Invariants of curves in $\mathbb{R}P^2$ and $\mathbb{R}^2$

ABIGAIL THOMPSON

There is an elegant relation [3] among the double tangent lines, crossings, inflections points, and cusps of a singular curve in the plane. We give a new generalization to singular curves in $\mathbb{R}P^2$. We note that the quantities in the formula are naturally dual to each other in $\mathbb{R}P^2$, and we give a new dual formula.

53A04; 14H50

1 Introduction

Let K be a smooth immersed curve in the plane. Fabricius-Bjerre [2] found the following relation among the double tangent lines, crossings, and inflections points for a generic K:

$$T_1 - T_2 = C + (1/2)I$$

where T_1 and T_2 are the number of exterior and interior double tangent lines of K, C is the number of crossings, and I is the number of inflection points. Here "generic" means roughly that the interesting attributes of the curve are invariant under small smooth perturbations. Fabricius-Bjerre remarks on an example due to Juel which shows that the theorem cannot be straightforwardly generalized to the projective plane. A series of papers followed. Halpern [5] re-proved the theorem and obtained some additional formulas using analytic techniques. Banchoff [1] proved an analogue of the theorem for piecewise linear planar curves, using deformation methods. Fabricius-Bjerre gave a variant of the theorem for curves with cusps [3]. Weiner [7] generalized the formula to closed curves lying on a 2–sphere. Finally Pignoni [6] generalized the formula to curves in real projective space, but his formula depends, both in the statement and in the proof, on the selection of a base point for the curve. Ferrand [4] relates the Fabricius-Bjerre and Weiner formulas to Arnold's invariants for plane curves. Note that any formula for curves in $\mathbb{R}P^2$ is more general than one for curves in $\mathbb{R}^2$, since one can specialize to curves in $\mathbb{R}^2$ by considering curves lying inside a small disk in $\mathbb{R}P^2$.

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There are two main results in this paper. The first is a generalization of the theorem in [3] to $\mathbb{R}P^2$, with no reference to a basepoint on the curve. The original theorem is transparently a special case of this result, which is not surprising as the techniques used to prove it are a combination of those found in [2] and in [7]. The difficulties encountered in the generalization are due to the problems in distinguishing between two "sides" of a closed geodesic in $\mathbb{R}P^2$. These are overcome by a careful attention to the natural metric on the space inherited from the round 2–sphere of radius one.

The main results are tied together by the observation that, in the version of the original formula which includes cusps [3], the quantities in the formula are naturally dual to each other in $\mathbb{R}P^2$. This leads to the second, more surprising, main result, which is a dual formula for generic curves in $\mathbb{R}P^2$. This specializes to a new formula for generic smooth curves in the plane. This new formula has the interesting property that it reveals delicate geometric distinctions between topologically similar planar curves, for example quantifying some of the differences between the two curves shown in Figure 1.

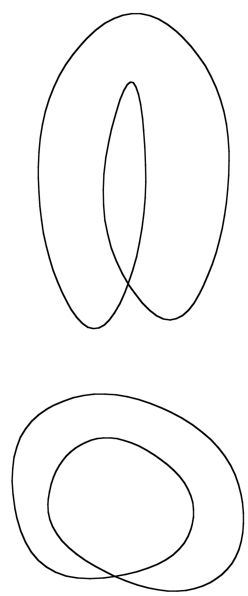


Figure 1

The outline of the paper is as follows: in Section 2 we state and prove the generalization of [3] to curves in $\mathbb{R}P^2$. In Section 3 we describe the duality between terms of the formula. In Section 4 we state and prove the dual formulation, and give its corollaries for planar curves.

2 A Fabricius-Bjerre formula for curves in $\mathbb{R}P^2$

Let $\mathbb{R}P^2$ be endowed with the spherical metric, inherited from its double cover, the round 2–sphere of radius one. With this metric, a simple closed geodesic (or projective line) in $\mathbb{R}P^2$ has length π. The figures will use a standard disk model for $\mathbb{R}P^2$, in which the boundary of the disk twice covers a closed geodesic. Let K be a generic oriented closed curve in $\mathbb{R}P^2$, which is smoothly immersed except for cusps of type 1, that is, cusps at which locally the two branches of K coming into the cusp are on opposite sides of the tangent geodesic. We postpone the definition of *generic* until the end of section 3. We will need some definitions.

Definitions

Let τ_p be the geodesic tangent to K at p, with orientation induced by K.

Let a_p, the *antipodal point to* p, be the point on τ_p a distance $\pi/2$ from p.

τ_p is divided by p and a_p into two pieces. Let ${\tau_p}^+$ be the segment from p to a_p and ${\tau_p}^-$ the segment from a_p to p. At cusp points ${\tau_p}^+$ and ${\tau_p}^-$ are not well-defined.

Let ν_p be the normal geodesic to K at p.

Let c_p (which lies on ν_p) be the center of curvature of K at p, that is, the center of the osculating circle to K at p.

We orient ν_p so that the length of the (oriented) segment from p to c_p is less than the length of the segment from c_p to p. This orientation is well-defined except at cusps and inflection points.

There is a natural duality from $\mathbb{R}P^2$ to itself. Under this duality simple closed geodesics, or projective lines, in $\mathbb{R}P^2$ are sent to points and vice versa. This duality is most easily described by passing to the 2–sphere S which is the double cover of $\mathbb{R}P^2$; in this view a simple closed geodesic in $\mathbb{R}P^2$ lifts to a great circle on S. If this great circle is called the equator, the dual point in $\mathbb{R}P^2$ is the image of the north (or south) pole.

Under this duality the image of K is a *dual curve* K'. To describe K' we need only observe that a point on K comes equipped with a tangent geodesic, τ_p. The *dual point to* p, called p', is the point dual to the tangent geodesic τ_p.

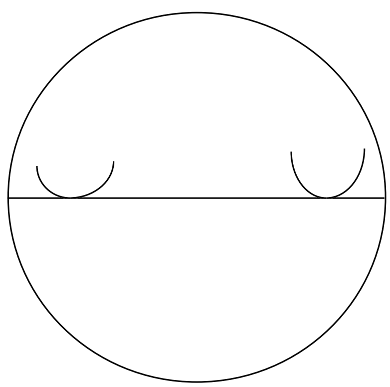
Another useful description is that p' is the point a distance $\pi/2$ along the normal geodesic to K at p. Notice that $\nu_p = \nu_{p'}$ and $c_p = c_{p'}$.

An ordered pair of points (p, q) on K is an *antipodal pair* if $q = a_p$.

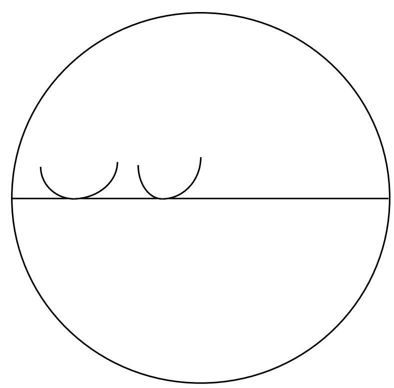
Let Y_p be the geodesic dual to the point c_p.

Let (p, q) be an antipodal pair. Then Y_p and τ_p intersect at q and divide $\mathbb{R}P^2$ into two regions, R_1 and R_2. One of the regions, say R_1, contains c_p. The geodesic τ_q lies in one of the two regions. An antipodal pair (p, q) is of type 1 if τ_q lies in R_1, type 2 if τ_q lies in R_2. Let A_1 be the number of type 1 antipodal pairs of K, A_2 the number of type 2.

T is a *double-supporting geodesic* of K if T is either a double tangent geodesic, a tangent geodesic through a cusp or a geodesic through two cusps. The two tangent or cusp points of K divide T into two segments, one of which has length less than $\pi/2$. We distinguish two types of double supporting geodesics, depending on whether the two points of K lie on the same side of this segment (type 1) or opposite sides (type 2). Let T_1 be the number of double supporting geodesics of K of type 1, T_2 the number of type 2 (see Figure 2).



Type 1 double supporting geodesic



Type 2 double supporting geodesic

Figure 2

The tangent geodesics at a crossing of K define four angles, two of which, α and β, are less than $\pi/2$. In a small neighborhood of a crossing there are four segments of K. The crossing is of type 1 if one of these segments lies in α and another in β, type 2 if two lie in α or two lie in β. Let C_1 be the number of type 1 crossings of K, C_2 the number of type 2 (see Figure 3).

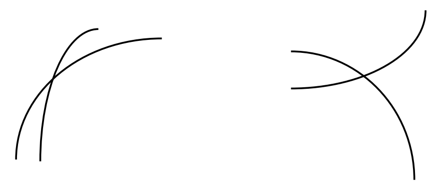


Figure 3

Let I be the number of inflection points of K.

Let U be the number of (type 1) cusps of K.

We are now ready to state the first main theorem, which is a generalization of the main theorem of [2] to the projective plane. We note that (unlike [6]) we do not need to choose a base-point for K.

Theorem 1 *Let K be a generic singular curve in $\mathbb{R}P^2$ with type 1 cusps. Then*

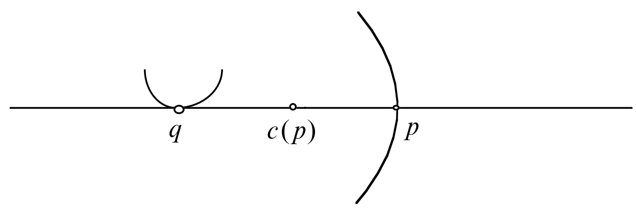
$$T_1 - T_2 = C_1 + C_2 + (1/2)I + U - (1/2)A_1 + (1/2)A_2$$

Proof The proof proceeds as in [3], with some caution being required at antipodal pairs and at cusp points. We choose a starting point p on K. Let ${M_p}^+$ be the number of times K intersects ${\tau_p}^+$, ${M_p}^-$ be the number of times K intersects ${\tau_p}^-$, and $M_p = {M_p}^+ - {M_p}^-$. We keep track of how M_p changes as we traverse the knot. Double-supporting geodesics, crossings, cusps and inflection points all behave as in [3]. Suppose p is a point of an antipodal pair (p,q). Let p_1 be a point immediately before p on K, p_2 a point immediately after. If (p,q) is of type 1 then τ_{p_1} intersects the arc of K containing q on ${\tau_{p_1}}^-$ and τ_{p_2} intersects the arc of K containing q on ${\tau_{p_2}}^+$, hence M_p increases by 2 as we pass through p. If (p,q) is of type 2 then τ_{p_1} intersects the arc of K containing q on ${\tau_{p_1}}^+$ and τ_{p_2} intersects the arc of K containing q on ${\tau_{p_2}}^-$, hence M_p decreases by 2 as we pass through p. This is easiest to see by approximating K near p by a circle centered at c_p. As we traverse a piece of this circle from p_1 through p to p_2, the antipodal points to p_1 and p_2 lie on the geodesic Y_p. Hence the critical distinction to be made at q is where the tangent geodesic at q lies in relation to Y_p and τ_p. This is the exactly the distinction between type 1 and type 2 antipodal pairs. □

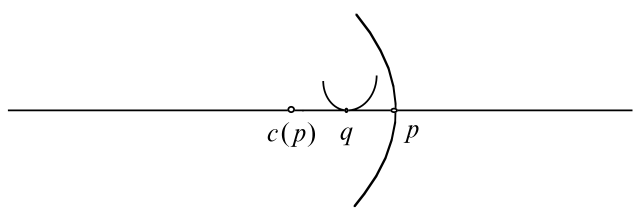
3 Duality in $\mathbb{R}P^2$

We describe the dual relations between crossings and double tangencies, cusps and inflection points, and antipodal points and normal-tangent pairs (defined below).

Definitions The points p and c_p divide ν_p into two pieces, ${\nu_p}^+$ from p to c_p and ${\nu_p}^-$ from c_p to p. An ordered pair of points (p,q) on K is a *normal-tangent pair* if $\tau_q = \nu_p$. A normal-tangent pair (p,q) is of type 1 if q lies on ${\nu_p}^-$, type 2 if q lies on ${\nu_p}^+$ (Figure 4). Let N_1 be the number of type 1 normal-tangent pairs of K, N_2 the number of type 2 .



(p,q) is a type 1 normal-tangent pair



(p,q) is a type 2 normal-tangent pair

Figure 4

Proposition 2 *Let K be a generic curve in $\mathbb{R}P^2$, with dual curve K'. Let $i = 1,2$. Then:*

(1) *A crossing of type i in K is dual to a double supporting geodesic of type i in K'.*

(2) *A cusp in K is dual to an inflection point in K'.*

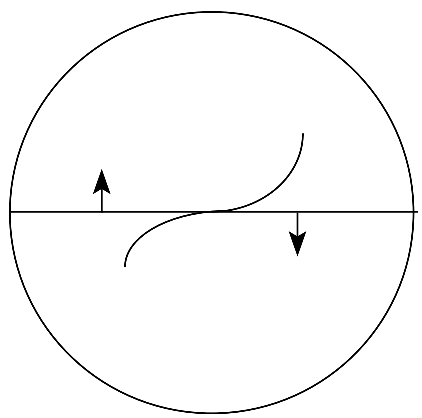
(3) *An antipodal pair of type i in K is dual to a normal-tangent pair of type i in K'.*

As the dual of K' is again K, these correspondences work in both directions.

Proof The proof is by construction in $\mathbb{R}P^2$. □

This correspondence breaks down slightly when we consider double supporting geodesics between cusps and tangents, or cusps and cusps. Fabricius-Bjerre suggests a small local alteration of K to understand his argument at a cusp point, replacing the cusp point by a small "bump". Just as the dual to a small round circle in $\mathbb{R}P^2$ is a (long) curve that is close to a geodesic, his local change at cusps induces a more global change at inflection points, and so in order to incorporate curves with inflection points we need to add *inflection geodesics* to our picture of K.

Let p be an inflection point of K, with tangent geodesic τ_p. Endow $\tau_p - p$ with a normal direction at each point (except the inflection point) by the convention shown in Figure 5.



inflection geodesic with normal direction

Figure 5

Definition Call this the *inflection geodesic to K at p*.

For crossings between K and an inflection geodesic τ_p, or between two inflection geodesics, the piece of τ_p in the neighborhood of the crossing should be construed as bending slightly towards its normal direction for the purposes of classifying the crossing type. This convention preserves the correct duality between crossing type in K and double supporting geodesic type in K'. A point on the inflection geodesic has center of curvature a distance $\pi/2$ in the normal direction, at p'. For α a point on an inflection geodesic τ_p, ν_α is the geodesic through α and p'.

Definition Denote by $\bar{K}$, K together with all its inflection geodesics. Crossings and normal tangencies are counted as described above. The inflection points of K (where the inflection geodesic intersects the curve) will still be counted as simply inflection points in $\bar{K}$, not as new crossing points.

If K is a generic curve with dual K', then double supporting geodesics in K involving cusp points correspond to crossings in $\bar{K}'$ involving inflection geodesics, and an antipodal pair (p,q) with p a cusp point will correspond to a normal-tangent pair (p',q') with p' a point on an inflection geodesic in $\bar{K}'$.

We end this section with the definition of what it means for K to be generic:

Definition K is *generic* if:

- K has a finite number of crossings, double tangent lines, cusps, inflection points, antipodal pairs, and normal-tangent pairs.
- Tangent geodesics at self-intersections of K are neither parallel nor perpendicular.
- The tangent geodesic through an inflection point or at a cusp is everywhere else transverse to K.
- A geodesic goes through at most two tangent points or cusps of K.
- No crossings occur at inflection points.
- A geodesic normal to K at one point is tangent to K at at most one point and everywhere else transverse to K.
- The distance between two points on a double-supporting geodesic is not $\pi/2$.
- If (p,q) is an antipodal pair, let Y_p be the geodesic dual to c_p. Then τ_q should be neither τ_p nor Y_p.
- If (p,q) on K is a normal-tangent pair, q is not c_p.

4 A dual formula, with applications

The simplest version of the dual theorem applies to curves with no inflection points.

Theorem 3 *Let K be a generic singular curve in $\mathbb{R}P^2$ with type 1 cusps and no inflection points. Then*

$$C_1 - C_2 = T_1 + T_2 + (1/2)U - (1/2)N_1 + (1/2)N_2$$

Since inflection points are dual to cusps, we also have:

Corollary 4 *Let K be the dual in $\mathbb{R}P^2$ of a smooth singular curve. Then Theorem 3 holds for K.*

Proof of Theorem 3 The theorem follows directly from duality on $\mathbb{R}P^2$, but it is illuminating to consider the dual of the proof of Theorem 1, as it provides a direct proof for curves in the plane. In Theorem 1 we count the number of intersections between the curve K and τ_p, with appropriate signs, as the curve is traversed once. Hence the main technical point is to understand the dual to M_p.

Assign an orientation to K. The geodesics τ_p and ν_p intersect in a single point (at p) and divide $\mathbb{R}P^2$ into two regions. We first define the *tangent-normal frame* F_p to K at p as follows: F_p is the union of τ_p and ν_p together with a black-and-white coloring of the two regions of $\mathbb{R}P^2$. We color them by the convention that if we think of τ_p and ν_p at P as being analogous to the standard $x-$ and $y-$ axes, the region corresponding to the quadrants where x and y have the same sign is colored white, the complementary region black (see Figure 6). The frame and its coloring are well-defined at points that are neither cusps nor inflection points. At cusps, the orientations of τ_p and ν_p *both* reverse as we traverse K, with the happy effect that the coloring of the normal-tangent frame is well-defined as we pass through a cusp point (notice that this is not true if we allow type 2 cusps).

We now describe how the tangent-normal frame for the dual curve K' is related to M_p for the curve K.

Let p be a point on K with tangent geodesic τ_p. Let r be a point of intersection between K and τ_p. r contributes either $+1$ or -1 to M_p, depending on where it lies relative to the antipodal point a_p. What does r correspond to in the dual picture? Under duality p is sent to the point p', and $\nu_p = \nu'_p$. The point r is mapped to a

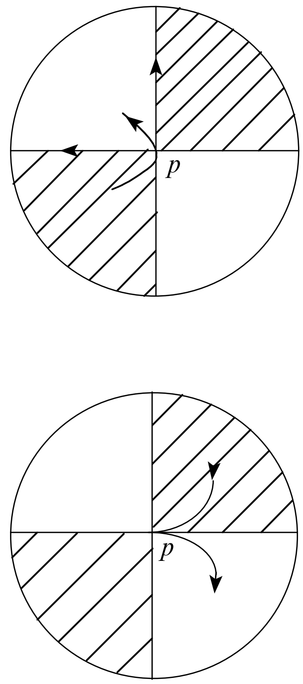


Figure 6

geodesic g_r through p'. A small neighborhood of r in K is mapped to an arc of K' tangent to g_r. If r contributes $+1$ to M_p, g_r lies in the white region of the tangent-normal frame at p'. If r contributes -1 to M_p, g_r lies in the black region of the tangent-normal frame at p'. This leads us to the following definition.

Definition At a given point p on K, we define W_p to be the number of geodesics through p and tangent to K which lie in the white region and B_p to be the number of geodesics through p and tangent to K which lie in the black region as defined by the tangent-normal frame at p. Let $V_p = W_p - B_p$. The proof consists of tracking how V_p changes as we traverse K once; each type of singularity contributes to V_p according to the following table:

Singularity	*Contribution*
C_1	$+4$
C_2	-4
T_i	-4
U	-2
N_1	$+2$
N_2	-2

□

We can use the natural duality directly for the general case:

Theorem 5 *Let K be a generic singular curve in $\mathbb{R}P^2$ with type 1 cusps. Then for $\bar{K}$,*

$$C_1 - C_2 = T_1 + T_2 + (1/2)U + I - (1/2)N_1 + (1/2)N_2$$

If K is a curve with no cusps, inflection points, or antipodal pairs (or for a smooth immersed curve in $\mathbb{R}^2$ with no inflection points), then the pair of formulas:

$$T_1 - T_2 = C_1 + C_2$$
$$C_1 - C_2 = T_1 + T_2 - (1/2)N_1 + (1/2)N_2$$

applies, and combining them we can obtain:

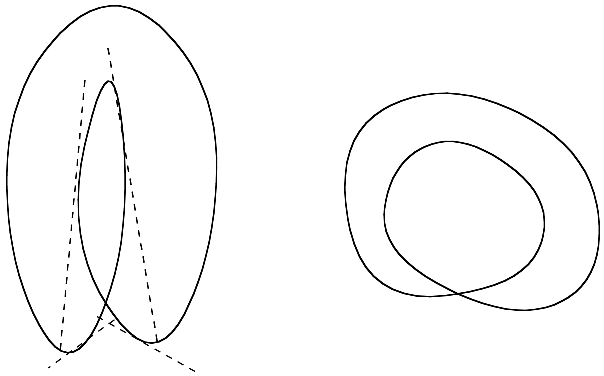
Corollary 6 *For K a curve with no cusps, inflection points, or antipodal pairs (or for a smooth immersed curve in $\mathbb{R}^2$ with no inflection points):*

$$4T_1 - 4C_1 = N_1 - N_2$$
$$4T_2 + 4C_2 = N_1 - N_2$$

Note that for the two curves shown in Figure 1 (redrawn in Figure 7) , we obviously have the values $T_1 = 1$, $T_2 = 0$. For the right-hand curve, $C_1 = 1$ and $C_2 = 0$, while for the left, $C_1 = 0$ and $C_2 = 1$. By observation, the right curve has no normal-tangent pairs, and the two equations in Corollary 6 are easily seen to be satisfied. Applying Corollary 6 to the left-hand curve, however, we obtain

$$4 = N_1 - N_2$$

and we can locate four normal-tangent pairs of type 1 (Figure 7).



$C_1 = 0,\ C_2 = 1,\ N_1 = 4$ $\qquad$ $C_1 = 1,\ C_2 = 0,\ N_1 = N_2 = 0$

Figure 7

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Mathematics Department, University of California
Davis, CA 95616, USA

thompson@math.ucdavis.edu

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