# The $D(2)$ property for $D_8$

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Wall's $D(2)$ problem asks if a cohomologically 2–dimensional geometric 3–complex is necessarily homotopy equivalent to a geometric 2–complex. We solve part of the problem when the fundamental group is dihedral of order $2^n$ and give a complete solution for the case where it is $D_8$ the dihedral group of order 8.

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## 1 Introduction

Wall introduced the $D(2)$ problem in [5]. This asks if a cohomologically 2–dimensional geometric 3–complex is necessarily homotopy equivalent to a geometric 2–complex. The answer depends only on the fundamental group and we say that a group *has the $D(2)$ property* if the answer is "yes" for complexes with this fundamental group. The $D(2)$ property has been verified for dihedral groups of order $4n+2$ by Johnson [2]. Therefore we concentrate on dihedral groups of order $4n$. Since these do not have periodic resolutions, not all the methods of [2] can be applied to them. Our main result is orthogonal to the result of Johnson in [3], in the sense that it concerns dihedral groups whose order is a power of 2, rather than twice an odd number.

We begin by recalling some of the theory of $k$–invariants. We work over a finite group $G$ of order $n$.

**Definition 1.1** (Algebraic complex) We define an algebraic $n$–complex, to be a sequence of maps and modules:

$$F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_1 \xrightarrow{d_1} F_0$$

where the $F_i$ are free finitely generated modules over $\mathbb{Z}[G]$, the cokernel of $d_1$ is $\mathbb{Z}$ (with trivial $G$ action) and the sequence is exact at $F_1$.

Let $(F_i, d_i)$ and $(F'_i, d'_i)$, $i=0,1,2$, denote algebraic 2–complexes. Suppose given $f_i\colon F_i \to F'_i$, $i=0,1,2$, which constitute a chain map $f$ between them.

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**Proposition 1.2** [3, Proposition 47.1] *$f$ is a homotopy equivalence if and only if it induces isomorphisms $\ker(d_2) \to \ker(d_2')$ and $\mathrm{coker}(d_1) \to \mathrm{coker}(d_1')$.*

**Definition 1.3** (Algebraic $\pi_2$) We define $\pi_2(F_i, d_i)$ to be $\ker(d_2)$.

Let $J$ denote the kernel of $d_2$ and let $J'$ denote the kernel of $d_2'$.

**Proposition 1.4**

(i) *Given $\alpha\colon J \to J'$, we may choose a chain map $f_\alpha\colon (F_i, d_i) \to (F_i', d_i')$ which induces $\alpha\colon J \to J'$. A map $\mathbb{Z} \to \mathbb{Z}$, is induced on the cokernels. Suppose that this map is given by multiplication by $k$.*

(ii) *The congruence class of $k$ modulo $n$ is independent of the choice of $f_\alpha$.*

(iii) *Given $k'$ congruent to $k$ modulo $n$, we may choose a chain map $f_\alpha'$, which also induces $\alpha\colon J \to J'$, and which induces multiplication by $k'$ on $\mathbb{Z}$.*

**Proof** (i) See [3, Proposition 25.3].

(ii) See [3, Propositions 25.3 33.3].

(iii) Let $\epsilon\colon F_0 \to F_0/\mathrm{Im}(d_i) \cong \mathbb{Z}$, $\epsilon'\colon F_0' \to F_0'/\mathrm{Im}(d_i') \cong \mathbb{Z}$ denote the natural quotient maps. Pick $x \in F_0'$ such that $\epsilon' x = 1$. Let $h\colon \mathbb{Z} \to F_0'$ be the map sending $1 \in \mathbb{Z}$ to $\sum_{g\in G} xg$. Then

$$\epsilon' h(1) = \epsilon'(\sum_{g\in G} xg) = \sum_{g\in G} (\epsilon' x)g = \sum_{g\in G} 1 = n$$

Let
$$(f_\alpha')_2 = (f_\alpha)_2$$
$$(f_\alpha')_1 = (f_\alpha)_1$$
$$(f_\alpha')_0 = (f_\alpha)_0 + (\frac{k'-k}{n})h\epsilon.$$

Then $f_\alpha'$ is a chain map since:

$$(f_\alpha')_0 d_1 = (f_\alpha)_0 d_1 + (\frac{k'-k}{n})h\epsilon d_1 = (f_\alpha)_0 d_1 + 0 = d_1'(f_\alpha)_1 = d_1'(f_\alpha')_1$$

Finally note: $\epsilon'(f_\alpha')_0 = \epsilon'(f_\alpha)_0 + \epsilon'(\frac{k'-k}{n})h\epsilon = k\epsilon + (k'-k)\epsilon = k'\epsilon$ □

**Definition 1.5** ($k$–invariant) Given $\alpha$ as in the proposition, we define $k_\alpha$ to be the congruence class of $k$ modulo $n$.

We have a ring homomorphism $\kappa\colon \mathrm{End}(J) \to \mathbb{Z}_n$ defined by $\alpha \mapsto k_\alpha$.

**Lemma 1.6** [3, Proposition 26.6] *The kernel of $\kappa$ comprises all maps which factor through a projective module.*

**Lemma 1.7** [3, Proposition 33.7] *$\kappa$ is independent of the choice of algebraic complex $(F_i, d_i)$.*

**Proof** $\kappa_1 = 1$, so $\kappa$ is surjective. Hence $\kappa$ is equal to the quotient map $\mathrm{End}(J) \to \mathrm{End}(J)/\mathrm{Ker}(\kappa)$ composed with a ring isomorphism $\mathbb{Z}_n \to \mathbb{Z}_n$. However, any ring isomorphism $\mathbb{Z}_n \to \mathbb{Z}_n$ must map $1 \mapsto 1$. Hence it must be the identity. □

**Definition 1.8** (Swan map) The Swan map is the homomorphism $\mathrm{Aut}(J) \to \mathbb{Z}_n^*$ which sends an automorphism to its $k$–invariant.

**Proposition 1.9** *If the Swan map $\mathrm{Aut}(J) \to \mathbb{Z}_n^*$ is surjective and we have an isomorphism $\alpha\colon J \to J'$, then $(F_i, d_i)$ and $(F_i', d_i')$ are chain homotopy equivalent.*

**Proof** By surjectivity we may choose $\beta\colon J \to J$, such that $k_\beta = k_\alpha^{-1}$. Then by Proposition 1.4(iii), we may pick $f_{\alpha\beta}$ which induces isomorphisms $J \to J'$ and the identity $\mathbb{Z} \to \mathbb{Z}$. Hence by Proposition 1.2, $f_{\alpha\beta}$ is a homotopy equivalence. □

**Lemma 1.10** *Given a map $\alpha\colon J \to J$, let $\alpha'\colon J \oplus \mathbb{Z}[G] \to J \oplus \mathbb{Z}[G]$ denote the map $\alpha \oplus 1$. Then $k_\alpha = k_{\alpha'}$.*

Hence it is sufficient to show that the Swan map is surjective for $J$, in order to deduce that it is surjective for $J \oplus \mathbb{Z}[G]^r$, for all natural numbers $r$. Consequently we have:

**Proposition 1.11** *If the Swan map is surjective for $J$, then, for each $r$, there is an algebraic 2–complex, unique up to chain homotopy equivalence, with algebraic $\pi_2$ equal to $J \oplus \mathbb{Z}[G]^r$.*

Here is an outline of the rest of the paper.

For $n$ coprime to 3, we will show that, in the case $G = \mathbb{Z}[D_{4n}]$, the unit $3 \in \mathbb{Z}_{4n}$ is in the image of the Swan map for $J$, where $J$ is the algebraic $\pi_2$ of a particular algebraic 2–complex. We will then show that $-1$ and $3$ generate the units of $\mathbb{Z}_{2^n}$, so the Swan map is surjective for $J$, for dihedral groups of order $2^n$. Thus, for each $r$, there is a algebraic 2–complex, unique up to chain homotopy equivalence, with algebraic $\pi_2$ equal to $J \oplus \mathbb{Z}[D_{4n}]^r$.

We then show that $J$ has minimal $\mathbb{Z}$–rank, for a module which occurs as an algebraic $\pi_2$. We use a cancellation result due to Swan [4] to show that for the group $D_8$, the only modules which arise as an algebraic $\pi_2$ of an algebraic 2–complex, are of the form $J \oplus \mathbb{Z}[D_{4n}]^r$. We have shown by this point, that, up to chain homotopy equivalence, there is only one algebraic 2–complex which has each of these algebraic $\pi_2$'s. We show each of these are geometrically realized.

Finally, we quote [3, Theorem I] which states that if every algebraic 2–complex over a finite group $G$ is geometrically realized, then $G$ satisfies the $D(2)$ property.

## 2 Surjectivity of the Swan map

Let $D_{4n}$ be the group given by the presentation, $\langle a, b \mid a^{2n} = b^2 = e,\ aba = b\rangle$. $\Sigma$ will denote $\sum_{i=0}^{2n-1} a^i$. This presentation has a Cayley complex, which in turn has an associated algebraic complex. This is an exact sequence over $\mathbb{Z}[D_{4n}]$:

$$J \hookrightarrow \mathbb{Z}[D_{4n}]^3 \xrightarrow{\partial_2} \mathbb{Z}[D_{4n}]^2 \xrightarrow{\partial_1} \mathbb{Z}[D_{4n}] \overset{\epsilon}{\twoheadrightarrow} \mathbb{Z} \tag{1}$$

$\epsilon$ is determined by mapping $1 \in \mathbb{Z}[D_{4n}]$ to $1 \in \mathbb{Z}$. $J$ is the kernel of $\partial_2$. Let $e_1$, $e_2$ denote basis elements of $\mathbb{Z}[D_{4n}]^2$. Then $\partial_1 e_1 = a - 1,\ \partial_1 e_2 = b - 1$.

Let $E_1,\ E_2,\ E_3$ be basis elements of $\mathbb{Z}[D_{4n}]^3$, which correspond to the relations in the presentation so that:

$$\begin{aligned}
\partial_2 E_1 &= e_1 \Sigma \\
\partial_2 E_2 &= e_2(1+b) \\
\partial_2 E_3 &= e_1 + e_2 a + e_1 ba - e_2 = e_1(1+ba) + e_2(a-1)
\end{aligned}$$

With respect to the basis $\{E_1,\ E_2,\ E_3\}$ and the basis $\{e_1,\ e_2\}$, $\partial_2$ is given by:

$$\begin{bmatrix} \Sigma & 0 & 1+ba \\ 0 & 1+b & a-1 \end{bmatrix}$$

Let
$$\begin{aligned}
\alpha_0 &= 1 + a + b \\
\alpha_1 &= \begin{bmatrix} 1+a-ba & b-1 \\ 0 & 1 \end{bmatrix} \\
\alpha_2 &= \begin{bmatrix} 1+a-ba & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

The following result is easily verified.

**Proposition 2.1** *The following diagram commutes*

$$\begin{array}{ccccccccc} J & \hookrightarrow & \mathbb{Z}[D_{4n}]^3 & \xrightarrow{\partial_2} & \mathbb{Z}[D_{4n}]^2 & \xrightarrow{\partial_1} & \mathbb{Z}[D_{4n}] & \stackrel{\epsilon}{\twoheadrightarrow} & \mathbb{Z} \\ \downarrow\theta & & \downarrow\alpha_2 & & \downarrow\alpha_1 & & \downarrow\alpha_0 & & \downarrow 3 \\ J & \hookrightarrow & \mathbb{Z}[D_{4n}]^3 & \xrightarrow{\partial_2} & \mathbb{Z}[D_{4n}]^2 & \xrightarrow{\partial_1} & \mathbb{Z}[D_{4n}] & \stackrel{\epsilon}{\twoheadrightarrow} & \mathbb{Z} \end{array}$$

*where $\theta$ is the restriction of $\alpha_2$.*

For the remainder we will assume $3$ coprime to $n$. Our goal is to show that $\theta$ is an isomorphism. As we know that $\kappa_\theta = 3$, this will suffice to show that 3 is in the image of the Swan map.

Note that, if we regard the above diagram as a diagram of commutative $\mathbb{Z}$–modules and $\mathbb{Z}$–linear maps, there are well defined integer determinants for all the maps in the chain map. A map is an isomorphism if and only if it has determinant $\pm 1$. (As the property of being an isomorphism is dependent only on surjectivity and injectivity, it does not depend on whether we are regarding modules as being over $\mathbb{Z}[D_{4n}]$, or $\mathbb{Z}$).

Note also that, over $\mathbb{Z}$, all the maps in the exact sequences above are given by quotienting a summand, followed by inclusion of a summand. Consequently, the following proposition holds:

**Proposition 2.2** $3\mathrm{Det}(\theta)\mathrm{Det}(\alpha_1) = \mathrm{Det}(\alpha_2)\mathrm{Det}(\alpha_0)$

**Proof** Let $u$ be the restriction of $\alpha_1$ to the kernel of $\partial_1$ and let $v$ be the restriction of $\alpha_0$ to the kernel of $\epsilon$. Then by the previous discussion, we have

$$3\mathrm{Det}(\theta)\mathrm{Det}(\alpha_1) = \mathrm{Det}(\theta)\mathrm{Det}(u)\mathrm{Det}(v)3 = \mathrm{Det}(\alpha_2)\mathrm{Det}(\alpha_0). \qquad \square$$

We will use this to show that $\mathrm{Det}(\theta) = 1$.

**Proposition 2.3** $\mathrm{Det}(1 + a + b) = -3$

**Proof** Let $A$ be the matrix for left multiplication by $1 + a + b$ in the regular representation, with basis $\{a^{2n-1}, a^{2n-2}, \ldots, a, 1, ba^{2n-1}, ba^{2n-2}, \ldots, ba, b\}$. Then the upper right quadrant of $A$ and the lower left quadrant of $A$ are copies of the identity matrix. The upper left quadrant has 1's along the diagonal and immediately above as well as a 1 in the bottom left corner. The lower right quadrant has 1's along the diagonal and immediately below, as well as a 1 in the top right corner. All the other entries in $A$ are $0$.

For example, if $n$ were equal to $4$, the matrix $A$ would be:

$$\begin{bmatrix}
1&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1\\[1ex]
1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1
\end{bmatrix}$$

Label the rows of $A$, $v_1, v_2 .....v_{4n}$. We will perform row operations.

First let $v'_{2n} = v_{2n} - v_1 + v_2 - v_3 .... - v_{2n-1}$. Now let $v''_{2n} = v_{4n}$ and $v''_{4n} = v'_{2n}$. Let the remaining $v''_i = v_i$. This swap causes a change of sign in the determinant, so the matrix with rows $v''_i$ has determinant $-\mathrm{Det}A$. In the case $n = 4$, this matrix is:

$$\begin{bmatrix}
1&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1\\[1ex]
1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&-1&1&-1&1&-1&1&-1&1
\end{bmatrix}$$

For each $2n+1 \leq i \leq 4n-2$, let $v_i''' = v_i'' + v_{i+1}'' - v_{i-2n}''$. Let $v_{4n-1}''' = v_{4n-1}'' + v_{2n}'' - v_{2n-1}''$ and for $i \leq 2n$ let $v_i''' = v_i''$.

When $n = 4$, the matrix with rows $v_i'''$ is:

$$\begin{bmatrix}
1&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1\\
\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&-1&1&-1&1&-1&1&-1&1
\end{bmatrix}$$

In general, the matrix with rows $v_i'''$ has an upper triangular top left quadrant, with 1's along the diagonal and a lower left quadrant with no non-zero entries. Let $B$ denote the lower right quadrant. Then $\mathrm{Det}(1+a+b) = -\mathrm{Det}(B)$.

Cycle the top $2n-1$ rows of $B$ upwards to get the matrix $B'$. As this is a cycle of odd length, $\mathrm{Det}(B') = \mathrm{Det}(B)$. When $n = 4$, $B'$ is:

$$\begin{bmatrix}
1&1&1&0&0&0&0&0\\
0&1&1&1&0&0&0&0\\
0&0&1&1&1&0&0&0\\
0&0&0&1&1&1&0&0\\
0&0&0&0&1&1&1&0\\
0&0&0&0&0&1&1&1\\
1&1&0&0&0&0&0&1\\
-1&1&-1&1&-1&1&-1&1
\end{bmatrix}$$

Label the rows of $B'$ as $w_1, \dots, w_{2n}$. Set $u_i = w_i - w_{i+1}$ for $i = 1, 2, \dots, 2n-3$. Let $B''$ denote the matrix with rows $u_i$. After these row operations, we have $\mathrm{Det}(1+a+b) = -\mathrm{Det}(B'')$.

When $n = 4$, $B''$ is:

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

We must consider two cases: $n$ congruent to 1 modulo 3 and $n$ congruent to 2 modulo 3.

If $n = 1$ modulo 3 then replace $u_{2n-1}$ with

$$u_{2n-1} - u_1 - u_2 - u_4 - u_5 - u_7 - u_8 \cdots - u_{2n-3}.$$

Also, replace $u_{2n}$ with

$$u_{2n} + (u_1 - u_2 + u_3) + (u_7 - u_8 + u_9) + (u_{13} - u_{14} + u_{15}) \cdots + (u_{2n-7} - u_{2n-6} + u_{2n-5}).$$

We are left with a matrix with 1's along the diagonal and 0's below, except in the last four columns. The 4 by 4 matrix in the bottom right corner is:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\mathrm{Det}(1 + a + b) = -\mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix} = -\mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} = -3$$

If $n = 2$ modulo 3 then replace $u_{2n-1}$ with

$$u_{2n-1} - u_1 - u_2 - u_4 - u_5 - u_7 - u_8 \cdots - u_{2n-5}.$$

Also, replace $u_{2n}$ with

$$u_{2n} + (u_1 - u_2 + u_3) + (u_7 - u_8 + u_9) + (u_{13} - u_{14} + u_{15}) \cdots + (u_{2n-9} - u_{2n-8} + u_{2n-7}).$$

We are left with a matrix with 1's along the diagonal and 0's below, except in the last four columns. The 4 by 4 matrix in the bottom right corner is:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

$$\mathrm{Det}(1+a+b) = -\mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} = -\mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$= -\mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & -1 \end{bmatrix} = -\mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} = -3$$ □

**Proposition 2.4** $\mathrm{Det}(2-b) = 3^{2n}$

**Proof** Let $A$ be the matrix for $2-b$ in the regular representation, with basis $\{1, b, a, ba, a^2, ba^2, \dots, a^{2n-1}, ba^{2n-1}\}$. Then $A$ consists of $2n$ two by two blocks of the form $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ along the diagonal. Hence $\mathrm{Det}(A) = 3^{2n}$. □

**Proposition 2.5** $\mathrm{Det}(1+a-ba) \neq 0$

**Proof** Let $\alpha'_2 = \begin{bmatrix} 2-b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The following diagram commutes:

$$\begin{array}{ccccccccc} J & \hookrightarrow & \mathbb{Z}[D_{4n}]^3 & \xrightarrow{\partial_2} & \mathbb{Z}[D_{4n}]^2 & \xrightarrow{\partial_1} & \mathbb{Z}[D_{4n}] & \overset{\epsilon}{\twoheadrightarrow} & \mathbb{Z} \\ \downarrow\eta & & \downarrow\alpha'_2 & & \downarrow\alpha_1 & & \downarrow\alpha_0 & & \downarrow 3 \\ J & \hookrightarrow & \mathbb{Z}[D_{4n}]^3 & \xrightarrow{\partial_2} & \mathbb{Z}[D_{4n}]^2 & \xrightarrow{\partial_1} & \mathbb{Z}[D_{4n}] & \overset{\epsilon}{\twoheadrightarrow} & \mathbb{Z} \end{array}$$

where $\eta$ is the restriction of $\alpha'_2$ Therefore $3\mathrm{Det}(\eta)\mathrm{Det}(\alpha_1) = \mathrm{Det}(\alpha'_2)\mathrm{Det}(\alpha_0)$.

So $3 * \mathrm{Det}(\eta)\mathrm{Det}(1+a-ba) = -3 * 3^{2n}$. Hence $\mathrm{Det}(1+a-ba)$ cannot be $0$. □

**Proposition 2.6** *$\theta$ is an isomorphism.*

**Proof** We have $3\mathrm{Det}(\theta)\mathrm{Det}(\alpha_1) = \mathrm{Det}(\alpha_2)\mathrm{Det}(\alpha_0)$. Therefore

$$3\mathrm{Det}(\theta)\mathrm{Det}(1+a-ba) = -3\mathrm{Det}(1+a-ba).$$

As $\mathrm{Det}(1+a-ba)$ is non-zero, we can conclude that $\mathrm{Det}(\theta) = -1$. Hence $\theta$ is an isomorphism. □

**Corollary 2.7** *If $3 \in (\mathbb{Z}_{4n})^*$ then $3$ is in the image of the Swan Map:* $\mathrm{Aut(J)} \to (\mathbb{Z}_{4n})^*$.

Let us now consider dihedral groups of order $2^m$ for $m \geq 2$. Clearly $2^m$ is divisible by $4$ and coprime to $3$. Hence we know that $3$ is in the image of the Swan Map.

**Lemma 2.8** *$2^m$ divides $3^{2^{m-3}} - 1 + 2^{m-1}$ for $m \geq 4$.*

**Proof** We proceed by induction. $3^{2^{4-3}} - 1 + 2^{4-1} = 16$. So the proposition holds for $m = 4$. Now suppose it holds for some $m$. Then $2^m z = 3^{2^{m-3}} - 1 + 2^{m-1}$ for some $z$. Rearranging gives $3^{2^{m-3}} = 1 - 2^{m-1} + 2^m z$. Then squaring gives:

$$3^{2^{m+1-3}} = (3^{2^{m-3}})^2 = (2^m z + 1 - 2^{m-1})^2$$

So
$$3^{2^{m+1-3}} - 1 + 2^{m+1-1} = (2^m z + 1 - 2^{m-1})^2 - 1 + 2^m$$
$$= 2^{2m} z^2 + 2^{2m-2} + 2^{m+1} z - 2^{2m} z = 2^{m+1}(2^{m-1}(z^2 - z) + 2^{m-3} + z).$$

So the proposition holds for $m+1$. Hence by induction it holds for all $m \geq 4$. □

**Proposition 2.9** *The elements $3, -1$ generate $(\mathbb{Z}/2^m)^*$ for $m \geq 2$.*

**Proof** The order of $(\mathbb{Z}/2^m)^*$ is $2^{m-1}$. $(\mathbb{Z}/4)^* = \{1, 3\}$ and $(\mathbb{Z}/8)^* = \{1, -1, 3, -3\}$, so only the case $m \geq 4$ remains. We know that the order of $3$ in $(\mathbb{Z}/2^m)^*$ is a power of $2$. The previous lemma shows us that for $m \geq 4$ it is at least $2^{m-2}$, as

$$3^{2^{m-3}} \equiv 1 + 2^{m-1} \bmod 2^m.$$

It remains to show that $-1$ is not a power of $3$, as then the $\pm 3^k$ give us all $2^{m-1}$ elements of $(\mathbb{Z}/2^m)^*$.

Suppose $3^k = -1 \bmod 2^m$ for some $m \geq 4$. Then $3^k = -1 \bmod 8$ which is impossible as $3^k$ only takes the values $1$ and $3$ modulo $8$. □

Combining this result with Corollary 2.7 we obtain:

**Corollary 2.10** *The Swan Map $\mathrm{Aut}(J) \to (\mathbb{Z}_{2^m})^*$ is surjective for all $m \geq 2$.*

From Proposition 1.11, we may conclude:

**Theorem 2.11** *Over $\mathbb{Z}[D_{2^m}]$ an algebraic 2–complex $X$ with $\pi_2(X) = J \oplus \mathbb{Z}[D_{2^m}]^r$ is unique up to chain homotopy equivalence.*

## 3 The $D(2)$ property for $\mathbb{Z}[D_8]$

Let $\mathbb{F}_2$ denote the two element module over $\mathbb{Z}[D_{4n}]$, on which the action of $\mathbb{Z}[D_{4n}]$ is trivial.

**Proposition 3.1** [1, page 127]

(i) $H^0(D_{4n}, \mathbb{F}_2) = \mathbb{F}_2$

(ii) $H^1(D_{4n}, \mathbb{F}_2) = \mathbb{F}_2^2$

(iii) $H^2(D_{4n}, \mathbb{F}_2) = \mathbb{F}_2^3$

Recall the sequence (1), from Section 2. By Schanuel's lemma, any module occurring as the algebraic $\pi_2$ of an algebraic 2–complex, over $\mathbb{Z}[D_{4n}]$, must be stably equivalent to $J$.

**Proposition 3.2** *$J$ has minimal $Z$–rank in its stable class.*

**Proof** Given any finite algebraic 2–complex, consider the cochain obtained by applying $\mathrm{Hom}_{Z[D_{4n}]}(\bullet, \mathbb{F}_2)$:

$$\mathbb{F}_2^{d_2} \xleftarrow{v_2} \mathbb{F}_2^{d_1} \xleftarrow{v_1} \mathbb{F}_2^{d_0}$$

where $d_0$, $d_1$, $d_2$, are the $\mathbb{Z}[D_{4n}]$ ranks of the modules in the complex. As $H^0(D_{4n}, \mathbb{F}_2) = \mathbb{F}_2$, the kernel of $v_1$ has $\mathbb{F}_2$–rank 1. Consequently, the image of $v_1$ has $\mathbb{F}_2$–rank $d_0 - 1$. $H^1(D_{4n}, \mathbb{F}_2) = \mathbb{F}_2^2$ so $v_2$ has kernel of $\mathbb{F}_2$–rank $2 + d_0 - 1 = d_0 + 1$. The image of $v_2$ is then seen to have rank $d_1 - d_0 - 1$. $H^2(D_{4n}, \mathbb{F}_2) = \mathbb{F}_2^3$ so we know that $d_2 \geq 3 + d_1 - d_0 - 1$. Rearranging gives $d_2 - d_1 + d_0 \geq 2$.

Exactness implies that the $\mathbb{Z}$–rank of the algebraic $\pi_2$ of the algebraic complex must be $4n(d_2 - d_1 + d_0) - 1$. Hence our inequality implies that this is at least $8n - 1$, which is the $\mathbb{Z}$–rank of $J$. □

We now restrict to the case $n = 2$.

**Proposition 3.3** *The only elements in the stable class of $J$ are modules of the form $J \oplus \mathbb{Z}[D_8]^k$.*

**Proof** We refer to [3, Theorem 6.1]. This states that over $\mathbb{Z}[D_8]$, $A \oplus C = B \oplus C$ implies $A = B$ for torsion free, finitely generated modules $A$, $B$, $C$.

If a module $M$ is in the stable class of $J$ then $M \oplus \mathbb{Z}[D_8]^r = J \oplus \mathbb{Z}[D_8]^s$. From proposition 3.2 we have $s \geq r$. From the theorem, we deduce that $M = J \oplus \mathbb{Z}[D_8]^{s-r}$. □

**Theorem 3.4** *The group $D_8$ satisfies the $D(2)$ property.*

**Proof** The only modules that can turn up as the algebraic $\pi_2$ of an algebraic 2–complex over $\mathbb{Z}[D_8]$ are ones of the form $J \oplus \mathbb{Z}[D_8]^s$ for some $s \geq 0$. Theorem 2.11 tells us that for each $s$, up to chain homotopy equivalence, there is a unique algebraic 2–complex with algebraic $\pi_2$ equal to $J \oplus \mathbb{Z}[D_8]^s$. Given any $r$, the chain homotopy equivalence class of this algebraic 2–complex is realized by the Cayley complex of the presentation:

$$\langle a, b \mid a^{2n} = b^2 = e,\ aba = b,\ r_1 = e,\ r_2 = e, \ldots r_s = e \rangle$$

where $r_i = e$ for $i = 1, \ldots, s$.

Hence we know that every algebraic 2–complex over $D_8$ is geometrically realized. By [3, Theorem I], this is equivalent to $D_8$ satisfying the $D(2)$ property. □

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