number of $[G:s]$ (note that there is no specific order on the tensor factors here). We define $C^\alpha(G)=A^\alpha(G)\otimes B^\alpha(G)$. Then we define the i^{th} chain group $C^i(G):=\oplus_{|\alpha|=i}C^\alpha(G)$. This defines the chain groups of our complex.

As a notational remark, we can also denote $C^\alpha(G)$ by $C^s(G)$, because of the one-to-one correspondence between s and α.

Next, we define the differential maps $d^i\colon C^i(G)\to C^{i-1}(G)$. We need to make use of the edges of the cube $\{0,1\}^E$. Each edge ξ of $\{0,1\}^E$ can be labeled by a sequence in $\{0,1,*\}^E$ with exactly one $*$. The tail of the edge is obtained by setting $*=0$ and the head is obtained by setting $*=1$. The height $|\xi|$ is defined to the height of its tail, which is also equal to the number of 1's in ξ.

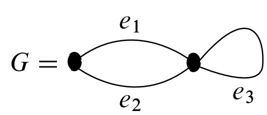
Given an edge ξ of the cube, let α_1 be its tail and α_2 be its head. Let e be the corresponding edge in G so that $s_2=s_1\cup\{e\}$. We define the per-edge map $d_\xi\colon C^{\alpha_1}(G)\to C^{\alpha_2}(G)$ based on the two cases in Lemma 2.3.

Case 1 e joins a component of $[G:s_1]$ to itself. The components of $[G:s_2]$ and $[G:s_1]$ naturally correspond to each other, and therefore $A^{\alpha_1}(G)=A^{\alpha_2}(G)$. Let $d^A_\xi\colon A^{\alpha_1}(G)\to A^{\alpha_2}(G)$ be the identity map. We also have $B^{\alpha_2}=B^{\alpha_1}\otimes B$. Let $d^B_\xi\colon B^{\alpha_1}(G)\to B^{\alpha_2}(G)$ be the homomorphism sending $b\in B^{\alpha_1}(G)$ to $b\otimes 1\in B^{\alpha_1}(G)\otimes B=B^{\alpha_2}(G)$. The per-edge map $d_\xi\colon C^{\alpha_1}(G)\to C^{\alpha_2}(G)$ is defined by $d_\xi=d^A_\xi\otimes d^B_\xi\colon A^{\alpha_1}(G)\otimes B^{\alpha_1}(G)\to A^{\alpha_2}(G)\otimes B^{\alpha_2}(G)$.

Case 2 e joins two different components of $[G:s_1]$. Let $E_1, E_2, \cdots, E_k$ be the components of $[G:s_1]$, where E_1 and E_2 are connected by e. Then the components of $[G:s_2]$ are $E_1 \cup E_2 \cup \{e\}, E_3, \cdots, E_k$. We define d_ξ^A to be the identity map on the tensor factors coming from $E_3, \cdots, E_k$, and d_ξ^A on the tensor factors coming from E_1, E_2 to be the multiplication map $A \otimes A \to A$ sending $a_1 \otimes a_2$ to $a_1 a_2$. For d_ξ^B, we have $B^{\alpha_2} = B^{\alpha_1}$ and we define d_ξ^B to be the identity map. Again, the per-edge map is defined by $d_\xi = d_\xi^A \otimes d_\xi^B$.

Now, we define the differential $d^i \colon C^i(G) \to C^{i+1}(G)$ by $d^i = \sum_{|\xi|=i} (-1)^\xi d_\xi$, where $(-1)^\xi = (-1)^{\sum_{i<j} \xi_i}$ and j is the position of $*$ in ξ.

To illustrate our construction consider the graph with two vertices, two parallel edges and a loop attached to one of its vertices. Let us label the edges of G as follows:



The representation of the 3–dimensional cube and the chain complex is given in Figure 1. In each rectangular box, the right upper corner has the sequence corresponding to α, the center has the graph $[G:s]$ and the algebra $C^\alpha(G)$ is at the bottom. Taking direct sum on each column gives the chain group $C^i(G)$ in the bottom row.

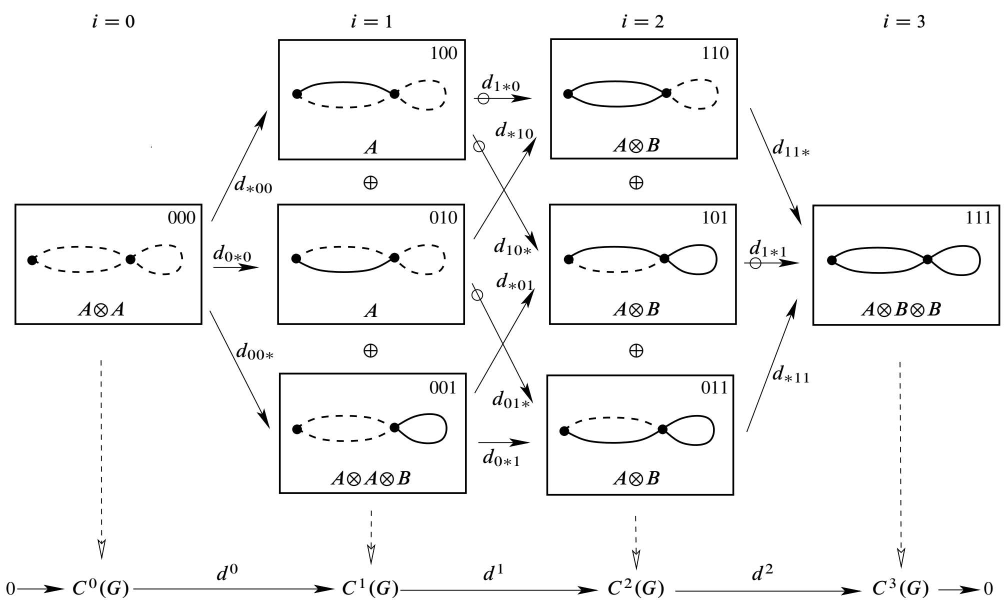


Figure 1

In Figure 1 we have represented the per-edge maps d_ξ. The arrows with a circle represent the maps for which $(-1)^\xi = -1$.

For this particular example we have that both d_{*00} and d_{0*0} map $a_1 \otimes a_2 \mapsto a_1 a_2$, d_{00*} maps $a_1 \otimes a_2 \mapsto a_1 \otimes a_2 \otimes 1_B$. Also d_{1*0}, d_{10*} and d_{01*} map $a \mapsto -a \otimes 1_B$; d_{*10} maps $a \mapsto a \otimes 1_B$; both d_{0*1} and d_{*01} map $a_1 \otimes a_2 \otimes b \mapsto a_1 a_2 \otimes b$. Finally d_{11*} and d_{*11} map $a \otimes b \mapsto a \otimes b \otimes 1_B$, d_{1*1} maps $a \otimes b \mapsto -a \otimes b \otimes 1_B$.

Theorem 3.1

(a) $0 \to C^0(G) \xrightarrow{d^0} C^1(G) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n(G) \to 0$ *is a chain complex of bigraded modules whose differential is degree preserving. Denote this chain complex by* $C(G)$.

(b) *The cohomology groups* $H^i(G)$ *are independent of the ordering of the edges of* G, *and therefore are invariants of the graph* G. *In fact, the isomorphism class of* $C(G)$ *is an invariant of* G.

(c) $\chi_q(C(G)) = \sum_{0 \le i \le n} (-1)^i q\dim(H^i) = \sum_{0 \le i \le n} (-1)^i q\dim(C^i) = \widehat{T}(G; x, y)$

Proof (a) The map d is obviously linear. It is also degree preserving since it is built on two basic maps: the multiplication on A and the map $b \to b \otimes 1$, both being degree preserving.

It remains to show that $d^2 = 0$. Let $s \subset E(G)$. Consider the result of adding two edges e_k and e_j to s where $k < j$. It is enough to show that the following diagram commutes.

(1)
$$\begin{tikzcd}
 & C^{s\cup\{e_k\}}(G) \arrow[dr, "d_{\ldots1\ldots*\ldots}"] & \\
C^S(G) \arrow[ur, "d_{\ldots*\ldots0\ldots}"] \arrow[dr, "d_{\ldots0\ldots*\ldots}"'] & & C^{s\cup\{e_k,e_j\}}(G) \\
 & C^{s\cup\{e_j\}}(G) \arrow[ur, "d_{\ldots*\ldots1\ldots}"'] &
\end{tikzcd}$$

where $d_{\ldots*\ldots0\ldots}$ means that the vector ξ has a star in the k-th position and a zero in the j-th position and the same convention applies to the remaining maps. We observe that in Diagram (1) all the per-edge maps will have the same sign except for exactly one. More precisely $d_{\ldots1\ldots*\ldots}$ will have a different sign if the number of 1's between the star and the zero is even, and $d_{\ldots0\ldots*\ldots}$ will differ in sign if the number of 1's is odd. This, along with the commutativity of Diagram (1), implies $d^{i+1}d^i = 0$.

The commutativity of Diagram (1) follows from some tedious but straight forward checking, based on various ways e_k and e_j join the components of $[G : s]$. Let e_k connect E_k to F_k, and e_j connect E_j to F_j, where E_k, E_j, F_k, F_j are (not necessarily distinct) components of $[G:s]$. Consider the set $C=\{E_k, F_k, E_j, F_j\}$. Up to symmetry (ie, interchange of k and j, and of E and F), there are seven possibilities shown below.

(1) $|C|=1$. We have $E_k=F_k=E_j=F_j$.

(2) $|C|=2$, and $E_k=F_k$, $E_j=F_j$.

(3) $|C|=2$, and $E_k=F_k=E_j$.

(4) $|C|=2$, and $E_k=E_j$, $F_k=F_j$.

(5) $|C|=3$, and $E_k=F_k$.

(6) $|C|=3$, and $F_k=E_j$.

(7) $|C|=4$.

As an example, we check the commutativity for Case 3. This is depicted in the following diagram.

$$\begin{tikzcd} & A\otimes A\otimes B \arrow[dr, "d_{\ldots 1\ldots *\ldots}"] & \\ A\otimes A \arrow[ur, "d_{\ldots *\ldots 0\ldots}"] \arrow[dr, "d_{\ldots 0\ldots *\ldots}"'] & & A\otimes B \\ & A \arrow[ur, "d_{\ldots 1\ldots *\ldots}"'] & \end{tikzcd} \qquad \begin{tikzcd} & a_1\otimes a_2\otimes 1_B \arrow[dr, "d_{\ldots 1\ldots *\ldots}"] & \\ a_1\otimes a_2 \arrow[ur, "d_{\ldots *\ldots 0\ldots}"] \arrow[dr, "d_{\ldots 0\ldots *\ldots}"'] & & a_1a_2\otimes 1_B \\ & a_1a_2 \arrow[ur, "d_{\ldots 1\ldots *\ldots}"'] & \end{tikzcd}$$

(b) The proof is similar to [3, Theorem 12]. Each permutation of the edges of G is a product of transpositions of the form $(k,k+1)$. An explicit isomorphism can be constructed for each such transposition. In fact, this shows that the isomorphism class of the chain complex is an invariant of the graph.

(c) First, a standard homological algebra argument shows that

$$\sum_{0\le i\le n}(-1)^i q\dim(H^i(G))=\sum_{0\le i\le n}(-1)^i q\dim(C^i(G)).$$

Next, each $C^i(G)$ is a direct sum of $C^\alpha(G)$ where α corresponds to $s\subset E(G)$ with $|s|=i$. We have $q\dim C^\alpha(G)=(q\dim A)^{b_0([G:s])}(q\dim B)^{b_1([G:s])}$ which is exactly the contribution of the state s in $\widehat{T}(G;x,y)$. This proves the equation. □

4 More general constructions

Our construction can be made more general. Let A be any commutative bigraded ring. Let B be any bigraded module over $\mathbb{Z}$. For both A and B, we assume that the dimension of the space of homogeneous elements at each degree is finite so that $q\dim A$ and $q\dim B$ are well-defined as 2–variable power series. For each integer $k \geq 0$, let $f_k\colon B^{\otimes k} \to B^{\otimes k+1}$ be a degree preserving module homomorphism. Given such A, B and f_k, we can construct homology groups in the following manner.

The chain groups are defined similarly as before. We fix an ordering on $E(G)$ and denote the edges by $e_1, \cdots, e_n$. Let α be a vertex of the cube $\{0,1\}^{E(G)} = \{0,1\}^n$. Let $s \subset E(G)$ be the edge set corresponding to α. We assign a copy of A to each component of $[G:s]$ and then take tensor product over the components. Let $A^{\alpha}(G)$ be the resulting graded $\mathbb{Z}$–module, with the induced grading from A. Let $B^{\alpha}(G) = B^{\otimes b_1([G:s])}$. We define $C^{\alpha}(G) = A^{\alpha}(G) \otimes B^{\alpha}(G)$. Then we define $C^i(G) := \oplus_{|\alpha|=i} C^{\alpha}(G)$.

The differential maps d^i are defined using the multiplication on A and the homomorphism f_k. First, we describe the per-edge map $d_{\xi}\colon C^{\alpha_1}(G) \to C^{\alpha_2}(G)$. Let e be the corresponding edge in G so that $s_2 = s_1 \cup \{e\}$. If e joins a component of $[G:s_1]$ to itself, we define $d_{\xi}^A\colon A^{\alpha_1}(G) \to A^{\alpha_2}(G)$ to be the identity map, and define $d_{\xi}^B\colon B^{\alpha_1}(G) \to B^{\alpha_2}(G)$ to be the homomorphism $f_k\colon B^{\otimes k} \to B^{\otimes k+1}$ where $B^{\alpha_1}(G) = B^{\otimes k}$ and $B^{\alpha_2}(G) = B^{\otimes k+1}$. The per-edge map $d_{\xi} : C^{\alpha_1}(G) \to C^{\alpha_2}(G)$ is defined to be $d_{\xi}^A \otimes d_{\xi}^B$. If e joins two different components, say E_1 and E_2, of $[G:s_1]$, we define d_{ξ} to be the multiplication map $A \otimes A \to A$ on tensor factors coming from E_1 and E_2, and d_{ξ} to be the identity map on the remaining tensor factors.

As before, we define the differential $d^i\colon C^i(G) \to C^{i+1}(G)$ by $d^i = \sum_{|\xi|=i} (-1)^{\xi} d_{\xi}$, where $(-1)^{\xi} = (-1)^{\sum_{i<j} \xi_i}$ and j is the position of $*$ in ξ.

A similar argument as before proves:

Theorem 4.1

(a) $0 \to C^0(G) \xrightarrow{d^0} C^1(G) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n(G) \to 0$ *is a chain complex of bigraded modules whose differential is degree preserving. Denote this chain complex by* $C(G) = C_{A,B,f_k}(G)$.

(b) *The cohomology groups* $H^i(G) (= H^i_{A,B,f_k}(G))$ *are independent of the ordering of the edges of* G*, and therefore are invariants of the graph* G*. In fact, the isomorphism type of the graded chain complex* $C(G)$ *is an invariant of* G.

(c) *The graded Euler characteristic*

$$\begin{aligned}\chi_q(C(G)) = \sum_{0 \le i \le n} (-1)^i q\dim(H^i) &= \sum_{0 \le i \le n} (-1)^i q\dim(C^i) \\ &= \widehat{T}(G; q\dim A - 1, q\dim B - 1).\end{aligned}$$

Remark 4.2 Some special choices of A, B and f_k are as follows.

(a) Given any A and B satisfying the basic conditions as above. Let b_0 be a fixed element in B with $\deg b_0 = (0,0)$. Then we can define $f_k\colon B^{\otimes k} \to B^{\otimes(k+1)}$ by $f_k(b) = b \otimes b_0$ for all $b \in B^{\otimes k}$. In particular, if A and B are given as in the previous section, and $b_0 = 1$, we obtain the construction in the previous section.

(b) Let A, B, b_0 be as in (a) above. A specific choice of b_0 is $b_0 = 0$.

(c) Let $B = \mathbb{Z}$, with $q\dim B = 1$ and $f_k(b) = b \otimes 1$. Then $\chi_q(C(G)) = P_G(q\dim A)$. The homology groups are isomorphic to the ones in [2].

5 Exact sequences

In this section, we show that our homology groups satisfy a long exact sequence which can be considered as a categorification for the deletion-contraction rule. Since we will work with $\widehat{T}(G)$ rather than $T(G)$, we need the deletion contraction rule for $\widehat{T}(G)$, which we establish in Section 5.1. This naturally leads us to two cases: adding an non-loop edge or adding a loop. The exact sequences for these two cases are discussed in Section 5.2 and Section 5.3. respectively.

5.1 Deletion-contraction rule for $\widehat{T}(G)$

Let e be a fixed edge of G. We wish to understand relations between $\widehat{T}(G)$, $\widehat{T}(G-e)$, and $\widehat{T}(G/e)$. Recall that $\widehat{T}(G;x,y) = \sum_{s \subset E(G)} (-1)^{|s|}(1+x)^{b_0([G:s])}(1+y)^{b_1([G:s])}$. We have $\widehat{T}(G;x,y) = \sum_1 + \sum_2$, where $\sum_1$ consists of terms with those s that do not contain e, and $\sum_2$ consists of the remaining terms.

For the first summation, we have

$$\textstyle\sum_1 = \sum_{e \notin s \subset E(G)} (-1)^{|s|}(1+x)^{b_0([G:s])}(1+y)^{b_1([G:s])}.$$

This summation is the same as the summation over all $s \subset E(G-e)$. Furthermore, for each such s, $[G:s] = [G-e:s]$. It follows that $\sum_1 = \widehat{T}(G-e;x,y)$. Note that this is true for all choices of e, including loops and isthmuses.

For the second summation, we have $\sum_2 = \sum_{e \in s \subset E(G)} (-1)^{|s|}(1+x)^{b_0([G:s])}(1+y)^{b_1([G:s])}$. Each such s can be written as $s = s_1 \cup \{e\}$, where s_1 corresponds to a subset of $E(G/e)$, which we also denote by s_1. Note that $(-1)^{|s|} = -(-1)^{|s_1|}$ since $|s| = |s_1| + 1$. If we assume that e is not a loop, we have $[G:s] \simeq [G/e:s_1]$ where $\simeq$ stands for homotopically equivalent. It follows that $\sum_2 = -\widehat{T}(G/e,x,y)$. Note that this is true for all e that are not loops (therefore e can be an isthmus).

We have proved (1) of the following:

Proposition 5.1 *Let e be an edge in a graph G.*

(1) *If e is not a loop (but possibly an isthmus), then $\widehat{T}(G;x,y)=\widehat{T}(G-e;x,y)-\widehat{T}(G/e;x,y)$.*

(2) *If e is a loop, then $\widehat{T}(G;x,y)=\widehat{T}(G-e;x,y)-(1+y)\widehat{T}(G/e;x,y)=-y\widehat{T}(G/e;x,y)$. Of course, we have $G-e=G/e$.*

(3) *If e is an isthmus, then*

$$\widehat{T}(G;x,y)=\widehat{T}(G-e;x,y)-\widehat{T}(G/e;x,y)=x\widehat{T}(G/e;x,y).$$

We also have $\widehat{T}(G-e;x,y)=(1+x)\widehat{T}(G/e;x,y)$.

Proof Part (1) is proved above. For (2), we claim $\sum_2=-(1+y)\widehat{T}(G/e,x,y)$. This is because $[G:s]$ is obtained from $[G/e:s_1]$ by adding the loop e to an existing vertex, which then implies $b_1([G:s])=b_1([G/e:s_1])+1$. For (3), We need only to show $\widehat{T}(G-e;x,y)=(1+x)\widehat{T}(G/e;x,y)$. Both sides can be written as a summation over $s\in E(G-e)=E(G/e)$. The only difference is that $b_0([G-e:s])=b_0([G/e:s])+1$. This implies our equation. □

5.2 Exact sequence for non-loop edges

Let e be an edge of a graph G. Assume that e is not a loop. We will construct degree preserving chain maps $\alpha: C^{i-1}(G/e)\to C^i(G)$, and β: $C^i(G)\to C^i(G-e)$ such that

$$0\to C^{i-1}(G/e)\xrightarrow{\alpha} C^i(G)\xrightarrow{\beta} C^i(G-e)\to 0$$

is exact.

First, we describe α. It is enough to define $\alpha|_{C^s(G/e)}$ for all $s\subset E(G/e)$ with $|s|=i-1$. Let $s\subset E(G/e)$ with $|s|=i-1$. Let $s_e=s\cup\{e\}$. Then $s_e\subset E(G)$ with $|s_e|=i$. Since e is not a loop, the two graphs $[G:s_e]$ and $[G/e:s]$ are homotopically equivalent under the natural map that contracts e to a point. This induces a natural isomorphism from $C^s(G/e)$ to $C^{s_e}(G)$. Define our map $\alpha|_{C^s(G)}$: $C^s(G/e)\to C^{s_e}(G)$ to be this isomorphism. Taking summation over s, we obtain the map α: $C^{i-1}(G/e)\to C^i(G)$.

Next, we describe the map β. Just as before, it is enough to define $\beta|_{C^s(G)}$ for all $s\subset E(G)$ with $|s|=i$. We consider two cases. If $e\in s$, we define $\beta|_{C^s(G)}$ to be the zero map. If $e\notin s$, the two graphs $[G-e:s]$ and $[G:s]$ are identical, and therefore the groups $C^s(G-e)$ and $C^s(G)$ are naturally identified. We define $\beta|_{C^s(G)}$ to be this identity isomorphism composed with the inclusion map.

A diagram chasing argument shows:

Theorem 5.2

(a) *If e is not a loop, the above defines an exact sequence of chain maps, and therefore*

(b) *it induces a long exact sequence:*

$$0 \to H^0(G) \xrightarrow{\beta^*} H^0(G-e) \xrightarrow{\gamma^*} H^0(G/e) \xrightarrow{\alpha^*}$$
$$H^1(G) \xrightarrow{\beta^*} H^1(G-e) \xrightarrow{\gamma^*} H^1(G/e) \xrightarrow{\alpha^*} \dots$$

It will be useful for further computations to understand the action of the map γ^*. This follows from tracing back the elements involved in the diagram chasing of the zig-zag lemma. The result is as follows.

Remark 5.3 The connecting homomorphism $\gamma^*\colon H^i(G-e) \to H^i(G/e)$ acts as follows. Each cycle $z \in C^i(G-e)$ is a linear combination of states for the graph $G-e$. That is, $z = \sum n_k(s_k, c_k)$ where $n_k \in \mathbb{Z}, s_k$ is a subset of $E(G-e)$, $c_k \in C^{s_k}(G-e)$. We add the edge e to each s_k to get $s_k \cup \{e\}$, and replace c_k by $d_\xi(c_k)$ where d_ξ is described in Section 3.2. More specifically, $d_\xi(c_k)$ is obtained using the multiplication on A if e joins two components of $G-e$, otherwise $d_\xi(c_k)$ is obtained using the map $B \to B \otimes B$ sending b to $b \otimes 1$.

5.3 Exact sequence for loops

Now we assume that e is a loop. We will define a new chain complex denoted by $C(G/e) \otimes B$. Then we will show that there is a short exact sequence of chain maps:

$$0 \to C^{i-1}(G/e) \otimes B \xrightarrow{\alpha} C^i(G) \xrightarrow{\beta} C^i(G-e) \to 0$$

Of course, $G-e = G/e$.

First, we describe $C(G/e) \otimes B$. For each i, define its i^{th} chain group to be $C^i(G/e) \otimes B$. Next, define differential $d\colon C^{i-1}(G/e) \otimes B \to C^i(G/e) \otimes B$ in terms of per-edge maps. Let $s \subset E(G/e)$, with $|s| = i-1$, and $f \in E(G/e) - s$. The per-edge map $d\colon C^s(G/e) \otimes B \to C^{s \cup \{f\}}(G/e) \otimes B$ is defined as follows. If f connects two different components of $[G/e : s]$, d is defined the same way as before by multiplying the "coloring" on these two components. If f joins a component of $[G/e : s]$ to itself, $C^{s \cup \{f\}}(G/e) \otimes B \cong (C^s(G/e) \otimes B) \otimes B$ via a natural isomorphism, and then per-edge map d is defined to be $d(x) = x \otimes 1$ for each $x \in C^s(G/e) \otimes B$. Note that this differential map is different from just taking tensor product with the identity, that is: $d_{C(G/e) \otimes B} \neq d_{C(G/e)} \otimes Id_B$.

We now describe the maps α and β. The map $\alpha : C^{i-1}(G/e) \otimes B \to C^i(G)$ is defined as follows. For each $s \subset E(G/e)$ with i edges, $s \cup \{e\} \in E(G)$ has $i+1$ edges. The graph $[G : s \cup \{e\}]$ is obtained from $[G : s]$ by adding the loop e. Thus $C^s(G/e) \otimes B$ is naturally isomorphic to $C^{s \cup \{e\}}(G)$. Define α to be this isomorphism summed over all s. The map $\beta : C^i(G) \to C^i(G-e)$ is defined the same way as in the above section. In other words, it is the projection map that kills all summands $C^s(G)$ where $e \in s$.

Theorem 5.4 *Let e be a loop in G. Then:*

(a) *There is a short exact sequence of chain maps*

$$0 \to C^{i-1}(G/e) \otimes B \xrightarrow{\alpha} C^i(G) \xrightarrow{\beta} C^i(G-e) \to 0$$

described above. Therefore:

(b) *This induces a long exact sequence:*

$$\begin{aligned} 0 \to H^0(G) &\xrightarrow{\beta^*} H^0(G-e) \xrightarrow{\gamma^*} H^0(C(G/e) \otimes B) \xrightarrow{\alpha^*} \\ H^1(G) &\xrightarrow{\beta^*} H^1(G-e) \xrightarrow{\gamma^*} H^1(C(G/e) \otimes B) \xrightarrow{\alpha^*} \ldots \end{aligned}$$

The proof consists of a standard but tedious diagram chasing argument, and is left as an exercise.

6 Other properties

We prove some other properties of our cohomology groups.

6.1 Adding a pendant edge

As an application of the above long exact sequence, let us consider the effect of adding a pendant edge on the cohomology groups. Recall that a *pendant vertex* in a graph is a vertex of degree one, and a *pendant edge* is an edge connecting a pendant vertex to another vertex. Let e be a pendant edge in a graph G, then Proposition 5.1 implies $\widehat{T}(G; x, y) = x\widehat{T}(G/e; x, y)$. On the level of our cohomology groups, we have the following:

Theorem 6.1 *If e is a pendant edge of the graph G, then $H^k(G) \cong H^k(G/e)\{(1,0)\}$, where $\{(1,0)\}$ is the operation that shifts the degree up by $(1,0)$.*

Remark 6.2 Let $A' = \mathbb{Z}x$ be the submodule generated by x. Then $A = \mathbb{Z}1 \oplus \mathbb{Z}x$, and $A' \cong \mathbb{Z}\{(1,0)\}$ as bigraded modules. The above can be rephrased as

$$H^k(G) \cong H^k(G/e) \otimes A'$$

This equation works for more general algebras A satisfying $A = \mathbb{Z}1 \oplus A'$.

Proof of Theorem 6.1 The proof is similar to the proof of an analogous result for the chromatic cohomology (Theorem 22 in [3]).

Consider the operations of contracting and deleting e in G. Denote the graph G/e by G_1. We have $G/e = G_1$, and $G - e = G_1 \sqcup \{v\}$, where v is the end point of e with $\deg v = 1$. Consider the exact sequence

$$\cdots \to H^{i-1}(G_1 \sqcup \{v\}) \xrightarrow{\gamma^*} H^{i-1}(G_1) \xrightarrow{\alpha^*} H^i(G) \xrightarrow{\beta^*} H^i(G_1 \sqcup \{v\}) \xrightarrow{\gamma^*} H^i(G_1) \to \cdots$$

We need to understand the map

$$H^i(G_1 \sqcup \{v\}) \xrightarrow{\gamma^*} H^i(G_1)$$

It is easy to understand the impact of adding an isolated vertex on the cohomology groups. We have

$$H^i(G_1 \sqcup \{v\}) \cong H^i(G_1) \otimes A.$$

This can be seen by first noting the same equation holds on the level of chain groups. Furthermore, the differential map restricted on the tensor factor A is the identity map. This implies that equation holds on the level of homology groups.

We therefore identify $H^i(G_1 \sqcup \{v\})$ with $H^i(G_1) \otimes A$. The map $\gamma^*\colon H^i(G_1 \sqcup \{v\}) \to H^i(G_1)$ sends $u \otimes 1$ to $(-1)^i u$. In particular, γ^* is onto. Therefore, the above long exact sequence becomes a collection of short exact sequences:

$$0 \to H^i(G) \xrightarrow{\beta^*} H^i(G_1 \sqcup \{v\}) \xrightarrow{\gamma^*} H^i(G_1) \to 0 \tag{2}$$

Hence, $H^i(G) \cong \ker \gamma^*$. We define a homomorphism:

$$f\colon H^i(G_1) \otimes A' \to \ker \gamma^* \text{ by } f(u \otimes a') = u \otimes a' - (-1)^i \gamma^*(u \otimes a') \otimes 1$$

One checks that f is an isomorphism of $\mathbb{Z}$–modules. Therefore, $H^i(G) \cong \ker \gamma^* \cong H^i(G_1) \otimes A'$. □

Remark 6.3 The above gives a complete description of the generators of $H^i(G)$. For the specific case of $A' = \mathbb{Z}x$, $H^i(G)$ is spanned by $\alpha \otimes x - (-1)^i \gamma^*(\alpha \otimes x) \otimes 1$, where α ranges over the generators of $H^i(G_1)$.

Using inductively Theorem 6.1 on the number of edges implies:

Corollary 6.4 *If $G = T_n$ is a tree with n edges, then $H^0(G) \cong \mathbb{Z}\{(n,0)\} \oplus \mathbb{Z}\{(n+1,0)\}$, and $H^i(G) = 0$ for all $i > 0$.*

We remark that, for the more general construction in Section 4, a statement similar to Corollary 6.4 still holds with A' being the direct summand of A satisfying $A = \mathbb{Z}1 \oplus A'$

6.2 Overlapping with the chromatic cohomology

Corollary 6.4 shows that, for all trees, the Tutte cohomology and the chromatic cohomology agree. This is extended to general graphs in part (a) below.

Theorem 6.5 *Let ℓ be the length of the shortest cycle in a graph G.*

(a) *For all $i < \ell - 1$, we have*

$$H^i_{\text{Tutte}}(G) \cong H^i_{\text{chromatic}}(G)$$

(b) *If G is the Tait graph of an unoriented framed alternating link diagram D, then for all $i < \ell - 1$,*

$$H^{i,j,0}_{\text{Tutte}}(G) \cong H_{p,q}(D)$$

where $p = |V(G)| - i - 2j$, $q = |E(G)| - 2|V(G)| + 4j$, and $H_{p,q}$ is the homology groups of the version of Khovanov cohomology theory for unoriented framed link defined by Viro in [9].

Proof (a) For each such i, the algebra B is not involved in the chain groups $C^0_{\text{Tutte}}(G), \cdots, C^{i+1}_{\text{Tutte}}(G)$ since subgraphs involved, $[G : s]$, contains no cycles. It follows that the chain groups and differentials in the Tutte cohomology and the chain groups in the chromatic cohomology are all the same. Therefore the corresponding cohomology groups are isomorphic.

(b) This is a consequence of Theorem 24 of Helme-Guizon, Przytycki and Rong [1] where a similar relation between chromatic cohomology and Khovanov's link homology was established. □

6.3 Functorial property

The classical homology theory is a functor: continuous maps between spaces induce homomorphism between homology groups. Khovanov's link homology Kh also satisfies a functorial property: a cobordism C between two links L_1 and L_2 induces a homomorphism $Kh(C)\colon Kh(L_1) \to Kh(L_2)$, well defined up to sign (Jacobsson [4], Khovanov [6]).

For our Tutte cohomology, we can associate homomorphisms between homology groups to each inclusion map of graphs. Essentially, it is the iteration of the map β^* in the above long exact sequence, with some additional attention to any possible additional vertices in the ambient graph. More specifically, let K be a subgraph of G, and we denote this relation by $K \subseteq G$. Define $\beta\colon C^i(G) \to C^i(K)$ as follows. Since $C^i(G) := \oplus_{|s|=i} C^s(G)$, it is enough to define $f|C^s(G)$.

(1) If $s \not\subseteq E(K)$, we define $\beta|C^s(G) = 0$.

(2) If $s \subseteq E(K)$, then $[G:s]$ is $[K:s]$ union l vertices of G where $l \geq 0$. We have $C^s(G) \cong C^s(K) \otimes A^{\otimes l}$. Since $A = \mathbb{Z}1 \oplus \mathbb{Z}x$, $A^{\otimes l} = A_0 \oplus A_1$, where $A_0 \cong \mathbb{Z}$ is the direct summand generated by $1 \otimes \cdots \otimes 1$. This implies $C^s(G) \cong C^s(K) \otimes A_0 \oplus C^s(K) \otimes A_1$. We define $\beta(g \otimes 1 \otimes \cdots \otimes 1) = g$, $\beta(g \otimes a) = 0$ for all $g \in C^s(K), a \in A_1$.

Theorem 6.6

(1) $\beta\colon C(G) \to C(K)$ *is a degree preserving chain map, therefore*

(2) *it induces a degree preserving homomorphism* $\beta^*\colon H^i(G) \to H^i(K)$.

(3) *This correspondence is natural. That is, if* $L \subseteq K$, $K \subseteq G$ *are subgraphs, then the diagram*

$$\begin{tikzcd} H^*(G) \arrow[rr, "\beta^*"] \arrow[dr, "\beta_2^*"'] & & H^*(L) \\ & H^*(K) \arrow[ur, "\beta_1^*"'] & \end{tikzcd}$$

commutes, where $\beta^*, \beta_1^*, \beta_2^*$ *are the homomorphisms induced by the inclusions* $L \subseteq G, L \subseteq K$, *and* $K \subseteq G$ *respectively.*

7 Examples

Example 1 Let L_1 be the graph with one vertex and one loop, that is, $L_1 =$ . Our construction yields the chain complex:

$$0 \to A \xrightarrow{d^0} A \otimes B \to 0$$

The differential d^0 maps $1 \mapsto 1_A \otimes 1_B$ and $x \mapsto x \otimes 1_B$. It is easy to see that

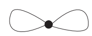
$H^1(L_1) = \langle 1_A \otimes y, x \otimes y \rangle \cong A \otimes B' \cong \mathbb{Z}\{(0,1)\} \oplus \mathbb{Z}\{(1,1)\}$, $H^i(L_1) = 0$ for $i \neq 1$.

where $B' = \mathbb{Z}\{y\}$ satisfies $B \cong \operatorname{span}\{1\} \oplus B'$. Hence we have $\chi(H^*(L_1)) = -y - xy = \widehat{T}(L_1; x, y)$

Example 2 Let $P_2 =$ , that is, the Polygon with two sides. The corresponding chain complex is:

$$0 \to A \otimes A \xrightarrow{d^0} A \oplus A \xrightarrow{d^1} A \otimes B \xrightarrow{d^2} 0$$

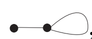
Where d^0 maps $x \otimes x \mapsto (0,0)$, $1_A \otimes 1_A \mapsto (1_A, 1_A)$, $x \otimes 1_A$ and $1_A \otimes x \mapsto (x, x)$. The kernel of d^0 is generated by the elements $x \otimes x$ and $x \otimes 1_A - 1_A \otimes x$. Thus $H^0(P_2) \cong A\{1\} \cong \mathbb{Z}(1,0) \oplus \mathbb{Z}(2,0)$. We also have d^1 maps $(1_A, 0) \mapsto -1_A \otimes 1_B$, $(x, 0) \mapsto -x \otimes 1_B$, $(0, 1_A) \mapsto 1_A \otimes 1_B$ and $(0, x) \mapsto x \otimes 1_B$. This implies $H^1(P_2) = 0$ and $H^2(P_2) \cong A \otimes B' \cong \mathbb{Z}\{(0,1)\} \oplus \mathbb{Z}\{(1,1)\}$. Clearly $H^i(P_2) = 0$, for $i \geq 3$. Hence $\chi(H^*(P_2)) = x^2 + x + y + xy = \widehat{T}(P_2; x, y)$.

Example 3 Let $L_2 =$  (ie one vertex and two loops). Its chain complex is:

$$0 \to A \xrightarrow{d^0} A \otimes B \oplus A \otimes B \xrightarrow{d^1} A \otimes B \otimes B \to 0$$

Where d^0 maps $1 \mapsto (1_A \otimes 1_B, 1_A \otimes 1_B)$ and $x \mapsto (x \otimes 1_B, x \otimes 1_B)$, d_1 maps $(a \otimes b, 0) \mapsto -a \otimes b \otimes 1_B$ and $(0, a \otimes b) \mapsto a \otimes b \otimes 1_B$. Hence, $H^1(L_2) = \langle (1_A \otimes y, 1_A \otimes y) \rangle \oplus \langle (x \otimes y, x \otimes y) \rangle \cong A \otimes B' \cong \mathbb{Z}\{(0,1)\} \oplus \mathbb{Z}\{(1,1)\}$, $H^2 \cong A \otimes B \otimes B' \cong \mathbb{Z}\{(0,1)\} \oplus \mathbb{Z}\{(1,1)\} \oplus \mathbb{Z}\{(0,2)\} \oplus \mathbb{Z}\{(1,2)\}$ and $H^i(L_2) = 0$ for $i \neq 1, 2$.

Clearly $\chi(H^*(L_2)) = xy^2 + y^2 = \widehat{T}(L_2; x, y)$.

Example 4 Let $G =$ , obtained from L_1 by adding a pendant edge. The computations in Example 1 and Remark 6.3 yield:

$$\begin{aligned} H^1(G) &= \langle 1_A \otimes x \otimes y - x \otimes 1_A \otimes y \rangle \oplus \langle x \otimes x \otimes y \rangle \\ &\qquad \cong A \otimes B'\{(1,0)\} \cong \mathbb{Z}\{(1,1)\} \oplus \mathbb{Z}\{(2,1)\}, \\ H^i(G) &= 0 \quad \text{for} \quad i \neq 1. \end{aligned}$$

We have that $\chi(H^*(G)) = -xy - x^2 y = \widehat{T}(G; x, y)$.

Example 5 Let $G =$ , whose construction of chain complex and differential was described in Section 3.2. An easy way to compute the cohomology groups for this graph is using the exact sequence given in Theorem 5.2. We choose the edge e to be one of the parallel edges on G : 

. Then $G - e$ is the graph in Example 4, and G/e is L_2 as in Example 3.

According to Theorem 5.2 we have the following long exact sequence:

$$0 \to H^0(G) \xrightarrow{\beta^*} H^0(G-e) \xrightarrow{\gamma^*} H^0(L_2) \xrightarrow{\alpha^*} H^1(G) \xrightarrow{\beta^*} H^1(G-e) \xrightarrow{\gamma^*} H^1(L_2) \xrightarrow{\alpha^*} \dots$$

If we use the computations in Examples 3 and 4, then the long exact sequence above decomposes into the following exact sequences:

$$0 \to H^0(G) \to 0$$

$$0 \to H^1(G) \xrightarrow{\beta^*} A \otimes B'\{(1,0)\} \xrightarrow{\gamma^*} A \otimes B' \xrightarrow{\alpha^*} H^2(G) \xrightarrow{\beta^*} 0$$

$$0 \to A \otimes B \otimes B' \xrightarrow{\alpha^*} H^3(G) \to 0$$

Hence we have $H^0(G) \cong 0$, $H^3(G) \cong A \otimes B \otimes B' \cong \mathbb{Z}\{(0,1)\} \otimes \mathbb{Z}\{(1,1)\} \otimes \mathbb{Z}\{(0,2)\} \otimes \mathbb{Z}\{(1,2)\}$. To compute $H^i(G)$ for $i = 1, 2$, we need to understand the map $\gamma^*\colon H^1(G-e) \to H^1(L_2)$. First note that, by Example 4, $H^1(G-e) \cong A \otimes B'\{(0,1)\}$ has basis $\{1 \otimes x \otimes y - x \otimes 1 \otimes y, x \otimes x \otimes y\}$ and, by Example 3, $H^1(L_2) \cong A \otimes B'$ has basis $\{(1 \otimes y, 1 \otimes y), (x \otimes y, x \otimes y)\}$. By Remark 5.3 we have $\gamma^*(1 \otimes x \otimes y - x \otimes 1 \otimes y) = x \otimes y - x \otimes y = 0$ and $\gamma^*(x \otimes x \otimes y) = x^2 \otimes y = 0$ (recall $x^2 = 0$ in A). Thus γ^* is the zero map. Hence $H^1(G) \cong H^1(G-e) \cong \mathbb{Z}\{(2,1)\} \oplus \mathbb{Z}\{(1,1)\}$ and $H^2(G) \cong A \otimes B' \cong \mathbb{Z}\{(0,1)\} \oplus \mathbb{Z}\{(1,1)\}$.

The graded Euler characteristic for the cohomology groups for G is $-(1+x)(1+y)y + (1+x)y - (1+x)xy = -y(1+x)(x+y)$, which agrees with $\widehat{T}(G; x, y)$.

Example 6 Let K_3 be the complete graph with 3 vertices. To compute its cohomology groups we will use the exact sequence given in 5.2. In this case $G - e = T_2$, the tree with two edges and $G/e = P_2$ as in Example 2 above. We get:

$$0 \to H^0(G) \xrightarrow{\beta^*} H^0(T_2) \xrightarrow{\gamma^*} H^0(P_2) \xrightarrow{\alpha^*} H^1(G) \xrightarrow{\beta^*} H^1(T_2) \xrightarrow{\gamma^*} H^1(P_2) \to \dots$$

Using the information of the previous examples, if we substitute the homology groups that are zero, we end up with the following exact sequences:

$$0 \to H^0(G) \xrightarrow{\beta^*} H^0(T_2) \xrightarrow{\gamma^*} H^0(P_2) \xrightarrow{\alpha^*} H^1(G) \to 0$$

$$0 \to H^2(P_2) \xrightarrow{\alpha^*} H^3(G) \to 0$$

As a consequence, we have that $H^0(G) \cong \ker \gamma^*$, $H^1(G) \cong H^0(P_2)/\mathrm{Im}\gamma^*$, $H^2(G) \cong 0$ and $H^3(G) \cong H^2(P_2) \cong \mathbb{Z}(0,1) \oplus \mathbb{Z}(1,1)$. So we need to understand the map $\gamma^*\colon H^0(P_2) \to H^0(T_2)$. The generators for $H^0(P_2)$ are the elements $1 \otimes x \otimes x - x \otimes 1 \otimes x + x \otimes x \otimes 1$ and $x \otimes x \otimes x$ which are mapped to $2x \otimes x$ and 0, respectively. Hence

$H^0(G) \cong \mathbb{Z}(3,0)$ and $H^1 \cong \langle x \otimes 1 - 1 \otimes x \rangle \oplus \langle x \otimes x \rangle / \langle 2x \otimes x \rangle \cong \mathbb{Z}(1,0) \oplus \mathbb{Z}_2(2,0)$
The graded Euler characteristic is $x^3 - x - xy - y = \widehat{T}(G; x, y)$

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