String homology of spheres and projective spaces

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We study a spectral sequence that computes the S^1–equivariant homology of the free loop space LM of a manifold M (the *string homology* of M). Using it and knowledge of the BV operations on $HH^*(H^*(M), H^*(M))$, we compute the (mod 2) string homology of M when M is a sphere or a projective space.

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1 Introduction

The free loop space $LM = \mathrm{Map}(S^1, M)$ of a closed n–manifold, M, admits an action of the circle S^1 by rotation of loops. This space and the homotopy orbit space (or Borel construction)

$$LM_{hS^1} = LM \times_{S^1} ES^1$$

were shown in Chas and Sullivan's article [4] on string topology to admit remarkable multiplicative structures inspired by conformal field theory. Furthermore, both spaces are entwined in the definition of topological cyclic homology given by Bökstedt, Hsiang and Madsen [1].

The goal of this paper is to compute the homology of these spaces for certain manifolds, namely spheres and projective spaces. Machinery for computing these (co-)homologies for general spaces does exist in the literature (see, for instance, Hess [9], Bökstedt and Ottosen [2], Chen [5]). Our purpose is to explore the link with Deligne's conjecture and to illustrate the power of the Batalin–Vilkovisky operations in making these computations "barehanded." As such, this paper is similar in spirit to Kallel and Salvatore [11] where string topology type operations are used to make computations of the homology of spaces of holomorphic maps.

The central idea is that the homology of LM_{hS^1} may be computed via a spectral sequence (essentially Connes' spectral sequence for cyclic homology) if one has knowledge of how the Batalin–Vilkovisky operator Δ acts on $HH^*(H^*(M), H^*(M))$. In turn, one may often compute Δ if one understands the other (Gerstenhaber) operations in Hochschild cohomology.

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In [21] we have computed this Hochschild cohomology when M is a sphere or real, complex, or quaternionic projective space. Furthermore, we computed certain homology operations: the one relevant to this paper is the Gerstenhaber bracket (or Browder operation), a Lie bracket that we will denote $[\cdot,\cdot]$. This arises in the presence of an action of the little disks operad C_2 on a space (or chain complex). In the case at hand, this action is given by McClure and Smith's proof [14] of Deligne's conjecture. A cyclic version of Deligne's conjecture (Kaufmann [12], McClure and Smith [15], Tradler and Zeinalian [19]) allows us to relate the actions of $[\cdot,\cdot]$ and Δ, through the BV formula:

$$\Delta(xy) = \Delta(x)y + (-1)^{|x|}x\Delta(y) + (-1)^{|x|}[x,y]$$

Therefore if we know the value of Δ and $[\cdot,\cdot]$ on the generators of the algebra, we may compute Δ for every class in $HH^*(H^*(M),H^*(M))$. As a result, we obtain the E_2 term of Connes' spectral sequence for these manifolds. A simple argument shows that for the manifolds under consideration the spectral sequence collapses at E_2.

In what follows, K denotes one of the division algebras $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and $d=\dim_{\mathbb{R}}(K)$. For brevity, define

$$\alpha_{d,n}(t) := t^{-d-1} + \frac{t^{d(2n)-3}}{1-t^2} \qquad \beta_{d,n}(t) := t^{d(2n+1)-3} + \frac{1+t^{d(2n)-2}+t^{-1}}{1-t^2}$$

Theorem 1.1 *A computation of the Poincaré series of $H_*^{S^1}(LM;\mathbb{F}_2)$:*

(1) *If $M=S^k$ and $k>1$, the Poincaré series is:*

$$\left(\frac{1}{1-t^{2(k-1)}}\right)\left(t^{k-1} + \frac{1+t^{2k-1}}{1-t^2}\right)$$

(2) *If $M=KP^{2n+1}$ and $n>0$ if $K=\mathbb{R}$, the Poincaré series is:*

$$\left(\frac{t^{d(2n+1)}}{1-t^{d(2n+2)-2}}\right)\left(\frac{1-t^{-2d(n+1)}}{1-t^{-2d}}\right)\left(t^{-1} + \frac{t^{d-1}+t^{-d}}{1-t^2}\right)$$

(3) *If $M=KP^{2n}$, the Poincaré series is:*

$$\left(\frac{t^{d(2n)}}{1-t^{2d(2n+1)-4}}\right)\left(\left(\frac{1-t^{-2dn}}{1-t^{-2d}}\right)\alpha_{d,n}(t) + \left(\frac{1-t^{-2d(n+1)}}{1-t^{-2d}}\right)\beta_{d,n}(t)\right)$$

We note that the Poincaré series for spheres agrees with answers obtained either through Carlsson–Cohen's splitting in [3] or the spectral sequence defined by Bökstedt and Ottosen in [2].

In Section 2 we introduce a cohomology theory for Frobenius algebras that is dual to negative cyclic homology. This allows us to dualize Jones' theorem [10] identifying $H^*(LX_{hS^1})$ with the negative cyclic homology of the singular cochain algebra $C^*(X)$. A defect of this result is its limited application to formal manifolds. Still, it allows a connection with Deligne's conjecture and string topology type operations which is employed throughout this article.

In Section 5 we use this construction to give a spectral sequence converging to $H_*(LM_{hS^1})$. Its origin in cyclic cohomology affords great control on the differentials. This allows us in Section 7 to show that the spectral sequence collapses at the E_2 term. In Section 6, we compute that E_2 term, proving Theorem 1.1.

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2 Cyclic Frobenius cohomology

In this section we will introduce a cohomology theory for associative Frobenius algebras A which we call *cyclic Frobenius cohomology*, $HC_F^*(A)$. It bears the same relation to cyclic homology that Hochschild cohomology does to Hochschild homology. We follow Kaufmann's definition [12] of a version (which we call $\check{B}$) of the B operator for Hochschild *cohomology* which is incorporated in the definition of $HC_F^*(A)$. The hypothesis that A is a Frobenius algebra is required in order to define $\check{B}$ by dualizing the usual definition of B in Hochschild homology. While we will see that $HC_F^*(A)$ encodes little more information than $HC_*^-(A)$, its virtue for our purposes will be in computing the homology of LM_{hS^1}.

For our purposes, a *Frobenius algebra* is an associative, unital, finite dimensional graded algebra A over a ring k, endowed with a non-degenerate inner product $\langle\cdot,\cdot\rangle$ which is symmetric and invariant:

$$\langle a,b\rangle=(-1)^{|a|+|b|}\langle b,a\rangle \ \ , \ \ \langle ab,c\rangle=\langle a,bc\rangle$$

Notice that these facts together imply that $\langle ab,c\rangle=\langle b,ca\rangle$. The inner product on A specifies an isomorphism from A to its dual A^*, via

$$a\mapsto\langle a,\cdot\rangle.$$

We will say that the *dualizing dimension* of A is d if $\langle\cdot,\cdot\rangle$ is a graded map of dimension d; that is, it restricts to a nondegenerate pairing $A_p\otimes A_{d-p}\to k$ for each dimension p.

Recall the Hochschild chain and cochain complexes of A:

$$CH_n(A,A) = A^{\otimes n+1} \ , \ CH^n(A,A) = \mathrm{Hom}_k(A^{\otimes n}, A)$$

The Hochschild chain complex is equipped with the differential b:

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}$$

We will write $\check{b}$ for the differential on the Hochschild cochain complex:

$$\begin{aligned}\check{b}(f)(a_0 \otimes \cdots \otimes a_n) = a_0 f(a_1 \otimes \cdots \otimes a_n) \\ + \sum_{i=1}^{n} (-1)^i f(a_0 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_n) \\ + (-1)^{n+1} f(a_0 \otimes \cdots \otimes a_{n-1}) a_n\end{aligned}$$

for $f \in CH^n(A,A)$ (see, eg, Loday [13]).

Define $\widetilde{f} \in \mathrm{Hom}_k(A^{\otimes n+1}, k) = CH_n(A,A)^*$ as:

$$\widetilde{f}(a_0 \otimes \cdots \otimes a_n) = \langle a_0, f(a_1 \otimes \cdots \otimes a_n) \rangle$$

Since $\langle \cdot, \cdot \rangle$ is nondegenerate, $f \mapsto \widetilde{f}$ is an isomorphism. This allows us to compare the Hochschild homology and cohomology differentials:

Lemma 2.1 *For $f \in CH^{n-1}(A,A)$, $\widetilde{\check{b}(f)} = b^*(\widetilde{f})$; here b^* is the linear dual of b.*

Proof We compute

$$\begin{aligned}\widetilde{\check{b}(f)}(a_0 \otimes \cdots \otimes a_n) &= \langle a_0, \check{b}(f)(a_1 \otimes \cdots \otimes a_n) \rangle \\ &= \langle a_0, a_1 f(a_2 \otimes \cdots \otimes a_n) \rangle \\ &\quad + \sum_{i=1}^{n-1} (-1)^i \langle a_0, f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \rangle \\ &\quad + (-1)^n \langle a_0, f(a_1 \otimes \cdots \otimes a_{n-1}) a_n \rangle\end{aligned}$$

Using the symmetry and invariance of the inner product, this is equal to

$$\begin{aligned}\langle a_0 a_1, f(a_2 \otimes \cdots \otimes a_n) \rangle + \sum_{i=1}^{n-1} (-1)^i \langle a_0, f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \rangle \\ + (-1)^n \langle a_n a_0, f(a_1 \otimes \cdots \otimes a_{n-1}) \rangle\end{aligned}$$

On the other hand,

$$\begin{aligned} b^*(\widetilde{f})(a_0 \otimes \cdots \otimes a_n) = & \sum_{i=0}^{n-1} (-1)^i \widetilde{f}(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ & + (-1)^n \widetilde{f}(a_n a_0 \otimes \cdots \otimes a_{n-1}) \\ = & \langle a_0 a_1, f(a_2 \otimes \cdots \otimes a_n) \rangle + \\ & + \sum_{i=1}^{n-1} (-1)^i \langle a_0, f(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \rangle \\ & + (-1)^n \langle a_n a_0, f(a_1 \otimes \cdots \otimes a_{n-1}) \rangle \end{aligned}$$ □

Using the operator B from cyclic homology, we may define an adjoint operator $\check{B}\colon CH^n(A,A) \to CH^{n-1}(A,A)$ by a similar equality:

$$\widetilde{\check{B}(f)} = B^*(\widetilde{f})$$

Here B^* is the linear dual of B. More explicitly, $\widetilde{\check{B}(f)}(a_0 \otimes \cdots \otimes a_{n-1})$ is:

$$\sum_{i=0}^{n-1} (-1)^{(n-1)i} \langle 1, f(a_{n-i} \otimes \cdots \otimes a_{n-1} \otimes a_0 \otimes \cdots \otimes a_{n-i-1}) \rangle +$$

$$\sum_{i=0}^{n-1} (-1)^{(n-1)(i+1)} \langle a_{n-i-1}, f(1 \otimes a_{n-i} \otimes \cdots \otimes a_{n-1} \otimes a_0 \otimes \cdots \otimes a_{n-i-2}) \rangle$$

Following the "B, b" definition of cyclic homology, we define a (homological) bicomplex $\mathcal{B}^{*,*}(A)$, the homology of whose total complex will be $HC_F^*(A)$:

Definition 2.2 Define the bicomplex $\mathcal{B}^{*,*}(A)$ using the Hochschild cochain complex $CH^*(A,A)$:

$$\mathcal{B}^{p,q}(A) := CH^{p-q}(A,A) = \mathrm{Hom}_k(A^{\otimes p-q}, A)$$

for $p-q \geq 0$ and $p \geq 0$. The vertical differential $\check{b}\colon \mathcal{B}^{p,q}(A) \to \mathcal{B}^{p,q-1}(A)$ is the Hochschild cohomology differential. The horizontal differential $\check{B}\colon \mathcal{B}^{p,q}(A) \to \mathcal{B}^{p-1,q}(A)$ is defined above.

This definition requires the following lemma:

Lemma 2.3 *$\mathcal{B}^{*,*}(A)$ is a bicomplex; that is, $\check{B}^2 = 0 = \check{b}^2$ and $\check{B}\check{b} + \check{b}\check{B} = 0$.*

Proof That $\check{b}^2 = 0$ is well-established. To see that $\check{B}^2 = 0$, note that the definition of $\check{B}$ implies

$$\widetilde{\check{B}(\check{B}(f))} = B^*(\widetilde{\check{B}(f)}) = B^* B^*(\widetilde{f}) = 0$$

Since $f \mapsto \widetilde{f}$ is an isomorphism, we conclude that $\check{B}^2 = 0$. Similarly, using the definition of $\check{B}$ and Lemma 2.1 we see that:

$$\widetilde{\check{B}(\check{b}(f))} + \widetilde{\check{b}(\check{B}(f))} = B^*(\widetilde{\check{b}(f)}) + b^*(\widetilde{\check{B}(f)}) = B^* b^*(\widetilde{f}) + b^* B^*(\widetilde{f}) = 0 \quad \square$$

Definition 2.4 The *cyclic Frobenius cochain complex of* A is defined to be the total complex

$$CC^*_F(A) := \mathrm{Tot}\,(\mathcal{B}^{*,*}(A))$$

and its homology, $HC^*_F(A)$, is the *cyclic Frobenius cohomology of* A.

It is necessary to make a few remarks about gradings. For two graded vector spaces A and B, $\mathrm{Hom}(A,B)$ is graded, where the dimension of a homomorphism $f\colon A \to B$ is $\dim(f)$ if

$$\dim(f(a)) = \dim(a) - \dim(f)$$

for every a. This equips $CH^p(A,A) = \mathrm{Hom}_k(A^{\otimes p}, A)$ with an internal grading. Therefore $\mathcal{B}^{*,*}(A)$ is triply graded. The *total* (or *topological*) degree of an element $f \in \mathcal{B}^{p,q}(A) = \mathrm{Hom}_k(A^{p-q}, A)$ is

$$|f| = \dim(f) + p + q.$$

This is the dimension of the class that f represents in $CC^*_F(A)$, after totalization. The grading on $\mathcal{B}^{*,*}(A)$ restricts to a grading on $\mathcal{B}^{0,*}(A) = CH^{-*}(A,A)$; for $f \in CH^q(A,A)$, the topological degree is $|f| = \dim(f) - q$.

3 Relation to negative cyclic homology

Recall the definition of the *negative cyclic homology* $HC^-_*(A)$ of an associative algebra A: it is the homology of the complex $CC^-_*(A)$ which is the totalization of a bicomplex $\mathcal{B}^-_{*,*}(A)$, with

$$\mathcal{B}^-_{p,q}(A) := CH_{q-p}(A,A) \ , \quad p \le 0.$$

The vertical differential is the Hochschild homology differential b, and the horizontal differential is the operator B.

Lemma 3.1 *There is an isomorphism of bicomplexes*

$$\mathcal{B}^{*,*}(A) \cong (\mathcal{B}^{-}_{*,*}(A))^*$$

for any Frobenius algebra A. If the dualizing dimension of A is d, then this isomorphism increases total dimension by d. Consequently there is a graded isomorphism

$$CC^*_F(A) \cong \Sigma^d (CC^-_*(A))^*$$

and, when working over a field k,

$$HC^*_F(A) \cong \Sigma^d (HC^-_*(A))^*.$$

Proof We will define an isomorphism of bicomplexes which negates bidegrees: for $p \geq 0$

$$c\colon \mathcal{B}^{p,q}(A) \to (\mathcal{B}^{-}_{-p,-q}(A))^*$$

The domain is $CH^{p-q}(A,A)$ and the range is $CH_{p-q}(A,A)^*$. So we may define $c(f) := \widetilde{f}$.

It is definitional that $c \circ \check{B} = B^* \circ c$, and Lemma 2.1 implies that $c \circ \check{b} = b^* \circ c$. So c is a map of bicomplexes. We have already seen that it is an isomorphism by the nondegeneracy of the inner product. The first isomorphism follows.

To obtain the degree shift, notice that, if $f \in \mathrm{Hom}(A^{\otimes n}, A)$, then $\widetilde{f} \in \mathrm{Hom}(A^{\otimes n+1}, k)$ has dimension

$$\dim(\widetilde{f}) = \dim(f) + d.$$

This implies the second isomorphism in the statement of the lemma; the third follows from the second via the Universal Coefficient Theorem. □

4 Topological Applications

Throughout this paper, for a space X, $C^*(X)$ will denote the singular cochain algebra, equipped with the cup product of cochains; this descends to the singular cohomology $H^*(X)$. We will grade $C^*(X)$ and $H^*(X)$ negatively. Poincaré duality is used to prove the following classical fact about the homology of manifolds.

Proposition 4.1 *The singular cohomology algebra $H^*(M;k)$ of a k–oriented closed n–manifold M is a Frobenius algebra of dualizing dimension $-n$.*

Proof It is well-established that $H^*(M)$ admits a graded commutative, associative cup product $\smile$. The inner product is the intersection form: Let $[M] \in H_n(M)$ be the fundamental class of M. Then for $a, b \in H^*(M)$, the inner product is defined to be the evaluation of the cup product on the fundamental class:

$$\langle a, b\rangle := (a \smile b)([M])$$

That $\langle \cdot, \cdot \rangle$ is graded symmetric and invariant follows from the graded commutativity and associativity of the cup product.

To show that the inner product is nondegenerate, for each $a \in H^k(M)$, we must produce a class $b \in H^{n-k}(M)$ for which $\langle b, a\rangle \neq 0$. Choose b to be any cohomology class which is nonzero on the homology class

$$a \frown [M] \in H_{n-k}(M)$$

Then $$\langle b, a\rangle = (b \smile a)([M]) = b(a \frown [M]) \neq 0$$ □

Recall that for a ring k, a manifold M is called *k–formal* (or just *formal* when k is understood) if there is a quasi-isomorphism of differential graded algebras

$$C^*(M; k) \simeq H^*(M; k)$$

where we give $H^*(M; k)$ the zero differential.

Theorem 4.2 *Let k be a field. If M is a simply connected, compact, k–formal manifold of dimension n, the cyclic Frobenius cohomology of its cohomology algebra is isomorphic to its string homology:*

$$HC_F^*(H^*(M; k)) \cong \Sigma^{-n} H_*^{S^1}(LM; k)$$

Proof Jones has shown in [10] that

$$HC_*^-(C^*(X)) \cong H^*_{S^1}(LX)$$

for any simply connected space X. Recall that negative cyclic homology is an invariant of the quasi-isomorphism type of a differential graded algebra over a field (see, eg [13, Theorem 5.3.5]). If X is k–formal, we may therefore replace $C^*(X)$ with $H^*(X)$. Taking X to be a compact manifold M (whose cohomology is of finite type), and using Lemma 3.1 above, we see that

$$HC_F^*(H^*(M; k)) \cong (HC_*^-(H^*(M; k))^* \cong (HC_*^-(C^*(M; k))^* \cong H_*^{S^1}(LM; k)$$

□

One would like a version of this result without an appeal to formality. It seems clear that for such a result, one needs to replace $H^*(M)$ with some version of the cochain complex of M. This suggests the need for a homotopy theoretic notion of a Frobenius algebra in which a version of $C^*(M)$ would be a prime example. Tradler and Zeinalian have introduced such a notion, V_k*–algebras*, in [18] and study Deligne conjecture type operations on the Hochschild cochain complex of a V_k–algebra. We expect that it is possible to extend the definition of cyclic Frobenius cohomology, given here for strict Frobenius algebras, to a class of V_k–algebras. For our purposes we shall only be considering formal manifolds, and therefore will not explore such subtleties.

We refer the reader to the work of Xiaojun Chen [5] where a very similar model for the S^1–equivariant chain complex of LM that may avoid such difficulties is developed using methods of rational homotopy theory and Brown's twisting cochains.

5 Connes' spectral sequence

In this section, we study a natural spectral sequence for cyclic Frobenius cohomology. One may filter $CC^*(A)$ by vertical stripes in the bicomplex $\mathcal{B}^{*,*}(A)$. This, in turn, produces a spectral sequence that computes $HC^*_F(A)$:

Proposition 5.1 *There is a spectral sequence converging to* $HC^*_F(A)$. *The* E_1*–term of this spectral sequence is given by*

$$E_1^{p,q} = HH^{p-q}(A, A);\ p \geq 0$$

with differential

$$d_1 : E_1^{p,q} = HH^{p-q}(A, A) \to HH^{p-q-1}(A, A) = E_1^{p-1,q}$$

given by the map $\Delta := \check{B}_*$ *induced by* $\check{B}$ *in Hochschild cohomology.*

In the dual case, the analogous spectral sequence for cyclic *homology* was considered by Connes and called *Connes' spectral sequence* in Weibel [20]. We keep that terminology here.

We collect information about the differentials in the spectral sequence that will allow us to prove that it collapses for the manifolds under consideration. The first statement below is standard; the second follows from the fact that $\check{B}$ raises topological degree by one.

Lemma 5.2 *The r^{th} differential in Connes' spectral sequence for $HC^*_F(A)$ is of bidegree $(-r, r-1)$. That is,*

$$d_r\colon E_r^{p,q} \to E_r^{p-r,q+r-1}$$

maps from a subquotient of $HH^{p-q}(A,A)$ to a subquotient of $HH^{p-q+1-2r}(A,A)$. Moreover, d_r is of topological degree $+1$ as a map between subquotients of $HH^*(A,A)$.

The first step in computing $HC^*_F(A)$ with these methods is to compute the differential $d_1 = \Delta$. Recall that, through the work of Gerstenhaber [8], $HH^*(A,A)$ is equipped with a product $\cdot$ and a Lie bracket $[\cdot,\cdot]$ which make it into a Gerstenhaber algebra. These operations interact with Δ in the following fashion:

Theorem 5.3 *If A is a Frobenius algebra, the operations $\cdot$, $[\cdot,\cdot]$ and Δ make $HH^*(A,A)$ into a Batalin–Vilkovisky algebra; that is, Δ satisfies the BV formula:*

$$\Delta(xy) = \Delta(x)y + (-1)^{|x|}x\Delta(y) + (-1)^{|x|}[x,y]$$

This is [12, Corollary 3.8], which Kaufmann describes as a folk theorem. See also [19, Corollary 3.4] or [17, Theorem 3.1] for a different approach to the cyclic Deligne conjecture which yields the same BV–structure.

Our interest is in the Frobenius algebra $A = H^*(M)$. To compute the E_2 term of Connes' spectral sequence, we must determine the action of the operator Δ on $HH^*(H^*(M), H^*(M))$. This is accomplished in the following section. For multiplicative generators of $HH^*(H^*(M), H^*(M))$, the action of Δ is determined through direct computations or filtration arguments. It is extended to all of the Hochschild cohomology by the BV formula and a computation of the bracket from [21].

It is worth emphasizing that we are using a particular BV algebra structure on $HH^*(A,A)$, where the product is the cup product of Hochschild cochains, the bracket is the Gerstenhaber bracket, and Δ is the map induced by $\check{B}$. An alternate BV algebra structure (which we will not use) employs Chas–Sullivan's string topology operations; these make $\Sigma^{-n}H_*(LM)$ a BV–algebra. Recall Cohen–Jones' isomorphism [6]:

$$HH^*(C^*(M), C^*(M)) \cong H_*(LM^{-TM}) \cong \Sigma^{-n}H_*(LM)$$

and the isomorphism $HH^*(C^*(M), C^*(M)) \cong HH^*(H^*(M), H^*(M))$ (of Gerstenhaber algebras) for formal manifolds (this uses the work of Felix–Menichi–Thomas [7]). One may thereby compare the BV structures coming from the cyclic Deligne conjecture and string topology. A recent paper by Menichi [16] implies that they are not equivalent, even for manifolds as simple as S^2.

6 A computation of the E_2 term of Connes' spectral sequence

We recall from [21] the following computations:

(1) If $k > 1$, $HH^*(H^*(S^k), H^*(S^k))$ is isomorphic as an algebra to $\mathbb{F}_2[x, v]/(x^2)$, where the dimensions of x and v are $-k$ and $k-1$, respectively. The Gerstenhaber bracket is given by $[x, v] = 1$.

(2) Let K be one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and let $d = \dim_{\mathbb{R}}(K)$. For n odd (and greater than 1 if $K = \mathbb{R}$),

$$HH^*(H^*(KP^n), H^*(KP^n)) = \mathbb{F}_2[x, v, t]/(x^{n+1}, v^2 - \frac{n+1}{2}tx^{n-1})$$

and for n even,

$$HH^*(H^*(KP^n), H^*(KP^n)) = \mathbb{F}_2[x, u, t]/(x^{n+1}, u^2, tx^n, ux^n)$$

where the topological dimensions of x, u, v, and t are $-d$, -1, $d-1$, and $d(n+1)-2$ respectively. Their Hochschild degrees are $0, 1, 1$, and 2. The bracket is given on generators by

$$[x, v] = 1,\ [x, u] = x,\ [x, t] = 0,\ [v, t] = 0,\ [u, t] = t.$$

In general, we will use the notation M to refer to any of the manifolds S^k $(k > 1)$, $\mathbb{R}P^n$ $(n > 1)$, $\mathbb{C}P^n$, or $\mathbb{H}P^n$.

Lemma 6.1 *Δ vanishes on algebra generators of $HH^*(H^*(M), H^*(M))$.*

Corollary 6.2

(1) *In $HH^*(H^*(S^k), H^*(S^k))$, $\Delta(x^a v^b) = abx^{a-1}v^{b-1}$.*

(2) *Depending upon the parity of n, monomials in $HH^*(H^*(KP^n), H^*(KP^n))$ may be written as $x^a v^b t^c$ or $x^a u^b t^c$ (where $b = 0, 1$). Then:*

$$\Delta(x^a v^b t^c) = abx^{a-1}t^c \ \text{ and } \ \Delta(x^a u^b t^c) = (a+c)bx^a t^c$$

Proof Assuming Lemma 6.1 we will prove part (1). Part (2) is somewhat tedious and proved in the same fashion.

First notice that from the Leibniz formula

$$[\alpha, \beta\gamma] = [\alpha, \beta]\gamma + [\alpha, \gamma]\beta,$$

it follows that

$$(*) \qquad [\alpha, \beta^p] = p[\alpha, \beta]\beta^{p-1}.$$

So taking $a = 0$ in part (1), we know that

$$\Delta(v^b) = \Delta(v)v^{b-1} + v\Delta(v^{b-1}) + [v, v^{b-1}]$$

The first term is 0 by Lemma 6.1, and the third by $(*)$. Therefore $\Delta(v^b) = 0$ by induction. So

$$\begin{aligned}\Delta(xv^b) &= \Delta(x)v^b + x\Delta(v^b) + [x, v^b] \\ &= b[x, v]v^{b-1} \\ &= bv^{b-1}\end{aligned}$$

Part (1) follows. □

Proof of Lemma 6.1 The operator Δ (induced by $\check{B}$) lowers the Hochschild degree by 1: if $\alpha \in HH^p(R, R)$, $\Delta(\alpha) \in HH^{p-1}(R, R)$. Automatically, we thereby obtain

$$\Delta(x) = 0,$$

since $x \in HH^0(H^*(M), H^*(M))$.

The element v has Hochschild degree 1 and topological degree $d - 1$, where d is as above if $M = KP^n$ and $d = k$ if $M = S^k$. Therefore $\Delta(v)$ has Hochschild degree 0 and topological degree $d > 0$. Since there are no elements in $HH^0(H^*(M), H^*(M))$ of positive topological degree, $\Delta(v) = 0$.

Similarly, $\Delta(t) \in HH^1(H^*(KP^n), H^*(KP^n))$ has topological degree $d(n+1) - 1$. If nonzero, $\Delta(t)$ may be written as

$$\Delta(t) = \begin{cases} x^k v & \text{if } n \text{ is odd,} \\ x^k u & \text{if } n \text{ is even.} \end{cases}$$

In the first case, the topological degree of $x^k v$ is $-kd + d - 1$, so we must have $k = -n < 0$, which is impossible. Similarly, if n is even, $k = -n - 1 < 0$.

Finally, to show that $\Delta(u) = 0$ we use the description of Δ as induced by the $\check{B}$ operator. In [21] we found that a representative for the class u is the function $\overline{u} \in CH^1(\mathbb{F}_2[x]/x^{n+1}, \mathbb{F}_2[x]/x^{n+1})$ given by

$$\overline{u}\colon x^m \mapsto mx^m$$

hence:

$$\begin{aligned}\widetilde{\check{B}(\overline{u})}(x^m) &= \langle 1, \overline{u}(x^m)\rangle + \langle x^m, \overline{u}(1)\rangle \\ &= m\langle 1, x^m\rangle + \langle x^m, 0\rangle \\ &= m \cdot x^m([KP^n])\end{aligned}$$

For $x^m([KP^n])$ to be nonzero, $m = n$. Since n is even, the product is zero. □

Proof of Theorem 1.1 Using Corollary 6.2 we will compute the Poincaré series of the E_2 term of Connes' spectral sequence. Lemma 7.1 then gives us Theorem 1.1. For brevity, we only do this computation for $M = KP^{2n+1}$. The computations for other manifolds are similar; the case for spheres is easier, the case for even projective spaces is more tedious.

Write $L = E_1^{0,*} = HH^{-*}(H^*(M), H^*(M))$ for the first column of the E_1 term of spectral sequence. Recall that $E_1^{p,q} = HH^{p-q}(H^*(M), H^*(M))$, so that the p^{th} column of the spectral sequence is $E_1^{p,*} = \Sigma^p L$. Therefore the E_1 term of the spectral sequence may be written

$$L \xleftarrow{\Delta} \Sigma L \xleftarrow{\Delta} \Sigma^2 L \xleftarrow{\Delta} \cdots$$

where each L is a column, and we are using the identification $d_1 = \Delta$ from Proposition 5.1.

Examine the action of the operator Δ on L. There are three types of classes:

(1) Classes a for which $\Delta(a) \neq 0$. We say these "survive alone."

(2) Classes b for which $\Delta(b) = 0$ and there is a class b' with $\Delta(b') = b$. We say these are "hit."

(3) Classes c for which $\Delta(c) = 0$ that are not in the image of Δ. We say these classes "propagate a stripe."

Classes a which survive alone give rise to an element of $E_2^{0,*}$ (of topological degree $|a|$). Classes b which are hit do not give rise to any element of E_2. Classes c which propagate a stripe give a class in $E_2^{p,*}$ (of topological degree $|c| + 2p$) for each $p \geq 0$.

Examining Corollary 6.2, we see that for every k and c, $x^{2k+1}vt^c$ survives alone, $x^{2k}vt^c$, and $x^{2k+1}t^c$ propagate a stripe, and all other monomials are hit.

The dimension of $x^{2k+1}vt^c$ is

$$|x^{2k+1}vt^c| = -1 + k(-2d) + c(d(2n+2) - 2)$$

so the Poincaré series of the space that they span is:

$$t^{-1} \sum_{k=0}^{n} \sum_{c=0}^{\infty} t^{k(-2d)} t^{c(d(2n+2)-2)} = t^{-1} \left(\frac{1 - t^{-2d(n+1)}}{1 - t^{-2d}} \right) \left(\frac{1}{1 - t^{d(2n+2)-2}} \right)$$

This is exactly the contribution to $E_2^{0,*}$ of the elements that survive alone.

The dimension of $x^{2k}vt^c$ is

$$|x^{2k}vt^c| = (2k)(-d) + d - 1 + c(d(2n+2) - 2).$$

Similarly, the Poincaré series of the family $\langle x^{2k} v t^c \rangle$ is

$$t^{d-1}\left(\frac{1-t^{-2d(n+1)}}{1-t^{-2d}}\right)\left(\frac{1}{1-t^{d(2n+2)-2}}\right)$$

and for the family $\langle x^{2k+1} t^c \rangle$:

$$t^{-d}\left(\frac{1-t^{-2d(n+1)}}{1-t^{-2d}}\right)\left(\frac{1}{1-t^{d(2n+2)-2}}\right)$$

Each element of dimension q in the latter two families gives rise to a sequence of classes in $E_2^{p,*}$ of dimension $q+2p$ for each $p \geq 0$. So in counting their contribution to E_2, we must multiply the answer by

$$\sum_{k=0}^{\infty} t^{2k} = \frac{1}{1-t^2}$$

Adding these three series gives us the Poincaré series for E_2:

$$\left(t^{-1} + \frac{t^{d-1}+t^{-d}}{1-t^2}\right)\left(\frac{1-t^{-2d(n+1)}}{1-t^{-2d}}\right)\left(\frac{1}{1-t^{d(2n+2)-2}}\right)$$

Recall that the spectral sequence computes a *desuspension* of $H_*^{S^1}(LM)$; one needs to multiply this series by $t^{\dim M} = t^{d(2n+1)}$ to get the correct answer. □

7 Collapse of the spectral sequence

We complete the proof of Theorem 1.1 with the following result:

Lemma 7.1 *For the manifolds M considered in this paper, Connes' spectral sequence for $HC_F^*(H^*(M))$ collapses at the E_2–term.*

Proof From Lemma 5.2 we know that all differentials d_r in the spectral sequence are of the form

$$\begin{array}{ccc} \text{subquotient} & & \text{subquotient} \\ \text{of} & \longrightarrow & \text{of} \\ HH^k(H^*(M), H^*(M)) & & HH^{k+1-2r}(H^*(M), H^*(M)) \end{array}$$

of topological degree $+1$. We will show that if $r > 1$, such a map is 0 by examining the range of topological degrees of the source and target. For simplicity, we take $M = KP^n$ with n odd; the proofs for even projective spaces and spheres are similar.

If $k=2l$ is even, then we showed in [21] that $HH^k(H^*(M),H^*(M))$ is concentrated in topological degrees

$$l(d(n+1)-2)-jd;\ j\in\{0,\dots,n\},$$

and if $k=2l+1$ is odd, it lies in dimensions

$$l(d(n+1)-2)+d-1-jd;\ j\in\{0,\dots,n\}.$$

Consider d_r as a mapping from a subquotient of HH^{2l} to a subquotient of $HH^{2l+1-2r}$, and let α lie in the domain. The smallest possible dimension for α is

$$l(d(n+1)-2)-nd=(l-1)nd+ld-2l$$

(realized by the element x^nt^l) and therefore, the dimension of $d_r(\alpha)$ is

$$|d_r(\alpha)|=|\alpha|+1\geq(l-1)nd+ld-2l+1.$$

We claim that this is larger than the dimension of any element in the range. The largest possible dimension in the range is

$$\begin{aligned}(l-r)(d(n+1)-2)+d-1 &= lnd+ld-2l+(-rnd-rd+2r+d-1)\\ &= (l-1)nd+ld-2l+1+(r-1)(2-d-nd)\end{aligned}$$

(realized by the element vt^{l-r}). Since $d,n\geq 1$, but not $d=n=1$ (in which case we would be considering $\mathbb{R}P^1=S^1$), and $r>1$,

$$(r-1)(2-d-nd)<0$$

so $|d_r(\alpha)|>|\beta|$ for every $\alpha\in HH^{2l}$ and $\beta\in HH^{2l+1-2r}$.

To check that d_r is 0 as a map from a subquotient of HH^{2l+1} to a subquotient of $HH^{2l+2-2r}$ takes only a little more work. First we notice that the element of the domain of lowest dimension, x^nvt^l does not lie in E_2, since

$$\Delta(x^nvt^l)=x^{n-1}t^l\neq 0$$

so the smallest dimension of an element α for which $d_r(\alpha)$ might be nonzero is

$$l(d(n+1)-2)+d-1-d(n-1)$$

corresponding to $x^{n-1}vt^l$. The class of largest possible dimension in the range of d_r is t^{l+1-r}, of dimension $(l+1-r)(d(n+1)-2)$. The difference in dimension between $d_r(\alpha)$ and the largest possible target is then

$$|t^{l+1-r}|-|d_r(\alpha)|\leq|t^{l+1-r}|-|d_r(x^{n-1}vt^l)|=(2dn-d-2)-r(dn+d-2).$$

Since $r \geq 2$,

$$\begin{aligned} |t^{l+1-r}| - |d_r(\alpha)| &= (2dn-d-2) - r(dn+d-2) \\ &\leq (2dn-d-2) - 2(dn+d-2) \\ &= 2-3d \\ &< 0 \end{aligned}$$

since $d > 1$. So again, $|d_r(\alpha)| > |\beta|$ for every $\alpha \in HH^{2l+1}$ and $\beta \in HH^{2l+2-2r}$. □

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