The FA_n Conjecture for Coxeter groups

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We study global fixed points for actions of Coxeter groups on nonpositively curved singular spaces. In particular, we consider property FA_n, an analogue of Serre's property FA for actions on CAT(0) complexes. Property FA_n has implications for irreducible representations and complex of groups decompositions. In this paper, we give a specific condition on Coxeter presentations that implies FA_n and show that this condition is in fact equivalent to FA_n for $n = 1$ and 2. As part of the proof, we compute the Gersten–Stallings angles between special subgroups of Coxeter groups.

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1 Introduction

A *Coxeter group* is a group W that has a presentation of the form

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $m_{ij} = m_{ji} \in \mathbb{N} \cup \{\infty\}$ and $m_{ij} = 1$ if and only if $i = j$. Recall that a CAT(0) *space* is a complete geodesic space which is nonpositively curved in the metric sense, ie, its geodesic triangles are no fatter than their Euclidean counterparts (see Section 3.1). We will consider isometric actions of Coxeter groups on CAT(0) spaces.

A fundamental notion of Bass–Serre theory is Serre's *property* FA. A group G has property FA if every G–action on every simplicial tree is *trivial*, ie, has a global fixed point. Such groups are "rigid" in the following sense: they do not split nontrivially as amalgamated free products or HNN extensions and all their irreducible $\mathrm{GL}_2(\mathbb{C})$–representations have algebraic integer traces. In addition to finite groups, Serre proved in [28] that $\mathrm{SL}_3(\mathbb{Z})$ and Coxeter groups with every $m_{ij} < \infty$ have FA.

A generalization of Serre's property is *property* FA_n, as defined by Farb in [19]. A group has FA_n if every action of it by cellular isometries on every CAT(0) n–complex is trivial. By CAT(0) n–complex, we mean a CAT(0) cell complex of piecewise constant curvature with only finitely many isometry types of cells (see Section 3.1). We emphasize that the actions are not assumed to be cocompact, properly discontinuous or faithful, and that the spaces are not assumed to be locally finite. Note that FA_1 is equivalent to FA.

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Properties FA_n and FA_m are distinct for $n \neq m$. For example, Farb proved in [19] that $\mathrm{SL}_n(\mathbb{Z}[1/p])$ has FA_{n-2}. However, $\mathrm{SL}_n(\mathbb{Z}[1/p])$ does not have FA_{n-1} since it acts nontrivially on the affine building for $\mathrm{SL}_n(\mathbb{Q}_p)$.

As with property FA (see [28]), groups with FA_n have certain strong properties.

(1) If Γ has property FA_n then Γ does not split nontrivially as a nonpositively curved *n–complex of groups* in the sense of Gersten–Stallings, Haefliger and Corson (see [29; 21; 14]).

(2) Suppose Γ has property FA_n. Let $\rho\colon \Gamma \to \mathrm{GL}_{n+1}(K)$ be any representation of degree $n+1$ over a field K. Then, the eigenvalues of each of the matrices in $\rho(\Gamma)$ are integral. In particular, they are algebraic integers if $\mathrm{char}(K) = 0$ and are roots of unity if $\mathrm{char}(K) > 0$. (As in the tree case, this follows from studying induced actions on the Bruhat–Tits buildings for $\mathrm{SL}_{n+1}(\mathbb{Q}_p)$, which are CAT(0) for all primes p.) In the language of Bass [2], Γ is thus of *integral $(n+1)$–representation type*. Consequently, there are only finitely many conjugacy classes of irreducible representations of Γ into $\mathrm{GL}_{n+1}(\mathrm{K})$ for any algebraically closed field K (see Farb [19]).

The following results are known about property FA_n for a Coxeter group W:

(1) (Serre [28]) If every m_{ij} is finite, then W has property FA.

(2) (Farb [19]) If W is a discrete group generated by reflections in the sides of a compact Euclidean or hyperbolic n–simplex, then W has property FA_{n-1} but does not have property FA_n.

In this paper, we generalize these results. We first consider natural conditions on Coxeter groups that imply property FA_n.

For $T \subset S$, let W_T denote the subgroup of W generated by T. It is well-known (see, for example, Bourbaki [4]) that (W_T, T) is a Coxeter system with a Coxeter presentation that is induced from the presentation for W. The group W_T is a *special subgroup* with *rank* equal to the size of T.

Applying techniques from [19], we prove the following theorem by considering the combinatorics of fixed sets of finite special subgroups.

Theorem 1.1 *Let (W, S) be a Coxeter system. If every special subgroup of W of rank at most $n+1$ is finite, then W has property FA_n.*

Suppose a group acts nontrivially on an n–dimensional CAT(0) space X. Then for $m \geq n$ it acts nontrivially on the m–dimensional CAT(0) space $X \times \mathbb{R}^{m-n}$. So, for $m \geq n$ we have $\mathrm{FA}_m \Rightarrow \mathrm{FA}_n$. In other words, for every group G that acts nontrivially on some finite-dimensional CAT(0) complex, there an integer n such that G has FA_m if and only if $m < n$. This n is the smallest dimension of a CAT(0) complex on which G acts nontrivially. As formulated in the following conjecture, we suspect that Theorem 1.1 gives this bound for Coxeter groups.

Conjecture 1.2 *Let (W, S) be a Coxeter system. The following are equivalent:*

(i) *The group W has property FA_n.*

(ii) *Every special subgroup of W of rank at most $n+1$ is finite.*

(iii) *For all $0 < m \leq n$, the group W does not split nontrivially as a nonpositively curved m–simplex of special subgroups.*

As noted by Mihalik and Tschantz in [24], Conjecture 1.2 is known for $n = 1$. In this paper, we reduce the proof of Conjecture 1.2 in general to proving that spaces arising from certain simplex of groups decompositions of W are CAT(0). The CAT(0) Conjecture (Section 6.2) posits that these spaces are indeed CAT(0). In Section 6.3, we prove the CAT(0) Conjecture in dimension 2 by computing the Gersten–Stallings angles between special subgroups of Coxeter groups. This implies the following theorem.

Theorem 1.3 *Conjecture 1.2 holds for $n \leq 2$.*

P Caprace observed that Conjecture 1.2 holds for all n if and only if it holds for $n \leq 8$ (see Remark 6.7). To prove Conjecture 1.2 in general, it therefore remains only to show that it holds for $3 \leq n \leq 8$.

In Section 7, we study the maximal FA_n subgroups of Coxeter groups. Special subgroups satisfying the condition of Theorem 1.1 are natural candidates, and we posit the following:

Maximal FA_n **Conjecture** *A subgroup $H \subset W$ is maximal FA_n if and only if $H = wAw^{-1}$ for some maximal FA_n special subgroup A of W and $w \in W$.*

This has been shown for $n = 1$ by Mihalik and Tschantz in [24]. In Section 7, we prove the following:

Theorem 1.4 CAT(0) *Conjecture* $\Rightarrow$ *Maximal* FA_n *Conjecture.*

In particular, since the CAT(0) Conjecture holds in dimension 2, the Maximal FA_2 Conjecture holds as well.

In Sections 2–4, we recall important results about Coxeter groups, CAT(0) spaces, and complexes of groups. We discuss in Section 5 background and techniques related to property FA_n. We present and prove our main results in Sections 6–8.

Recently, Luis Paris has independently discovered results similar to those in Section 6.

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2 Coxeter groups

We briefly recall key definitions and results on Coxeter groups that we will need. See Bourbaki [4], Davis [16] or Humphreys [22] for further details.

2.1 Definitions

Let S be a finite set. A *Coxeter matrix* on S is a symmetric $|S|\times|S|$ matrix M with entries in $\mathbb{N}\cup\{\infty\}$ such that each diagonal entry is 1 and each off-diagonal entry is at least 2. Associated to M is a group W with presentation $W=\langle S \mid (st)^{m_{st}}=1\rangle$, where the relation $(st)^{m_{st}}=1$ is omitted if $m_{st}=\infty$. The pair (W,S) is a *Coxeter system*, W is a *Coxeter group*, and the group presentation is a *Coxeter presentation* for W. The W–conjugates of elements of S are called *reflections*.

Given a Coxeter system (W,S), the *Coxeter diagram* Γ associated to (W,S) is a labeled graph with vertex set S and with an edge labeled m_{st} connecting s to t if and only if $m_{st}\neq 1,2$. Note that the Coxeter diagram encodes the same information as the Coxeter matrix. A Coxeter system (W,S) is *irreducible* if its Coxeter diagram is a connected graph.

2.2 Reduced words and the word problem

Given a Coxeter system (W, S) and $w \in W$, we denote by $\ell(w)$ the *length* of w, which is given by

$$\ell(w) = \min\{k : w = s_1 s_2 \dots s_k \text{ for some } s_1, s_2, \dots, s_k \in S\}.$$

An expression for w which achieves its length $\ell(w)$ is called a *reduced* or *geodesic* expression for w. Note that this notion depends on the choice of Coxeter generating set S.

We denote by $\widehat{s_i}$ the omission of s_i from an expression. Below is a standard characterization of Coxeter groups (see, for example, [11; 16; 22]).

Theorem 2.1 (Deletion and Exchange Conditions) *Let W be a group generated by a set S of involutions. The following are equivalent:*

(i) *(W, S) is a Coxeter system.*

(ii) (Deletion Condition) *For all $w \in W$, if $\ell(w) < k$ and $w = s_1 s_2 \dots s_k$ for some generators $s_1, s_2, \dots s_k \in S$, then there exist indices $1 \le j < l \le k$ such that $w = s_1 s_2 \dots \widehat{s_j} \dots \widehat{s_l} \dots s_k$.*

(iii) (Strong Exchange Condition) *Let $w \in W$ and let $w = s_1 s_2 \dots s_k$ ($s_i \in S$) be an expression for w. If a reflection r in W satisfies $\ell(rw) < \ell(w)$, then there is an index i for which $w = r s_1 \dots \widehat{s_i} \dots s_k$. Moreover, if $k = \ell(w)$, then i is unique.*

An immediate consequence of the Deletion Condition is the following:

Corollary 2.2 *Let (W, S) be a Coxeter system and $w \in W$. Then every unreduced expression for w can be reduced to a geodesic expression for w by omitting an even number of generators.*

We now define a standard set of operations for reducing words in Coxeter groups.

Definition 2.3 Let M be the Coxeter matrix associated to a Coxeter system (W, S). An *elementary M–operation* on a word w in the alphabet S is an operation of *Type (I)* or *Type (II)*, which are defined as:

Type (I) Delete a subword of the form ss for some $s \in S$.

Type (II) Replace an alternating subword $sts\dots$ of length m_{st} for some $s, t \in S$ with the alternating word $tst\dots$ of length m_{st}.

Note that elementary M–operations do not change the image of the word in W. An *M–reduction* of a word is a sequence of elementary M–operations. A word is *M–reduced* if its length cannot be reduced via elementary M–operations.

The *word problem* is a fundamental problem in combinatorial and geometric group theory. The Deletion Condition implies that Coxeter groups have solvable word problem. In particular, Tits (see [30; 11]) proved the following theorem.

Theorem 2.4 *Let (W, S) be a Coxeter system with associated Coxeter matrix M. Then an expression for $w \in W$ is reduced if and only if it is M–reduced. Moreover, given two reduced expressions for w, one can be transformed to the other via a sequence of Type (II) elementary M–operations.*

Since elementary M–operations do not increase word length, Theorem 2.4 solves the word problem for Coxeter groups.

Remark 2.5 An important special case of Theorem 2.4 is for alternating products. In particular, for $a, b, \in S$, an alternating product of a and b of length strictly less than m_{ab} is the *unique* reduced representative of the corresponding element of W.

2.3 Parabolic subgroups

Given a Coxeter group W, we will be particularly interested in certain natural subgroups of W.

Note that for elements or subsets $A_1, A_2, \ldots, A_k$ of a group G we denote by $\langle A_1, A_2, \ldots, A_k \rangle$ the subgroup of G generated by the union of the A_i's.

Definition 2.6 A *special* subgroup of W is a subgroup W_T of W given by $W_T = \langle T \rangle$ for some $T \subset S$. We say $W_\varnothing = 1$ and we define the *rank* of W_T to be $|T|$. The W–conjugates of special subgroups of W are called *parabolic* subgroups of W.

Theorem 2.7 (See, for example, [22]) *Let (W, S) be a Coxeter system with Coxeter matrix M, and let $T \subset S$.*

(i) *(W_T, T) is a Coxeter system with Coxeter matrix the submatrix of M corresponding to T.*

(ii) *If $w \in W_T$, and $w = s_1 s_2 \ldots s_k$ is a reduced expression for w with respect to S, then $s_i \in T$ for all i. In particular, $W_T \cap S = T$. Moreover, the length of w with respect to S equals the length of w with respect to T.*

(iii) *S is a minimal generating set for W, and more generally, T is a minimal generating set for W_T.*

Given $T \subset S$, consider the Coxeter system (W_T, T). Denote by Γ_T the associated Coxeter diagram. Note that by Theorem 2.7, this is the labeled subgraph of Γ spanned by the vertices corresponding to the elements of T.

Remark 2.8 Given a special subgroup $G \subset W$, we denote by S_G the generators of G as a special subgroup. By Theorem 2.7, we have $S_G = G \cap S$ and $G = W_{S_G}$. In this notation, statement (ii) of Theorem 2.7 implies the well-known result that for special subgroups A and B, we have $A \cap B = \langle S_A \cap S_B \rangle$.

For $T, U \subset S$, we say that w is *(T,U)–reduced* if it is of minimal length in the double coset $W_T w W_U$. The following is standard (see, for example, [4]).

Proposition 2.9 *Let T and U be (possibly empty) subsets of S.*

(i) *Let $w \in W$. There is a unique (T, U)–reduced element $d \in W_T w W_U$. Moreover, every element $w' \in W_T w W_U$ can be written as $w' = xdy$ for some $x \in W_T$ and $y \in W_U$ so that $\ell(w') = \ell(x) + \ell(d) + \ell(y)$.*

(ii) *An element $w \in W$ is (T, U)–reduced if and only if $\ell(tw) > \ell(w)\ \forall\ t \in T$ and $\ell(wu) > \ell(w)\ \forall\ u \in U$*

Given two parabolic subgroups of W, we will be interested in their intersection. In fact, the intersection of two parabolic subgroups is again a parabolic subgroup. In particular, we have the following result of Kilmoyer (see [24]).

Proposition 2.10 *Suppose A and B are special subgroups of W, with corresponding generating sets $S_A, S_B \subset S$. For $w \in W$, let d be the unique (S_A, S_B)–reduced element of AwB and let $a \in A$, $b \in B$ so that $w = adb$. Then*

$$A \cap wBw^{-1} = a\langle S_A \cap dS_B d^{-1}\rangle a^{-1}.$$

Corollary 2.11 *Suppose $S_A, S_B \subset S$ and let $w \in W$. Let d be the unique (S_A, S_B)–reduced element in AwB. If $A \subset wBw^{-1}$, then $S_A \subset dS_B d^{-1}$.*

Proof of Corollary 2.11 We have $A = A \cap wBw^{-1}$, so for $a \in A$ and $b \in B$ such that $w = adb$, we find $A = a\langle S_A \cap dS_B d^{-1}\rangle a^{-1}$ by Proposition 2.10. Since $a \in A$, we know $a^{-1}Aa = A$ and hence $A = \langle S_A \cap dS_B d^{-1}\rangle$. Now S_A is a minimal generating set of A by statement (iii) of Theorem 2.7, so A cannot be generated by a proper subset of S_A. We therefore find that $S_A = S_A \cap dS_B d^{-1}$ so in particular $S_A \subset dS_B d^{-1}$. □

2.4 Classification

Let W be a Coxeter group arising as a discrete group generated by reflections in the sides of a compact Euclidean or hyperbolic simplex. Note that every proper special subgroup of W is finite since every such subgroup stabilizes a point under the discrete and proper action of W on Euclidean or hyperbolic space. In fact, this property distinguishes these groups (see, for example, [11; 16; 22]).

Theorem 2.12 *Let (W, S) be a Coxeter system of rank $n+1$. Then, every proper special subgroup of W is finite if and only if one of the following holds:*

(i) *W is finite.*

(ii) *W is an irreducible Euclidean or hyperbolic reflection group with fundamental domain a compact simplex.*

In case (ii), W acts cocompactly on a Euclidean or hyperbolic space of dimension n, and we can recover the dihedral angles of the fundamental domain from the Coxeter presentation. Moreover, the hyperbolic case only occurs for $n \leq 4$.

Remark 2.13 This is part of the general classification of Coxeter groups. Irreducible finite Coxeter groups and those described in case (ii) of Theorem 2.12 are listed in standard books on Coxeter groups, such as [4] and [22]. In particular, determining whether Coxeter groups (or their special subgroups) are finite reduces to verifying whether their irreducible components appear on the list of irreducible finite Coxeter groups.

3 CAT(0) spaces

We recall a few key facts about CAT(0) spaces. See Bridson and Haefliger [10] for full details.

3.1 Definitions, properties and constructions

Given $\kappa \in \mathbb{R}$, let M_κ^n denote $\mathbb{E}^n$ (Euclidean n–space), $\mathbb{H}_\kappa^n$ (hyperbolic n–space of constant curvature κ) or $\mathbb{S}_\kappa^n$ (the n–sphere of constant curvature κ) as κ is 0, negative or positive, respectively. Denote by d_κ the distance function on M_κ^2.

Definition 3.1 Assume that (X, d) is a geodesic metric space. Let $T \subset X$ be a geodesic triangle. A κ*–comparison triangle* for T is a triangle $T' \subset M^2_\kappa$ with the same edge lengths as those in T. We say T satisfies the *CAT(κ) inequality* if for all points $x, y \in T$ with corresponding points $x', y' \in T'$ (see Figure 1), we have $d(x, y) \leq d_\kappa(x', y')$. The space X is a *CAT(κ) space* if every triangle $T \subset X$ (of perimeter at most $\frac{2\pi}{\sqrt{\kappa}}$ if $\kappa > 0$) satisfies the CAT(κ) inequality. Note that CAT(κ) implies CAT(κ') for all $\kappa' \geq \kappa$.

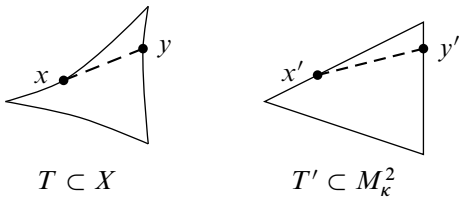

Figure 1: Comparison triangle

Remark 3.2 We will be most interested in the case $\kappa = 0$. CAT(0) spaces are a generalization of complete, simply connected, nonpositively curved Riemannian manifolds. CAT(0) spaces can be singular and even locally infinite, but the CAT(0) condition implies many strong properties, including convexity of the distance function, unique geodesics and contractibility.

We now briefly discuss a natural construction that yields many interesting examples of CAT(0) spaces.

An M_κ*–polyhedral complex* is a cell complex formed by taking the disjoint union of convex polyhedral cells in M^n_κ which are then glued along isometric faces. An M_κ–polyhedral complex is called piecewise-Euclidean, piecewise-hyperbolic or piecewise-spherical if κ is 0, -1 or 1, respectively. (See [10] for details.)

Bridson showed in [7] that if an M_κ–polyhedral complex is composed of only finitely many isometry types of cells, then it is a complete geodesic space with respect to the naturally defined metric. Note that this condition is often satisfied in cases that arise naturally. For example, if a group acts cocompactly by isometries on a metric polyhedral complex X, then X has only finitely many isometry types of cells.

Definition 3.3 An M_κ–polyhedral complex X satisfies the *link condition* if for every vertex $v \in X$, the link of v (with its natural piecewise-spherical structure) is a CAT(1) space.

The importance of this condition is given by the following theorem, due to Gromov, Ballman and Bridson (see [10]).

Theorem 3.4 *Let X be an M_κ–polyhedral complex with only finitely many isometry types of cells. If $\kappa \leq 0$, then X is* CAT(κ) *if and only if X satisfies the link condition and is simply connected. If $\kappa = 1$, then X is* CAT(1) *if and only if X satisfies the link condition and contains no isometrically embedded circles of length less than 2π.*

If $\kappa \leq 0$, to prove that an M_κ–complex is CAT(0), it thus suffices to show that it is simply connected and has CAT(1) links. Applying Theorem 3.4 to the links then reduces the problem to one of ruling out short loops in successive links. In dimension 2, links of vertices are graphs, where checking for embedded loops is straightforward.

Proposition 3.5 [10, Lemma II.5.6] *A 2–dimensional M_κ–complex satisfies the link condition if and only if every injective loop in the link of every vertex of X has length at least 2π.*

3.2 Isometries and group actions

Definition 3.6 By an *action* of a group G on a space X, we mean a homomorphism $\phi\colon G \to \mathrm{Isom}(X)$. (We therefore consider only isometric actions, but we will not, in general, assume that actions are proper or faithful.) A group action on X is *trivial* if a point of X is fixed by every element of the group. An isometry $\gamma \in \mathrm{Isom}(X)$ is *semisimple* if $d(\cdot, \gamma(\cdot))$ attains a minimum on X and an isometry is *elliptic* if it fixes a point. For $H \subset G$, we say H is *ϕ–elliptic* if the action of H on X given by $\phi|_H$ is trivial.

The following lemma consists of standard facts about fixed sets.

Lemma 3.7 *Suppose a group G acts by isometries on a geodesic metric space (X, d) via $\phi\colon G \to \mathrm{Isom}(X)$.*

(i) *If d is convex (eg if X is* CAT(0)*), then for any ϕ–elliptic subset $H \subset G$, the set of global fixed points of H under ϕ, denoted* $\mathrm{Fix}_\phi(H)$*, is contractible.*

(ii) *If $H_1, H_2 \subset G$, then* $\mathrm{Fix}_\phi(\langle H_1 \cup H_2\rangle) = \mathrm{Fix}_\phi(H_1) \cap \mathrm{Fix}_\phi(H_2)$.

The next result [10, Corollary II.2.8] is crucial for our arguments.

Proposition 3.8 (Bruhat–Tits Fixed Point Theorem) *Let X be a complete, connected* CAT(0) *space. Any action of a finite group on X is trivial. More generally, any action on X with a bounded orbit is trivial.*

The proof of this result depends on the fact that the center of a bounded set is well-defined in complete CAT(0) spaces. Since every orbit is preserved, the center of a bounded orbit, which is metrically defined, is preserved as well.

A standard consequence is the following:

Corollary 3.9 *Suppose a group G acts by isometries on a complete, connected* CAT(0) *space X via $\phi\colon G \to \mathrm{Isom}(X)$. If H is a finite index subgroup of G and H is ϕ–elliptic, then G is ϕ–elliptic.*

Proof Since H is finite index in G, there is a normal subgroup N of finite index in G such that $N < H < G$. Now N is ϕ–elliptic since H is. Denote the (nonempty, contractible) fixed set of N by X^N. Then G/N acts on X^N. This action has a fixed point x_G by Proposition 3.8. The point x_G is fixed by the group G. □

3.3 Actions of Coxeter groups

Coxeter groups act on CAT(0) spaces. In this section, we consider some of these actions. Fix a Coxeter system (W, S) and suppose W is infinite.

- The *Davis–Moussong complex* Σ_{DM} associated to (W, S) (see, for example, [17; 16]) has a simplicial structure similar to that of the classical Coxeter complex. The dimension of Σ_{DM} is given by the maximal rank of finite special subgroups of W. In his thesis [25], Moussong proved that by assigning appropriate Euclidean metrics to the cells of a particular cellular structure on Σ_{DM}, the resulting metric on Σ_{DM} is CAT(0). The group W acts on Σ_{DM} properly by isometries.
- Given another Coxeter system (W', S), there is a surjection of W to W' if m'_{ij} divides m_{ij} for all i, j. In particular, W acts nontrivially on the Davis–Moussong complex of each such infinite quotient group.
- Niblo and Reeves construct in [26] a locally finite, finite-dimensional CAT(0) cube complex on which W acts properly discontinuously. For right-angled Coxeter groups, their complex is isometric to the Davis–Moussong complex.
- Since W is generated by torsion elements, it has no nontrivial homomorphisms to $\mathbb{Z}$. However, Cooper, Long and Reid in [13] and Gonciulea in [20] showed that W has a finite index subgroup that surjects to $\mathbb{Z}$. Such a finite index subgroup then acts nontrivially by translation on the real line. However, W itself acts nontrivially on a tree if and only if some m_{ij} is infinite (see Remark 5.1 below).

4 Group decompositions

4.1 Amalgamated products and normal form length

Recall from Bass–Serre Theory (see [27] or [28]) that actions on trees correspond to decompositions of groups as graphs of groups. We denote by $A *_C B$ the *amalgamated product* of A and B along C. Thus $A *_C B$ is the pushout of the diagram of groups $A \hookleftarrow C \hookrightarrow B$. That is, it is the fundamental group of the graph of groups with vertices A and B and a single edge C.

Definition 4.1 Let $G = A *_C B$, and let $g \in G$. The *normal form length* of g is defined to be the following:

$$\min\{k : g = \underbrace{a_1 b_2 a_3 \dots}_{k} \text{ or } \underbrace{b_1 a_2 b_3 \dots}_{k} \text{ with } a_i \in A \text{ and } b_j \in B \ \forall\ 1 \le i, j \le k\}.$$

Note that this is equal to the length of the normal form representative of g in $A *_C B$ (see [27] or [28] for details on normal forms in amalgamated products). We will be particularly interested in the normal form length of words in amalgamated products of Coxeter groups.

Let (W, S) be a Coxeter system with Coxeter matrix M and special subgroups A and B. Let M_{AB} denote the submatrix of M corresponding to the generators $S_A \cup S_B$. Define M' to be the matrix M_{AB} with only the following change: for $s_a \in S_A - S_B$ and $s_b \in S_B - S_A$, the corresponding entry in M' is ∞. Then $(A *_{A\cap B} B, S_A \cup S_B)$ is a Coxeter system with Coxeter matrix M'.

Let $w = s_1 s_2 \cdots s_k$ be a word in the alphabet $S_A \cup S_B$. We denote by $\mathrm{alt}(w)$ the minimum number k such that there are indices $1 = i_1 < i_2 < \cdots < i_k \le r$ with $s_{i_j}, s_{i_j+1}, \dots, s_{i_{j+1}-1} \in S_A$ for j odd and $s_{i_j}, s_{i_j+1}, \dots, s_{i_{j+1}-1} \in S_B$ for j even (or vice versa). That is, $\mathrm{alt}(w)$ is the number of "alternations" between elements of A and elements in B in the word w. The normal form length of an element $g \in G$ is thus $\min\{\mathrm{alt}(w) : w \text{ is a word representing } g\}$.

Note that the only Type (II) elementary M'–operations in G are those in A and B themselves. Together with Theorem 2.4, this implies the following:

Lemma 4.2 *Let W be a Coxeter group with special subgroups A and B, and let $G = A *_{A\cap B} B$. Let M' denote the Coxeter matrix of $(G, S_A \cup S_B)$ as defined above.*

(i) *Let w be a word in the alphabet $S_A \cup S_B$ and let w' be any subword of a word obtained from w via a sequence of elementary M'–operations. Then* $\mathrm{alt}(w') \le \mathrm{alt}(w)$.

(ii) *For* $g \in G$, *every* M'*–reduced word representing* g *realizes the normal form length of* g. *That is, if the normal form length of* g *in* G *is* k *then* $\mathrm{alt}(w) = k$ *for every* M'*–reduced word* w *representing* g.

(iii) *For* $a \in S_A - S_B$ *and* $b \in S_B - S_A$, *the normal form length in* G *of an alternating product of* a *and* b *is the length of the product as a word in the alphabet* $\{a, b\}$.

4.2 Triangles of groups

The study of triangles of groups is a 2–dimensional analogue of Bass–Serre Theory due originally to S Gersten and J Stallings (see [29]). In the next section, we will consider a more general construction, but the case of triangles will be of particular interest.

Suppose a group G acts on a 2–dimensional simplicial complex with quotient a triangle. By choosing a fundamental domain for the action, we can assign stabilizer groups to the vertices, edges, and face of this triangle. This process yields a triangle of groups, a 2–dimensional analogue of the graph of groups corresponding to $A *_C B$. Formally, a *triangle of groups* is a commutative diagram of groups and monomorphisms, as in Figure 2 below. The groups A, B and C are *vertex groups*, D, E and F are *edge groups*, and K is the *face group*. The vertex, edge and face groups are all called *local groups*.

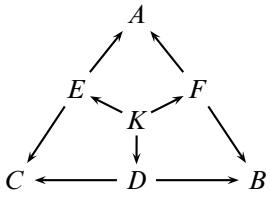

Figure 2: Triangle of Groups

The *fundamental group* G of the triangle of groups is the colimit of the diagram, ie, the unique group (up to isomorphism) satisfying the following universal mapping property:

> Given any group H and homomorphisms from the vertex groups to H, there is a unique homomorphism from G to H such that the resulting diagram of groups commutes.

Definition 4.3 Let A and K be groups, E and F be subgroups of A, and $K \to E$ and $K \to F$ be homomorphisms. Let ϕ be the natural surjection $\phi\colon E *_K F \to \langle E, F\rangle \subset A$.

The *Gersten–Stallings angle between E and F over K*, denoted $\angle_A(E, F; K)$, is $\frac{2\pi}{n}$ where

$$n = \min\{\text{normal form lengths of nontrivial elements in } \ker(\phi)\}.$$

Equivalently, n is the length of the shortest loop in the graph whose vertices are the cosets of E and F in A and whose edges are the cosets of K in A, with incidence given by inclusion. Note that n is *even* since this coset graph is bipartite. In the triangle of groups of Figure 2, we say that the *Gersten–Stallings angle at the vertex A* is $\angle_A(E, F; K)$.

Given a triangle of groups with fundamental group G such that all the vertex groups inject into G, there is a natural 2–complex X, the *universal cover* of the triangle of groups, on which G acts with quotient a triangle. The 2–simplices of X are given by cosets of K in G, with incident edges given by the corresponding cosets of D, E and F, and vertices the cosets of A, B and C. We may give X a metric by assigning a metric on the fundamental domain that agrees with the Gersten–Stallings angles. In particular, each triangle can be given the metric of a triangle in $\mathbb{E}^2$, $\mathbb{H}^2$ or $\mathbb{S}^2$, according to the sum of the Gersten–Stallings angles at each vertex of the triangle of groups. The resulting metric on X is then piecewise-Euclidean, piecewise-hyperbolic or piecewise-spherical.

The assigned metric ensures that X satisfies the link condition (see Definition 3.3). This implies the following theorem of Gersten and Stallings (see [29]).

Theorem 4.4 *If the sum of the Gersten–Stallings angles in a triangle of groups is less than or equal to π, then its universal cover X is* CAT(0)*. If the sum is strictly less than π, then X is* CAT(−1)*.*

4.3 Simple complexes of groups

The theory of *complexes of groups* was studied by Haefliger in [21] and independently (in dimension 2) by Corson in [14]. We will follow the development in [10] but will only need the theory of simple complexes of groups.

Definition 4.5 A *simple complex of groups* over a poset $\mathcal{Q}$, denoted $G(\mathcal{Q})$, consists of:

(1) For each $\sigma \in \mathcal{Q}$, a group G_σ (the *local group at σ*).

(2) For each $\tau < \sigma$, a monomorphism $\iota_{\tau\sigma}\colon G_\sigma \to G_\tau$ so that $\iota_{\tau\sigma} = \iota_{\tau\rho}\iota_{\rho\sigma}$ whenever $\tau < \rho < \sigma$.

Denote by $\widehat{G(\mathcal{Q})}$ the colimit of the diagram of groups $G(\mathcal{Q})$. We will take $\mathcal{Q}$ to be the poset associated to the cells of a cell complex. Note that if the geometric realization of $\mathcal{Q}$ is simply connected, as is true in the cases we will consider, then $\widehat{G(\mathcal{Q})}$ is also what is known as the *fundamental group* of the corresponding complex of groups.

Remark 4.6 Given presentations for the local groups of $G(\mathcal{Q})$, we have a presentation for the colimit $\widehat{G(\mathcal{Q})}$. The generators are given by the generators of the local groups, and along with the relations coming from the local groups, elements of different local groups are related via the monomorphisms $\iota_{\tau\sigma}$. Specifically, if the local groups have presentations $G_\sigma = \langle S_\sigma \mid R_\sigma \rangle$, then we have

$$\widehat{G(\mathcal{Q})} = \left\langle \bigcup_{\sigma\in\mathcal{Q}} S_\sigma \;\middle|\; \bigcup_{\sigma\in\mathcal{Q}} \{R_\sigma, \{g = \iota_{\tau\sigma}(g) : g \in G_\sigma,\ \tau < \sigma\}\} \right\rangle.$$

Suppose $\mathcal{Q}$ is the poset corresponding to the simplices of a 2–simplex, ordered by inclusion. A simple complex of groups over $\mathcal{Q}$ is a triangle of groups. More generally, if $\mathcal{Q}$ is the poset corresponding to the simplices of an n–simplex Δ^n, ordered by inclusion, we refer to a simple complex of groups over $\mathcal{Q}$ as an *n–simplex of groups*.

As was the case for triangles, if the local groups inject into $\widehat{G(\mathcal{Q})}$, there is a simply connected space, the *universal cover of* $G(\mathcal{Q})$, upon which $\widehat{G(\mathcal{Q})}$ acts with quotient the underlying cell complex. In particular, the underlying cell complex is a *strict* fundamental domain for the action. Assigning a metric to the underlying cell complex of $\mathcal{Q}$ yields a metric on the universal cover and the resulting action is by isometries. We will call a simplex of groups *nonpositively curved* if its universal cover is $\mathrm{CAT}(0)$.

The following is an analogue of the graph of groups version given by Mihalik and Tschantz in [24].

Proposition 4.7 *Let $\mathcal{Q}$ be the poset of cells of a complex $\overline{\mathcal{Q}}$ ordered by inclusion. Suppose (W, S) is a Coxeter system and $G(\mathcal{Q})$ is a simple complex of special subgroups of W with monomorphisms $\iota_{\tau\sigma}$ given by natural inclusions. Then, $\widehat{G(\mathcal{Q})} = W$ (with all the resulting homomorphisms from the local groups to W the natural inclusions) if both of the following hold:*

(i) *For $s \in S$, the set $\{\sigma : \sigma \in \mathcal{Q},\ s \in G_\sigma\}$ corresponds to a nonempty connected subcomplex of $\overline{\mathcal{Q}}$.*

(ii) *If $s, t \in S$ and $m_{st} < \infty$, then $\{s, t\} \subset G_\sigma$ for some $\sigma \in \mathcal{Q}$.*

Proof For each local group G_σ, let $\langle S_\sigma \mid R_\sigma \rangle$ be the induced Coxeter presentation of G_σ as a special subgroup of W.

By Remark 4.6, we have

$$\widehat{G(\mathcal{Q})} = \left\langle \bigcup_{\sigma\in\mathcal{Q}} S_\sigma \;\middle|\; \bigcup_{\sigma\in\mathcal{Q}} \{R_\sigma, \{g = \iota_{\tau\sigma}(g) : g \in G_\sigma,\ \tau < \sigma\}\} \right\rangle.$$

Suppose that (i) and (ii) hold. Since the monomorphisms of $G(\mathcal{Q})$ are the natural inclusions, by (i) we can write

$$\widehat{G(\mathcal{Q})} = \left\langle S \;\middle|\; \bigcup_{\sigma \text{ vertex of } \overline{\mathcal{Q}}} R_\sigma \right\rangle.$$

So, it suffices to show that the relations appearing in the vertex groups are precisely the relations of W. By our choice of presentations for the local groups (as special subgroups of W), every relation $r_\sigma \in R_\sigma$ is a relation from our Coxeter presentation for (W, S). Moreover, by (i), every relation $s^2 = 1$ appears in some local group (hence some vertex group). Finally, by (ii), every relation of the form $(st)^{m_{st}} = 1$ appears in some local group (so in a vertex group). □

5 Property FA$_n$

5.1 Definition and examples

A group G has Serre's *property* FA (see [28]) if for every tree T, every action (without inversions) of G on T is trivial (has a global fixed point). In particular, if G has FA, then G does not split nontrivially as a graph of groups. Since trees are CAT(0), the Bruhat–Tits Theorem (Proposition 3.8) applies, so finite groups have FA. Other groups known to have FA include the following:

- Every finite index subgroup of $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ (Margulis–Tits, see [28])
- Finitely generated torsion groups (Serre [28])
- Coxeter groups such that $m_{ij} < \infty$ for all i and j (Serre [28])
- Finitely generated groups with Kazhdan's property (T) (Watatani [31])
- $\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$ for $n \geq 3$ (Bogopolski [3], Culler–Vogtmann [15])
- Mapping class groups of higher genus surfaces (Culler–Vogtmann [15])

Remark 5.1 Mihalik and Tschantz note in [24] that a Coxeter group has FA if and only if every m_{ij} is finite. In particular, if (W, S) is a Coxeter system and $m_{ij} = \infty$, then $W \cong \langle S - \{s_i\}\rangle *_{\langle S-\{s_i,s_j\}\rangle} \langle S - \{s_j\}\rangle$.

Definition 5.2 A CAT(0) *n–complex* is an M_κ–polyhedral n–complex (see Section 3.1) that is complete, connected, CAT(0), and has only finitely many isometry types of cells. A group G has *property* FA_n if for every CAT(0) n–complex X, every action of G on X by cellular isometries has a global fixed point. The group G has *strong* FA_n if for every complete, connected CAT(0) space X of topological dimension n, every action of G on X by semisimple isometries has a global fixed point.

Bridson showed in [8] that if X is a connected M_κ–polyhedral complex with only finitely many isometry types of cells, then every cellular isometry of X is semisimple. Therefore, any group with strong FA_n also has FA_n.

The following, as noted by Farb in [19], is an immediate consequence of Proposition 3.8.

Corollary 5.3 *Let G be a finite group. Then G has strong FA_n for all n.*

Farb also showed in [19] that the following groups have property FA_n:

- Finite index subgroups of $\mathrm{SL}_m(\mathbb{Z})$ and $\mathrm{SL}_m(\mathbb{Z}[1/p])$ for $m \geq n+2$
- More generally, arithmetic or S-arithmetic subgroups of K–simple algebraic K–groups of K–rank at least $n+1$ for K a global field
- Discrete groups generated by reflections in the sides of Euclidean or hyperbolic $(n+1)$–simplices

Recently, Bridson [6] has also studied property FA_n for automorphism groups of free groups.

5.2 Homological techniques and implications

Note that throughout we consider homology with $\mathbb{Z}$ coefficients.

We will prove that certain actions are trivial by studying the combinatorics of fixed sets. Consider a collection of sets $\{S_\alpha\}_{\alpha\in I}$. Recall that the *nerve* of this collection, denoted $\mathcal{N}(\{S_\alpha\}_{\alpha\in I})$, is the simplicial complex whose vertices are indexed by the set I and such that the set of vertices corresponding to $J \subset I$ span a simplex if and only if $\bigcap_{\alpha\in J} S_\alpha \neq \varnothing$.

Notation We will be particularly interested in the nerves of collections of fixed sets. Let G be a group, and suppose $\phi\colon G \to \mathrm{Isom}(X)$ is the homomorphism describing an action of G on X. Let Σ be a finite collection of ϕ–elliptic subsets of G (see Definition 3.6). We denote by $\mathcal{N}(\Sigma,\phi)$ the nerve of the collection of fixed sets of the elements of Σ. That is $\mathcal{N}(\Sigma,\phi) = \mathcal{N}\left(\{\mathrm{Fix}_\phi(S)\}_{S\in\Sigma}\right)$.

Remark 5.4 Suppose Σ is finite. Then $\mathcal{N}(\Sigma,\phi)$ is a simplex if and only if there is a global fixed point for the action the subgroup $\langle\Sigma\rangle$ of G generated by the union of the elements of Σ. By Corollary 3.9, if $\langle\Sigma\rangle$ has finite index in G, then this holds if and only if the action of G has a global fixed point.

Our main technique is based on the following two results.

Theorem 5.5 (Leray [12, Theorem VII.4.4]) *Suppose X is a CW complex and is the union of subcomplexes X_α such that the intersection of any finite subcollection of the X_α is either empty or acyclic. Then $H_*(X) = H_*\left(\mathcal{N}(\{X_\alpha\})\right)$.*

Theorem 5.6 (McCord [23, Theorem 2]) *Let X be a space and $\mathcal{U}$ a locally finite open cover of X such that the intersection of any finite subcollection of $\mathcal{U}$ is either empty or homotopically trivial. Then, there is a weak homotopy equivalence $\mathcal{N}(\mathcal{U}) \to X$ so $H_*(\mathcal{N}(\mathcal{U})) = H_*(X)$.*

Definition 5.7 Let K be a simplicial complex. We say that K is *n–allowable* if $H_m(K) = 0$ for all $m \geq n$.

The motivation for this definition can be found in the following, which is implicit in [19]. We include a proof here for completeness.

Proposition 5.8 *Let G be a group. Suppose G acts on a complete* CAT(0) *space X of dimension n via $\phi\colon G \to \mathrm{Isom}(X)$. Let Σ be a finite set of ϕ–elliptic subsets of G. Then $\mathcal{N}(\mathcal{T},\phi)$ is n–allowable for all $\mathcal{T} \subset \Sigma$.*

Proof Let Y be the union of the fixed sets $\mathrm{Fix}_\phi(S)$ for $S \in \mathcal{T}$. By Lemma 3.7, nonempty intersections of the sets $\mathrm{Fix}_\phi(S)$ are also fixed sets so are contractible. Taking regular neighborhoods of the sets $\mathrm{Fix}_\phi(S)$ that preserve the intersection data, we apply Theorem 5.6 to the resulting open cover to find that $H_*(\mathcal{N}(\mathcal{T},\phi)) = H_*(Y)$.

Let $m \geq 1$. Applying the long exact homology sequence for pairs, we have the exact sequence $H_{m+1}(X) \to H_{m+1}(X,Y) \to H_m(Y) \to H_m(X)$. Since X is contractible (see Remark 3.2), we find that $H_{m+1}(X,Y) \cong H_m(Y)$. Now $H_{m+1}(X,Y) = 0$ for $m \geq n$ as X is n–dimensional. Thus for $m \geq n$ we have $H_m(Y) = 0$ so $H_m(\mathcal{N}(\mathcal{T},\phi)) = 0$ as well. □

Note that in the above proof we only used the CAT(0) assumption for contractibility of fixed sets, so the result holds more generally.

Remark 5.9 Let K be an n–allowable simplicial complex with 0–skeleton consisting of the vertices $v_0, \dots, v_k$ for some $k \geq n$. Then $H_k(K) = 0$ by the definition of n–allowable. So, if the $(k-1)$–skeleton of K is the boundary of a k–simplex, then the k–skeleton (and hence the entire complex K) is a k–simplex.

Using a topological form of Helly's Theorem, Farb applies certain cases of the following result in [19]. We include here a proof of the general result using the techniques from above.

Corollary 5.10 *Let G be a group and Σ a finite collection of subsets of G whose union generates a finite index subgroup of G. Suppose G acts on a complete n–dimensional* CAT(0) *space X via $\phi\colon G \to \mathrm{Isom}(X)$. If $n < |\Sigma|$ and if every $n+1$ elements of Σ generate a ϕ–elliptic subgroup of G, then G is ϕ–elliptic.*

Proof By assumption, every $n+1$ elements of Σ generate a ϕ–elliptic subgroup. So, by definition of $\mathcal{N} = \mathcal{N}(\Sigma, \phi)$, every $n+1$ vertices in $\mathcal{N}$ span an n–simplex. If $|\Sigma| = n+1$, then G is ϕ–elliptic by Remark 5.4. Otherwise, consider a subset $\mathcal{T} \subset \Sigma$ of cardinality $n+2$. Then, the n–skeleton of $\mathcal{N}(\mathcal{T}, \phi)$ is the boundary of an $(n+1)$–simplex. By Proposition 5.8, we know that $\mathcal{N}(\mathcal{T}, \phi)$ is n–allowable, so by Remark 5.9, its $(n+1)$–skeleton is actually an $(n+1)$–simplex. Inductively, we see that the $(|\Sigma|-1)$–skeleton of $\mathcal{N}$ is an $(|\Sigma|-1)$–simplex, so $\mathcal{N}$ is a simplex. Thus G is ϕ–elliptic by Remark 5.4. □

As an immediate consequence we have the following:

Corollary 5.11 *Let G be a group and let Σ be a finite collection of subsets of G whose union generates a finite index subgroup of G. If for every subcollection $\{S_0, S_1, \dots, S_n\} \subset \Sigma$ there is an* FA_n *(resp. strong* FA_n*) subgroup $H \subset G$ such that $S_i \subset H$ for $0 \leq i \leq n$, then G has* FA_n *(resp. strong* FA_n*). In particular, if every $n+1$ elements of Σ generate a group with property* FA_n *(resp. strong* FA_n*), then G has property* FA_n *(resp. strong* FA_n*).*

Definition 5.12 (Notation as in Section 4.3.) We say that an n–simplex of groups is *minimal* if for all $k < n$, every local group G_σ corresponding to a k–simplex σ is generated by the (images of the) local groups G_τ such that $\sigma \subsetneq \tau$. We will call such a G_σ a *local group of codimension* $n-k$.

A version of the following was proved for FA_1 by R Alperin in [1].

Corollary 5.13 *Let $n \geq 1$. Suppose a group G has a decomposition as a minimal $(n+1)$–simplex of groups Λ such that every local group of Λ has FA_n (resp. strong FA_n). Then, G has FA_n (resp. strong FA_n).*

Remark 5.14 Note that the simplex of groups decomposition does not have to be nonpositively curved. This result implies, for example, that the fundamental group of any realizable minimal n–simplex of finite groups has strong FA_{n-1}. So, if a group acts nontrivially by semisimple isometries on any complete, connected $\mathrm{CAT}(0)$ space of dimension n, then it does not decompose as a minimal m–simplex of finite groups for any $m > n$.

Proof of Corollary 5.13 Let $\Sigma = \{\text{local groups of codimension } 1\}$. By assumption, H has FA_n for all $H \in \Sigma$ and by minimality, the union of the elements of Σ generates G. Moreover, for all $k \leq n+1$, every collection of k elements of Σ generates a local group of codimension k, which in turn has FA_n. Thus G has FA_n by Corollary 5.11. The case of strong FA_n is analogous. □

6 Strong FA_n for Coxeter groups

6.1 n–spherical Coxeter groups

Definition 6.1 Let (W, S) be a Coxeter system. The Coxeter group W is *n–spherical* if every special subgroup of W of rank less than or equal to n is a finite group. (Note that for $n \leq |S|$, the group W is n–spherical if and only if all of its special subgroups of rank n are finite.) A special subgroup W_T is *n–spherical* if it is n–spherical with respect to the generating set T.

Finite groups have FA_n by Corollary 5.3, so Theorem 1.1 follows from Corollary 5.11 by setting $\Sigma = \{\{s\} : s \in S\}$. In fact, Corollary 5.10 implies the following:

Corollary 6.2 *Let W be a Coxeter group. If W is $(n+1)$–spherical, then every action of W on every complete connected $\mathrm{CAT}(0)$ space of dimension n has a global fixed point. In particular, W has strong FA_n.*

The converse is true in dimension 1 (see [24]), and we will prove it also holds in dimension 2. For higher dimensions, we have the following reformulation of Conjecture 1.2.

Conjecture 6.3 (Coxeter FA_n Conjecture) *Let (W,S) be a Coxeter system. The following are equivalent:*

(i) *W is $(n+1)$–spherical.*

(ii) *Every action of W on every complete connected* CAT(0) *space of dimension n has a global fixed point.*

(iii) *W has strong FA_n.*

(iv) *W has property FA_n.*

(v) *W does not split nontrivially as a nonpositively curved m–simplex of special subgroups, for all $0 < m \leq n$.*

We have already seen that (i) $\Rightarrow$ (ii). That (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) is clear by definition. It remains only to show that (v) $\Rightarrow$ (i). We describe an approach to this in the next section.

Note that property FA_n is, by definition, a property of a group rather than of a presentation of the group. However, by the classification of Coxeter groups, the property of being n–spherical is easily verified by looking at a Coxeter presentation (see Remark 2.13). Thus, if the Coxeter FA_n Conjecture holds in general, then property FA_n can be detected by considering any Coxeter presentation.

6.2 The CAT(0) Conjecture

We now describe an approach to completing the proof of the Coxeter FA_n Conjecture (Conjecture 6.3). In particular, the goal is to prove that if W is not $(n+1)$–spherical, then W splits nontrivially as a nonpositively curved simplex of special subgroups of dimension at most n. Note that if W is finite, then W is $(n+1)$–spherical for all n. We thus only need consider infinite Coxeter groups in what follows.

Fix a Coxeter system (W,S) with W infinite. Let

$$v = v(W) = \max\{m : W \text{ is } m\text{–spherical}\}.$$

Our goal is to construct a decomposition of W as a v–simplex of groups. To do this, we first consider a "natural" decomposition of a subgroup of W.

Let $S' \subset S$ be the generating set of an infinite special subgroup of W of rank $v+1$. Note that such a subset exists by the definition of v. Moreover, every proper subset of S' generates a finite group. By Theorem 2.12, the Coxeter group W' is thus an irreducible Euclidean or hyperbolic reflection group with fundamental domain a compact simplex. Hence, W' has a natural decomposition as a Euclidean or hyperbolic

v–simplex of groups Λ' in which the vertex groups are the special subgroups of W' of rank v. We assign to the simplex Λ' the metric given by the Euclidean or hyperbolic metric on a fundamental domain for the natural action of W'. Then, the universal cover X' of Λ' is isometric to the original hyperbolic or Euclidean space, and the action of W' on X' is the original action.

Following the motivation from dimension one (see Remark 5.1), we construct a new simplex of groups Λ by adding the missing generators. In particular, Λ is given by the same metric simplex as was Λ', but for $A \subsetneq S'$, the local group $\langle A \rangle$ is replaced by the group $\langle A \cup (S - S') \rangle$. Note that the local group associated to the maximal simplex in Λ is thus $\langle S - S' \rangle$.

Example 6.4 Let $S = \{a, b, c, x, y, z\}$ and let W be the Coxeter group given by the Coxeter diagram in Figure 3.

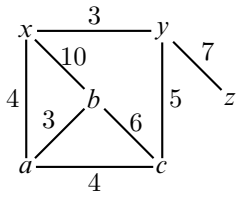

Figure 3: Coxeter diagram with $v(W) = 2$

Then W is 2–spherical since every m_{ij} is finite. Let $S' = \{a, b, c\}$. Then $W' = \langle S' \rangle$ is infinite, so $v(W) = 2$. Moreover, W' is a hyperbolic triangle group with angles $\frac{\pi}{3}$, $\frac{\pi}{4}$ and $\frac{\pi}{6}$, and the action of W' on $\mathbb{H}^2$ gives the triangle of groups Λ' shown in Figure 4. The resulting triangle of groups Λ is shown in Figure 5. The metric on Λ is that of a hyperbolic $(3, 4, 6)$–triangle.

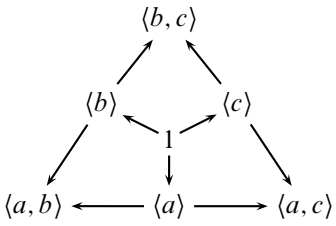

Figure 4: Natural 2–splitting Λ' of W'

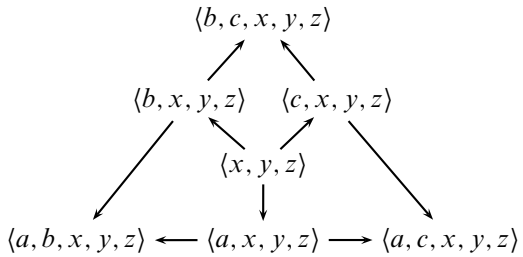

Figure 5: 2–splitting Λ determined by S'

As an immediate consequence of Proposition 4.7 we have the following:

Corollary 6.5 *Let* Λ *be the simplex of groups associated to a rank* $v+1$ *infinite special subgroup of* W *as constructed above. Then* Λ *is a splitting of* W*. That is* $W = \widehat{\Lambda}$ *(notation as in Section 4.3).*

We will refer to the decomposition Λ of W as the *v–splitting of W determined by S'*. The group W acts on the simply connected universal cover of Λ with quotient a v–simplex.

CAT(0) Conjecture *In the above construction, the universal cover of the v–splitting of W determined by S' is* CAT(0).

The preceding discussion actually proves the following:

Theorem 6.6 CAT(0) *Conjecture* $\Rightarrow$ *Coxeter* FA_n *Conjecture (Conjecture 6.3).*

Remark 6.7 The following observation of P Caprace implies that the CAT(0) Conjecture holds for $v \geq 9$. Let (W, S) be a Coxeter system with $v(W) \geq 9$. Then W' is an irreducible Euclidean simplex reflection group of rank at least 10, and by inspection of the standard list of these groups, we find that W decomposes as $W = W' \times W_{S-S'}$. Thus W acts on $\mathbb{E}^v$ via projection onto W'. In particular, this action yields a decomposition of W as a nonpositively curved v–simplex of special subgroups. This decomposition is the v–splitting Λ constructed above.

In the next section, we will prove the CAT(0) Conjecture for $v = 2$. For $3 \leq v \leq 8$, there are only finitely many different possibilities for Λ' since, by the classification of irreducible Coxeter groups, there are only finitely many isomorphism types of subgroups W'. These are the remaining open cases.

6.3 Proof of the CAT(0) Conjecture in dimension 2

We will now prove the CAT(0) Conjecture in the case $v = 2$, ie, when the v–splitting of W is a triangle of groups decomposition of W. To do so, we will show that the Gersten–Stallings angles in this triangle of groups are the same as those for the natural decomposition of W'. In particular, we will prove the following result (notation as in Section 4.2). Note that π/∞ means 0.

Proposition 6.8 *Suppose (W, S) is a Coxeter system with distinct special subgroups A and B.*

(1) *Let $m = \min\{m_{ij} : s_i \in S_A - S_B,\ s_j \in S_B - S_A\}$. Then,*

$$\angle_W(A, B; A \cap B) = \begin{cases} 0 & \text{if } A \subset B \text{ or } B \subset A; \\ \pi/m & \text{otherwise.} \end{cases}$$

(2) *If C is a special subgroup of W such that $A \cap C = \{1\} = B \cap C$, then*

$$\angle_{\langle A,B\rangle}(A, B; A \cap B) = \angle_W(\langle A, C\rangle, \langle B, C\rangle; \langle A \cap B, C\rangle).$$

In particular, for $a \neq b \in S$ and $T \subset S - \{a, b\}$, we have

$$\angle_W(\langle a, T\rangle, \langle b, T\rangle; \langle T\rangle) = \frac{\pi}{m_{ab}}.$$

Note that statement (2) of this proposition implies that the Gersten–Stallings angle at a vertex in a triangle of special subgroups of a Coxeter group does not change when (the same) additional Coxeter generators are added to every local group.

Theorem 6.9 *The CAT(0) Conjecture is true for $v = 2$.*

By Theorem 6.6 above, this will complete the proof of Theorem 1.3.

Proof of Theorem 6.9 As in Section 6.2, let Λ be the triangle of groups decomposition of W determined by $S' \subset S$, and let X be the universal cover of Λ. By statement (2) of Proposition 6.8, the Gersten–Stallings angles of Λ are the same as those of Λ'. Since W' is a hyperbolic or Euclidean triangle group, the sum of the angles of Λ is therefore at most π. Thus X is CAT(0) by Theorem 4.4. □

Corollary 6.10 *The Coxeter FA_n Conjecture (Conjecture 6.3) is true for $n = 2$. In particular, a Coxeter group W acts nontrivially on a 2–dimensional complete connected* CAT(0) *space if and only if W has an infinite special subgroup of rank 3 with respect to some (hence any) Coxeter generating set.*

Note that this result gives a complete characterization of FA_n for many Coxeter groups, including all Coxeter groups of large, even and odd type.

If Λ' is hyperbolic (ie, if W' is a hyperbolic triangle group), then X is piecewise-hyperbolic and hence CAT(-1). Thus, we have also shown the following:

Corollary 6.11 *Suppose* (W,S) *is a Coxeter system with* $m_{ij}<\infty$ *for all* i *and* j. *Suppose further that* W *has a special subgroup of rank* 3 *that is a hyperbolic triangle group. Then* W *acts nontrivially on a* $\mathrm{CAT}(-1)$ 2*–complex.*

Before proceeding to the proof of Proposition 6.8, we introduce the following:

Notation Let (W,S) be a Coxeter system.

(i) For $s,\ t\in S$ with $m_{st}<\infty$ we define α_{st} to be the alternating word in s and t of length m_{st}. That is,

$$\alpha_{st}=\begin{cases}\underbrace{st\ldots st}_{m_{st}} & \text{if } m_{st} \text{ is even;}\\ \underbrace{st\ldots ts}_{m_{st}} & \text{if } m_{st} \text{ is odd.}\end{cases}$$

(ii) We denote by $\widehat{w}$ the word w with one letter removed.

(iii) To denote equality of words in the alphabet S, we will use the notation $=$ and for equality of the corresponding group elements in a group H we will use $=_H$.

The following lemmas will be used in the proof of Proposition 6.8. The first is an immediate consequence of the Deletion Condition along with Remark 2.5.

Lemma 6.12 *Let* (W,S) *be a Coxeter system. Suppose* $s_1s_2\ldots s_k$ *is a word such that for some* $r,t\in S$ *and* $j\le m_{rt}$, *we have*

$$s_1s_2\ldots s_k=_W \underbrace{rtr\ldots}_{j}.$$

Then, for some indices $i_1<i_2<\ldots<i_j$, *we have* $s_{i_j}=r$ *for* j *odd and* $s_{i_j}=t$ *for* j *even (or perhaps vice versa if* $j=m_{rt}$*). That is, as a word in the alphabet* S, *we have* $s_{i_1}s_{i_2}\ldots s_{i_j}=\underbrace{rtr\ldots}_{j}$ *(or possibly* $\underbrace{trt\ldots}_{j}$ *if* $j=m_{rt}$*).*

The notation used below is as in Proposition 2.9.

Lemma 6.13 *Let* (W,S) *be a Coxeter system. Let* $r,t,\in S$ *and let* $w(r,t)$ *be an alternating word in* r *and* t *of length strictly less than* m_{rt}. *Suppose* w *is the element of* W *represented by the word* $w(r,t)$. *Let* $i(w)$ *(resp.* $j(w)$*) be the first (resp. last) letter in the word* $w(r,t)$. *Then* w *is* $(I-i(w),J-j(w))$*–reduced for all* $I,J\subset S$.

Proof of Lemma 6.13 By Proposition 2.9, it suffices to show that for all $s \in I - i(w)$ and all $s' \in J - j(w)$, we have $\ell(sw) > \ell(w)$ and $\ell(ws') > \ell(w)$. Note that the Deletion Condition implies that $\ell(sw) \neq \ell(w)$. Suppose that $\ell(sw) < \ell(w)$ for some $s \in I - i(w)$. By the Strong Exchange Condition (Theorem 2.1), we have $w(r,t) =_W s\widehat{w(r,t)}$. However, since $w(r,t)$ is the unique reduced word representing the element w of W (see Remark 2.5), we have $w(r,t) = s\widehat{w(r,t)}$ as words in the alphabet S. Thus $s = r$ or $s = t$.

Without loss of generality, we may assume $w(r,t)$ begins with r. Now $s \neq r$ since $r = i(w)$ and $s \in I - i(w)$. So $s = t$ and $sw(r,t)$ is an alternating product of r and t beginning with t and of length $\ell(w)+1$. Note that $sw(r,t)$ is reduced since $\ell(w) < m_{rt}$, so $\ell(sw) = \ell(w)+1 > \ell(w)$. This is a contradiction.

The case of $s' \in J - j(w)$ is analogous. □

We will now prove Proposition 6.8.

Proof of Proposition 6.8 First note that statement (2) follows immediately from statement (1) since Theorem 2.7 implies that $\langle A, C\rangle \cap \langle B, C\rangle = \langle A \cap B, C\rangle$ and $S_{\langle A,C\rangle} - S_{\langle B,C\rangle} = S_A - S_B$. We proceed now to prove statement (1).

Let $G = A *_{A\cap B} B$ and let $\rho\colon G \twoheadrightarrow \langle A, B\rangle \subset W$ be the natural surjection. If $A \cap B = A$ or $A \cap B = B$, then $\angle_W(A, B; A \cap B) = 0$ since the induced map $G \to \langle A, B\rangle$ is an isomorphism. The same is true if $m_{ab} = \infty$ for all $a \in S_A - S_B$ and $b \in S_B - S_A$. Assume otherwise.

By definition, $\angle_W(A, B; A \cap B) = \frac{\pi}{k}$ where $2k$ is the minimal normal form length among nontrivial elements in $\ker(\rho)$. Let $a \in S_A - S_B$ and $b \in S_B - S_A$ such that m_{ab} is finite. Then, $(ab)^{m_{ab}} =_W 1$ but $(ab)^{m_{ab}}$ is nontrivial in G (see Section 4.1). Therefore $(ab)^{m_{ab}}$ is a nontrivial element in $\ker(\rho)$. Moreover, by Lemma 4.2 (iii), the normal form length in G of $(ab)^{m_{ab}}$ is $2m_{ab}$, so $k \leq m_{ab}$ for all such choices of a and b. To prove part (1), it thus suffices to show that there exist elements $a \in S_A - S_B$ and $b \in S_B - S_A$ such that $k \geq m_{ab}$.

Let M be the Coxeter matrix of (W, S) and, as described in Section 4.1, let M' be the Coxeter matrix of $(G, S_A \cup S_B)$. Then every elementary M'–operation is an elementary M–operation (see Definition 2.3). Moreover, the only elementary M–operations for elements in $\langle A, B\rangle$ which are not also elementary M'–operations are those which replace α_{ab} by α_{ba} (or vice versa) for some $a \in S_A - S_B$ and $b \in S_B - S_A$.

Let $g \in \ker(\rho)$. By Theorem 2.4, since $g =_W 1$ there is an M–reduction of g to the identity. Since $g \neq_G 1$ we know that g does not M'–reduce to the identity. So, the

M–reduction of g to the identity in W involves at least one elementary M–operation which is not also an elementary M'–operation. In particular, for some $a \in S_A - S_B$ and $b \in S_B - S_A$ with m_{ab} finite, some M'–reduced expression of g must have a subword of the form α_{ab} or α_{ba}.

Hence we may choose g nontrivial in $\ker(\rho)$ along with $a \in S_A - S_B$ and $b \in S_B - S_A$ so that m_{ab} is minimal among all possible such choices also satisfying both of the following:

(i) The normal form length of g in G is $2k$.

(ii) Some M'–reduced expression of g has α_{ab} or α_{ba} as a subword.

Without loss of generality, we may assume that the alternating expression of (ii) begins with a, ie, that α_{ab} is a subword of an M'–reduced word representing g. For notational convenience we will also use g to denote this particular choice of M'–reduced word.

Let $m = m_{ab}$. To complete the proof of part (1), we will show $k \geq m$. We consider separately the cases when m is even and odd.

Case 1 Suppose m is even.

Since g is M'–reduced, by Lemma 4.2 (ii), we may assume

$$g = a_1 b_2 \dots b_{j-1} a_j \alpha_{ab} b_{m+j-1} a_{m+j} \dots a_{2k-1} b_{2k}$$

with $a_j a, a_i \in A - \{1\}$ for $i \neq j$ and similarly $b b_{m+j-1}, b_i \in B - \{1\}$ for $i \neq m+j-1$.

If $m = 2$ then we need only show $k \neq 1$. If $k = 1$ then $g = a_1 a b b_2$ for some $a_1 \in A$ and $b_2 \in B$. Then $a_1 a =_W b_2^{-1} b$ so $a_1 a$ and $b b_2$ are elements of $A \cap B$ and thus $g \in A \cap B$. In particular, the fact that $g =_W 1$ implies that $g =_G 1$ in G since $\rho|_{A \cap B}$ is injective. This is a contradiction as g is a nontrivial element in $\ker(\rho) \subset G$.

Assume now that $m > 2$. Conjugation in G yields another nontrivial element h of $\ker(\rho)$ represented by the word

$$h = \alpha_{ab} b_{m+j-1} a_{m+j} \dots a_{2k-1} b_{2k} a_1 b_2 \dots b_{j-1} a_j .$$

Since $h =_W 1$ and $\alpha_{ba} =_W \alpha_{ab}^{-1}$, we have

$$\alpha_{ba} =_W b_{m+j-1} a_{m+j} \dots a_{2k-1} b_{2k} a_1 b_2 \dots b_{j-1} a_j . \tag{1}$$

As $a \notin S_B$, the generator a cannot appear in any M'–reduced expression of b_i for any i by Theorem 2.7 (and similarly for b and a_i). Applying Lemma 6.12 we therefore find that $m \leq j + (2k - (m + j - 1) + 1) = 2k - m + 2$ so $k \geq m - 1$. If $k > m - 1$ then we are done, so we assume $k = m - 1$.

Let $\widetilde{h}$ denote the word on the right side of equation (1). Since $k = m-1$, the word $\widetilde{h}$ consists of an alternating sequence of exactly m elements of A and B, ie, $\mathrm{alt}(\widetilde{h}) = m$ in the notation of Section 4.1. Thus the normal form length of the group element $\widetilde{h}$ in G is at most m.

Since α_{ba} is M–reduced, Theorem 2.4 implies that there is an M–reduction of $\widetilde{h}$ to α_{ba}. Since h is nontrivial in G and $h = \alpha_{ab}\widetilde{h}$, it follows that $\widetilde{h} \neq_G \alpha_{ba}$ as $\alpha_{ab}\alpha_{ba} =_G 1$. Thus, by the same argument as above, there is an M'–reduced form of $\widetilde{h}$ containing a subword of the form $\alpha_{a'b'}$ or $\alpha_{b'a'}$ for some $a' \in S_A - S_B$ and $b' \in S_B - S_A$. By Lemma 4.2 (iii), the normal form length of $\widetilde{h}$ in G is thus at least $m_{a'b'}$. However $m_{a'b'} \geq m$ by the minimality assumption on m, so we find that $m = m_{a'b'}$. Moreover, the normal form length of $\widetilde{h}$ is exactly $m_{a'b'} = m$ and there exist elements $\widetilde{a} \in A$ and $\widetilde{b} \in B$ so that $\widetilde{h} =_G \widetilde{b}\alpha_{b'a'}\widetilde{a}$. It follows that equation (1) can be rewritten as $\alpha_{ba} =_W \widetilde{b}\alpha_{b'a'}\widetilde{a}$. By Lemma 6.12, since $m > 2$ we have $a' = a$ and $b' = b$. Thus, we find $\widetilde{h} =_G \widetilde{b}\alpha_{ba}\widetilde{a}$. Hence, equation (1) can be rewritten as $\alpha_{ba} =_W \widetilde{b}\alpha_{ba}\widetilde{a}$. It follows that

$$b\widetilde{b}b\underbrace{ab\cdots ab}_{m-2} =_W \underbrace{ab\cdots ab}_{m-2}a\widetilde{a}^{-1}a. \tag{2}$$

Since $b\widetilde{b}b$ is a subword of h, it is M'–reduced so is also M–reduced. Then Lemma 6.13 implies that the left side of equation (2) is M–reduced. So, by Theorem 2.4, there is a sequence of elementary M–operations taking the right side to the left side. By the nontriviality of h, equation (2) does not hold in G, so at least one operation must be used which is not an M'–operation. Hence a finite number of M'–operations transforms the right side of equation (2) into a word w' containing a subword of the form $\alpha_{a'b'}$ or $\alpha_{b'a'}$ for some $a' \in S_A - S_B$ and $b' \in S_B - S_A$. Note that $m_{a'b'} \geq m$ by the minimality assumption on m. Then, in the notation of Section 4.1, we find

$$\mathrm{alt}(w') \geq \mathrm{alt}(\alpha_{a'b'}) = m_{a'b'} \geq m > \mathrm{alt}(\underbrace{ab\cdots ab}_{m-2}a\widetilde{a}^{-1}a).$$

This contradicts Lemma 4.2 (i) and thus completes the even case.

Case 2 Suppose m is odd.

By an argument analogous to the one in the even case, we find a nontrivial M'–reduced $h \in \ker(\rho)$ so that $h =_G \alpha_{ab}\widetilde{a}_1\alpha_{ab}\widetilde{a}_2$ for some $\widetilde{a}_1, \widetilde{a}_2 \in A$. This implies that

$$a\widetilde{a}_1 a\underbrace{ba\cdots ab}_{m-2} =_W \underbrace{ba\cdots ab}_{m-2}a\widetilde{a}_2^{-1}a. \tag{3}$$

As in Case 1, the left side of equation (3) is M–reduced by Lemma 6.13 since $a\widetilde{a}_1 a$ is a subword of h. Again there is a sequence of M'–operations that when applied to

the right side of equation (3) results in a subword $\alpha_{a'b'}$ or $\alpha_{b'a'}$ with $m_{a'b'} \geq m$. This again contradicts Lemma 4.2 (i). □

7 Maximal FA_n subgroups

Let W be a Coxeter group. We say that a subgroup $H \subset W$ is *maximal* FA_n if H has property FA_n and H is not properly contained in any other subgroup of W with FA_n. Recall that every $(n+1)$–spherical special subgroup of W (with respect to *any* Coxeter generating set of W) has FA_n by Corollary 6.2. We conjecture that all maximal FA_n subgroups of W arise in this way.

Maximal FA_n **Conjecture** *Let (W,S) be a Coxeter system. A subgroup $H \subset W$ is maximal FA_n if and only if $H = wAw^{-1}$ for some maximal $(n+1)$–spherical special subgroup A of W and $w \in W$.*

Note that if the Coxeter FA_n Conjecture is true, then the above statement is equivalent to the formulation given in the Introduction.

The Maximal FA_1 Conjecture was proven by Mihalik and Tschantz in [24]. In this section, we use a modification of their arguments. We prove the following reformulation of Theorem 1.4.

Theorem 7.1 *Suppose the* CAT(0) *Conjecture holds for all $v \leq n$. Then the Maximal* FA_n *Conjecture holds for every Coxeter group.*

By Theorem 1.3, we will thus have the following:

Corollary 7.2 *The Maximal* FA_2 *Conjecture is true for every Coxeter group.*

Because FA_n is a property of the group rather than of a particular presentation, we immediately conclude the following:

Corollary 7.3 *For all Coxeter groups W, and for all n for which the Maximal* FA_n *Conjecture holds, the set of conjugates of maximal $(n+1)$–spherical special subgroups of W is independent of the Coxeter presentation. In particular, this is true for $n = 1$ and $n = 2$.*

The next proposition shows that the only candidates for maximal FA_n subgroups are indeed conjugates of $(n+1)$–spherical special subgroups.

Proposition 7.4 *Let (W,S) be a Coxeter system. Suppose a subgroup H of W has property FA_n. If the CAT(0) Conjecture holds for all $v \le n$, then there is a $w \in W$ such that $H \subset wBw^{-1}$ for some $(n+1)$–spherical special subgroup B of W.*

Proof Let $v = \max\{m : W \text{ is } m\text{–spherical}\}$ as in Section 6.2. If $n < v$, then W is $(n+1)$–spherical, and we are done. So, we may assume $n \ge v$. Note then that H has FA_v. By our construction in Section 6.2, there is a special v–splitting Λ of W with vertex groups special subgroups of W of rank $|S|-1$. Since H has FA_v and the CAT(0) Conjecture is assumed true for v, we know that H fixes a point in the action of W on the universal cover of Λ. Therefore, $H \subset w_1 B_1 w_1^{-1}$ for some special subgroup $B_1 \subset W$ of rank $|S|-1$ and some $w_1 \in W$. If B_1 is $(n+1)$–spherical, then we are done. Otherwise, let $v_1 = \max\{m : B_1 \text{ is } m\text{–spherical}\}$. Then $v_1 \le n$ and there is a special v_1–splitting of B_1. As before, $H \subset w_2 B_2 w_2^{-1}$ where B_2 is a special subgroup of rank $|S|-2$ and $w_2 \in W$. Once again, if B_2 is $(n+1)$–spherical, then we are done. Otherwise, we can continue splitting in this way. Since S is finite and the number of generators of B_k is $|S|-k$, the process terminates. So, we find $H \subset w_k B_k w_k^{-1}$ for some $(n+1)$–spherical B_k. □

Remark 7.5 Proposition 7.4 implies in particular that the only possible subgroups of W which could be maximal FA_n are conjugates of $(n+1)$–spherical special subgroups. Moreover, if $B \subset A$ are $(n+1)$–spherical subgroups of W, then $wBw^{-1} \subset wAw^{-1}$ for all $w \in W$, so if H is a maximal FA_n subgroup of W, then H is a conjugate of a *maximal* $(n+1)$–spherical subgroup. Since all $(n+1)$–spherical subgroups have FA_n, it remains only to show that all conjugates of maximal $(n+1)$–spherical subgroups are *maximal* FA_n subgroups.

The lemma below follows from Proposition 5.5 of Deodhar [18] together with Corollary 2.11.

Lemma 7.6 *Let (W,S) be a Coxeter system and suppose A is a maximal $(n+1)$–spherical special subgroup of W. If B is any $(n+1)$–spherical special subgroup of W such that $A \subset wBw^{-1}$ for some $w \in W$, then $S_A = S_B$.*

We now complete the proof of Theorem 7.1.

Proof of Theorem 7.1 As described in Remark 7.5, it remains only to prove that all conjugates of maximal $(n+1)$–spherical subgroups are maximal FA_n subgroups. Suppose now that A is a maximal $(n+1)$–spherical subgroup and $w \in W$. To show that A is maximal FA_n, it suffices to show that A is not properly contained in a conjugate of any $(n+1)$–spherical subgroup.

Suppose $A \subset wBw^{-1}$ for some $(n+1)$–spherical subgroup B. We will show that $A = wBw^{-1}$. By Lemma 7.6, we have $S_A = S_B$, so $A = B$ and $A \subset wAw^{-1}$. By Corollary 2.11, this implies that $S_A = dS_Ad^{-1}$ where d is an (S_A, S_A)–reduced element of AwA. Thus we find

$$wAw^{-1} = dAd^{-1} = d\langle S_A\rangle d^{-1} = \langle dS_Ad^{-1}\rangle = \langle S_A\rangle = A.$$

Hence $A = wAw^{-1} = wBw^{-1}$ so A is indeed a maximal FA_n subgroup. It immediately follows that the conjugates of A are all maximal FA_n as well. □

Remark 7.7 Note that the above arguments also apply to maximal strong FA_n subgroups. So, in particular, if the CAT(0) Conjecture holds, the maximal strong FA_n subgroups of W are the same as its maximal FA_n subgroups.

8 Proper actions

In this section, we briefly consider the case of proper actions of Coxeter groups. The CAT(0) *dimension* of a group G, denoted $\dim_{ss}(G)$, is defined to be the minimal dimension of a complete CAT(0) space on which G acts properly by semisimple isometries. CAT(0) dimension has been studied by Bridson [9] and Brady and Crisp [5], among others.

The following lemma is immediate from the definitions.

Lemma 8.1 *Suppose G is a group and H is an infinite subgroup of G with strong* FA_n. *Then* $\dim_{ss}(G) > n$.

Together with Corollary 6.2, this gives the following lower bound on the CAT(0) dimension of Coxeter groups.

Corollary 8.2 *Let (W, S) be a Coxeter system. Then*

$$\dim_{ss}(W) > \max\{n : W \text{ has an infinite } (n+1)\text{–spherical special subgroup}\}.$$

In fact, we obtain a better lower bound by applying Proposition 5.8 to the set of finite special subgroups of W. Let $L(W, S)$ denote the simplicial complex with vertices corresponding to the elements of S and such that a subset $U \subset S$ spans a simplex in $L(W, S)$ if and only if W_U is finite. The complex $L(W, S)$ is called the *nerve* of the Coxeter system (W, S).

Corollary 8.3 *Let (W, S) be a Coxeter system. Then*

$$\dim_{ss}(W) > \max\{k : H_k(L(W_T, T)) \neq 0 \text{ for some } T \subset S\}.$$

Proof (Notation as in Section 5.2.) Let $T \subset S$. Suppose $\phi\colon W \to \mathrm{Isom}(X)$ gives an action of W on a complete CAT(0) space X via semisimple isometries. Then the Bruhat–Tits Fixed Point Theorem (Proposition 3.8) implies that $L(W_T, T) \subset \mathcal{N}(T, \phi)$. On the other hand, Proposition 5.8 implies that $H_k(\mathcal{N}(T, \phi)) = 0$ for $k \geq \dim(X)$. So, if $H_k(L(W_T, T)) \neq 0$ for some $k \geq \dim(X)$, then $L(W_T, T) \neq \mathcal{N}(T, \phi)$. In particular, some infinite subgroup of W must fix a point, so the action is not proper. □

On the other hand, a Coxeter group W acts properly on its Davis–Moussong Complex Σ_{DM} (see Section 3.3). It follows that $\dim(\Sigma_{DM})$ is an upper bound for $\dim_{ss}(W)$. In particular,

$(*)$ $\qquad \dim_{ss}(W) \leq \max\{n : W \text{ has a finite special subgroup of rank } n\}.$

A natural question then is how the bounds on $\dim_{ss}(W)$ given by the above inequalities are related. The following example shows that the upper bound given by $(*)$ is not optimal.

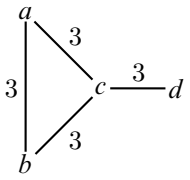

Figure 6:
Coxeter diagram of W

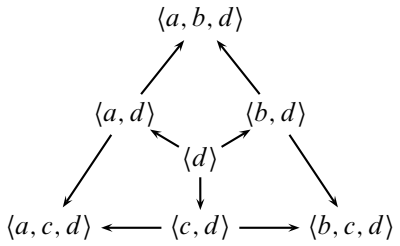

Figure 7: Triangle of
finite groups

Example 8.4 Consider the Coxeter group W with Coxeter diagram given by Figure 6. Note that W is 2–spherical but not 3–spherical since $W_{\{a,b,c\}}$ is infinite. So by Corollary 6.10 we know that W has strong FA_1 but not strong FA_2. In particular, the triangle of groups decomposition of W in Figure 7 gives a CAT(0) 2–complex on which W acts nontrivially by semisimple isometries. Moreover, the point stabilizers under this action are conjugates of the local groups in Figure 7. In particular, all point stabilizers are finite, so the action is proper. Hence $\dim_{ss}(W) \leq 2$. Moreover, by Corollary 8.2, since $W_{\{a,b,c\}}$ is infinite and 2–spherical, we have $\dim_{ss}(W) \geq 2$. So, the CAT(0) dimension of W is 2. However, the Davis–Moussong complex Σ_{DM} is 3–dimensional since, for example, $W_{\{a,b,d\}}$ is finite.

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