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On non-compact Heegaard splittings

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A Heegaard splitting of an open 3–manifold is the partition of the manifold into two non-compact handlebodies which intersect on their common boundary. This paper proves several non-compact analogues of theorems about compact Heegaard splittings. The main result is a classification of Heegaard splittings of those open 3–manifolds obtained by removing boundary components (not all of which are 2–spheres) from a compact 3–manifold. Also studied is the relationship between exhaustions and Heegaard splittings of eventually end-irreducible 3–manifolds. It is shown that Heegaard splittings of end-irreducible 3–manifolds are formed by amalgamating Heegaard splittings of boundary-irreducible compact submanifolds.

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1 Introduction

To what extent do non-compact 3-manifolds share the structures and properties of their compact cousins? Investigating this question has long been a central concern in the study of non-compact 3–manifolds. Given the importance of Heegaard splittings in the topology and geometry of compact 3–manifolds, it seems natural to consider them in this exploration. Non-compact Heegaard splittings, however, rarely appear in the literature. This is, perhaps, surprising since every open 3–manifold has a Heegaard splitting: Heegaard splittings can be constructed from triangulations or, if the open manifold covers a closed manifold, by lifting a Heegaard splitting of the closed manifold. Heegaard splittings of an open manifold may also be constructed from an exhaustion of the the manifold (Section 2).

Exhaustions have traditionally been the main tool for studying non-compact 3–manifolds; this paper is no exception. Indeed, the interplay between exhaustions and Heegaard splittings is the focus of much of this present work. Considering the interaction between Heegaard splittings and exhausting sequences leads to the first main theorem: a non-compact analogue of Casson and Gordon's theorem on weakly reducible Heegaard splittings.

Applying this result to manifolds such as $M =$ (closed orientable surface)$\times \mathbb{R}$ leads to a non-compact version of the classification of Heegaard splittings of (closed surface)$\times I$

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by Scharlemann and Thompson and (closed surface) $\times S^1$ by Schultens. If the closed surface is not a 2–sphere we discover that, in fact, Heegaard splittings of M are unique up to proper ambient isotopy. The next section explains these results in more detail.

This paper builds upon methods used by Frohman and Meeks in their program (completed in [14]) of topologically classifying minimal surfaces in $\mathbb{R}^3$. In [13], they show that every complete one-ended minimal surface in Euclidean 3–space is a Heegaard surface and prove two theorems about Heegaard splittings of $\mathbb{R}^3$. The first is a non-compact analogue of the Reidemeister–Singer theorem and the second is an analogue of Waldhausen's classification of Heegaard splittings of S^3. Perhaps our analogues of compact Heegaard splitting theorems will also be useful for studying minimal surfaces in non-compact 3–manifolds.

1.1 Main results

This paper focuses on the two most tractable types of non-compact 3–manifolds: eventually end-irreducible 3–manifolds and deleted boundary 3–manifolds. A non-compact 3–manifold M is *eventually end-irreducible* if there is a compact set $C \subset M$ and an exhaustion $\{K_i\}$ for M where the frontier of each K_i is incompressible in $M - C$. We say that M is end-irreducible (rel C). The class of eventually end-irreducible 3–manifolds includes uncountably many manifolds with infinitely generated fundamental group and uncountably many simply connected 3–manifolds (such as the Whitehead manifold) but excludes uncountably many others. A deleted boundary 3–manifold is a particular type of eventually end-irreducible 3–manifold. M is a *deleted boundary* 3–manifold if it is obtained by removing at least one boundary component from a compact 3–manifold. For example, $F \times \mathbb{R}$ is a deleted boundary manifold for any closed surface F since it can be obtained by removing the boundary from $F \times [0, 1]$.

Even though these classes of 3–manifolds are relatively manageable their Heegaard splittings can still exhibit strange behavior. Section 2.4 constructs a Heegaard splitting of the Whitehead manifold which contains infinitely many stabilizing balls, but where no infinite collection of stabilizing balls is locally finite. This is similar in spirit (though not in method) to Peter Scott's construction [26] of a 3–manifold which does not have a prime decomposition.

The tractability of eventually end-irreducible 3–manifold Heegaard splittings is shown by our first main theorem, Theorem 5.1. This theorem shows that, for these manifolds, every Heegaard splitting is built from smaller Heegaard splittings in way analogous to the construction of compact weakly reducible Heegaard splittings.

Weakly reducible Heegaard splittings of compact 3-manifolds were first studied by Casson and Gordon [9]. They prove that if a compact Heegaard splitting is weakly reducible (ie, there are disjoint essential discs in the opposing handlebodies of the splitting) then the manifold contains a closed incompressible surface (other than an inessential 2–sphere). Every non-compact Heegaard splitting (other than the genus 0 splitting of $\mathbb{R}^3$) is weakly reducible, so we might hope that if an open manifold (other than $\mathbb{R}^3$) has a non-stabilized Heegaard splitting then it contains a closed incompressible surface (other than an inessential S^2). It is, however, unclear if such a result holds.

Casson and Gordon's result can be rephrased as the claim that a weakly reducible splitting is either stabilized or was created by amalgamating Heegaard splittings of submanifolds across a separating incompressible surface. The first main result of this paper is a non-compact analogue of this statement. Notice that, although we assume the existence of incompressible surfaces, we can make a strong conclusion about the structure of the Heegaard splitting.

Simplified Version of Theorem 5.1 *If M is orientable and end-irreducible (rel C) with Heegaard splitting $U \cup_S V$ then there is an exhaustion $\{K_i\}$ of M with the frontier of each K_i incompressible in $M - C$ such that $U \cup_S V$ is obtained by the amalgamation of splittings of the submanifolds* $\mathrm{cl}(K_{i+1} - K_i)$.

If M is a deleted boundary manifold, each end of M is a copy of (closed surface)$\times\mathbb{R}_+$. In this case, each submanifold $\mathrm{cl}(K_{i+1} - K_i)$ is a copy of (closed surface)$\times I$ whose Heegaard splittings were classified by Scharlemann and Thompson [23]. Using their classification and a theorem, due essentially to Frohman and Meeks, we classify the Heegaard splittings of nearly every deleted boundary 3–manifold:

Simplified Version of Theorem 6.4 *If $\overline{M}$ is an orientable compact 3–manifold such that $\partial\overline{M}$ is non-empty and contains no S^2 component then any two Heegaard splittings of $M = \overline{M} - \partial\overline{M}$ are properly ambient isotopic. If $\overline{M}$ is $S^3 -$ (3–balls) or contains at least one boundary component which is not S^2 then the Heegaard splittings of $\overline{M} - \partial M$ can also be classified.*

Just as the Frohman–Meeks classification of Heegaard splittings of $\mathbb{R}^3$ is analogous to, and depends on, Waldhausen's classification of the splittings of S^3, so the classification of splittings of deleted boundary 3-manifolds is analogous to, and depends on, the Scharlemann–Thompson classification [23] of splittings of (closed surface)$\times I$. Since, (closed surface)$\times\mathbb{R}$ is a deleted boundary manifold which covers (closed surface)$\times S^1$, our result may also be viewed as a non-compact analogue of Schulten's classification [25] of Heegaard splittings of (closed surface)$\times S^1$.

The proof of Theorem 6.4 relies on the following theorem of Frohman and Meeks:

Simplified Version of Theorem A.1 (Frohman–Meeks) *If $U_1 \cup_{S_1} V_1$ and $U_2 \cup_{S_2} V_2$ are two Heegaard splittings of a one-ended manifold M such that for each splitting there is a properly embedded collection of infinitely many disjoint stabilizing balls then S_1 and S_2 are properly ambient isotopic in M.*

This theorem says that any two Heegaard splittings are equivalent after, possibly infinitely many, stabilizations. It is, therefore, a non-compact analogue of the Reidemeister-Singer theorem for compact 3–manifolds. A complete proof of this theorem (and its trivial extension to the case where M has multiple ends and compact boundary) is given in the Appendix. We also correct a misstatement[1] in their proof. The correction is not difficult, but does require some work and an additional hypothesis for one of their propositions.

For simplicity, the previous statements have been for open orientable 3–manifolds. Since we require the use of compressionbodies throughout the paper, it requires no extra work to prove all of our results for orientable non-compact 3–manifolds with compact boundary. Most of the work in this paper occurs in the ends of the manifold; requiring compact boundary allows us to, for the most part, ignore the boundary altogether. It is likely that the situation where M has infinitely many compact boundary components could be handled using the methods of this paper.

1.2 Acknowledgements

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1.3 Outline

- Section 2 provides several examples of non-compact Heegaard splittings, shows how to construct a Heegaard splittings by amalgamation and proves that the inclusion of a Heegaard surface into a non-compact 3–manifold induces a homeomorphism of ends.

[1] The error occurs in the last sentence of Prop. 2.2. After including the collars of $J_i - L_i$ and $L_i - J_i$ you have arranged for K_i to have a relative (hollow) Heegaard splitting, but $K_{i+1} - K_i$ may not. For example, the frontier of $K_i \cap H_1$ may not be incompressible in $H_1 \cap \text{cl}(K_{i+1} - K_i)$. This error affects the proof of Proposition 2.3. In our correction of the proof of Prop 2.2 we need to use the assumption that the splitting is end-stabilized.

- Sections 3 and 4 provide preliminary work. Section 3 defines and studies handle-slides of boundary-reducing discs in compressionbodies. Section 4 examines a certain type of submanifold which is "balanced" on a non-compact Heegaard surface. We discuss the type of Heegaard splittings (called "relative Heegaard splittings") which these submanifolds inherit from the splitting of the manifold. Both balanced submanifolds and relative Heegaard splittings are central in the work of Frohman and Meeks.
- Section 5 proves the non-compact analogue of Casson and Gordon's theorem. While there does not seem to be a way to usefully quote Casson and Gordon's theorem, we do rely heavily on the proof of their result given in [21].
- Section 6 provides the classification of Heegaard splittings of deleted boundary 3–manifolds.
- Appendix A proves Theorem A.1, the non-compact analogue of the Reidemeister–Singer theorem.

1.4 History

Scharlemann, in his survey paper [21], gives an overview of the history of Heegaard splittings of compact 3–manifolds. As he notes in that paper, very few types of compact 3–manifolds are known to have unique Heegaard splittings of a given genus and partition of the boundary. The 3-sphere [28], lens spaces [4], (closed orientable surface) $\times I$ [2; 23], and (closed orientable surface) $\times S^1$ [2; 25] are among these. $\mathbb{R}^3$ and manifolds which are homeomorphic to $\mathbb{R}^3$ minus closed 3–balls, on the other hand, are the only non-compact manifolds whose Heegaard splittings (of the type considered here) have received attention. Frohman and Meeks [13] show that $\mathbb{R}^3$ has (up to proper ambient isotopy) a unique Heegaard splitting of given (finite or infinite) genus. Meeks and Rosenberg [17] observe that the work of [13] can also be applied to $S^2 \times \mathbb{R}$.

Pitts and Rubinstein [18] have also considered Heegaard splittings of non-compact 3–manifolds. They, however, consider only deleted boundary 3–manifolds and compact Heegaard surfaces which split the manifold into two "hollow handlebodies". For them, a hollow handlebody is simply a compact compressionbody with ∂_- removed. Frohman and Meeks also use the term "hollow handlebody", but they refer to what Canary and McCullough [8] call "relative compressionbodies", terminology which has become standard. In an effort to avoid confusion with Pitts and Rubinstein's use of "hollow handlebody", this paper uses "relative compressionbody". Both Frohman–Meeks and Pitts–Rubinstein use Heegaard surfaces in non-compact 3–manifolds to study minimal surfaces from a topological point of view. The main appearances of non-compact handlebodies and Heegaard splittings have been in minimal surface theory, for example Freedman [11], Froman [12], Froman and Meeks [13], and Meeks and Rosenberg [17].

This paper focuses on "eventually end-irreducible" 3–manifolds. These manifolds were first studied by Brown and Tucker [7]. They are an important class of 3–manifolds since some questions about arbitrary non-compact 3–manifolds can be reduced to questions about eventually end-irreducible 3–manifolds [5].

1.5 Definitions

Notation If X is a subcomplex of a complex Y, then $\eta(X)$ denotes a closed regular neighborhood of X in Y. The term "submanifold" will be reserved for codimension 0 submanifolds. If X is a submanifold of a manifold Y then $\mathrm{cl}(X)$ indicates the closure of X in Y and $\mathrm{int}(X)$ indicates the interior of X in Y. The number of components of a complex X is denoted $|X|$. The spaces $[0,1]$ and $[0,\infty)$ are denoted by I and $\mathbb{R}_+$ respectively. $\mathbb{R}^n$ denotes n–dimensional Euclidean space and S^n denotes the sphere of dimension n. The closed unit disc in $\mathbb{R}^2$ is denoted by D^2. The integers and natural numbers are indicated by $\mathbb{Z}$ and $\mathbb{N}$ respectively. All homology groups use $\mathbb{Z}$ coefficients.

3–manifold topology We work in the PL category and use, with a few exceptions, standard terminology from 3–manifold theory (see [15; 16]). All 3–manifolds and surfaces are assumed to be orientable. A map $\rho\colon X \to Y$ between complexes is *proper* if the preimage of each compact set is compact. If X is a surface and Y is 3–manifold, ρ is a *proper embedding* if, in addition to being proper and an embedding, $\rho^{-1}(\partial Y) = \partial X$. To say that a graph is properly embedded in a 3–manifold means that the inclusion map is proper and an embedding. In particular, for a graph we do not require that the valence one vertices of the graph be on the boundary of the manifold. A homotopy $\rho\colon X \times I \to Y$ is *proper* if it is proper as a map. If X is a surface and Y is a 3–manifold we also require that $\rho^{-1}(\partial Y) = \partial(X \times I)$. The homotopy ρ is *ambient* if $X \subset Y$ and there is an extension of ρ to a proper homotopy $\rho\colon Y \times I \to Y$. An isotopy $\rho\colon X \times I \to Y$ is a homotopy where for each $t \in I$, $\rho(\cdot, t)\colon X \to Y$ is an embedding. An ambient isotopy $\rho\colon Y \times I \to Y$ is required to be a homeomorphism at each time $t \in I$. To say that a homotopy ρ is *fixed* on a set C means that, for each $t \in I$, ρ restricted to $C \times \{t\}$ is the identity map.

A loop on a surface is *essential* if it is not null-homotopic in the surface. An embedded 2–sphere in a 3–manifold is *essential* if it does not bound a 3–ball. A *compressing disc* for a surface F in a 3–manifold is an embedded disc D for which $D \cap F = \partial D$ and ∂D is an essential loop on F. A surface F properly embedded in M is *incompressible* if there are no compressing discs for F in M.

Remark Note that our definition considers inessential 2–spheres and discs to be incompressible surfaces. This is slightly non-standard, but it makes the statements and proofs of some of the results easier. We will emphasize places where this observation matters.

If $S \subset M$ is a surface embedded in a 3–manifold and if Δ is the union of pairwise disjoint compressing discs for S then $\sigma(S;\Delta)$ will denote the surface obtained from F by compressing along Δ. If $R \subset S$ is a topologically closed subsurface (ie $\mathrm{cl}(R) = R$) with each component of ∂R either contained in or disjoint from $\partial\Delta$ then $\sigma(R;\Delta)$ will denote the surface obtained from R by compressing along those discs of Δ with boundary in R. If $S \subset \partial M$ then the manifold obtained by boundary-reducing M along Δ is denoted $\sigma(M;\Delta)$. As it will always be clear when we have a surface and when we have a 3–manifold this should not cause confusion.

A manifold (2- or 3–dimensional) is *open* if it is non-compact and without boundary. It is *closed* if it is compact and without boundary. A 3–manifold is *irreducible* if every embedded 2–sphere bounds an embedded 3–ball. As much as possible, we do not assume irreducibility. A submanifold of a 3–manifold is a *product region* if it is homeomorphic to $F \times I$ where F is a surface. A *fiber* of $F \times I$ is $\{x\} \times I$ where $x \in F$. A set $X \subset F \times I$ is *vertical* if it is the union of fibers.

Heegaard splittings The survey article [21] is a good reference for compact Heegaard splittings – particularly for the proof of Casson and Gordon's theorem which will be referred to later in this paper. Since we are interested in splittings of non-compact 3–manifolds, some of our definitions differ from conventions in the compact setting.

Let F be either a compact, orientable surface (possibly disconnected) or the empty set. A *compressionbody* H is formed by taking the disjoint union of $F \times I$ and countably (finitely or infinitely) many disjoint 3–balls and then attaching 1–handles. 1–handles are attached to $F \times I$ on the interior of $F \times \{1\}$ and to the boundaries of the 3–balls. Only finitely many 1–handles are to be attached to each 3–ball and only finitely many may be attached to $F \times \{1\}$. We usually require that the result be connected. The surface $F \times \{0\}$ is denoted $\partial_- H$ and the surface $\mathrm{cl}(\partial H - \partial_- H)$ is denoted $\partial_+ H$ and is called the *preferred surface* of H. If F is a closed surface then H is an *absolute compressionbody*; if F has non-empty boundary then H is a *relative compressionbody*. If $H = F \times I$ then H is a *trivial compressionbody*. If F is empty, then H is a *handlebody*. We will generally require that F contain no S^2 components, as then H is irreducible. At one point in Section 5 we will need to allow S^2 components. This will be explicitly pointed out. A *subcompressionbody* A of H is a submanifold of H whose frontier in H consists of properly embedded discs. (We do not require these discs to be essential. Thus, for example, $\mathbb{R}^3_+$, which is a handlebody, has an exhaustion consisting of subcompressionbodies.) We denote $\partial A \cap \partial_+ H$ by $\partial_{\partial_+ H} A$. There is a proper strong deformation retraction of a compressionbody H onto $\partial_- H \cup \Sigma$ where Σ is a properly embedded graph in H attached at valence one vertices to $\partial_- H$. $\Sigma \cup \partial_- H$ is called the *spine* of the compressionbody.

A properly embedded collection Δ of disjoint discs in a compressionbody H with boundary on $\partial_+ H$ will be called a *disc set* for H or for $\partial_+ H$. If the union of some components of $\sigma(H;\Delta)$ is $\partial_- H \times I$, then Δ is *collaring*. If $\sigma(H;\Delta)$ consists of 3–balls and $\partial_- H \times I$ the disc set is *defining*.

Remark The discs in a (defining) disc set are not required to be essential in the compressionbody. Thus, for example, upper half space (which is a handlebody) has a defining disc set.

Remark Although every defining disc set is collaring, we use the term "collaring" to focus attention on the property that is used most often. In Section 4, for example, we use collaring disc sets which may not be defining disc sets.

A *Heegaard splitting* of a 3–manifold M is a decomposition of M into two compressionbodies U and V glued along $\partial_+ U = \partial_+ V$ = S. If U and V are absolute compressionbodies the splitting is an *absolute Heegaard splitting*. If U and V are relative compressionbodies then the splitting is a *relative Heegaard splitting*. The surface S is called the *Heegaard surface*. We write $M = U \cup_S V$. If the term "Heegaard splitting" is used without either the adjective "absolute" or "relative", we will mean "absolute Heegaard splitting". Usually, relative Heegaard splittings will be of compact submanifolds of a non-compact 3–manifold.

A Heegaard splitting of a manifold $M = U \cup_S V$ is *reducible* if there is an essential simple closed curve on S which bounds embedded discs in U and V. To *stabilize* a Heegaard surface, push the interior of an embedded arc on the surface into one of the compressionbodies and include a regular neighborhood of the arc into the other compressionbody. A Heegaard splitting has been stabilized if there is, in M, an embedded 3–ball which intersects the Heegaard surface in a properly embedded, unknotted, once-punctured torus. Such a ball is called a *stabilizing ball*. A Heegaard splitting $M = U \cup_S V$ is *end-stabilized* if for every compact set $C \subset M$ and every non-compact component W of $\mathrm{cl}(M - C)$ there is a stabilizing ball for S entirely contained in W.

Non-compact 3–manifolds An *exhaustion* for a non-compact 3–manifold M is a sequence $\{K_i\}$ of compact, connected 3-submanifolds such that $K_i \subset \text{int}(K_{i+1})$ and $M = \cup_i K_i$. A 3–manifold M is *end-irreducible (rel C)* for a compact subset C if there is an exhaustion for M such that the frontier of each element of the exhaustion is incompressible in $M - C$. If C can be taken to be the empty set, then M is simply *end-irreducible*. If M is end-irreducible (rel C) for some C then M is *eventually end-irreducible*. If a non-compact 3–manifold is obtained by removing at least one boundary component from a compact 3–manifold then the non-compact 3–manifold is a *deleted boundary 3–manifold*. Deleted boundary 3–manifolds are eventually end-irreducible. Except for compressionbodies, all 3–manifolds considered in this paper will have compact boundary. When the manifold is end-irreducible (rel C) we will assume that C contains ∂M.

2 Examples

Some examples of non-compact Heegaard splittings are in order. When thinking about non-compact Heegaard splittings, keep in mind that an absolute handlebody is the closed regular neighborhood of a properly embedded, locally finite graph in $\mathbb{R}^3$. Frohman and Meeks [13] (adapting an example of Fox and Artin) give an example of a non-compact 3–manifold whose interior is an open infinite genus handlebody but where the closure of the interior is not a handlebody. A handlebody has a properly embedded disc set which cuts it into 3–balls. Another observation, which may help the reader's intuition, is that no essential loop in an absolute compressionbody can be homotoped out of every compact set. This is easily proved using the proper deformation retraction of the compressionbody to its spine. This implies, for example, that if $F \neq D^2$ is a compact connected surface then $F \times \mathbb{R}$ is not a handlebody.

2.1 Heegaard splittings of $\mathbb{R}^3$

Heegaard splittings of $\mathbb{R}^3$ are easy to construct. Since the upper and lower half spaces are each homeomorphic to a closed regular neighborhood of the positive z-axis, $\mathbb{R}^3$ has a genus zero Heegaard splitting. Obviously, this splitting can be stabilized any given (finite) number of times. By choosing an infinite, properly embedded collection of arcs in the surface, it can also be stabilized an infinite number of times simultaneously to give an infinite genus Heegaard surface. Frohman and Meeks prove that these are, up to proper ambient isotopy, the only infinite genus Heegaard splittings of $\mathbb{R}^3$.

2.2 Finite genus Heegaard splittings

Let $\overline{M}$ be a compact 3–manifold with Heegaard splitting $\overline{U} \cup_{\overline{S}} \overline{V}$. Let B be an embedded closed 3–ball in M which intersects $\overline{S}$ in a properly embedded disc. Let $X = \overline{X} - B$ for $X = M, U, S, V$. Then $M = U \cup_S V$ is a finite genus Heegaard splitting of the deleted boundary manifold M. (Infinitely many discs parallel to $\partial B \cap \overline{U}$ ($\partial B \cap \overline{V}$) are in any defining set of discs for U (V).) Classifying such Heegaard splittings would be equivalent to classifying all Heegaard splittings of compact manifolds. No such simple classification is to be hoped for, and so our classification of Heegaard splittings for deleted boundary 3–manifolds does not address such examples. Fortunately, this is the only type not covered by our classification.

2.3 Amalgamating Heegaard splittings

Heegaard splittings of non-compact manifolds can be created by amalgamating splittings of compact submanifolds. We describe a way to do this, beginning with a description of amalgamation. See [25] for the definition of amalgamation. Let N_0 and N_1 be two compact 3–manifolds with absolute Heegaard splittings $N_0 = U_0 \cup_{S_0} V_0$ and $N_1 = U_1 \cup_{S_1} V_1$ and collections of components $F_0 \subset \partial_- V_0$ and $F_1 \subset \partial_- V_1$ which are homeomorphic via a homeomorphism $h\colon F_1 \to F_0$. In the *amalgamated manifold* $N = N_0 \cup_h N_1$ we can *amalgamate* the Heegaard splittings of N_0 and N_1 as follows:

In V_i there are collaring discs δ_i which cut off a product region $F_i \times I$ contained in V_i. Choose labels so that $F_i = F_i \times \{0\}$. Let P denote the product region $(F_1 \times I) \cup (F_0 \times I)$ in N. Identify P with $F_1 \times [0,2]$. Note that it is contained in $V_0 \cup V_1$. Perform an isotopy of N_1 so that, in P, $A_1 = \delta_1 \times [0,2]$ is disjoint from δ_0. Let $U = U_0 \cup (V_1 - (F_1 \times I)) \cup A_1$, $V = \mathrm{cl}(N - U)$ and $S = V \cap U$. Then $M = U \cup_S V$ is a Heegaard splitting of genus equal to $\mathrm{genus}(S_0) + \mathrm{genus}(S_1) - \mathrm{genus}(F_1)$. Note that there there are disjoint discs $\delta_1 \subset U$ and $\delta_0 \subset V$ which, when we compress S along them, leave us with a surface parallel to $F_0 = F_1$ in N.

Here is a method of producing an infinite genus Heegaard splitting of a non-compact 3–manifold M. Let $\{K_i\}$ be an exhaustion for M with the properties that $\partial M \subset K_1$, that no component of $\mathrm{cl}(M - K_i)$ is compact for any i, and that for each i and for each component J of $\mathrm{cl}(K_{i+1} - K_i)$ the intersection $J \cap K_i$ is connected. For each i, let $L_i = \mathrm{cl}(K_{i+1} - K_i)$ and $F_i = L_i \cap K_i$. K_{i+1} is formed by amalgamating K_i and each component of L_i along a single component of the surface F_i.

We now carefully choose absolute Heegaard splittings of K_1 and each component of L_i for each $i \geq 1$. Choose a Heegaard splitting $K_1 = U_1 \cup_{S_1} V_1$ of K_1 so that every boundary component of K_1 is contained in V_1. Let δ_1' be a set of collaring discs for V_1. Now for each component of L_i choose a Heegaard splitting so that $L_i = X_i \cup_{T_i} Y_i$. Choose the splitting so that $\partial L_i \subset Y_i$. Figure 1 provides a schematic depiction of our choices. Inductively, form a Heegaard splitting of $K_n = U_n \cup_{S_n} V_n$ for $n \geq 2$ by amalgamating the Heegaard splittings of K_{n-1} and L_{n-1}. Let V_n be the compressionbody which contains δ'_1 and let U_n be the other.

Recall from the definition of amalgamation that if $F_n \subset U_n$ then $U_{n+1} \cap U_n$ can be created by removing 1–handles in U_n which join F_i to S_i and are vertical in the product structure of U_n compressed along a defining set of discs. Denote these 1–handles by A_n. The surface F_n is contained in U_n whenever n is even (by our choice of Heegaard splitting for L_n). If n is odd then F_n is not in U_n, so for odd n, let $A_n = \varnothing$. If n is even then $U_n \subset U_{n+1}$. Define $U'_n = \mathrm{cl}(U_n - A_n)$. Since for each n, $\partial L_n \subset Y_n$, $U'_n \subset U'_{n+1}$ for all n. In particular, when we extend the 1–handles from Y_n into K_{n-1} they do not not need to reach into L_{n-2}. Let $U = \cup_{\mathbb{N}} U'_n$. Figure 2 depicts the 1–handles A_1.

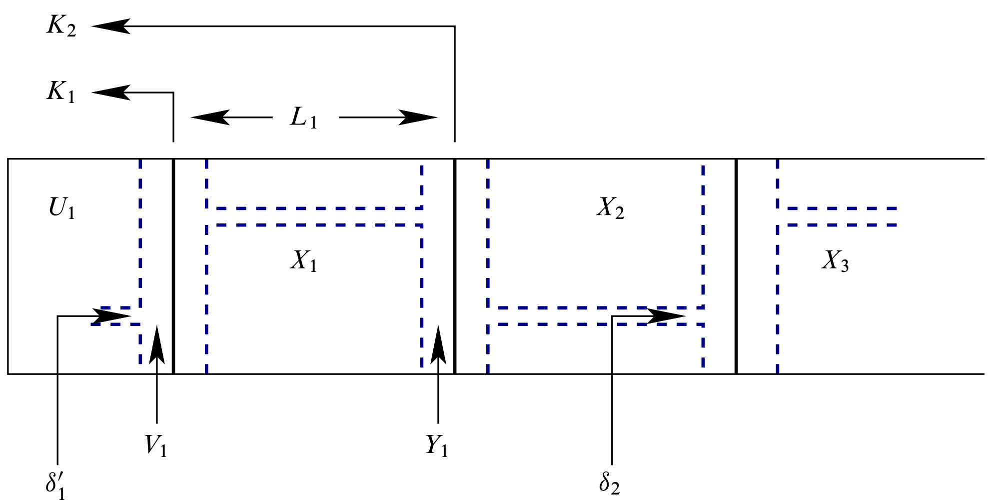

Figure 1: Choosing the splittings of L_i

We desire to show that U is an absolute compressionbody. Since $\partial M \subset V$, U will be an absolute handlebody. To prove this we will produce a properly embedded collection of discs in U which cut U into compact handlebodies. Let δ_n be a collaring set of discs contained in L_{n-1} for U_n for each even n. We may assume that δ_n is disjoint from A_n and so δ_n is a properly embedded finite collection of discs in U, for each even n. Furthermore, since $\delta_n \subset L_{n-1}$ the infinite collection of discs $\delta = \cup \delta_n$ is properly embedded in M. The discs δ_n cut off a compact submanifold $U'_n - (\partial K_{n+2} \times I)$. As $U = \cup U'_n$ every component of $\sigma(U;\delta)$ is compact. Let H be a component of

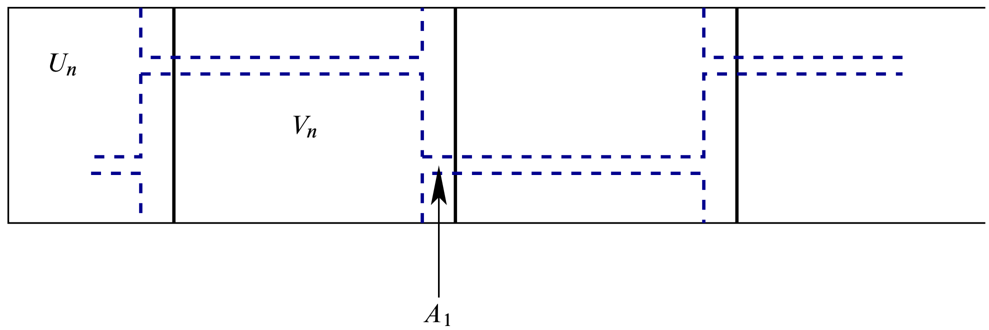

Figure 2: After the amalgamations

$\sigma(U;\delta)$. Choose an even n large enough so that $H \subset U'_{n-2}$. H is thus a component of $\sigma(U'_n;\delta)$ which is not contained in $\partial U'_n \times I$. As such, it must be a handlebody as U'_n, for n even, is an absolute compressionbody. Hence, U is a handlebody.

Let $V = \mathrm{cl}(M - U)$. The argument to show that V is an absolute compressionbody is similar, except that the disc set δ will cut V into compact handlebodies and, if $\partial M \neq \varnothing$, a compact absolute compressionbody H with $\partial_- H = \partial M$. Letting $S = U \cap V$, we have shown that $U \cup_S V$ is an absolute Heegaard splitting of M.

It is instructive to examine this construction in the case when M is a deleted boundary 3–manifold. Let M_0 be a compact, orientable 3–manifold with non-empty boundary component $\partial_1 M_0 = F \neq S^2$. Let $M_0 = U_0 \cup_{S_0} V_0$ be a Heegaard splitting of M_0 with $F \subset V_0$. Let M_i for $i \geq 1$ be homeomorphic to $F \times I$ and choose a Heegaard splitting $M_i = U_i \cup_{S_i} V_i$ of M_i which is obtained by tubing together two copies of F in M_i. Such a Heegaard splitting has both boundary components, $\partial_0 M_i$ and $\partial_1 M_i$, contained in V_i and has genus which is twice the genus of F. (Heegaard splittings of $F \times I$ are classified by Scharlemann and Thompson in [23]. This classification will be important for our work in Section 6.) Build a 3–manifold M, homeomorphic to $M_0 - F$ by glueing $\partial_0 M_i$ to $\partial_1 M_{i-1}$ for $i \geq 1$. At stage n of the glueing process we can obtain a Heegaard splitting of the new manifold by amalgamating the splittings of the previously constructed manifold and M_n. The new Heegaard splitting will have genus equal to $\mathrm{genus}(S_0) + n \cdot \mathrm{genus}(F)$. This produces an infinite genus splitting of M. It is easy to verify that the splitting is end-stabilized. The content of Proposition 6.8 is that, up to proper ambient isotopy, this is the only Heegaard splitting of M.

2.4 Infinite genus splittings which are not end-stabilized

Theorem 6.4 shows that all infinite genus splittings of one-ended deleted boundary 3–manifolds are end-stabilized. It is then natural to ask:

Question Are there examples of one-ended, irreducible 3–manifolds which have infinite genus Heegaard splittings that are not stabilized? Are there such examples where the manifold has finitely generated fundamental group? where the manifold contains no incompressible surfaces (other than inessential 2–spheres)? What if we simply require that the splitting not be end-stabilized?

In this subsection, we give two examples of splittings which are not end-stabilized. The first example is a non-stabilized splitting of a one-ended, irreducible 3–manifold M with infinitely generated fundamental group. The second example, which is obtained from the first, is a splitting of the Whitehead manifold W which is stabilized, not end-stabilized, and which cannot be made non-stabilized by finitely many destabilizations. The key point is that, although there are infinitely many "inequivalent" stabilizing balls, they are not properly embedded in W. I do not know of a one-ended, irreducible manifold with finitely generated fundamental group which has an infinite genus non-stabilized splitting or of an open manifold which contains no incompressible surfaces (other than inessential 2–spheres) and a non-stabilized positive genus Heegaard splitting.

We begin by constructing the splitting of M. Let W_0 be the exterior of the Whitehead link in S^3. W_0 is a compact 3–manifold which contains no essential annuli or essential tori[2]. W_0 is hyperbolic (Example 3.3.9 of Thurston [27]). As the Whitehead link is a 2-bridge link, it has tunnel number one, and therefore W_0 has a genus 2 Heegaard splitting which does not separate ∂W_0. Let $\partial_0 W_0$ and $\partial_1 W_0$ be the two boundary components of W_0. Let λ_j and μ_j be the longitude and meridian of $\partial_j W_0$ (for $j = 0, 1$). The choice should be made so that λ_j and μ_j correspond to the longitude and meridian of the corresponding component of the Whitehead link in S^3. In particular, λ_0 and λ_1 are homologically trivial in W_0 and μ_0 and μ_1 included into W_0 generate the first homology of W_0. Let $f\colon \partial_0 W_0 \to \partial_1 W_0$ be a homeomorphism which takes λ_0 to μ_1 and μ_0 to λ_1.

For each $i \in \mathbb{N}$ let W_i be a copy of W_0. Denote the boundary components of W_i by $\partial_0 W_i$ and $\partial_1 W_i$ in such a way that the labelling corresponds to the labelling of the boundary components of W_0. Let S_i be a genus 2 Heegaard surface for W_i which does not separate the boundary components. Let $f_i\colon \partial_0 W_i \to \partial_1 W_{i-1}$ be the map f. Let $M_1 = W_1$ and, inductively, let $M_n = M_{n-1} \cup_{f_n} W_n$, $\partial_0 M_n = \partial_0 W_1$, and $\partial_1 M_n = \partial_1 W_n$ for $n \geq 2$. Let S'_n be the Heegaard surface of M_n and S the Heegaard surface of $M = \cup_{i \in \mathbb{N}} M_i$ obtained by amalgamating the surfaces S_i, as described previously. Figure 3 shows the construction of M_4 and S'_4.

Next we show that S is not stabilized. If it were, then some S'_n would be stabilized, as stabilizing balls are compact. Without loss of generality, we may assume that n is odd,

[2]This is easy to prove directly, or see Muñoz and Uchida [10].

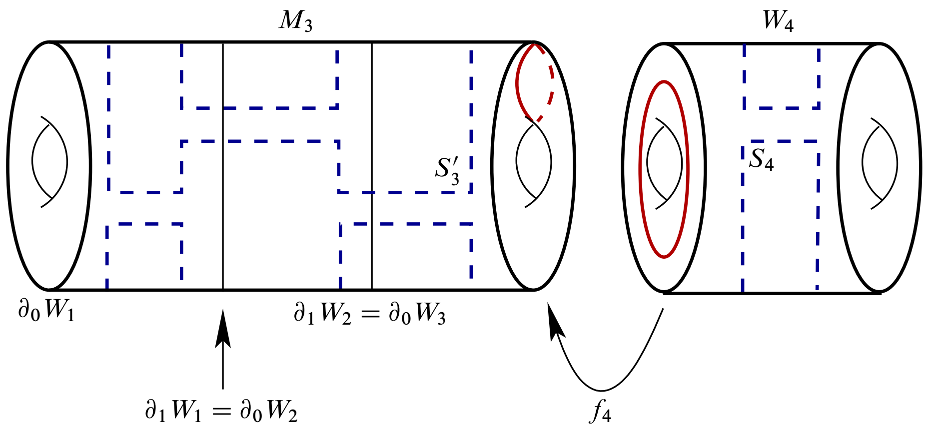

Figure 3: Forming the Heegaard surface S_4' of M_4

so that S_n' does not separate the boundary components of M_n. It will be beneficial to work with a closed 3–manifold: glue a copy of W_0 to M_n to obtain a closed 3–manifold M'. Use the glueing maps $f\colon \partial_0 W_0 \to \partial_1 M_n$ and $f^{-1}\colon \partial_1 W_0 \to \partial_0 M_n$. We may form a Heegaard splitting of M' by amalgamating a genus 2 splitting, which does not separate ∂W_0, of W_0 to S_n' across ∂M_n to obtain a Heegaard surface T. As neither splitting separates the boundary components of the respective manifolds, this operation gives a well-defined Heegaard splitting T of M', a closed 3–manifold. Figure 4 shows the process of forming M' and T. The genus of S_n' is $(2n-(n-1)) = n+1$. The splitting given by T is obtained from S_n' by adding a single one-handle to the handlebody in the splitting of M_n. Thus, the genus of T is one more than the genus of S_n'; that is, the genus of T is $n+2$. By assumption, S_n' is stabilized, and so T is, as well. Thus, M' has an irreducible Heegaard splitting of genus $g \leq n+1$.

We now appeal to a theorem of Scharlemann and Schultens. A consequence of Theorem 4.7 of [22] is that if M' (a closed, orientable, irreducible 3–manifold) has a JSJ-decomposition with q non-Seifert fibered submanifolds, then $q \leq g-1$. Let Θ be the union of the boundary tori of W_i for $i \leq n$. As each W_i contains no essential annuli or tori, Θ is the union of the canonical tori in the JSJ-decomposition of M'. None of the W_i are Seifert fibered, so $q = n+1$. Therefore, $q = n+1 \leq g-1 \leq n$, a contradiction. We conclude that S is not stabilized.

We have just shown that the manifold M has an infinite genus Heegaard surface S which is not stabilized. M has infinitely generated fundamental group as the tori ∂W_i for $i \geq 1$ are all incompressible and non-parallel.

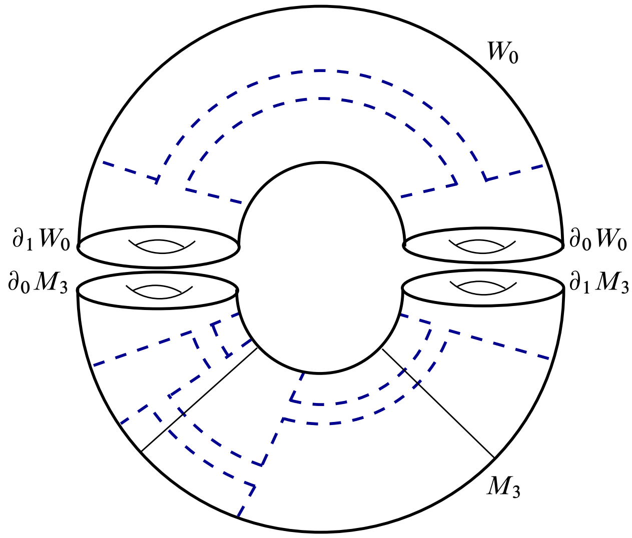

Figure 4: Forming the Heegaard surface T of M' when $n = 3$

Remark There are many other similar constructions of 3–manifolds with infinitely generated fundamental group which have non-stabilized splittings. By allowing arbitrary glueing maps between boundary tori of the compact pieces, one can use a theorem of Bachman, Schleimer, and Sedgewick [1] to show that the amalgamated splittings are not stabilized. We do not pursue this route further in this paper.

Finally, we use the splitting of M to obtain a splitting of the Whitehead manifold W. The manifold M has a single boundary component $\partial_0 W_1$. By attaching a solid torus V to ∂M so that the meridian of the solid torus is equal to the meridian μ_0 of $\partial_0 W_1$, we obtain the Whitehead manifold $W \supset M$. As this same process can be achieved by attaching first a 2–handle and then a 3–ball to ∂M, the surface S is still a Heegaard surface for W. As S is not stabilized in $W - V$, every stabilizing ball of S in W must intersect the compact set V. Thus, S is not end-stabilized. S is, however, stabilized. To see this, recall that S is formed by amalgamating the splitting S'_n (for any given n) to the splittings S_i for $i \geq n$. Interpreted in W, S'_n (for any n) is a splitting of a solid torus. The genus of S'_n is $n+1$ and so, by the classification of splittings of handlebodies, S'_n can be destabilized n times in W. This means, then, that S can be destabilized infinitely many times in W. The stabilizing balls are not properly embedded in W and so only finitely many destabilizations can occur at once.

2.5 Ends of Heegaard surfaces

The remainder of this section is devoted to showing that the inclusion of a Heegaard surface into M induces a homeomorphism of end spaces. Informally this means that the Heegaard surface has one end for each end of the manifold. This theorem is used implicitly in all of the work which follows and may serve to give the reader some feel for the properties of noncompact Heegaard splittings. Before stating the results, we recall the definition of the set of ends of a manifold. (See, for example, Brown and Tucker [7].)

Definition A *ray* in a connected manifold M is a proper map $r\colon \mathbb{R}_+ \to M$. An *end* of a non-compact manifold M is an equivalence class of rays. Two rays $r, s\colon \mathbb{R}_+ \to M$ are equivalent if for every compact set $C \subset M$ there is a number $t_C \in \mathbb{R}_+$ such that the images of $[t_C, \infty)$ under r and under s are in the same component of $M - C$. The set of ends is topologized by declaring that for any compact set C and any non-compact component A of the closure of $M - C$ the set of equivalence classes $\{[r] : \exists t \in \mathbb{R}_+ \text{ with } r([t, \infty)) \subset A\}$ is an open set. These open sets form a basis for the topology on the end space of M. The set of ends of M with this topology is 0–dimensional, compact, and Hausdorff (Raymond [19]).

The proofs of the following lemma and proposition follow suggestions by Martin Scharlemann.

Lemma 2.1 *Let Γ be a locally finite graph properly embedded in an open 3–manifold M. Then the inclusion of $M - \text{int}(\eta(\Gamma))$ into M induces a homeomorphism of ends.*

Proof Let $X = M - \text{int}(\eta(\Gamma))$. Let r and s be two rays determining the same end of X. Let $C \subset M$ be a compact set. X is a closed subset of M. As such, $C \cap X$ is a compact subset of X. Hence, there exists a $t \in [0, \infty)$ such that the images of $[t, \infty)$ under r and s are contained in the same component of $X - C$. This means that the images of $[t, \infty)$ under r and s are contained in the same component of $M - C$. Thus, r and s are rays in M and determine the same end of M. Hence, there is a well-defined map on ends induced by the inclusion of X into M.

We next prove that the induced map on ends is surjective. Suppose that $[r]$ is an equivalence class of ends of M. By general position, there is a representative of this equivalence class which is disjoint from Γ and, hence, there is a representative r which is contained in X. Under the induced map the equivalence class $[r]$ in the set of ends of X is sent to the equivalence class $[r]$ in the set of ends of M. Thus, the induced map on ends is surjective.

Now suppose that $[r]$ and $[s]$ are equivalence classes in the set of ends of X which have the same image in the set of ends of M under the map induced by the inclusion of X into M. Let r and s be representatives of these equivalence classes in the set of ends of X. Since r and s represent the same equivalence class in the set of ends of M, for any compact set $C \subset M$ there is a $t_C \in [0, \infty)$ such that the images of $[t_C, \infty)$ under r and s are contained in the same component of $M - C$. Let $K \subset X$ be a compact set. As X is closed in M, K is a compact subset of M. The images $r([t_K, \infty))$ and $s([t_K, \infty))$ are contained in the same component of $M - K$. The components of $M - K$ are also the path components of $M - K$, so there is a path γ contained in $M - K$ joining $r([t_K, \infty))$ and $s([t_K, \infty))$. By general position, we may homotope γ so that its image is contained in $M - (K \cup \eta(\Gamma))$. That is, γ is a path in $X - K$ joining $r([t_K, \infty))$ and $s([t_K, \infty))$. Thus, $r([t_K, \infty))$ and $s([t_K, \infty))$ are contained in the same component of $X - K$. Since K was an arbitrary compact subset of X, $[r] = [s]$ in the set of ends of X and the induced map on ends is injective.

We now prove that the induced map is bicontinuous. To show continuity, it suffices to show that the preimage of a basis element in the topology of ends of M is open in the ends of X. Let A' be a basis element in the topology of the set of ends of M. By definition, there is a compact set $C \subset M$ and a non-compact component A of $M - C$ such that for each ray r for which $[r] \in A'$ there is $t_r \in [0, \infty)$ such that $r([t_r, \infty))$ is contained in A. By replacing C with $\eta(C)$, we may assume that C and A are submanifolds of M. Since X is closed in M, $C \cap X$ is compact and so by choosing representatives r for each $[r] \in A'$ such that r is a ray in X, we see that $r([t_r, \infty))$ is contained in $A \cap X$.

We claim that $A \cap X$ is connected and non-compact. It is easy to see that $A \cap X$ is path-connected: choose two points $x, y \in A \cap X$. Since A is path-connected, there is a path in M joining them. By general position we may assume that the path is disjoint from Γ. Thus, there is a path in A disjoint from $\eta(\Gamma)$. Hence, $A \cap X$ is path-connected and therefore connected. $A \cap X$ is also non-compact since r is a proper map and the image of $[t_r, \infty)$ under r is contained in $A \cap X$. The preimage of A' is, therefore, contained in the set $A'' = \{[s] : \exists t \in \mathbb{R}_+ \text{ with } s([t, \infty)) \subset (A \cap X)\}$. Suppose, now, that s is a representative for $[s] \in A''$. Since $A \cap X \subset A$, $s([t, \infty)) \subset A$. Thus, the image of $[s]$ under the inclusion map of ends of X into ends of M is contained in A'. Thus, the preimage of A' is A''. A'' is, by definition, an open set in the topology of the set of ends of X. Hence, the induced map on ends is continuous. Since the set of ends of a connected manifold is compact and Hausdorff the induced map also has continuous inverse. Thus, the induced map is a homeomorphism. □

Proposition 2.2 *Let $M = U \cup_S V$ be an absolute Heegaard splitting of a non-compact manifold with compact boundary. Then the inclusion of S into M induces a homeomorphism of ends.*

Proof If $\partial M \neq \varnothing$ we can attach finitely many 2 and 3-handles to ∂M to obtain an open 3–manifold M' containing M. An absolute Heegaard splitting for M is also a Heegaard splitting for M', since the 2 and 3-handles were attached to ∂_- of the compressionbodies. Since we attached only finitely many 2 and 3-handles, the inclusion of M into M' induces a homeomorphism of ends. So, without loss of generality, we may assume that M is open.

Choose spines Σ_U and Σ_V for U and V respectively. Let $\Gamma = \Sigma_U \cup \Sigma_V$. Γ is a locally finite graph properly embedded in M. Let X be the complement of an open regular neighborhood of Γ in M. Since Σ_U and Σ_V are spines of handlebodies giving a Heegaard splitting of M, X is homeomorphic to $S \times I$. By Lemma 2.1, the inclusion of X into M induces a homeomorphism of ends. Since X is homeomorphic to $S \times I$ there is a proper deformation retraction of X onto $S \times \{\frac{1}{2}\}$. Thus the inclusion of S into X is a proper homotopy equivalence and so induces a homeomorphism on ends. Therefore, the inclusion of S into M induces a homeomorphism of ends. □

Remark In [13], Frohman and Meeks prove by algebraic means that a Heegaard surface in a 1–ended 3–manifold is 1–ended.

3 Slide-moves

3.1 Handle-slides

Let H be a compressionbody (absolute or relative) with preferred surface $S = \partial_+ H$. Suppose that we are given a disc set Δ for H (with $\partial\Delta \subset \partial_+ H$). We now describe a process which transforms Δ into a new disc set Δ'.

Let $\alpha \subset \partial_+ H$ be an oriented arc such that $\alpha \cap \partial\Delta = \partial\alpha$. Suppose that the endpoints of α are on distinct discs of Δ. Let D_1 and D_2 be the discs of Δ containing $\partial\alpha$ so that α joins D_1 to D_2. A regular neighborhood of $D_1 \cup \alpha \cup D_2$ has frontier in H consisting of three discs. Two of these discs are parallel to D_1 and D_2, the other has arcs in its boundary which are subarcs of $\eta(\alpha)$. Let $D_1 \underset{\alpha}{\curvearrowright} D_2$ denote this disc. See Figure 5. Let $\Delta' = (\Delta - D_1) \cup (D_1 \underset{\alpha}{\curvearrowright} D_2)$.

Definition The disc set Δ' is obtained from Δ by a *handle-slide* of Δ along α. If D_1, D_2 and α are all disjoint from a closed set X then the handle-slide is said to be done *relative* to X or *(rel X)*.

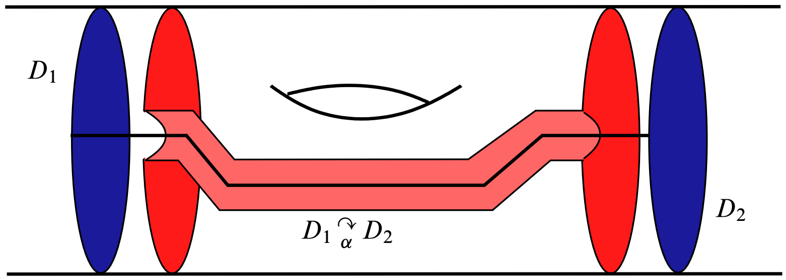

Figure 5: The disc $D_1 \overset{\curvearrowright}{\underset{\alpha}{}} D_2$

Suppose that $A \subset H$ is a subcompressionbody with the property that $\operatorname{fr} A \subset \Delta$. There is a subcompressionbody A' of H with frontier contained in Δ' which we say is *obtained from A by a handle-slide*. The definition of A' depends on the location of D_1 and α:

- If D_1 is not in the frontier of A then A' is equal to A.
- If D_1 is in the frontier of A and α is contained in $\partial_S A$ then we remove the interior of a collar neighborhood of $\alpha \cup D_2$ from A. (The neighborhood of D_2 should be taken to be just on the side of D_2 which α intersects. This way, if $D_2 \subset \operatorname{int} A$, the disc D_2 itself is not removed.) If D_2 wasn't in the frontier of A, it is now contained in $\operatorname{fr} A'$.
- If D_1 is in the frontier of A and α is not contained in $\partial_S A$ then to form A', we add the closure of a regular neighborhood of $\alpha \cup D_2$ to A. (Again, the neighborhood of D_2 should be taken to be just on the side of D_2 which intersects α.)

Remark The subcompressionbodies A and A' may not be homeomorphic (if, for example, both D_1 and D_2 are contained in $\operatorname{fr} A$ and α is not in $\partial_S A$). We do have, however, that $\sigma(A;\Delta)$ is homeomorphic to $\sigma(A';\Delta')$.

Likewise, if R is a topologically closed subsurface of $\partial_+ H$ with the following three properties:

- ∂D_1 is either a component of ∂R or disjoint from ∂R.
- ∂D_2 is either a component of ∂R or disjoint from ∂R.
- The interior of α is disjoint from ∂R.

then we can form a new surface R' which is *obtained from R by a handle-slide*. If $\partial D_1 \cap \partial R = \varnothing$ then R' is defined to be R. If $\partial D_1 \subset \partial R$ and $\alpha \subset R$ then R' is defined to be $\mathrm{cl}(R-\eta(\alpha\cup\partial D_2))$ where the neighborhood of ∂D_2 is a one-sided neighborhood on the side of D_2 which α meets. This way if $\partial D_2\subset\mathrm{int}(R)$ then $\partial D_2\subset\partial R'$. If $\partial D_1\subset\partial R$ and α is not contained in R then R' is defined to be $R\cup\eta(\alpha\cup\partial D_2)$. As before, the neighborhood of ∂D_2 should be taken to be a one-sided neighborhood on the side of ∂D_2 which α meets. Figure 6 shows an example of how to obtain R' from R.

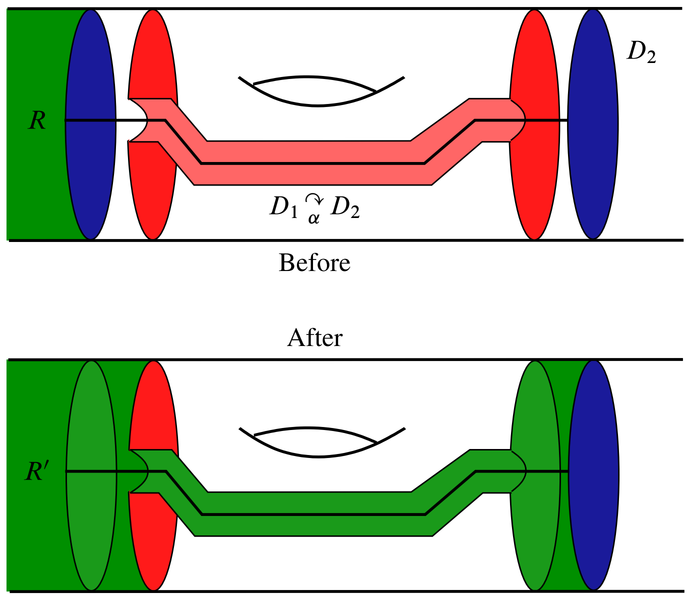

Figure 6: Obtaining R' from R by a handle-slide

Bonahon developed the use of handle-slides to prove results about compressionbodies. The following proposition and its corollaries are based on his work in [3]. For proofs see Appendix B of that paper. The essence of the proof of Proposition 3.1 shows up in Step 6 of the proof of Proposition 5.2 of this paper.

Proposition 3.1 *If D is a boundary-reducing disc for H then there is a collection of defining discs for H which are disjoint from D.*

Corollary 3.2 *Boundary-reducing a compressionbody along a finite disc set results in compressionbodies.*

Corollary 3.3 *Given any finite disc set for a compressionbody, there is a defining collection of discs for the compressionbody which contains the given disc set.*

Corollary 3.4 *A subcompressionbody with compact frontier is a compressionbody.*

The following definition will be useful later. We include it here since Corollary 3.5 follows from Corollary 3.3.

Definition If A and B are relative compressionbodies with $A \subset B$, we say that A is *correctly embedded* in B if $\partial_+ A \subset \partial_+ B$ and if every closed component of $\partial_- A$ is also a component of $\partial_- B$.

Another way of stating the definition is that $A \subset B$ is correctly embedded if each component of $\mathrm{fr}\, A$ is a component of $\partial_- A$ which has non-empty boundary and is properly embedded in B. Figure 7 schematically depicts an example of a relative compressionbody A correctly embedded in a handlebody B.

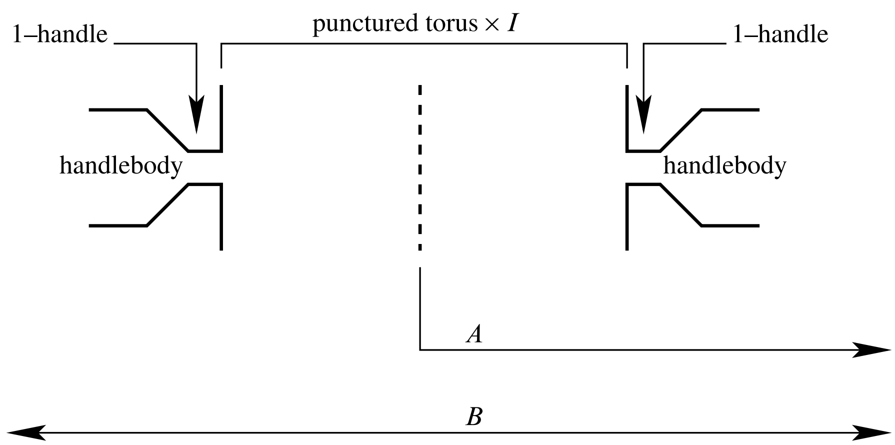

Figure 7: A is correctly embedded in B

Remark The notion of "correctly embedded" is similar to Canary and McCullough's "normally imbedded" [8, Section 3.4].

Corollary 3.5 *Suppose that A is a compact relative compressionbody correctly embedded in a relative compressionbody B. Then $\mathrm{cl}(B - A)$ is a relative compressionbody. In particular, if each closed component of $\partial_- B$ is contained in A then $\mathrm{cl}(B - A)$ is a handlebody.*

Proof Choose a defining set of discs Δ_A for A. Boundary-reduce B along Δ_A to obtain B'. Corollary 3.2 implies that each component of B' is a (relative) compressionbody. Each component of B' was either contained in A or contains a copy of $\mathrm{fr}\, A \times I$

with $\operatorname{fr} A \times \{1\}$ a subsurface of $\partial B'$. Subtracting $\operatorname{fr} A \times I$ from those components of B' simply removes a collar of a subsurface of $\partial B'$ from B' and hence leaves us with a compressionbody. But this is exactly $\operatorname{cl}(B - A)$. If each closed component of $\partial_- B$ is contained in A then, if C is a component of B' which is not contained in A, $\partial_- C$ contains no closed components. By our definition of "compressionbody", ∂C is compact and so C is formed by adding one-handles to $F \times I$ where F is a compact surface, no component of which is boundary-less. Thus, C is obtained by adding one-handles to handlebodies and, so, is a handlebody. We then form a component of $\operatorname{cl}(B - A)$ by removing a neighborhood of $\operatorname{fr} A \cap C$ from C. The result is homeomorphic to C and so is a handlebody. □

Remark We may not be able to choose $\operatorname{cl}(\partial_+ B - \partial_+ A)$ to be the preferred surface of $\operatorname{cl}(B - A)$. For example, if B is a compact relative compressionbody, push each non-closed component of $\partial_- B$ slightly into B and take A to be the closure of the complement of the product regions.

3.2 Slide-moves and isotopies

For the remainder of this section, let S be an absolute Heegaard surface dividing a 3–manifold M with compact boundary into absolute compressionbodies U and V.

If we have disc sets $\overline{\Delta}_1$ for U and $\overline{\Delta}_2$ for V which are disjoint from each other we can perform handle-slides on each disc set individually. The remainder of this section studies how these handle-slides affect the surface S.

Definition A *2–sided disc family* $\overline{\Delta}$ for S in M is the union of disc sets $\overline{\Delta}_1$ and $\overline{\Delta}_2$ for U and V with the property that the discs of $\overline{\Delta} = \overline{\Delta}_1 \cup \overline{\Delta}_2$ are pairwise disjoint.

We can expand the notion of a handle-slide to that of a slide-move on the 2–sided disc family $\overline{\Delta} = \overline{\Delta}_1 \cup \overline{\Delta}_2$:

Definition A *slide-move* of $\overline{\Delta}$ is one of the following operations:

(M1) Perform a handle-slide (rel $\partial\overline{\Delta}_2$) of $\overline{\Delta}_1$.

(M2) Add to $\overline{\Delta}_1$ a boundary-reducing disc for U which is disjoint from $\overline{\Delta}_1 \cup \overline{\Delta}_2$.

(M3) Perform a handle-slide (rel $\partial\overline{\Delta}_1$) of $\overline{\Delta}_2$.

(M4) Add to $\overline{\Delta}_2$ a boundary-reducing disc for V which is disjoint from $\overline{\Delta}_1 \cup \overline{\Delta}_2$.

Suppose that A is a subcompressionbody of U or V with $\operatorname{fr} A \subset \overline{\Delta}$. If we perform a slide-move on $\overline{\Delta}$ to obtain a 2–sided disc family Δ we can obtain from A a subcompressionbody A' with frontier contained in Δ: If slide-move (M2) or (M4) is performed, A' is defined to be equal to A. If $A \subset U$ and slide-move (M1) is performed, A' is defined to be the subcompressionbody obtained from A by the handle-slide (see Section 3.1). Similarly, if $A \subset V$ and slide-move (M3) is performed, A' is defined to be the subcompressionbody obtained from A by the handle-slide. If we perform a finite sequence of slide-moves to obtain Δ from $\overline{\Delta}$ there is a subcompressionbody A' with $\operatorname{fr} A' \subset \Delta$ obtained from A by a finite number of handle-slides. We say that Δ is *obtained from* $\overline{\Delta}$ *by slide-moves* and that A' is *obtained from* A *by slide-moves*.

Suppose that R is a topologically closed subsurface of S with $\partial R \subset \partial \overline{\Delta}$. The boundary components of R may bound discs in either U or V (ie, discs which are in $\overline{\Delta}_1$ or $\overline{\Delta}_2$). If we perform a finite sequence of slide-moves on $\overline{\Delta}$ to obtain Δ we may define a subsurface R' of S which is *obtained from* R *by slide-moves* and has boundary contained in $\partial\Delta$. The definition is basically the same as the definition when a single handle-slide is performed: If slide-moves (M2) or (M4) are performed, R is left unchanged. If (M1) or (M3) is performed, so that a disc D_1 is slid over a disc D_2 via a path α, we can define R' as before (see Section 3.1).

The following proposition is an integral part of the proof of Theorem 5.1.

Proposition 3.6 *Suppose that* $\overline{\Delta}$ *is a 2–sided disc family for* S *and that* Δ *is obtained from* $\overline{\Delta}$ *by slide-moves. Then there is a finite collection of disjoint discs* $\mathcal{D}$ *with* $\partial\mathcal{D} \subset \sigma(S;\overline{\Delta})$ *and a proper ambient isotopy of* $\sigma(S;\Delta)$*, fixed outside a compact subset of* M*, with the following properties:*

(i) *The discs* $\mathcal{D} = D_1 \cup \ldots \cup D_p$ *have an ordering such that the disc* D_i *intersects only on its boundary the surface* $\sigma(S;\overline{\Delta})$ *compressed along* $D_1, \ldots, D_{i-1}$*. (See the remark below.)*

(ii) *The isotopy takes* $\sigma(S;\Delta)$ *to* $\sigma(S;\overline{\Delta})$ *compressed along* $\mathcal{D}$*.*

(iii) *Let* R *be a topologically closed subsurface of* S *such that* $\partial R \subset \partial\overline{\Delta}$ *and* R' *the subsurface of* S *obtained from* R *by that sequence of slide-moves. The isotopy takes* $\sigma(R';\Delta)$ *to the surface obtained from* $\sigma(R;\overline{\Delta})$ *by compressing along whatever discs of* $\mathcal{D}$ *have boundary in* R*.*

Remark The discs $\mathcal{D}$ may intersect S on their interiors, so part of the conclusion of the theorem is that when we compress $\sigma(S;\overline{\Delta})$ along the discs $D_1, \ldots, D_{i-1}$ we have chosen the regular neighborhoods of $D_1, \ldots, D_{i-1}$ so that D_i, although it may intersect S, does not intersect $\sigma(S;\overline{\Delta})$ compressed along $D_1, \ldots, D_{i-1}$. We will abuse notation and write $\sigma(S;\overline{\Delta}\cup\mathcal{D})$ for the surface obtained from $\sigma(S;\overline{\Delta})$ by compressing along the discs $\mathcal{D}$ in the order given. Similarly, if R is a topologically closed subsurface of S with $\partial R\subset\partial\overline{\Delta}$ we will use $\sigma(R;\overline{\Delta}\cup\mathcal{D})$ to indicate the surface obtained from R by compressing along the discs of $\overline{\Delta}$ and then $\mathcal{D}$ in the given order (rather, compressing along those discs which have boundary on R).

The proof of Proposition 3.6 will make use of the following lemma:

Lemma 3.7 *If $\overline{\Delta}_i$ is a disc family for S with $\overline{\Delta}_i\subset U$ or $\overline{\Delta}_i\subset V$ and if Δ_i is obtained from $\overline{\Delta}_i$ by a single handle-slide of the disc D_1 over the disc D_2 via a path α, then there is a proper ambient isotopy of M, fixed off a compact set, with the following properties:*

(a) *the isotopy takes $\sigma(S;\Delta_i)$ to $\sigma(S;\overline{\Delta}_i)$.*

(b) *if the handle-slide is relative to a closed set X then we can choose the isotopy to be relative to X.*

(c) *if R is a subsurface of S with all of the following properties:*
   - *∂D_1 is either a component of ∂R or disjoint from ∂R.*
   - *∂D_2 is either a component of ∂R or disjoint from ∂R.*
   - *The interior of α is disjoint from ∂R.*

   then if R' is the subsurface of S obtained from R by the handle-slide, the isotopy takes $\sigma(R';\Delta_i)$ to $\sigma(R;\overline{\Delta}_i)$.

Proof of Lemma 3.7 Recall that Δ_i is obtained from $\overline{\Delta}_i$ by removing the disc D_1 and replacing it with the disc $D_1 \overset{\curvearrowright}{\underset{\alpha}{}} D_2$. Let $S'=\sigma(S;\Delta_i)$. When we compress along the discs D_2 and $D_1 \overset{\curvearrowright}{\underset{\alpha}{}} D_2$ we end up with a situation as depicted in Figure 8. Note that the figure depicts four discs parallel to D_2 since a disc parallel to D_2 makes up part of $D_1 \overset{\curvearrowright}{\underset{\alpha}{}} D_2$ and both $D_1 \overset{\curvearrowright}{\underset{\alpha}{}} D_2$ and D_2 are in Δ_i.

After compressing along $D_1 \overset{\curvearrowright}{\underset{\alpha}{}} D_2$ we see that there is regular neigborhood N (in the compressionbody containing $\overline{\Delta}_i$) of α homeomorphic to $D^2\times I$ with $D^2\times\{0\}$ glued to a copy of D_1 and $D^2\times\{1\}$ glued to a copy of $D_2\times I$. Take a regular neighborhood in the compressionbody containing $\overline{\Delta}_i$ of $N\cup(D_2\times I)$ which misses the rest of the surface S'. This regular neighborhood is a 3–ball B. Choose the regular neighborhoods so that $B\cap S'\subset\partial B$. The intersection of B with D_1 is a disc which is a regular neighborhood (in the compressionbody) of the point $\alpha\cap D_1$. Slightly enlarge B in M to a ball B' and perform an ambient isotopy supported on B' and which takes $B-D_1$ to $B\cap D_1$. Next use the regular neighborhood of α to isotope back to S the portion of S' which forms part of the boundary of a regular neighborhood of α

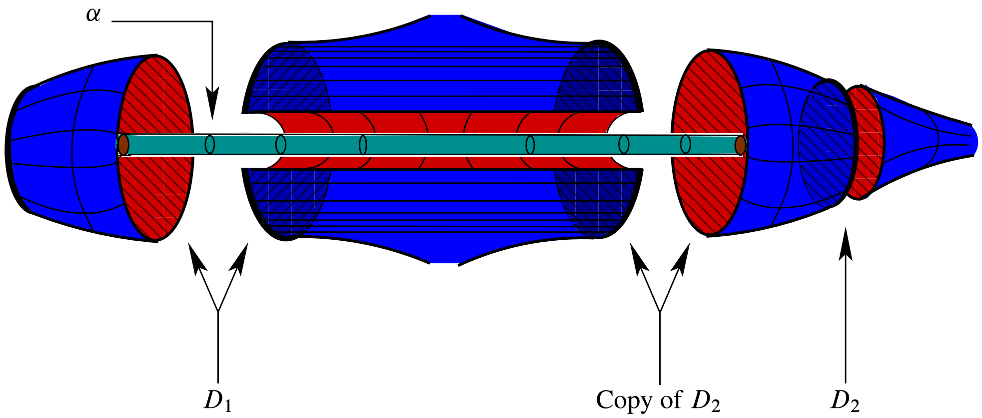

Figure 8: Compressing along Δ_i

(the "trough"). The result is the same as if we had compressed along $\overline{\Delta}_i$. This proves statement (a). The isotopy described is the identity off a neighborhood of $D_1 \cup \alpha \cup D_2$ and so is a proper isotopy.

If the handle-slide was relative to a closed set X, then by choosing the neighborhoods of D_1, D_2, and α to be disjoint from X, the isotopy described is relative to X. This proves statement (b).

To prove conclusion (c), we examine the possibilities. Suppose that R is a subsurface of S as in the statement and suppose that R' is obtained from R by the handle-slide. Recall that B is the ball which is a regular neighborhood of α and D_2. The important observation is that the isotopy takes $\partial B - D_1$ into D_1.

- Suppose that $\partial D_1 \subset \partial R$ and that $\alpha \subset R$. In this case, R' equals $R - \partial B$. The isotopy fixes $R' - \eta(\partial D_1 \underset{\alpha}{\curvearrowright} D_2 \cup D_2)$. And so the isotopy takes $\sigma(R'; \Delta_i)$ into $\sigma(R; \overline{\Delta}_i)$
- Suppose that $\partial D_1 \subset \partial R$ and that α is not contained in R. Then $\sigma(R'; \Delta_i)$ equals $\sigma(R; \overline{\Delta}_i) \cup \partial B$. The isotopy described takes $\partial B - D_1$ into $\sigma(R; \overline{\Delta}_i)$.
- Suppose that ∂D_1 is not contained in R. The previous case shows that $\sigma(\mathrm{cl}(S - R'); \Delta_i)$ is taken into $\sigma(\mathrm{cl}(S - R); \overline{\Delta}_i)$ and by part (a) we must have that $\sigma(R'; \Delta_i)$ is taken into $\sigma(R; \overline{\Delta}_i)$.
- Suppose that $\partial D_1 \subset \mathrm{int}\, R$. In this case, $\partial B - D_1$ is contained in $\sigma(R'; \Delta_i)$ and $(\partial B - D_1) \cap S$ in $\sigma(R; \overline{\Delta}_i)$. The isotopy clearly satisfies (c). □

We now turn to the proof of Proposition 3.6.

Proof of Proposition 3.6 Suppose that the 2–sided disc family Δ is obtained from the 2–sided disc family $\overline{\Delta}$ by a finite sequence $\{\mu_1, \dots, \mu_n\}$ of slide-moves. Each μ_i is a slide-move of type (M1), (M2), (M3), or (M4). We prove the proposition by induction on the length of the sequence. If the sequence is of length 0 the result is immediate so suppose that $n \geq 1$ and that the proposition is true for all sequences with $n-1$ elements.

Let δ be the 2–sided disc family obtained from $\overline{\Delta}$ by the sequence $\nu = \{\mu_1, \dots, \mu_{n-1}\}$. Using the notation from the statement of the proposition: let r be the subsurface of S obtained from the subsurface R by the sequence ν.

By the induction hypothesis, there is a collection of disjoint discs $\mathcal{E}$ with boundary on $\sigma(S; \overline{\Delta})$ and there is an ambient isotopy f, fixed off a compact set, which takes $\sigma(S;\delta)$ to $\sigma(S; \overline{\Delta} \cup \mathcal{E})$ and which takes the surface $\sigma(r;\delta)$ into the surface $\sigma(R; \overline{\Delta} \cup \mathcal{E})$. (Recall that this means R compressed along those discs of $\overline{\Delta} \cup \mathcal{E}$ with boundary on R.) We assume that f also satisfies conclusions (i), (ii), and (iii).

The 2–sided disc family Δ is obtained from the 2–sided disc family δ by a single slide-move μ_n of type (M1), (M2), (M3), or (M4). We divide the proof into the case when μ_n is of type (M2) or (M4) and the case when the slide-move is of type (M1) or (M3).

Case 1: μ_n is of type (M2) or (M4) If μ_n is a slide-move of type (M2) or (M4), Δ is obtained from δ by adding a single disc D' to δ. In this case, $R' = r$. The ambient isotopy f takes the disc D' to a disc D with boundary on $\sigma(S; \overline{\Delta} \cup \mathcal{E})$. By a further isotopy, if necessary, we may arrange that the disc D has boundary disjoint from the remnants of $\mathcal{E}$ and so has boundary on $\sigma(S; \overline{\Delta})$ and that D is disjoint from the discs of $\mathcal{E}$, though it may intersect S in a neighborhood of $\mathcal{E}$. Let $\mathcal{D}$ equal $\mathcal{E} \cup D$. We need to show that we have satisfied the conclusions of the proposition.

To prove (i), recall that the discs $\mathcal{E}$ are numbered. The disc D should be given the next number. Since $\operatorname{int} D'$ is disjoint from $\sigma(S;\delta)$ and the isotopy is an ambient isotopy the disc D has interior disjoint from $\sigma(S; \overline{\Delta} \cup \mathcal{E})$. Thus, D intersects $\sigma(S; \overline{\Delta} \cup \mathcal{E})$ only on ∂D.

Conclusion (ii) is clear, since the isotopy f took $\sigma(S; \Delta' - D')$ to $\sigma(S; \overline{\Delta} \cup \mathcal{E})$ and also took D' to D which is a disc with boundary on the surface obtained from $\sigma(S; \overline{\Delta})$ by compressing along $\mathcal{E}$.

To prove (iii), recall that since μ_n is the slide-move consisting of adding the disc D' to δ, the surface R' equals the surface r. The induction hypothesis says that f takes $\sigma(r;\delta)$ to the surface $\sigma(R; \overline{\Delta} \cup \mathcal{E})$. Conclusion (ii) shows that the isotopy f takes the surface $\sigma(R'; \Delta = \delta \cup D')$ to $\sigma(R; \overline{\Delta} \cup \mathcal{D})$.

Case 2: μ_n is of type (M1) or (M3) If the slide-move μ_n is of type (M1) or (M3) we have obtained Δ from δ by a single handle-slide of δ_1 in U or δ_2 in V. Without loss of generality, assume that μ_n is a slide-move of type (M3), so that Δ is obtained from δ by the slide-move μ_n of δ_2. By Lemma 3.7, there is an ambient isotopy g of M, fixed off a compact set, which satisfies properties (a), (b), and (c). In particular, g takes $\sigma(S;\Delta)$ to $\sigma(S;\delta)$ because it takes $\sigma(S;\Delta_2)$ to $\sigma(S;\delta_2)$ and is performed relative to $\partial\delta_1$ (property (b)). Let h be the ambient isotopy formed by performing g and then performing f. Let $\mathcal{D} = \mathcal{E}$. We show that h satisfies conclusions (i), (ii), and (iii).

Conclusions (i) and (ii) follow immediately from the induction hypothesis on f and property (a) of Lemma 3.7.

To prove conclusion (iii), recall that r denotes the surface obtained from R by the sequence of slide-moves $\{\mu_1, \dots, \mu_{n-1}\}$. The surface R' is obtained from r by the handle-slide μ_n. Property (c) from Lemma 3.7 shows that g takes $\sigma(R';\Delta_2)$ to $\sigma(r;\delta_2)$. The isotopy g is an ambient isotopy which was performed relative to $\partial\delta_1$, so g also takes $\sigma(R';\Delta)$ to $\sigma(r;\delta)$. By induction, the isotopy f takes $\sigma(r;\delta)$ to $\sigma(R;\overline{\Delta} \cup \mathcal{D})$. And so h satisfies (iii). □

4 Relative Heegaard splittings

4.1 The outer collar property

Recall from Section 3.1 that a compact relative compressionbody A is correctly embedded in a compressionbody B if the frontier of A in B consists only of components of $\partial_- A$ which have boundary. Corollary 3.5 states that, in this case, $\mathrm{cl}(B - A)$ is a relative compressionbody with some preferred surface. However, $\mathrm{cl}(B - A)$ may not be correctly embedded in B as we may not be able to choose $\mathrm{cl}(\partial_+ B - \partial_+ A)$ to be the preferred surface of $\mathrm{cl}(B - A)$.

In this section we explore situations in which we can "come close" to having $\partial_+ \mathrm{cl}(B - A)$ equal $\mathrm{cl}(\partial_+ B - \partial_+ A)$. These situations will arise when we have exhaustions of noncompact absolute compressionbodies.

Definition Suppose that $\{K_i'\}$ is an exhaustion for a noncompact absolute compressionbody U. If each K_i' is a relative compressionbody correctly embedded in U and each K_i' is correctly embedded in K_{i+1}' then $\{K_i'\}$ is a *correctly embedded exhaustion* for U.

The following definition is somewhat technical, but will be useful for statements of results in Section 5. Recall that a collaring set of discs for a compressionbody H is a set of discs which cuts off a copy of $\partial_- H \times I$.

Definition Suppose that $\{K_i'\}$ is a correctly embedded exhaustion for U. Suppose also that for each $i \geq 2$ there is an embedding of $(\mathrm{fr}\, K_i' \times I, (\partial\, \mathrm{fr}\, K_i') \times I)$ into $(\mathrm{cl}(K_i' - K_{i-1}'), \partial_+ U \cap \mathrm{cl}(K_i' - K_{i-1}'))$ so that $\mathrm{fr}\, K_i' = \mathrm{fr}\, K_i' \times \{0\}$ and so that $\mathrm{fr}\, K_i' \times \{1\}$ is a subsurface of $\partial_+ U$ except at a finite number of open discs. Then $\{K_i'\}$ is said to have the *outer collar property*.

Remark The open discs of $\mathrm{fr}\, K_i' \times \{1\}$ which are not contained in $\partial_+ U$ are the interiors of a set of collaring discs for K_i'.

Definition Suppose that $\{K_i'\}$ is a correctly embedded exhaustion for U. Additionally, suppose that for each $i \geq 2$ there is an embedding of $(\mathrm{fr}\, K_{i-1}' \times I, (\partial\, \mathrm{fr}\, K_{i-1}') \times I)$ into $(\mathrm{cl}(K_i' - K_{i-1}'), \partial_+ U \cap \mathrm{cl}(K_i' - K_{i-1}'))$ so that $\mathrm{fr}\, K_{i-1}' = \mathrm{fr}\, K_{i-1}' \times \{0\}$ and so that $\mathrm{fr}\, K_{i-1}' \times \{1\}$ is a subsurface of $\partial_+ U$ except at a finite number of open discs. Then $\{K_i'\}$ is said to have the *inner collar property*.

In this paper, it is the outer collar property which is most used. The inner collar property makes an appearance in the appendix. It may, therefore, be helpful to give an example of an exhaustion of a handlebody with the outer collar property. Our example is, in fact, an exhaustion of a one-ended, infinite genus handlebody which has both the inner and outer collar properties.

Example For each natural number i, let F_i be a compact, connected surface with non-empty boundary. Let $P_i = F_i \times I$. Recall that P_i is a handlebody. For each $i \geq 2$ join $F_i \times \{0\}$ to $F_{i-1} \times \{1\}$ by a one-handle H_i. Denote the union of all the product regions and all the one-handles by H. See Figure 9 for a schematic depiction of H. Let D_i be a disc which is a cocore of the one-handle H_i. Let

$$K_i' = P_1 \cup H_2 \cup P_2 \cup \ldots \cup P_{2i-1} \cup H_{2i} \cup (F_{2i} \times [0, 1/2]).$$

The construction makes clear that $\{K_i'\}$ is a correctly embedded exhaustion of the handlebody H. The frontier of K_i' is $F_{2i} \times \{\frac{1}{2}\}$ which is an incompressible surface in H. For $i \geq 2$, compressing K_i' along D_{2i} leaves two components, one of which is $F_{2i} \times [0, \frac{1}{2}] = \mathrm{fr}\, K_i' \times I$. This component is disjoint from K_{i-1}'. From the construction, it is clear that $\{K_i'\}$ has the outer collar property. For $i \geq 2$, boundary-reducing $\mathrm{cl}(K_i' - K_{i-1}')$ along the disc D_{2i-1} leaves two components, one of which is $F_{2i-2} \times [\frac{1}{2}, 1] = \mathrm{fr}\, K_{i-1}' \times I$. Again, from the construction it is clear that $\{K_i'\}$ has the inner collar property.

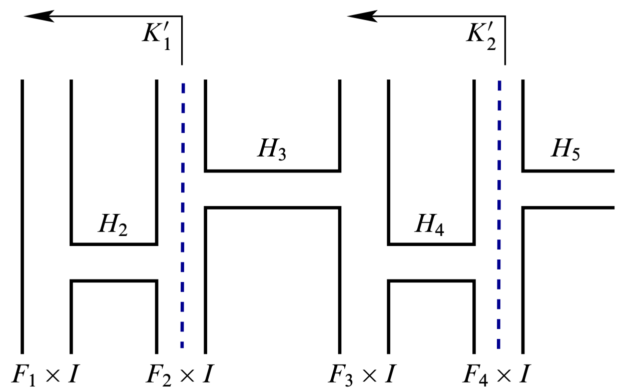

Figure 9: Example of an exhaustion with the inner and outer collar properties

Certainly not every correctly embedded sequence has the outer collar property. If, for example, for some i, $\partial_- K'_{i-1}$ was not a disc and bounded a product region with $\partial_- K'_i$, the sequence would not have the outer collar property. If a sequence has both the inner and outer collar properties we can take $\mathrm{cl}(\partial_+ K'_{i+1} - \partial_+ K'_i)$ to be the preferred surface for the relative compressionbody $\mathrm{cl}(K'_{i+1} - K'_i)$. It is in this sense that a sequence with the outer collar property "comes close" to having $\mathrm{cl}(\partial_+ K'_{i+1} - \partial_+ K'_i)$ the preferred surface for the relative compressionbody $\mathrm{cl}(K'_{i+1} - K'_i)$.

Exhaustions with the outer collar property are fairly common:

Lemma 4.1 *Suppose that $\{K'_i\}$ is a correctly embedded exhaustion of the absolute compressionbody U. Then there is a subsequence with the outer collar property.*

Proof Let $\{L_i\}$ be an exhaustion of U by subcompressionbodies. Recall that the frontier of a subcompressionbody consists of properly embedded discs. Take subsequences of $\{K'_i\}$ and $\{L_i\}$ so that for all i, $L_i \subset K'_i \subset L_{i+1}$. Each inclusion should be into the interior of the succeeding submanifold. Fix some $i \in \mathbb{N}$.

By Corollary 3.3, we may choose a defining collection of discs Δ for K'_i which includes the discs $\mathrm{fr}\, L_i$. Boundary-reducing K'_i along Δ leaves us with 3–balls and products $\partial_- K'_i \times I$. Since $K'_{i-1} \subset L_i$ and $\mathrm{fr}\, L_i$ separates U we have that the remnants of K'_{i-1} are completely contained in the 3–balls. Thus the product regions $\partial_- K'_i \times I$ are contained completely in $\mathrm{cl}(K'_i - K'_{i-1})$. Label $\partial_- K'_i$ with $\partial_- K'_i \times \{1\}$. The discs of Δ which show up on $\partial_- K'_i \times \{0\}$ can be taken to be our collaring set of discs. This collaring set of discs and the product region $\partial_- K'_i \times I$ are contained in $K'_i - K'_{i-1}$ so the sequence $\{K'_i\}$ now has the outer collar property. □

4.2 Relative Heegaard splittings

Suppose that $M = U \cup_S V$ is an absolute Heegaard splitting of a non-compact 3–manifold with compact boundary, containing no S^2 components. If $N \subset M$ is a compact submanifold, the surface $S \cap N$ cannot possibly give an absolute Heegaard splitting of N as S is non-compact and N is compact. It can, however, give a relative Heegaard splitting of N.

We will eventually look at the relationship between relative Heegaard splittings and absolute Heegaard splittings, but first we show how exhaustions of M by compact submanifolds which inherit relative Heegaard splittings from S give rise to correctly embedded exhaustions of U.

Definition A submanifold N contained in M is *adapted* to S if $(U \cap N) \cup_{S \cap N} (V \cap N)$ is a relative Heegaard splitting of N and $(U \cap N)$ is correctly embedded in U and $(V \cap N)$ is correctly embedded in V. An exhaustion $\{K_i\}$ is *adapted* to S if each K_i is adapted to S. It is *perfectly adapted* to S if it is adapted to S and, additionally, each $\mathrm{cl}(K_{i+1} - K_i)$ is adapted to S.

Remark The requirement that $(U \cap N)$ and $(V \cap N)$ are correctly embedded in U and V respectively means that $\mathrm{fr}\, N$ can have no closed components which are contained entirely in U or V: such a component would have to be a component of $\partial_-(U \cap N)$ or $\partial_-(V \cap N)$ as it would not be a subsurface of S. This, however, means that $U \cap N$ or $V \cap N$ is not correctly embedded in U.

In constructing an exhaustion that is adapted to S, the requirement that $U \cap N$ and $V \cap N$ are correctly embedded in U and V is a minor one. To see this, suppose that a compact submanifold $N \subset M$ containing ∂M has the property that $N \cap U$ and $N \cap V$ are relative compressionbodies with preferred surfaces $S \cap N$. It is easy to adjust N so that $U \cap N$ and $V \cap N$ are correctly embedded. If $U \cap N$, say, is not correctly embedded there must be a component F of $\partial_-(U \cap N) - \partial M$ which is a closed surface. Since U is an absolute compressionbody, $\mathrm{H}_2(U, \partial_- U) = 0$. Thus either F, or F and components of $\partial_- U \cap \partial M$, bound(s) a compact submanifold L of U. L cannot be interior to N as $N \cap U$ is a relative compressionbody with non-empty preferred surface and $F \cup \partial_- U$ is contained in $\partial_-(U \cap N)$. Thus L is exterior to N. Since $\partial_- U \subset \partial M \subset N$, we have that $\partial L = F$. In fact, $(N \cap U) \cup L$ must still be a relative compressionbody: Note that F must be compressible in L as F is incompressible in $N \cap U$. (∂_- of a compressionbody is incompressible in the compressionbody). Every closed incompressible surface in U is parallel to $\partial_- U$. Boundary-reducing L is the same as adding a 2–handle to $N \cap U$ along a curve in $F \subset \partial_-(N \cup L)$.

Adding a 2–handle to ∂_- of a compressionbody preserves the fact that we have a relative compressionbody (up to the introduction of spherical boundary components). Eventually, our surface is a collection of spheres, which, since U is irreducible, bound balls in U. Including these balls into N (with the 2–handles attached) also preserves the fact that we have a relative compressionbody.

Lemma 4.2 *If $\{K_i\}$ is an exhaustion of M adapted to S with $\partial M \subset K_1$ then $\{K_i \cap U\}$ and $\{K_i \cap V\}$ are correctly embedded exhaustions of U and V respectively.*

Proof Let X denote either U or V. Since $\{K_i\}$ is adapted to S, by definition each $K_i \cap X$ is correctly embedded in X. Thus, $\partial_+(K_i \cap X) \subset \partial_+(K_{i+1} \cap X)$. Furthermore, any closed component of $\partial_-(K_i \cap X)$ is a component of $\partial_- X$ which is contained in $\partial_-(K_{i+1} \cap X)$. Thus, each closed component of $\partial_-(K_i \cap X)$ is a component of $\partial_-(K_{i+1} \cap X)$. Hence, $K_i \cap X$ is correctly embedded in $K_{i+1} \cap X$. □

Definition If $\{K_i\}$ is an exhaustion of M adapted to S with $\partial M \subset K_1$ and such that $\{K_i \cap U\}$ has the outer collar property we say that $\{K_i\}$ has *the outer collar property with respect to U*.

Corollary 4.3 *If $\{K_i\}$ is an exhaustion of M adapted to S with $\partial M \subset K_1$ then there is a subsequence which has the outer collar property with respect to U.*

Proof By Lemma 4.2, $\{K_i \cap U\}$ is a correctly embedded exhaustion of U. By Lemma 4.1, there is an infinite subset $\mathcal{N}$ of $\mathbb{N}$ such that $\{K_i \cap U\}_{i\in\mathcal{N}}$ has the outer collar property. Hence, $\{K_i\}_{i\in\mathcal{N}}$ has the outer collar property with respect to U. □

4.3 Balanced exhaustions

We've shown so far that if M has an exhaustion adapted to S we can find one which has the outer collar property. We've not yet addressed the question of the existence of an exhaustion adapted to S. We do that now. This construction is a variation of the construction given by Frohman and Meeks in [13].

Recall that $M = U \cup_S V$ is an absolute Heegaard splitting of a non-compact 3–manifold with compact boundary. Let A and B be compact subcompressionbodies of U and V respectively with the property that $\partial_S A \subset \operatorname{int}(\partial_S B)$. Let C be a regular neighborhood of $A \cup B$.

Definition A set C constructed in such a manner will be called a *balanced submanifold* of M (with respect to S). An exhaustion $\{C_i\}$ of M will be called a *balanced exhaustion* for M (with respect to S) if each $C_i = \eta(A_i \cup B_i)$ is a balanced submanifold and if, for all i, $\partial_S B_i \subset \operatorname{int}(\partial_S A_{i+1})$.

The next lemma guarantees that balanced submanifolds are adapted to the Heegaard surface. Consequently, we will say that such a set C is a *balanced submanifold of M (adapted to S)*.

Lemma 4.4 (Frohman and Meeks [13, Proposition 2.2]) *If C is a balanced submanifold of M with respect to S then C is adapted to S.*

Proof We must show that $(U \cap C) \cup_{S \cap C} (V \cap C)$ is a relative Heegaard splitting of C. In other words, we must show that $U \cap C$ and $V \cap C$ are both relative compressionbodies with preferred surface $S \cap C$.

Assume that C is a regular neighborhood of $A \cup B$ where A and B are compact subcompressionbodies of U and V respectively and $\partial_S A \subset \partial_S B$. We have $C \cap V = \eta(B)$ so $C \cap V$ is a relative compressionbody with preferred surface $S \cap C$. To obtain $C \cap U$ we take a regular neighborhood of $\mathrm{cl}(\partial_S B - \partial_S A)$ in U and glue it to A. An alternative way of performing the construction is as follows.

Let D be the collection of discs which make up the frontier of A. Take a regular neighborhood of D and let D' be the components of the frontier of the neighborhood which are not in A. Let $F = \mathrm{cl}(\partial_S B - (\partial_S A \cup \eta(D)))$. Take a regular neighborhood of F in $U - A$. Consider F to be $F \times \{1\}$. Since $D' \subset F$, this regular neighborhood contains $D' \times I$. See Figure 10. This revised neighborhood is $\partial_- C \times I$. We may then add one-handles so that one end of each one-handle is on a disc of $D' \times \{1\}$ and the other end is on the corresponding disc of D. It is clear that $S \cap C$ is the preferred surface of this compressionbody. □

To obtain a balanced exhaustion of M, start by taking exhaustions $\{A_i\}$ and $\{B_i\}$ of U and V by subcompressionbodies. Since each A_i and each B_i are compact we may take subsequences of $\{A_i\}$ and $\{B_i\}$ so that, for all i, $\partial_S A_i \subset \partial_S B_i \subset \partial_S A_{i+1}$. Each of the inclusions should be into the interior of the succeeding surface.

A component of the frontier of a balanced submanifold C can be thought of as being a compact subsurface of S with discs, each contained entirely in U or V, glued onto the boundary components. In fact, since each component of the frontier of each balanced submanifold intersects S, neither $\partial_-(C \cap U)$ nor $\partial_-(C \cap V)$ have components which are closed surfaces not contained in ∂M. Thus, if we have a balanced exhaustion $\{C_i\}$ of M adapted to S with $\partial M \subset C_1$, it is adapted to S in the sense of the definition given at the beginning of this section. By Corollary 4.3, we can take a subsequence of $\{C_i\}$ so that it has the outer collar property.

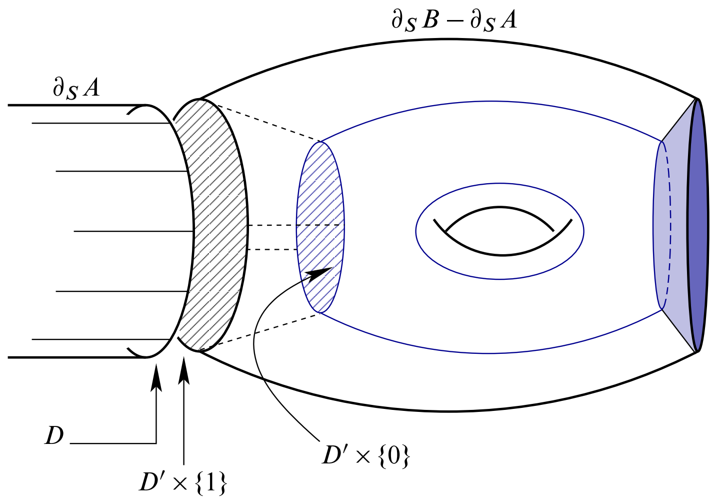

Figure 10: Adding a regular neighborhood of $\partial_S B - \partial_S A$ to A gives us a relative compressionbody

Remark Even though we have a balanced exhaustion $\{C_i\}$ of M which is adapted to S and has the outer collar property, there is no reason to suppose that it is perfectly adapted to S. The difficulty is in the fact that $\mathrm{cl}(C_{i+1} - C_i) \cap U$ may not be a relative compressionbody with preferred surface $S \cap \mathrm{cl}(C_{i+1} - C_i)$.

Let $\{C_i\}$ be a balanced exhaustion for M adapted to S. Each C_i is the neighborhood of $A_i \cup B_i$ where A_i and B_i are subcompressionbodies of U and V respectively. As such, the collection of discs $\overline{\Delta} = \cup_i (\mathrm{fr}\, A_i \cup \mathrm{fr}\, B_i)$ is a 2–sided disc family for S in M. (The notation means the frontier of A_i in U and the frontier of B_i in V.) We can perform a finite sequence of slide-moves (Section 3.2) on $\overline{\Delta}$ to obtain a new 2–sided disc family Δ. This sequence also gives us, for each i, subcompressionbodies A'_i and B'_i obtained from A_i and B_i respectively by slide-moves. The important observation is that since the slide-moves are done relative to $\cup_i(\partial\, \mathrm{fr}\, A_i \cup \partial\, \mathrm{fr}\, B_i)$ we still have, for each i, that $\partial_S A'_i \subset \partial_S B'_i \subset \partial_S A'_{i+1}$. Thus, $C'_i = \eta(A'_i \cup B'_i)$ is a balanced submanifold of M adapted to S. And so, $\{C'_i\}$ is a balanced exhaustion of M adapted to S. These observations provide the key to the proof of Theorem 5.1.

Definition The balanced submanifold $C' = \eta(A' \cup B')$ is obtained from the balanced submanifold $C = \eta(A \cup B)$ *by slide-moves* if there is a finite sequence of slide-moves by which A' is obtained from A and B' is obtained from B.

4.4 Comparing absolute and relative Heegaard splittings

In the remainder of this section, we look at the relationship between absolute and relative Heegaard splittings of a compact manifold. These results will help us to translate facts about absolute Heegaard splittings to relative Heegaard splittings. Let N denote a 3–manifold, compact or non-compact, with non-empty compact boundary.

Suppose that $N = U \cup_S V$ is a relative Heegaard splitting. Let $\mathcal{B}$ denote the boundary components of N which intersect S. Define $\widehat{U}$ to be U together with a regular neighborhood of $\mathcal{B}$. Define $\widehat{V}$ to be the closure of the complement of $\widehat{U}$ in N and let $\widehat{S} = \widehat{U} \cap \widehat{V}$.

Lemma 4.5 *$N = \widehat{U} \cup_{\widehat{S}} \widehat{V}$ is an absolute Heegaard splitting of N.*

Proof If $\mathcal{B} = \varnothing$ there is nothing to prove, so assume that $\mathcal{B}$ is non-empty. U is a relative compressionbody and so is obtained from $F \times I$ by adding 1–handles to $F \times \{1\}$ and countably many 3–balls. F is a compact surface with boundary. Let B be a component of $\mathcal{B}$ and let $B_U = B \cap U$ and $B_V = B \cap V$. In the process of obtaining $\widehat{U}$ we glue $B_V \times I$ to $B_U \times I$ along $\gamma \times I$ where $\gamma = \partial B_V = \partial B_U$. So $\widehat{U}$ is $B \times I$ attached by 1–handles to the preferred surface of a compressionbody. Hence, performing this operation for each boundary component of N which intersects S, leaves us with $\widehat{U}$, an absolute compressionbody. On the other hand, to form $\widehat{V}$ we have removed a collar neighborhood of each component of $\partial_- V$ which intersected $\partial_+ V$. Let $\mathcal{D}$ be a collaring set of discs for V. The discs $\mathcal{D}$ are also discs in $\widehat{V}$. Let $\mathcal{E}$ be the collection of components of $\sigma(V;\mathcal{D})$ which contain $\partial_- V \cap \mathcal{B}$. Each of these components is a (surface with boundary) $\times\, I$. As such, each component is a handlebody. Removing a collar neighborhood of $\partial_- V \cap \mathcal{B}$ from these components does not change the homeomorphism type. The space $\widehat{V}$ is formed by attaching these handlebodies to the preferred surface of the absolute compressionbody $\sigma(V;\mathcal{D}) - \mathcal{E}$ by 1–handles dual to the discs $\mathcal{D}$. Thus, $\widehat{V}$ is an absolute compressionbody. □

If we know that V intersects ∂N in discs, the relationship is stronger.

Lemma 4.6 (The Marionette Lemma) *Suppose that $U_S \cup_S V_S$ and $U_T \cup_T V_T$ are two relative Heegaard splittings of a 3–manifold N. Suppose also that for each component of ∂N which intersects S, V_S and V_T intersect that component in discs. If, for each such boundary component of N, V_S and V_T intersect that boundary component in the same number of discs, then S and T are properly ambient isotopic if and only if $\widehat{S}$ and $\widehat{T}$ are properly ambient isotopic.*

We form $\widehat{U}_S$ and $\widehat{U}_T$ by including a regular neighborhood of $V_S \cap \partial N$ and $V_T \cap \partial N$ into U_S and U_T. If we want to undo this operation we can remember the cocores of the discs $V_S \cap \partial N$ and $V_T \cap \partial N$. These give us finite collections of arcs in $\widehat{U}_S$ and $\widehat{U}_T$ joining ∂N to $\widehat{S}$ and $\widehat{T}$ respectively. To prove the lemma, we need to understand how these arcs can be isotoped within the compressionbodies $\widehat{U}_S$ and $\widehat{U}_T$. We will show that if $\widehat{S}$ and $\widehat{T}$ are isotopic, we can isotope $\widehat{S}$ and $\widehat{T}$ to coincide and then isotope the arcs to coincide.

Definition Let ψ be a finite collection of arcs in an absolute compressionbody H with at least one endpoint of each arc on $\partial_+ H$. If H is a 3–ball then ψ is *standard* if it is isotopic to a collection of arcs which lie in $\partial_+ H = \partial H$. If $H = F \times I$ where F is a closed connected surface, then ψ is *standard* if there is an isotopy of ψ so that each spanning arc is vertical in the product structure and each non-spanning arc is contained in $F \times \{1\} = \partial_+ H$. For a generic absolute compressionbody, ψ is *standard* if there is a defining collection of discs Δ for H which is disjoint from ψ and such that ψ is standard in each component of $\sigma(H;\Delta)$.

We need the following two results which are slightly rephrased from [23]. We are allowing our compressionbody to be non-compact, but since the number of arcs is finite the results are still true, as we may restrict attention to a compact subcompressionbody.

Lemma 4.7 (Scharlemann and Thompson [23, Lemma 6.4]) *If σ and τ are standard collections of arcs in an absolute compressionbody H, then for any defining collection of discs Δ for H there is an isotopy of σ and an isotopy of τ so that σ and τ are standard in $\sigma(H;\Delta)$.*

Lemma 4.8 (Scharlemann and Thompson [23, Corollary 6.7]) *Let ψ be a collection of arcs properly embedded in a compressionbody H such that for every subcollection $\psi' \subset \psi$, the complement of ψ' is a compressionbody. Then ψ is standard.*

Proof of the Marionette Lemma If S and T are ambient isotopic, it is clear that $\widehat{S}$ and $\widehat{T}$ are. So suppose that $\widehat{S}$ and $\widehat{T}$ are ambient isotopic.

As mentioned earlier, we can recover S and T from $\widehat{S}$ and $\widehat{T}$ by remembering the cocores of the 2–handles that were added to U_S and U_T. Let σ be the collection of arcs coming from $V_S \cap \partial N$ and let τ be the collection of arcs coming from $V_T \cap \partial N$.

Isotope $\widehat{S}$ onto $\widehat{T}$. Now we have $\widehat{U}_S = \widehat{U}_T$. This isotopy takes σ to some collection of arcs which we continue to call σ. If we can show that there is an isotopy of σ onto τ which keeps $\widehat{S}$ mapped onto $\widehat{T}$ for all time, we will be done. The isotopy is allowed to move the endpoints of the arcs, but it must keep them on $\partial N \cup \widehat{S}$.

We claim, first, that for each subcollection σ' of arcs in σ the complement of σ' in $\widehat{U}_S = \widehat{U}_T$ is a compressionbody. Let σ' be a subcollection of arcs from σ. Let s' denote the arcs of $\sigma - \sigma'$. Let $D_{s'}$ be the 2–handles of $\eta(V_T \cap \partial N)$ which have cocores s'. Consider the relative compressionbody U_S. U_S is formed by taking a surface F with boundary, forming $F \times I$ and adding 1–handles to $F \times \{1\}$. The surface F has one boundary component for each component of $S \cap \partial N$. Let γ denote the boundary components of $F \times \{0\}$ which correspond to s'. Adding the 2–handles $D_{\sigma'}$ to U_S is achieved by attaching copies of $D^2 \times I$ to F along $\gamma \times I$. It's clear that the result is still a compressionbody. But this is exactly $\mathrm{cl}(\widehat{U}_S - \eta(\sigma'))$. Thus, the complement of every subcollection of σ in $\widehat{U}_S$ is a compressionbody. The same result holds for τ.

By Lemma 4.8, both σ and τ are standard. By Lemma 4.7, there is a proper isotopy of σ and a proper isotopy of τ so that both σ and τ are disjoint from a defining disc set Δ for $\widehat{U}_S = \widehat{U}_T$ and both are standard in $\sigma(U_S; \Delta)$. Since each arc of $\sigma \cup \tau$ has an endpoint on a component of ∂N, we may assume that the isotopy has made each arc of σ and each arc of τ vertical in the product structure of $(\partial N \times I) \cap \widehat{U}_S$. Since for each component of ∂N the arcs of σ and τ with an endpoint on that component are in one-to-one correspondence, there is the required isotopy taking σ onto τ. □

The following is a version of Haken's Lemma for relative Heegaard splittings. It is, perhaps, well-known. It appears in similar versions as Lemma 5.2 in [1] and as a remark following Definition 2.1 in [13].

Lemma 4.9 (Haken's Lemma) *Suppose that $U \cup_S V$ is a relative Heegaard splitting of N with the property that each component of $V \cap \partial N$ is a disc. Then if ∂N is compressible in N there is a compressing disc for ∂N whose intersection with S is a single simple closed curve. Furthermore, boundary reducing N along this disc leaves us with a relative Heegaard splitting $\mathrm{cl}(U - \eta(D)) \cup_{\mathrm{cl}(S - \eta(D))} \mathrm{cl}(V - \eta(D))$ of the resulting manifold.*

Proof Let $\widehat{U} \cup_{\widehat{S}} \widehat{V}$ be the absolute Heegaard splitting for N obtained by including $\eta(V \cap B)$ into U for each component $B \subset \partial N$ which intersects S. Since ∂N is compressible, by Casson and Gordon's version of Haken's Lemma [9], there is a compressing disc D for ∂N which intersects $\widehat{S}$ in a single simple closed curve.

To obtain $U \cup_S V$ from $\widehat{U} \cup_{\widehat{S}} \widehat{V}$ we include into $\widehat{V}$ the neighborhood of a certain collection of arcs σ. The arcs σ are the cocores of the 2–handles which we added to U in order to obtain $\widehat{U}$.

If ∂D is on a component of ∂N contained in $\widehat{V}$, then by Lemma 4.7 we may isotope σ to be disjoint from the disc $D \cap \widehat{U}$. Thus, there is a compressing disc for ∂N which intersects S in a single simple closed curve.

If ∂D is on a component of ∂N contained in $\widehat{U}$ then $D \cap \widehat{U}$ is an annulus. By performing handle-slides, we may obtain a defining collection of discs Δ for $\widehat{U}$ which are disjoint from that annulus. We may assume that the annulus $D \cap \widehat{U}$ is vertical in the product structure of the component of $\sigma(\widehat{U};\Delta)$ containing it. By Lemma 4.7, there is an isotopy of the arcs σ so that σ is disjoint from Δ and is vertical in the product structure of the components of $\sigma(\widehat{U};\Delta)$ containing it. It is then easy to isotope σ to be disjoint from the annulus $D \cap \widehat{U}$. Hence, when we remove an open regular neighborhood of σ from $\widehat{U}$ to obtain U we have the disc D intersecting S in a single simple closed curve. Thus S divides D into a disc and an annulus.

Boundary-reducing N along D leaves us with a 3–manifold $\overline{N} = \sigma(N;D)$. We have boundary-reduced the relative compressionbody (U or V) containing the disc part of D along a disc with boundary in the preferred surface. Thus, by Corollary 3.2 it is still a relative compressionbody. In the other compressionbody X (equal to V or U), there is a defining set of discs Δ disjoint from D and the annulus $D \cap X$ is vertical in the product structure of the component of $\sigma(X;\Delta)$ containing it. That component is homeomorphic to $F \times I$ where F is a compact surface, possibly with boundary. Removing the open neighborhood of a vertical annulus in such a component leaves us with a manifold homeomorphic to $G \times I$ where G is a compact surface obtained from F by removing an open annulus. Thus, $X - \mathrm{int}(\eta(D \cap X))$ is still a relative compressionbody with preferred surface $S - \mathrm{int}(\eta(D))$. This implies that $\overline{N} = \mathrm{cl}(U - \eta(D)) \cup_{\mathrm{cl}(S - \eta(D))} \mathrm{cl}(V - \eta(D))$ is a relative Heegaard splitting. □

5 Heegaard splittings of eventually end-irreducible 3–manifolds

5.1 Introduction

Recall that a non-compact 3–manifold M is *end-irreducible rel* C for a compact set $C \subset M$ if there is an exhaustion $\{K_i\}_{\mathbb{N}}$ for M such that $C \subset K_1$ and, for all i, $\mathrm{fr}\, K_i$ is incompressible in $M - C$. Inessential spheres count as incompressible surfaces, so, for example, $\mathbb{R}^3$ is end-irreducible rel $\varnothing$. Other examples of eventually end-irreducible 3–manifolds are deleted boundary 3–manifolds. A deleted boundary 3–manifold M contains a compact set C so that $\mathrm{cl}(M - C)$ is homeomorphic to $F \times \mathbb{R}_+$ for some closed surface F.

For the remainder of this section, assume that M is an orientable non-compact 3–manifold which is end-irreducible rel C and that $\partial M \subset C$. Let $M = U \cup_S V$ be an absolute Heegaard splitting of M.

Since we will be dealing with a variety of exhaustions for M we collect the following definitions here:

Definition Let $\{K_i\}$ be an exhaustion for M with $C \subset K_1$. We say that:

- $\{K_i\}$ is *frontier-incompressible rel C* if, for each i, fr K_i is incompressible in $M - C$.
- $\{K_i\}$ is *adapted* to S if, for all i, $(U \cap K_i) \cup_{(S \cap K_i)} (V \cap K_i)$ is a relative Heegaard splitting of K_i and if $(X \cap K_i)$ is correctly embedded in X for $X = U, V$. If $\{K_i\}$ is adapted to S there is a subsequence which has the *outer collar property* (Lemma 4.1).
- $\{K_i\}$ is *perfectly adapted* to S if it is adapted to S and, in addition, each $\mathrm{cl}(K_{i+1} - K_i)$ is adapted to S. That is, each $\mathrm{cl}(K_{i+1} - K_i)$ inherits a relative Heegaard splitting with Heegaard surface $S \cap \mathrm{cl}(K_{i+1} - K_i)$.
- $\{K_i = \eta(A_i \cup B_i)\}$ is a *balanced exhaustion* for M (adapted to S) if each K_i is a regular neighborhood of $A_i \cup B_i$ where A_i and B_i are subcompressionbodies of U and V respectively with $\partial_S A_i \subset \partial_S B_i \subset \partial_S A_{i+1}$.
- $\{K_i\}$ is *well-placed on S rel C* if it is a frontier-incompressible (rel C) exhaustion for M which is adapted to S and, in addition, has the following properties:

(WP1) For each i, V intersects each component of fr K_i in a single disc.
(WP2) For each i, fr $K_i \cap U$ is incompressible in U.
(WP3) $\{K_i\}$ has the outer collar property with respect to U.
(WP4) For each i, no component of $\mathrm{cl}(M - K_i)$ is compact.

The main result of this section is:

Theorem 5.1 *Suppose that M is a non-compact orientable 3–manifold with compact boundary which is end-irreducible (rel C) where C is a compact set containing ∂M. Suppose also that $U \cup_S V$ is an absolute Heegaard splitting of M. Then there is an exhaustion of M which is well-placed on S rel C.*

The most difficult part of the proof is in showing that there is a frontier-incompressible (rel C) exhaustion which is adapted to S.

5.2 Balanced sequences and the weakly reducible theorem

We begin by showing that there is a balanced exhaustion of M adapted to S so that the compressing discs for the frontiers of the exhausting elements are in a "good position" relative to the Heegaard surface.

Proposition 5.2 *There is a balanced exhaustion $\{C_i = \eta(A'_i \cup B'_i)\}$ for M adapted to S and a 2–sided disc family Ψ for S which contains $\cup_i(\mathrm{fr}\, A'_i \cup \mathrm{fr}\, B'_i)$ such that, for each i, $\sigma(\mathrm{cl}(\partial_S B'_i - \partial_S A'_i); \Psi)$ is incompressible in $M - C$.*

Remark In [9] Casson and Gordon prove that if a Heegaard splitting of a compact 3–manifold is weakly reducible then there is a 2–sided disc family for the Heegaard surface such that when the surface is compressed along that family, the result is a collection of incompressible surfaces (possibly inessential spheres)[3]. Since the frontiers of balanced submanifolds consist of surfaces which are obtained from the Heegaard surface by compressions along disjoint discs, it is natural to try to harness the power of the Casson and Gordon theorem.

It is unclear, however, if the Casson and Gordon theorem can be extended to non-compact 3–manifolds in a way that is directly useful in this situation. Nonetheless, the proof of our theorem is based on the outline of a proof of Casson and Gordon's theorem given in [21]. We will also need to use Casson and Gordon's version of Haken's Lemma.

The proof is rather long so we begin with an outline of the proof:

Step 1 Take a balanced exhaustion $\{K_i = \eta(A_i \cup B_i)\}$. For each K_n show how to replace K_{n-2}, K_{n-1}, and K_n with "better" balanced submanifolds $K_k^L = \eta(A_k^L \cup B_k^L)$ for $k = n-2, n-1, n$. Each of these better balanced submanifolds is still contained in K_{n+1} and still contains K_{n-3}. Let $C_n = K_n^L$. The new manifolds will be obtained from the old ones by a finite sequence of slide-moves L. The process of obtaining C_n will also leave us with a 2–sided disc family Δ for $S \cap K_{n+1}$.

Step 2 Suppose that there is a compressing disc D for $\sigma(\mathrm{cl}(\partial_S B_n^L - \partial_S A_n^L); \Delta)$.

Step 3 Show that we can assume that D is contained in $K_{n+1} - K_{n-2}^L$. This step is where we use the eventual end-irreducibility of M.

Step 4 Replace D by a compressing disc of $\sigma(S \cap K_{n+1}; \Delta)$ which intersects $\sigma(S \cap K_{n+1}; \Delta)$ only on ∂D. We continue calling the disc D.

[3] This is not how the result is usually stated, but see the proof given in [21].

Step 5 Use Haken's Lemma to replace D by a disc which intersects a certain Heegaard surface exactly once and is contained in $K_{n+1} - K_{n-3}$. We continue calling the disc D.

Step 6 Follow the arguments of Casson and Gordon's Weakly Reducible theorem to obtain from C_n by slide-moves a balanced submanifold which is even better than C_n. This will contradict the construction of C_n.

Step 7 Use this replacement technique on each element of a subsequence of $\{K_i\}$ to obtain the desired $\{C_i\}$. Construct the 2–sided disc family Ψ from the 2–sided disc families Δ which were created in each replacement operation.

Proof of Proposition 5.2 Let $\{K_i = \eta(A_i \cup B_i)\}_{i \geq 0}$ be a balanced exhaustion for M adapted to S and let $\{P_i\}$ be a frontier incompressible exhaustion (rel C). Choose the exhaustions so that $C \subset K_0 \subset P_{i-1} \subset K_i \subset P_i$ for all $i \geq 1$. Each of the inclusions should be into the interior of the succeeding submanifold. Figure 11 is a schematic of the exhaustions. The frontiers of the submanifolds in $\{P_i\}$ may have a very complicated intersection with the Heegaard surface. The frontier of each submanifold in the balanced exhaustion consists of discs and compact surfaces parallel to subsurfaces of S.

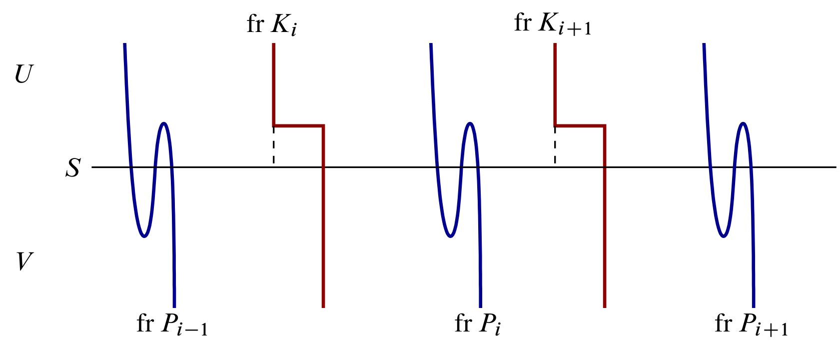

Figure 11: A schematic of the exhaustions

We will show that given a $q \in \mathbb{N}$ and $n \geq q+3$, K_n can be replaced by a compact submanifold $C_n = \eta(A'_n \cup B'_n)$ with the following properties:

(1) C_n is obtained from K_n by slide moves.

(2) There is a 2–sided disc family Δ for S in K_{n+1} containing $\text{fr}\, A'_n \cup \text{fr}\, B'_n$ such that $\sigma(\text{cl}(\partial_S B'_n - \partial_S A'_n); \Delta)$ is incompressible in $M - C$.

(3) We still have $K_q \subset C_n$ and the discs $\text{fr}\, A_q \cup \text{fr}\, B_q$ are contained in Δ.

Choose some $n \geq q + 3$.

Let $\overline{\Delta} = \bigcup_{q \leq i \leq n+1} (\mathrm{fr}\, A_i \cup \mathrm{fr}\, B_i)$. Recall from Section 3.2 that a slide-move of this 2–sided disc family consists of either adding a compressing disc for S to $\overline{\Delta}$ which is disjoint from all other discs of $\overline{\Delta}$ or performing a 2–handle slide of one disc of $\overline{\Delta}$ over another disc of $\overline{\Delta}$. The arc over which a 2–handle slide is performed must have its interior disjoint from all discs of $\overline{\Delta}$.

Recall from just after Lemma 4.4 that each slide-move performed on $\overline{\Delta}$ leaves us with new balanced submanifolds obtained from the submanifolds $\{K_i\}_{i \leq n+1}$ by slide-moves. After performing a slide-move, we still have $K_i \subset K_{i+1}$ for all i, since all the slides are performed relative to $\overline{\Delta}$.

Let $\mathcal{L}$ denote the set of all finite sequences of slide-moves of $\overline{\Delta}$ subject to the following restrictions:

(1) Every time a disc is added to $\overline{\Delta}$, the disc has boundary lying on $S \cap K_{n+1}$.

(2) No disc of $\mathrm{fr}\, K_{n+1} \cup \mathrm{fr}\, K_q$ is ever slid over another disc.

These restrictions mean that performing a sequence of slide-moves in $\mathcal{L}$ preserves the ordering of submanifolds K_i for $q \leq i \leq n+1$. Furthermore, the manifolds K_{n+1} and K_q are left unchanged.

Step 1 Each sequence $L \in \mathcal{L}$ leaves us with new balanced submanifolds K_i^L for $q < i < n+1$. The submanifolds K_q and K_{n+1} are left unchanged. For ease of notation, let $K_q^L = K_q$ and $K_{n+1}^L = K_{n+1}$. Let A_i^L be the subcompressionbody of U obtained from A_i by the slide-moves L and let B_i^L be the subcompressionbody of V obtained from B_i by the slide-moves L so that $K_i^L = \eta(A_i^L \cup B_i^L)$.

Recall from [9] that the complexity of a closed, connected surface F is defined to be $1 - \chi(F)$, unless F is a two-sphere, in which case, it is 0. The complexity of a disconnected closed surface is the sum of the complexities of the components.

Performing L on $\overline{\Delta}$ leaves us with a disc family $\overline{\Delta}_L$ which contains the discs $\mathrm{fr}\, A_i^L \cup \mathrm{fr}\, B_i^L$ for $q \leq i \leq n+1$. Define the complexity of $\overline{\Delta}_L$ to be the complexity of $\sigma(S \cap K_{n+1}; \overline{\Delta}_L)$. Since complexity is invariant under handle-slides (Lemma 3.7), the complexity of a 2–sided disc family cannot increase under slide-moves.

Choose an $L \in \mathcal{L}$ so that $\overline{\Delta}_L$ has minimal complexity. Let $\Delta = \overline{\Delta}_L$ and $C_n = K_n^L$. Let Δ_1 be those discs of Δ which lie in U and Δ_2 those discs which lie in V.

Recall that if $R \subset S$ is a compact subsurface of S with $\partial R \subset \partial \Delta$, the notation $\sigma(R; \Delta)$ signifies the surface obtained from R by compressing along those discs of Δ which have boundary on R. Let $R_i = \mathrm{cl}(\partial_S B_i - \partial_S A_i)$ and let $R'_i = \mathrm{cl}(\partial_S B_i^L - \partial_S A_i^L)$ for $q \le i \le n+1$. Note that R'_i is obtained from R_i by the sequence of slide-moves L. We claim that $\sigma(R'_n; \Delta)$ is incompressible in $M - C$. The surface R'_i is a subsurface of S which is parallel in K_i^L to $\mathrm{cl}(\mathrm{fr}\, K_i^L - (\mathrm{fr}\, A_i^L \cup \mathrm{fr}\, B_i^L))$.

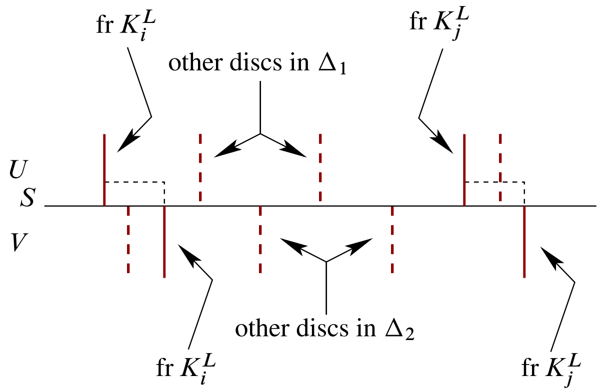

Figure 12: A schematic of Δ_1 and Δ_2

Step 2 Let $S_k = \sigma(S \cap K_{n+1}; \Delta_k)$ for $k = 1, 2$. Let W_1 be $U \cap K_{n+1}$ together with the 2–handles coming from Δ_2 minus the 2–handles coming from Δ_1. Let W_2 be the closure of the complement of W_1 in K_{n+1}. Let $\overline{S} = \sigma(S \cap K_{n+1}; \Delta)$. We are trying to show that $\sigma(R'_n; \Delta)$ is incompressible in $M - C$. We assume the contradiction: suppose that a component $\overline{B}$ of $\sigma(R'_n; \Delta)$ is compressible in $M - C$.

Step 3 Our next task is to show that there is a compressing disc for $\overline{B}$ which lies entirely in $K_{n+1} - K_{n-2}^L$. Recall that $\{P_i\}$ is the frontier-incompressible (rel C) exhaustion for M which is interlaced with $\{K_i\}$. Let $\Sigma = \mathrm{fr}\, P_{n-1} \cup \mathrm{fr}\, P_n$. The key technique is an application of Proposition 3.6 and the incompressibility in $M - C$ of Σ.

By Proposition 3.6, there is a proper ambient isotopy f taking $\sigma(S; \Delta)$ to the surface obtained from $\sigma(S; \overline{\Delta})$ by compressing along a certain collection of discs. In particular, there are disjoint collections of disjoint ordered discs $\mathcal{E}$ and $\mathcal{G}$ so that the discs of $\mathcal{E}$ have boundary on $\sigma(R_n; \overline{\Delta})$ and the discs of $\mathcal{G}$ have boundary on $\sigma(R_{n-1}; \overline{\Delta})$ and the isotopy f takes $\sigma(R'_n; \Delta)$ to $\sigma(R_n; \overline{\Delta} \cup \mathcal{E})$ and $\sigma(R'_{n-1}; \Delta)$ to $\sigma(R_{n-1}; \overline{\Delta} \cup \mathcal{G})$. The notation $\sigma(R_n; \overline{\Delta} \cup \mathcal{E})$ means the surface obtained from $\sigma(R_n; \overline{\Delta})$ by compressing along the discs of $\mathcal{E}$ in the order given. Similarly, we write $\sigma(R_{n-1}; \overline{\Delta} \cup \mathcal{G})$ for the surface obtained from $\sigma(R_{n-1}; \overline{\Delta})$ by compressing along $\mathcal{G}$. The surfaces $\sigma(R_n; \overline{\Delta} \cup \mathcal{E})$ and $\sigma(R_{n-1}; \overline{\Delta} \cup \mathcal{G})$ are disjoint.

The discs $\mathcal{E}$ have boundary on $\sigma(R_n;\overline{\Delta}) \subset (P_n - P_{n-1})$. As Σ is incompressible in $M - C$ the intersections of $\mathcal{E}$ with Σ are inessential on Σ. Similarly, the discs of $\mathcal{G}$ have boundary on $\sigma(R_{n-1};\overline{\Delta}) \subset P_{n-1}$ and so $\mathcal{G}$ intersects Σ in loops which are inessential on Σ. The surface $\overline{B}$ is taken by the isotopy f to a surface $\overline{K}$ which is a component of $\sigma(R_n;\overline{\Delta} \cup \mathcal{E})$. Since $\overline{B}$ is compressible in $M - C$ so is $\overline{K}$.

Since the intersections of $\overline{K}$ with the incompressible (in $M - C$) Σ come from the intersections of $\mathcal{E}$ with $\overline{K}$, the loops $\overline{K} \cap \Sigma$ are inessential on both surfaces. There is, therefore, a surface $K' \subset (P_n - P_{n-1})$ which is obtained from $\overline{K}$ by cutting and pasting along the intersections $\overline{K} \cap \Sigma$. (Start with innermost discs of intersection on Σ and replace the corresponding discs of $\overline{K}$ with copies of the discs on Σ which have been pushed slightly into $(P_n - P_{n-1})$.) As $\overline{K}$ is compressible in $M - C$, K' is also compressible in $M - C$. Since Σ is incompressible in $M - C$ there is a compressing disc F for K' which is contained in $P_n - P_{n-1}$. Our goal is to use F to construct a compressing disc for $\overline{K}$ which is disjoint from $\sigma(R_{n-1};\overline{\Delta} \cup \mathcal{G})$.

Since $\partial\mathcal{E}$ consists of inessential loops on K' we may assume that $\partial F \cap \partial\mathcal{E} = \varnothing$. The disc F may intersect the discs $\mathcal{E}$. It may also intersect the discs of $\mathcal{G}$ in simple closed curves. Since each loop of $F \cap \mathcal{E}$ is inessential on both F and $\mathcal{E}$ we may, by cutting and pasting F along the intersections, obtain a compressing disc F' for $\overline{K}$. Since both K' and $\mathcal{E}$ were disjoint from $\sigma(R_{n-1};\overline{\Delta} \cup \mathcal{G})$, any intersections of the disc F' with the surface $\sigma(R_{n-1};\overline{\Delta} \cup \mathcal{G})$ occur because F' intersects $\mathcal{G}$ in simple closed curves. These intersections are inessential on both F' and on $\sigma(R_{n-1};\overline{\Delta} \cup \mathcal{G})$. We may cut and paste F' along these intersections to produce a compressing disc E for $\overline{K}$ which is disjoint from $\sigma(R_{n-1};\overline{\Delta} \cup \mathcal{G})$. The disc E may intersect Σ, but that is not of concern.

Reversing the isotopy f takes E to a compressing disc D for $\overline{B}$. D is contained in K_{n+1}. The disc D is disjoint from $\sigma(R'_{n-1};\Delta)$ since E was disjoint from $\sigma(R_{n-1};\overline{\Delta} \cup \mathcal{G})$.

Recall that we are trying to construct a compressing disc for $\overline{B}$ which is contained in $K_{n+1} - K^L_{n-2}$. Each disc of Δ which had boundary on R'_{n-1} was disjoint from R'_{n-2} since no disc of Δ intersects S except at its boundary and the discs of Δ are pairwise disjoint. Thus K^L_{n-2} is contained inside some component of $\sigma(K^L_{n-1};\Delta)$. But since D is disjoint from $\sigma(R'_{n-1};\Delta)$ which is parallel to $(\mathrm{fr}\,\sigma(K^L_{n-1};\Delta))$, D can be isotoped so as to not intersect K^L_{n-2}. Hence, there is a compressing disc D for $\overline{B}$ which is contained in $K_{n+1} - K^L_{n-2}$.

Step 4 The compressing disc D may intersect the surface $\overline{S} \cap (K_{n+1} - K^L_{n-2})$. By revising the disc D we may assume that no loops of $D \cap \overline{S}$ are inessential on $\overline{S}$. Replace D by an innermost disc, which we will continue to call D, that intersects $\overline{S}$

only on ∂D. By our construction D is now a compressing disc for $\overline{S}$. The boundary of D may no longer be on $\overline{B}$. D lies in either W_1 or W_2 and is completely contained in $(K_{n+1} - K^L_{n-2})$. Recall that $W_1 = [(U \cap K_{n+1}) - \eta(\Delta_1) \cup \eta(\Delta_2)]$ and that $W_2 = [(V \cap K_{n+1}) - \eta(\Delta_2) \cup \eta(\Delta_1)]$.

Step 5 Our goal is to use the disc D to construct a sequence $L' \in \mathcal{L}$ such that $\overline{\Delta}_{L'}$ has lower complexity than $\Delta = \overline{\Delta}_L$. This will contradict our original choice of L. As mentioned in the remark preceding this proof, the strategy is to follow the outline of the proof of Casson and Gordon's Weakly Reducible theorem given in [21]. We will view S_1 as a Heegaard surface for W_1 or S_2 as a Heegaard surface for W_2 depending on which side the disc D lies. In the Casson and Gordon theorem the two cases had identical arguments. Here, however, the relationship of W_1 and W_2 to $K_{n+1} - K_q$ is not symmetric due to the asymmetry in the construction of balanced submanifolds. We will briefly need to consider the two cases separately. We will eventually be able to combine arguments.

Remark Some care is needed when we consider S_1 or S_2 as a Heegaard surface, as $\overline{S}$ may contain spheres. This means that the compressionbodies we are considering may not be irreducible. This does not really affect the proofs as the only times we would want to use the irreducibility of a compressionbody is when we isotope (in a compressionbody) one disc past another which shares its boundary. If $\overline{S}$ contains spherical components which get in the way of the isotopy, we may first perform a surgery on the disc we want to isotope so that the two discs with common boundary bound a 3–ball and then perform the isotopy. We will refer to this process as *revising and isotoping* the disc which, if $\overline{S}$ were irreducible, we would have merely isotoped.

Suppose, first, that D lies in W_1. By pushing $\overline{S}$ slightly into W_2 we can view S_1 as a Heegaard surface for the disconnected 3–manifold W_1. S_1 divides W_1 into (disconnected) absolute compressionbodies U' and V'. Let V' be the absolute compressionbody containing $\overline{S}$. See Figure 13. The disc D is a compressing disc for ∂W_1.

We can apply Haken's Lemma to obtain a compressing disc D', a compressing disc for $\overline{S}$ in W_1, which intersects S_1 in a single loop and is such that $\partial D' = \partial D$. $W_1 \subset K_{n+1}$ by the definition of W_1, so D' does not intersect $\operatorname{fr} K_{n+1}$. The discs $\operatorname{fr} A^L_{n-2}$ are in Δ_1 and separate U. Thus no component of W_1 intersects both $\operatorname{fr} K^L_{n-2}$ and $\operatorname{fr} K^L_{n-3}$. Hence, since ∂D is in $W_1 \cap (K_{n+1} - K^L_{n-2})$ the disc D' is in $K_{n+1} - K^L_{n-3}$. Summarizing: D' is a compressing disc for $\overline{S}$ which intersects S_1 in a single loop and is contained in $K_{n+1} - K^L_{n-3}$.

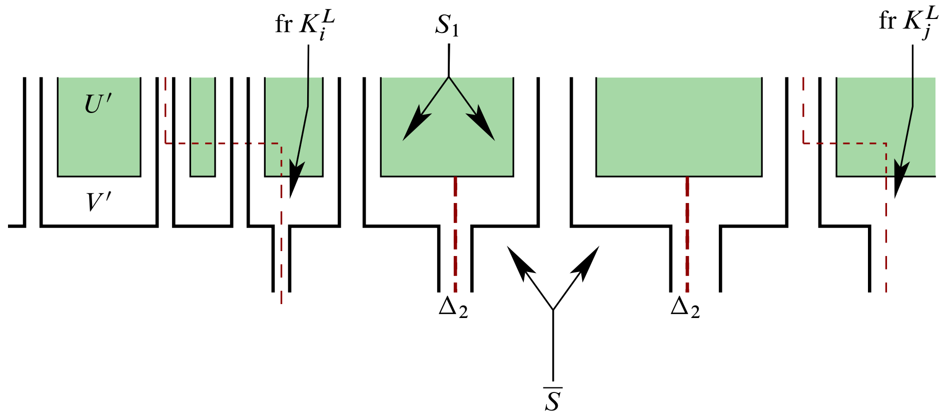

Figure 13: S_1 as a Heegaard surface for W_1

We now turn to the case when $D \subset W_2$. Push $\overline{S}$ slightly into W_1 and view S_2 as a Heegaard surface for the 3–manifold W_2. The disc D is a compressing disc for ∂W_2. Let U' and V' be the submanifolds of W_2 into which S_2 divides W_2. U' is the submanifold which has $\overline{S}$ as its boundary. See Figure 14.

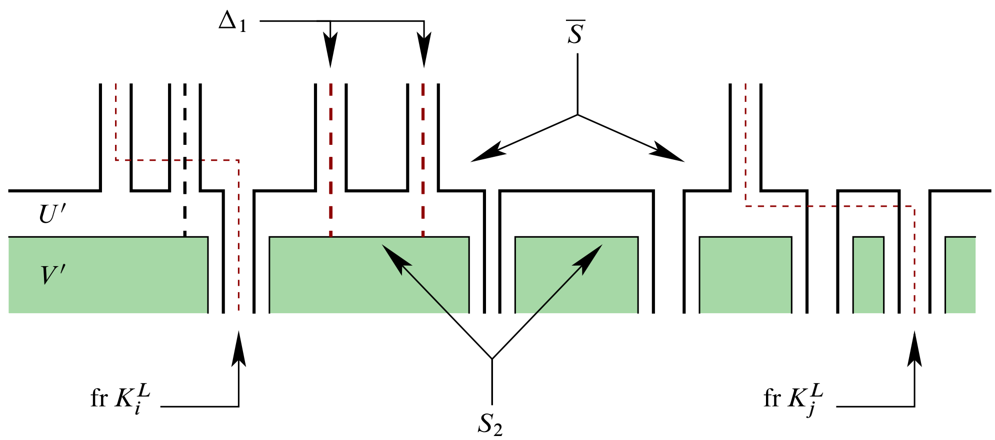

Figure 14: S_2 as a Heegaard surface for W_2

The discs of fr B_{n-2}^L are contained in Δ_2 and separate V. Thus no component of W_2 intersects both $(K_{n+1} - K_{n-2}^L)$ and int K_{n-2}^L. The disc D is a compressing disc for $\overline{S} \subset \partial W_2$ which is contained in a component of W_2 disjoint from int K_{n-2}^L. Applying Haken's Lemma, we can replace D with a disc D' such that $\partial D' = \partial D$ and D' intersects S_2 in a single loop. Since D and D' are in the same component of W_2, $D' \cap \text{int}\, K_{n-2}^L = \varnothing$. Summarizing: The disc D' is a compressing disc for $\overline{S}$ which intersects S_2 in a single loop and is contained in $K_{n+1} - K_{n-2}^L$.

Step 6 Recall that $2 < q \leq (n-3)$. We may now combine arguments. In the previous step, we showed that there was a compressing disc for $\overline{S}$ which is located in either $W_1 \cap \mathrm{cl}(K_{n+1} - K_q)$ or $W_2 \cap \mathrm{cl}(K_{n+1} - K_q)$ and intersects S_1 or S_2 (respectively) in a single loop γ. We will now produce a sequence of slide-moves l such that the sequence of slide-moves L followed by l is in $\mathcal{L}$ and the such that the sequence L followed by l has lower complexity than L. This will contradict our choice of L. The difficult part of this step is nearly identical to Bonahon's proof of Proposition 3.1. This is Proposition B.1 of [3]. We include the proof here because we need to pay careful attention to the type of slide-moves which are required.

Without loss of generality, suppose that D is a compressing disc for $\overline{S}$ which is located in $W_1 \cap \mathrm{cl}(K_{n+1} - K_q)$ and intersects S_1 in a single loop γ. (We were calling this disc D' in the previous step.) We continue to view S_1 as a Heegaard surface for W_1. Recall that V' denotes the compressionbody which is the region between $\overline{S}$ and S_1 and that U' is the closure of the complement of V' in W_1. See Figure 13.

We may assume that D is disjoint from the discs of Δ_1; it may, however, intersect the discs Δ_2 (including the frontiers of some B_i^L (for $q < i < n+1$). Let A denote the annulus $D \cap V'$ and D' the disc $D \cap U'$. Consider how A intersects Δ_2.

By an innermost disc argument we may assume that the annulus A intersects the discs of Δ_2 entirely in arcs with both endpoints on γ. Let a be an outermost arc of intersection on A. Let b be the arc of γ with endpoints ∂a which intersects no disc of Δ_2. Let G be the disc of Δ_2 such that $a \subset G \cap A$. Let c be an arc of ∂G which has endpoints ∂a. The arc c, of course, may have other intersections with γ.

Combining the subdiscs of A and G with boundaries $a \cup b$ and $a \cup c$ respectively and pushing off Δ_2 a little, we obtain a compressing disc for S_1 in V' which is disjoint from the complete collection of discs Δ_2 for V'. Thus $b \cup c$ is a loop bounding a disc Q in $\sigma(S_1; \Delta_2)$. (We are not calling this surface $\overline{S}$ since we have pushed $\overline{S}$ into W_2.)

We now adapt Bonahon's proof of Proposition 3.1 to show that we can perform 2–handle slides of G over the discs of Δ_2 which have boundary in Q and then revise and isotope D to remove all intersections of D with G (see the remark in Step 5 about the term "revise and isotope"). When we compress S_1 along Δ_2, the remnants of Δ_2 show up as spots, some of which are in the interior of the disc Q. Each disc of Δ_2 contributes two spots to $\sigma(S_1; \Delta_2)$. For each spot F_i from Δ_2 which shows up in Q, excluding a possible spot coming from G, choose oriented arcs α_i contained in Q joining G to the discs of Δ_2 giving rise to those spots. If a disc of Δ_2 produces two spots contained in Q then we have two oriented arcs joining G to that disc. Choose the arcs α_i so that $\alpha_i \cap \Delta_2 = \partial \alpha_i$ and so that the $\{\alpha_i\}$ are pairwise disjoint. The arcs α_i lie on S_1 and for each arc α_i we may perform a handle-slide of G over the the disc to which it is joined by α_i. Continue calling this disc G. By performing these slides, we may have increased the number of intersections between ∂G and γ. These handle-slides convert Q into a new disc as the arc c is changed by the handle-slides. We continue calling the disc Q. After these handle-slides Q contains no spots from Δ_2, except perhaps one coming from G. Revise and isotope D (rel b) so that γ has minimal intersection with ∂G. Suppose, now, that the disc Q contains a spot arising from G. Let G_1 and G_2 denote the two spots. Since they both arise from G we have that $|\gamma \cap \partial G_1| = |\gamma \cap \partial G_2|$. Since any arc of γ with both endpoints on ∂G_2 would bound a disc in S_1 and could, therefore, be removed by revising and isotoping D, each arc of γ with an endpoint on ∂G_2 also has an endpoint on ∂G_1. However $\partial b \subset \partial G_1$ and thus $|\gamma \cap \partial G_1| = |\gamma \cap \partial G_2| + 2$. This, however, contradicts the earlier equation and so the spot G_2 cannot exist in Q. The disc Q, therefore, is now a disc in S_1 and we can revise and isotope D to remove the intersection a from $D \cap \Delta_2$. Since we have previously removed all other intersections, including the intersections introduced earlier, of c with $\partial\Delta_2$ we have decreased $|D \cap \Delta_2|$ by at least one. Hence, by induction, we can remove all intersections of D with Δ_2 by revising and isotoping D (rel ∂D) and handle-sliding Δ_2.

This produces a new disc set Δ_2' which is disjoint from $\Delta_1 \cup \{D'\}$. At the beginning of the process the curve γ does not intersect any disc of $\operatorname{fr} K_{n+1} \cup \operatorname{fr} K_q$. The set of discs with boundary in R may contain discs that are associated to discs of $\operatorname{fr} K_{n+1} \cup \operatorname{fr} K_q$, but we were able to choose our sliding arcs so that they only intersected the discs of $\operatorname{fr} K_{n+1} \cup \operatorname{fr} K_q$ in at most one endpoint. The only slides we performed were of the disc G over other discs, and since γ intersected G, G was not a disc of $\operatorname{fr} K_{n+1} \cup \operatorname{fr} K_q$. Furthermore, since the discs of Δ_1 show up as spots on S_1 it is easy to arrange these slides to be relative to Δ_1. Thus, these handle-slides are of the sort allowed in sequences in $\mathcal{L}$. Let l denote the sequence of these handle-slides followed by the slide-move (M2) where we add the disc D' to Δ_1. The sequence of slide-moves consisting of L followed by l does, therefore, give us a sequence of slide-moves in the collection $\mathcal{L}$. As D was a compressing disc for $\overline{S}$ this sequence of slide-moves has lower complexity than our original choice from L. This, however, contradicts our initial choice L to be such that the complexity of $\sigma(S \cap K_{n+1}; \Delta)$ was minimal. The contradiction arises from our assumption that $\overline{B}$ is compressible: therefore, $\overline{B}$ is incompressible in $M - C$.

Step 7 Recall that $\{K_i\}$ is our balanced exhaustion adapted to S which is interspersed with a frontier-incompressible (rel C) exhaustion. Let $q_n = 5n$ for $n \geq 2$. We have shown how to replace K_{q_n} with a balanced submanifold $C_n = \eta(A_n' \cup B_n')$ which contains $K_{q_n - 3}$. The sequence $\{C_n\}$ is a balanced exhaustion adapted to S. In the construction of each C_n we also constructed a 2–sided disc family Δ so that

$\sigma(\mathrm{cl}(\partial_S B'_n - \partial_S A'_n); \Delta)$ is incompressible in $M - C$. Let Δ_n denote those discs of Δ with boundary on $\mathrm{cl}(\partial_S B'_n - \partial_S A'_n)$. Note that Δ_n is disjoint from Δ_i for all $i < n$. Let $\Psi = \cup_n \Delta_n$. Ψ is a 2–sided disc family for S where each disc of Ψ has boundary on the frontier of some C_n. When we compress $\cup\,\mathrm{cl}(\partial_S B'_n - \partial_S A'_n)$ along Ψ we obtain surfaces which are incompressible (rel C). □

5.3 The proof of Theorem 5.1

Recall that M is end-irreducible rel $C \supset \partial M$ and that $U \cup_S V$ is an absolute Heegaard splitting for M.

Proposition 5.3 *M has a frontier-incompressible (rel C) exhaustion which is adapted to S. Furthermore, V intersects the frontier of each element of the exhaustion in discs.*

Proof Let $\{C_i = \eta(A'_i \cup B'_i)\}$ be the balanced exhaustion guaranteed by Proposition 5.2. By the construction of balanced submanifolds, V intersects each $\mathrm{fr}\, C_i$ in discs.

Proposition 5.2 guarantees the existence of a 2–sided disc family Ψ for S such that $\cup_i(\mathrm{fr}\, A'_i \cup \mathrm{fr}\, B'_i) \subset \Psi$ and each $\sigma(\mathrm{cl}(\partial_S B'_i - \partial_S A'_i); \Psi)$ is incompressible in $M - C$. Let $\Psi_1 = \Psi \cap U$ and $\Psi_2 = \Psi \cap V$. We may use the product region between $(\mathrm{fr}\, C_i - (\mathrm{fr}\, A_i \cup \mathrm{fr}\, B_i))$ and $\partial_S B'_i - \partial_S A'_i$ to extend the discs in Ψ_2 with boundary on $(\partial_S B'_i - \partial_S A'_i)$ to have boundary on $\mathrm{fr}\, C_i$.

If we boundary-reduce C_i along Ψ_2 and add the 2–handles $\eta(\Psi_1)$ to $\mathrm{fr}\, C_i$ we end up with a new submanifold $\overline{C}_i$ of M. By construction, the discs of Ψ with boundary on $\mathrm{fr}\, C_i$ are disjoint from $\mathrm{fr}\, C_{i-1} \cup \mathrm{fr}\, C_{i+1}$. Hence, C_i is contained in a single component of $\overline{C}_{i+1}$ and so $M = \cup \overline{C}_i$. Let K_1 be the component of $\overline{C}_2$ containing C and, for each $n > 1$, let K_n be the component of $\overline{C}_{n+1}$ containing C_n. Since $C_n \subset K_n$ the sequence $\{K_n\}$ is an exhaustion for M. Since the frontier of each K_i is incompressible in $M - C$, the sequence $\{K_i\}$ is frontier-incompressible rel C.

When we boundary-reduce C_i along Ψ_2 we are boundary-reducing C_i along disjoint discs which each intersect the relative Heegaard surface $S \cap C_i$ in a single simple closed curve. By Haken's Lemma (Lemma 4.9), the resulting submanifold still has its intersection with S a relative Heegaard surface. When we add the 2–handles Ψ_1 to $\mathrm{fr}\, C_i$ we are adding 2–handles to $\partial_-(U \cap C_i)$. Hence, the resulting submanifold still has a relative Heegaard splitting coming from its intersection with S, apart from the introduction of 2–sphere components to $\partial_-(U \cap C_i)$. If there are any, we may add to $U \cap K_i$ the 3–balls bounded by those 2–spheres in U. After we have added these 3–balls, $\{K_i\}$ is a correctly embedded exhaustion. Therefore, $\{K_i\}$ is adapted to S. Since the sequence is also frontier-incompressible (rel C) the proposition is proved. □

We now embark on proving that there is a frontier-incompressible (rel C) exhaustion for M which is adapted to S and has properties (WP1), (WP2), (WP3), and (WP4) in the definition of "well placed exhaustion".

Lemma 5.4 *Let $\{K_i\}$ be a frontier-incompressible (rel C) exhaustion for M which is adapted to S. Suppose that V intersects $\operatorname{fr} K_i$ in discs for each i. Then after taking a subsequence of $\{K_i\}$ and performing a proper ambient isotopy of $\cup_i \operatorname{fr} K_i$ we may arrange that V intersects each component of each $\operatorname{fr} K_i$ in a single disc. Additionally, $\{K_i\}$ has the outer collar property.*

Proof Begin by taking a subsequence of $\{K_i\}$ such that $\{K_i\}$ has the outer collar property. Let $K = K_j$ (for $j \geq 2$) be an element of this revised exhaustion. Suppose that B is a component of $\operatorname{fr} K$ such that $|V \cap B| \geq 2$. We will describe an ambient isotopy of $\operatorname{fr} K$ which is the identity outside of $\operatorname{cl}(K_{j+1} - K_{j-1})$ to reduce the number of components of $|B \cap V|$ by one. We may then perform this ambient isotopy on each element of $\{K_{2i}\}$ as needed in order to arrange that V intersects each component of $\operatorname{fr} K_{2i}$ in a single disc. The union of these isotopies is a proper ambient isotopy of $\{K_{2i}\}$. After performing this isotopy, it will be clear that $\{K_{2i}\}$ still has the outer collar property.

Let $B' = U \cap B$. Since $V \cap B$ consists of discs, B' is connected and has at least two boundary components. B' makes up part of the frontier of the relative compressionbody $K \cap U$. B' is a component of $\partial_-(K \cap U)$ since $\{K_i\}$ is a correctly embedded exhaustion. Since $\{K_i\}$ has the outer collar property, there is a product region $P = B' \times I$ which is embedded in $\operatorname{cl}((K - (K_{j-1}) \cap U))$ such that $B' = B' \times \{0\}$ and $B' \times \{1\}$ is a subsurface of $S \cap K$ except at a finite number of open discs δ. Choose an arc $\alpha \subset B' \times \{1\}$ so that $\alpha \cap \partial(B' \times \{1\}) = \partial\alpha$, α joins different components of $\partial(B' \times \{1\})$, and α is disjoint from the discs δ. Let $D = \alpha \times I \subset P$ so that $\alpha = \alpha \times \{1\}$. D is an embedded disc in P such that ∂D is composed of two arcs, one on B' and one on $S \cap K$. Isotope $B \cap \eta(D)$ across the disc D. After this isotopy, the number of intersections $B \cap S$ has been reduced by one.

We now inspect the effect of this isotopy on $V \cap K$ and $U \cap K$. In $V \cap K$ we have changed $\partial_- V$ by banding together two discs. Since $V \cap K$ was a relative compressionbody with $\partial_-(V \cap K)$ consisting of discs, we have not changed the homeomorphism type of $V \cap K$, we have changed only the preferred surface.

The effect of the isotopy on $U \cap K$ is to replace $B' \times I$ with $C' \times I$ where C' is the surface obtained from B' by removing a neighborhood of an arc joining two components of $\partial B'$. Clearly, $U \cap K$ is still a relative compressionbody with preferred surface $S \cap K$. Furthermore, the presence of the product region $C' \times I$ shows that the sequence $\{K_i\}$ still has the outer collar property. The isotopy we have described is the identity outside of $K_{j+1} - K_{j-1}$. □

Proof of Theorem 5.1 Take the exhaustion $\{K_i\}$ given by Lemma 5.4. The only properties we have left to achieve are (WP2) and (WP4). We now prove that we have, in fact, already achieved (WP2) and that we can achieve (WP4) without ruining the others.

Suppose that B is some component of $\mathrm{fr}\, K_i$ such that $B \cap U$ has a compressing disc D which is contained in U. Since $K_i \cap U$ is a relative compressionbody and $(B \cap U) \subset \partial_-(K_i \cap U)$, the compressing disc D must be on the outside of K_i. The curve ∂D bounds a disc $E \subset B$ since B is incompressible in $M - C$ and $C \subset K$. Since D is a compressing disc for $B \cap U$, the disc E is not contained in $B \cap U$. Thus $(V \cap B) \subset E$. Forming K_i' by adding $\eta(D)$ to K_i cuts B into two surfaces: B' which is homeomorphic to B and B'' which is a 2–sphere. Note that both B' and B'' are components of $\partial K_i'$. The surface B' is contained in U and the sphere B'' intersects V in a single disc.

Since B was incompressible in $M - C$ and B' was obtained from B by cutting off a 2–sphere, B' is also incompressible in $M - C$. The surface $B' \subset U$ is closed and incompressible in U. Hence, B' is parallel to a component of $\partial_- U \subset \partial M$. This product region has boundary consisting of two components both of which are components of $\partial K_i'$. Thus the product region is actually K_i'. But B'' is also a component of $\partial K_i'$, so this is a contradiction. Hence, $B \cap U$ is incompressible in U. Thus $\{K_i\}$ satisfies (WP2).

Finally, we need to achieve (WP4). Suppose that $\mathrm{cl}(M - K_1)$ has a compact component L. There is some K_n so that every compact component of $\mathrm{cl}(M - K_1)$ is contained in K_n. By Corollary 3.5, $U \cap L$ and $V \cap L$ are relative compressionbodies. Since there are no closed components of $\partial_-(U \cap L)$ or $\partial_-(V \cap L)$, both are also handlebodies. Let $Q = L \cap K_1$. $Q \cap U$ is an incompressible surface in U which makes up part of $\partial_-(U \cap K_1)$. Choose a collaring set of discs δ for $U \cap K_1$. Boundary-reducing $K_1 \cap U$ along δ leaves us with components homeomorphic to $(Q \cap U) \times I$. Let $L' = (L \cap U) \cup ((Q \cap U) \times I)$. This does not change the homeomorphism type of $L \cap U$, so L' is a handlebody. We may now reassemble $K_1 \cap U$ by attaching 1–handles corresponding to the discs δ. When we do this, we are attaching the handlebody L' to the ∂_+ of a relative compressionbody and so the result is a relative compressionbody with preferred surface $S \cap ((K_1 \cap U) \cup L')$. Since V intersected each component of B in a single disc, $V \cap L$ is a handlebody and so $V \cap (K_1 \cup L)$ is also a relative compressionbody with preferred surface $S \cap (K_1 \cup L)$.

Thus, if we include each compact component of $\mathrm{cl}(M - K_1)$ into K_1 to form K_1' we still have a relative Heegaard splitting $K_1' = (U \cap K_1') \cup_{S \cap K_1'} (V \cap K_1')$. Assume that we have defined K_j' for $j \geq 1$. There exists an n_j so that $K_j' \subset K_{n_j}$. Let K_{j+1}' be the union of K_{n_j} and all of the compact components of $\mathrm{cl}(M - K_{n_j})$. By the previous argument, S gives a relative Heegaard splitting of K_{j+1}. In such a way we obtain an exhaustion $\{K_n'\}$ for M with property (WP4). It is clear from the construction that $\{K_n'\}$ is, in fact, an exhaustion well-placed on S. □

Remark Theorem 5.1 tells us that there is a frontier-incompressible (rel C) exhaustion $\{K_i\}$ for M such that each K_i inherits a relative Heegaard splitting from $U \cup_S V$. An examination of the structure of the absolute Heegaard splitting of K_{j+1} induced by the relative Heegaard splitting coming from S, shows that this absolute Heegaard splitting is obtained by amalgamating Heegaard splittings of K_j and each component of $\mathrm{cl}(K_{j+1} - K_j)$.

6 Heegaard splittings of deleted boundary 3–manifolds

6.1 Introduction

Definition A 3–manifold M is *almost compact* if there is a compact 3–manifold $\overline{M}$ with non-empty boundary and a non-empty closed set $J \subset \partial \overline{M}$ such that M is homeomorphic to $\overline{M} - J$. If J is the union of components of $\partial \overline{M}$ then M is a *deleted boundary manifold*.

Let M be a deleted boundary manifold obtained from the compact manifold $\overline{M}$ by removing the union J of boundary components. By removing an open collar neighborhood of J from $\overline{M}$ we obtain a compact manifold C which resides in M. The closure of $M - C$ is homeomorphic to $J \times \mathbb{R}_+$. Since J is the union of components of $\partial \overline{M}$, J is a closed, possibly disconnected, surface. M is obviously end-irreducible (rel C) and $\partial M \subset C$. We will also assume that ∂M contains no spherical components, but, except where noted, J may have spherical components. If $|J| \geq 2$ and if at least one component is a sphere, M has Heegaard splittings which have infinitely many properly embedded stabilizing balls but are not end-stabilized. The following definitions (which make sense even when M is not a deleted boundary 3–manifold) assist the classification in this case.

Definition Let e be an end of M represented by submanifolds $\{W_i\}$ such that $\mathrm{cl}(W_i)$ is non-compact, $W_{i+1} \subset W_i$ for all i, and $M = \cup(M - W_i)$. A Heegaard splitting $M = U \cup_S V$ is *e-stabilized* if for each i there is a stabilizing ball for S contained in W_i. Recall that M is *infinitely-stabilized* if it is e-stabilized for some end e and *end-stabilized* if it is e-stabilized for every end e.

The notion of being e-stabilized is a proper ambient isotopy invariant, as the next lemma shows.

Lemma 6.1 *Suppose that S and T are Heegaard surfaces for M. If there is an end e of M such that S is e-stabilized but T is not then S and T are not properly ambient isotopic.*

Proof This follows directly from the fact that including a Heegaard surface into M induces a homeomorphism on ends (Proposition 2.2) and that proper ambient isotopies fix each end of a manifold. □

Definition Suppose that $U_S \cup_S V_S$ and $U_T \cup_T V_T$ are two absolute Heegaard splittings of M. Then they are *approximately isotopic* if for any compact set C there are proper ambient isotopies of S and T so that $S \cap C = T \cap C$.

The goal of this section is to completely classify Heegaard splittings of M up to proper ambient isotopy and up to approximate isotopy. In particular, if J contains no spherical components, M has, up to proper ambient isotopy, exactly one Heegaard splitting and that splitting is end-stabilized.

The following three theorems provide key ingredients in the classification.

Theorem 6.2 (Reidemeister-Singer) *After finitely many stabilizations, any two absolute Heegaard splittings of a compact 3–manifold which have the same partition of boundary are ambient isotopic.*

The next is a version of Theorem 2.1 of [13]. A proof is provided in the Appendix (Theorem A.1).

Theorem 6.3 (Frohman–Meeks) *Any two end-stabilized absolute Heegaard splittings with the same partition of ∂M are properly ambient isotopic. Any two infinitely-stabilized Heegaard splittings with the same partition of ∂M are approximately isotopic.*

The following is the most involved result of this section. Its proof uses Scharlemann and Thompson's classification of splittings of (closed surface) $\times I$.

Let $W_1, \ldots, W_n$ denote the components of $\mathrm{cl}(M - C)$ and let $X_1, \ldots, X_n$ denote the components of J so that W_i is homeomorphic to $X_i \times \mathbb{R}_+$. Let $e_1, \ldots, e_n$ denote the ends of M corresponding to $W_1, \ldots, W_n$ respectively.

Theorem 6.4 *Let S be any Heegaard surface for M. If $S \cap W_i$ is of infinite genus then S is e_i-stabilized. Furthermore, if X_i is not a sphere $S \cap W_i$ is of infinite genus and, therefore, S is e_i-stabilized.*

The promised classification is contained in the following propositions. The proofs of these propositions use Theorem 6.4 to give information about stabilizations and then appeal to Frohman and Meeks' theorem for the existence of the desired isotopies.

In Section 2, it was explained how to obtain finite genus splittings of non-compact 3–manifolds: remove some finite number of closed balls from a compact 3–manifold. All such 3–manifolds are deleted boundary 3–manifolds. One consequence of Theorem 6.4 is that these are the only deleted boundary 3–manifolds with finite genus Heegaard splittings. All others have only infinite genus splittings and we can classify them up to approximate isotopy and up to proper ambient isotopy.

The following propositions provide the classification. Recall that $M = \overline{M} - J$ is a deleted boundary 3–manifold:

Proposition 6.5 (2–sphere boundary) *Suppose that J consists of 2–spheres and that M' is obtained from $\overline{M}$ by attaching 3–balls to J. Then, up to proper ambient isotopy of M, any finite genus Heegaard surface in M is the intersection of a Heegaard surface for M' with M. The Heegaard surface in M' intersects each attached 3–ball in a properly embedded disc. If two such splittings of M' are isotopic then the resulting splittings of M are properly ambient isotopic.*

Proposition 6.6 (Approximate isotopy) *Suppose that S and T are infinite genus Heegaard surfaces for M whose splittings have the same partition of ∂M. Then S and T are approximately isotopic.*

Proposition 6.7 (Proper ambient isotopy) *Suppose that S and T are infinite genus Heegaard surfaces for M with the same partition of ∂M. Consider the following condition:*

$(*)$ *For each i, $S \cap W_i$ has infinite genus if and only if $T \cap W_i$ is of infinite genus.*

Then $(*)$ holds if and only if S and T are properly ambient isotopic.

Proposition 6.8 (No 2–sphere boundary components) *If no X_i is a 2–sphere then any two Heegaard splittings of M with the same partition of ∂M are equivalent up to proper ambient isotopy.*

Before we prove the theorem and the classifications, we review a technique developed by Scharlemann and Thompson [24] which was inspired by work of Otal. We also need to review the classification of Heegaard splittings of $G \times I$ where G is a closed surface.

6.2 Edge-slides of reduced spines

Definition Suppose that Q is a compact 3–manifold and that Σ is a finite graph in Q such that Σ intersects ∂Q in valence one vertices. Let B denote the components of ∂Q which intersect Σ. If $\mathrm{cl}(Q-\eta(B\cup\Sigma))$ is a compressionbody then Σ is a *reduced spine*.

Choose an edge $e\subset\Sigma$ and a path $\gamma\subset\partial Q\cup\Sigma$ with γ beginning at an endpoint of e but otherwise disjoint from e. An edge-slide of e over γ replaces e with the union of e and a copy of $\mathrm{int}(\gamma)$ pushed slightly away from $\Sigma\cup B$. See [20; 23; 24] for more detail. Edge slides give isotopies of the surface $S=(B-\mathrm{int}(\eta(\Sigma)))\cup\partial\eta(\Sigma)$. Conversely, an isotopy of a Heegaard surface can be converted into a sequence of edge-slides and isotopies of a reduced spine for one of the compressionbodies. The correspondence between edge-slides of reduced spines and isotopies of the Heegaard surface will be useful for the proof of Theorem 6.4. The reason that this viewpoint is helpful is that if Q is a compact submanifold of a non-compact manifold and if $(\partial Q-\mathrm{int}(\eta(\Sigma)))\cup\partial\eta(\Sigma)$ is part of a Heegaard surface S for M then the isotopies described by edge-slides in Q of Σ are fixed off a regular neighborhood of Q and so describe a proper isotopy of S.

To increase the genus of the Heegaard surface obtained from the reduced spine, we may stabilize a reduced spine by choosing an edge $e\subset\Sigma$. The edge e is homeomorphic to $[0,1]$ and, choosing some homeomorphism, let e' denote the subarc $[\frac{1}{4},\frac{3}{4}]$. Introduce new vertices on e at $\frac{1}{4}$ and $\frac{3}{4}$ and push the interior of e' slightly off of e to form a new edge e'' with endpoints on e at the vertices $\frac{1}{4}$ and $\frac{3}{4}$. The new edges e'' and e' of Σ bound a disc D whose interior is disjoint from Σ. The induced Heegaard splitting is stabilized in the usual sense as the boundary of the disc D intersects a meridian disc of $\eta(\Sigma)$ exactly once.

The final lemma of this section produces a reduced spine for $(\text{surface})\times I$ with particular properties. The spine gives rise to a relative version of a standard splitting of $(\text{surface})\times I$.

Lemma 6.9 *Let G be a closed surface of positive genus. Let G' and G'' be the surfaces $G\times\{\frac{1}{4}\}$ and $G\times\{\frac{3}{4}\}$ in $G\times I$. Let n be an fixed integer bigger than or equal to twice the genus of G. Let $P_0=G\times[0,\frac{1}{4}]$. Then there is a connected reduced spine $\Sigma=\Sigma(G,n)$ in $G\times I$ such that Σ intersects both boundary components of $G\times I$, Σ intersects P_0 in a vertical arc, the rank of $H_1(\Sigma)=n$, and $\partial\eta(\Sigma)$ is a relative Heegaard surface for $G\times[\frac{1}{4},1]$.*

Proof Consider $Q' = (G \times [\frac{7}{16}, \frac{9}{16}]) - (\eta(* \times [\frac{7}{16}, \frac{9}{16}]))$ where $*$ is a point on G. Then Q' is a handlebody of genus twice the genus of G. Choose genus(G) loops L based at a point $b \in \operatorname{int} Q'$ which represent generators of $\pi_1(Q', b)$. Let a be the arc $b \times I$ in $G \times I$ and assume, by general position, that the interior of each loop of L is disjoint from a. Since $\partial Q'$ is a Heegaard surface for $G \times I$, $\partial(Q' \cup \eta(a))$ is a relative Heegaard surface for $G \times I$. Stabilize the reduced spine Σ enough times so that the rank of its first homology is n. Be sure that the stabilizations take place in the interval $[\frac{1}{4}, 1]$. Then $a \cup L$ is a reduced spine for $G \times I$ satisfying the desired properties. □

6.3 Heegaard splittings of (closed surface)×I

Scharlemann and Thompson classified Heegaard splittings of $G \times I$, where G is a closed connected surface. In Theorem 6.1 of [23] they give a way of interpreting their classification in terms of edge slides of spines (reduced or non-reduced). The following are the versions of their results which we will need.

Theorem 6.10 (Scharlemann–Thompson [23]) *Suppose that Σ and Ψ are connected reduced spines for $G \times I$ which intersect both boundary components of $G \times I$ and whose first homology groups have the same rank. Then there is a finite sequence of edge-slides and isotopies taking Σ to Ψ.*

Theorem 6.11 (Scharlemann–Thompson [23]) *If a Heegaard splitting of $G \times I$ has both boundary components of $G \times I$ contained in the same compressionbody and if the splitting surfaces has genus greater than twice the genus of G then the splitting is stabilized.*

6.4 The proofs

Before beginning each proof, the theorem or proposition has been repeated for the convenience of the reader.

Theorem 6.4 *If $S \cap W_i$ is of infinite genus then S is e_i-stabilized. Furthermore, if X_i is not a sphere $S \cap W_i$ is of infinite genus and, therefore, S is e_i-stabilized.*

Proof of Theorem 6.4 Since M is end-irreducible (rel C) and $\partial M \subset C$, Theorem 5.1 guarantees that there is an exhaustion $\{K_n\}$ which is well-placed on S. In particular, $\operatorname{fr} K_n$ is incompressible in $M - C$ and no component of $\operatorname{cl}(M - K_n)$ is compact. Recall that W_i is a component of $\operatorname{cl}(M - C)$ and is homeomorphic to $X_i \times \mathbb{R}_+$ where X_i is a closed connected surface. For each n, the surface $\operatorname{fr} K_n \cap W_i$ is an incompressible surface in W_i. Furthermore, as $H_2(W_i, \partial W_i) = 0$ and $\operatorname{cl}(M - K_n)$ has no compact components, $\operatorname{fr} K_n \cap W_i$ is connected and is not a 2–sphere which is inessential in W_i.

Lemma 6.12 *For each i and for each n the submanifold $\mathrm{cl}(K_{n+1} - K_n) \cap W_i$ is homeomorphic to $X_i \times I$.*

Proof The proof is well-known, but we include it for completeness. Let $F = \mathrm{fr}\, K_{n+1} \cap W_i$. F is incompressible in W_i. Let $N_n = \mathrm{cl}(K_{n+1} - K_n) \cap W_i$. Suppose first that $X_i = S^2$. In this case, F is also homeomorphic to S^2. As F is essential it does not bound a ball in W_i. By [6, Theorem 3.1], N_n is homeomorphic to $S^2 \times I$.

Now suppose that $X_i \neq S^2$. As W_i is irreducible, $F \neq S^2$. The inclusion map of F into N_n induces an injective map on fundamental groups. Since W_i is homeomorphic to $X_i \times \mathbb{R}_+$, each loop in N_n with basepoint on F is homotopic (rel basepoint) to a loop outside of N_n. Hence, each loop is homotopic into F. Thus, the inclusion of F into N_n induces an isomorphism of fundamental groups and, so by the h-cobordism theorem [15, Theorem 10.2], N_n is homeomorphic to $F \times I$. A similar argument shows that the submanifold bounded by X_i and F is homeomorphic to $F \times I$ and so F is homeomorphic to X_i. □

Fix some i. Let $W = \mathrm{cl}(W_i - K_2)$. We will show that there is a subsequence of $\{K_n\}$ and a proper ambient isotopy of S which is fixed off $\mathrm{cl}(W_i - K_1)$ so that either $W \cap \mathrm{cl}(K_{n+1} - K_n)$ is homeomorphic to $S^2 \times I$ and $S \cap W \cap \mathrm{cl}(K_{n+1} - K_n)$ is a genus 0 relative Heegaard surface or $S \cap W \cap \mathrm{cl}(K_{n+1} - K_n)$ is a stabilized relative Heegaard surface of $W \cap \mathrm{cl}(K_{n+1} - K_n)$.

We deal first with the case when $X_i = S^2$. Let $N_n = W \cap \mathrm{cl}(K_{n+1} - K_n)$ for each $n \geq 2$.

Lemma 6.13 *If $X_i = S^2$ then $S \cap N_n$ is a relative Heegaard surface for N_n.*

Proof Recall that for each n, $\mathrm{fr}\, K_n \cap W$ is an essential 2–sphere and, by property (WP1) of well-placed exhaustions, $V \cap (\mathrm{fr}\, K_n \cap W)$ is a single disc. This implies that $U \cap (\mathrm{fr}\, K_n \cap W)$ is a single disc. Thus, for each $n \geq 2$, $U \cap N_n$ is a relative compressionbody with preferred surface $S \cap N_n$. Similarly, for each $n \geq 2$, $V \cap N_n$ is a relative compressionbody with preferred surface $S \cap N_n$. Thus $S \cap N_n$ is a relative Heegaard surface for N_n. □

By Lemma 6.12, N_n is homeomorphic to $S^2 \times I$. By the classification of Heegaard splittings of $S^2 \times I$, if $S \cap N_n$ has positive genus, there is a stabilizing ball for $S \cap N_n$ which is contained in N_n. If $S \cap W_i$ is of infinite genus, there are infinitely many n so that $S \cap N_n$ is of positive genus, and hence S is e_i-stabilized. If $S \cap W_i$ is of finite genus, we can take a subsequence of $\{K_i\}$ so that $S \cap N_n$ has genus 0. This concludes the case when $X_i = S^2$.

Suppose, for the remainder, that X_i is a closed orientable surface of positive genus g. We do not begin by supposing that $S \cap W_i$ is of infinite genus but, rather, draw that as our first conclusion.

Recall that since $\{K_n\}$ is well-placed on S, V intersects each $\operatorname{fr} K_n \cap W$ is a single disc. Let $N_n = W \cap \operatorname{cl}(K_{n+1} - K_n)$ for each $n \geq 1$. Since $\{K_n\}$ is well-placed on S the sequence $\{K_n\}$ has the outer collar property with respect to U. This means that in each $U \cap N_n$ there is a collection of discs δ_n with boundary on $S \cap N_n$ so that $\sigma(U \cap N_n; \delta_n)$ has a component which is $(\operatorname{fr} K_{n+1} \cap U \cap N_n) \times I$. The frontier of $K_{n+1} \cap U \cap N_n$ is $(\operatorname{fr} K_{n+1} \cap U \cap N_n) \times \{0\}$. On the other hand, $(\operatorname{fr} K_{n+1} \cap U \cap N_n) \times \{1\}$ is a subsurface of S except at the remnants of δ_n. Since $V \cap N_n \cap \operatorname{fr} K_{n+1}$ is a single disc and since $\operatorname{fr} K_{n+1} \cap N_n$ is homeomorphic to X_i, the surface $\operatorname{fr} K_{n+1} \cap N_n \cap U$ is homeomorphic to X_i with a single puncture. As X_i has positive genus g, the surface $\sigma(S \cap N_n; \delta_n)$ has positive genus, and, therefore, $S \cap N_n$ has positive genus for all $n \geq 1$. This implies that $S \cap W$ has infinite genus.

Take a subsequence of $\{K_n\}$ so that the first two terms of the new exhaustion are still K_1 and K_2 but so that the genus of $S \cap \operatorname{cl}(K_{n+1} - K_n) \cap W$ is at least $3g$ for $n \geq 1$. We continue referring to $\operatorname{cl}(K_{n+1} - K_n) \cap W$ as N_n.

Fix some $n \geq 2$ and let $N = N_n$. By Lemma 6.12, N is homeomorphic to $X_i \times I$. Let $F_0 = \operatorname{fr} K_n \cap N$ and $F_1 = \operatorname{fr} K_{n+1} \cap N$. V intersects F_i in a single disc D_i for $i \in \{0, 1\}$. Since $\{K_i\}$ has the outer collar property with respect to U, there is a collection of boundary-reducing discs δ_0 for $U \cap K_n \cap W$ with boundary on S and such that $\sigma(U \cap K_n \cap W; \delta_0)$ contains a component P_0^U with boundary containing $F_0 \cap U$ and which is homeomorphic to $(F_0 \cap U) \times I$. Since S is the preferred surface of $U \cap K_n$, there is a copy of $D^2 \times I$ embedded in V so that $D^2 \times \{0\} = V \cap F_0$ and $\partial D^2 \times I = S \cap P_0^U$. Let P_0 be the union of P_0^U and this $D^2 \times I$. Note that P_0 is homeomorphic to $F_0 \times I$, has F_0 as a boundary component, and has V running through P_0 as the neighborhood of an arc which is vertical in the product structure. Let $F_0' = \partial P_0 - F_0$.

We can perform a similar construction on K_{n+1} to obtain, embedded in N_n, a submanifold P_1 homeomorphic to $F_1 \times I$, with $\partial P_1 = F_1 \cup F_1'$ and $V \cap P_1$ a neighborhood of a vertical arc. Let $N' = N \cup P_0$ and $N'' = \operatorname{cl}(N' - P_1)$. Note that N' and N'' are homeomorphic to $X_i \times I$, since F_0, F_1, F_0', and F_1' are all homeomorphic to X_i. See Figure 15.

Let Σ_V be a spine for V in M which intersects each surface F_0', F_0, F_1', F_1 exactly once and which is a vertical arc in P_0 and P_1. Let $\Sigma_S = \Sigma_V \cap N'$. Note that Σ_S is a reduced spine for a Heegaard splitting of N''. To see this, recall that $U \cap (K_{n+1} - K_n)$ is a handlebody (Corollary 3.5) and notice that $N'' - \eta(\Sigma_S \cup \partial N')$ is homeomorphic

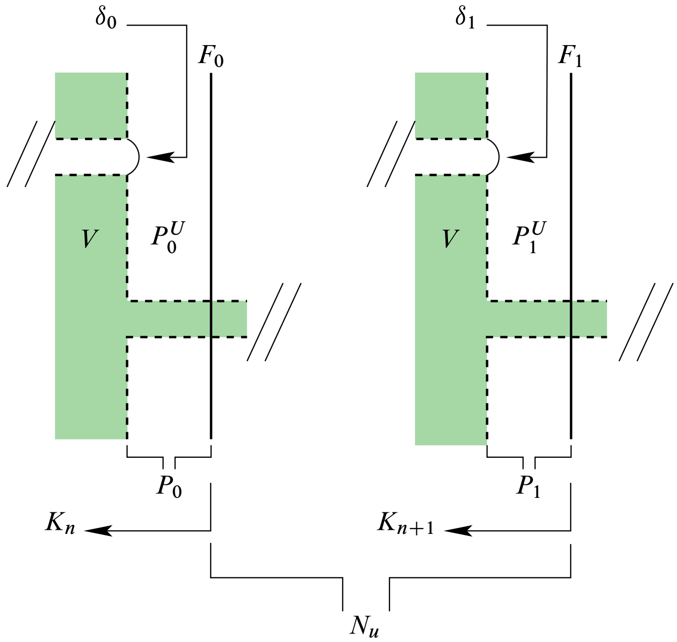

Figure 15: A schematic representing N

to $U \cap (K_{n+1} - K_n)$. We wish to show that after a proper ambient isotopy of S which is the identity off $\eta(N')$, $S \cap N$ is a Heegaard surface for N.

Choose a connected reduced spine Σ_T for a Heegaard splitting of N'' such that Σ_T intersects P_0 in a vertical arc, the rank of $H_1(\Sigma_T)$ is the same as the rank of $H_1(\Sigma_S)$, $\Sigma_T \cap F_1' \neq \varnothing$, $\Sigma_T \cap F_0' \neq \varnothing$, and $\partial\eta(\Sigma_T)$ is a hollow Heegaard surface for N. Such a spine exists by Lemma 6.9. We call Σ_T the *model spine*.

By the Scharlemann–Thompson classification of Heegaard splittings of (surface) $\times I$ (Theorem 6.10) since Σ_S and Σ_T are both reduced spines with first homologies of the same rank and since they have the same partition of $\partial N''$ there is a sequence of edge-slides and isotopies which takes Σ_S to Σ_T. It is easy to arrange these slides to be away from $\delta_0 \cup \delta_1$. The sequence of edge slides thus describes an isotopy of the surface $S \cap \eta(N'')$. By the choice Σ_T, we have that after the isotopy, $S \cap N$ is a relative Heegaard surface of genus at least $3g$ for N'.

The next corollary follows from our work so far; it is a technical result which will be useful for the classifications.

Corollary 6.14 *If X_i is a closed surface of positive genus then after a proper ambient isotopy of S which is supported on a neighborhood of $W_i' = \mathrm{cl}(M - K_1) \cap W_i$ we* *have that $S \cap K_1$ is the same before and after the isotopy and afterwards $S \cap W_i'$ is a relative Heegaard surface for W_i'.*

Proof Perform the isotopy just described so that $N_2 = \mathrm{cl}(K_2 - K_1)$ inherits a relative Heegaard splitting from S. This isotopy is fixed off a neighborhood of $W_i' = \mathrm{cl}(M - K_1) \cap W_i$ and $S \cap K_1$ is the same before and after the isotopy. Since $N_2 \subset W_i'$, there are now discs in $U \cap W_i'$ with boundary on S so that boundary reducing $U \cap W_i'$ along those discs leaves a component homeomorphic to $(\mathrm{fr}\, W_i' \cap U) \times I$. Since $V \cap W_i'$ is a disc we have that $U \cap W_i'$ and $V \cap W_i'$ are relative compressionbodies with preferred surface $S \cap W_i'$. Thus, $S \cap W_i'$ is a relative Heegaard surface for W_i'. □

We now continue the proof of Theorem 6.4. For each even n, perform this ambient isotopy on N_n. By construction, the union of these ambient isotopies is a proper ambient isotopy of $S \cap W_i$. After the isotopy, for each even n, $S \cap N_n$ is a relative Heegaard surface of genus at least $3g$ for a space homeomorphic to $X_i \times I$ where the genus of X_i is g. By the Scharlemann–Thompson classification of splittings of (surface)$\times I$ (Theorem 6.11), there is a stabilizing ball for S in each N_n for n even. Hence S is e_i-stabilized. This concludes the proof of Theorem 6.4. □

Proof of classification

Proposition 6.5 (2–sphere boundary) *Suppose that J consists of 2–spheres and that M' is obtained from $\overline{M}$ by attaching 3–balls to J. Then, up to proper ambient isotopy of M, any finite genus Heegaard surface in M is the intersection of a Heegaard surface for M' with M. The Heegaard surface in M' intersects each attached 3–ball in a properly embedded disc. If two such splittings of M' were isotopic then the resulting splittings of M are properly ambient isotopic.*

Proof of Proposition 6.5 Suppose that $U_S \cup_S V_S$ and $U_T \cup_T V_T$ are both finite genus Heegaard splittings of M. Let M' be the compact 3–manifold obtained from $\overline{M}$ by removing only the interiors of the 3–balls whose removal created M.

There is an exhaustion $\{K_i\}$ for M which is well-placed on S and an exhaustion $\{L_i\}$ which is well-placed on T. We may assume that K_1 and L_1 are homeomorphic to M' and that $S \cap (M - K_1)$ and $T \cap (M - L_1)$ have genus zero. The frontiers of the exhausting elements are essential spheres in $S^2 \times \mathbb{R}_+$ so, after taking a subsequence of each, there is a proper ambient isotopy of M which takes $\mathrm{fr}\, L_i$ to $\mathrm{fr}\, K_i$ for each i and so that $V_S \cap \mathrm{fr}\, K_i$ equals $V_S \cap \mathrm{fr}\, L_i$.

Let $N_n = \mathrm{cl}(K_{n+1} - K_n)$. By Lemma 6.13, each component of N_n inherits a genus zero relative Heegaard splitting from S and also from T. By Waldhausen's classification of splittings of $S^2 \times I$, there is a proper ambient isotopy taking $S \cap N_n$ to $T \cap N_n$ which is fixed on $\operatorname{fr} N_n$. The union of these isotopies over all n is a proper ambient isotopy of M taking $S \cap \mathrm{cl}(M - K_1)$ to $T \cap \mathrm{cl}(M - K_1)$. In particular, we may assume that $S \cap \mathrm{cl}(M - K_1)$ and $T \cap \mathrm{cl}(M - K_1)$ are vertical annuli in $S^2 \times \mathbb{R}_+$.

When we compactify M to $\overline{M}$, $S \cap \mathrm{cl}(M - K_1)$ and $T \cap \mathrm{cl}(M - K_1)$ compactify to compact annuli. $V_S \cap \mathrm{cl}(M - K_1) = V_T \cap \mathrm{cl}(M - K_1)$ compactifies to $V' = D^2 \times I$. Let V'_S and V'_T be the compactified versions of V_S and V_T respectively. Attach the 3–balls to $\overline{M}$ to create M' and let $U'_S = \mathrm{cl}(M' - V'_S)$ and $U'_T = \mathrm{cl}(M' - V'_T)$. It is clear from the construction that U'_S, U'_T, V'_S and V'_T are absolute compressionbodies and that the splittings $U_S \cup_S V_S$ and $U_T \cup_T V_T$ are obtained from splittings of M' in the correct fashion.

Furthermore, if we remove the open 3–balls from $\overline{M}$ to create M' we can extend the splittings $U'_S \cup_S V'_S$ and $U'_T \cup_T V'_T$ of $\overline{M}$ to be relative Heegaard splittings of M'. By the Marionette Lemma the relative Heegaard splittings of M' are isotopic if and only if the absolute splittings of $\overline{M}$ are isotopic. If the splittings of M' are isotopic then since K_1 is homeomorphic to M', the surfaces $S \cap K_1$ and $T \cap K_1$ are isotopic. Thus, since we already have $S \cap \mathrm{cl}(M - K_1) = T \cap \mathrm{cl}(M - K_1)$ we can arrange by a proper ambient isotopy for S to be equal to T. □

Proposition 6.6 (Approximate isotopy) *Suppose that S and T are infinite genus Heegaard surfaces for M whose splittings have the same partition of ∂M. Then S and T are approximately isotopic.*

Proof of Proposition 6.6 If S and T have infinite genus then $S \cap W_i$ and $T \cap W_j$ have infinite genus for some i, j. Since each W_k is homeomorphic to $X_k \times I$ where X_k is a closed surface, Theorem 6.4 shows that S must be e_i-stabilized and T must be e_j-stabilized. Theorem 6.3 then shows that S and T are approximately isotopic. □

Proposition 6.7 (Proper ambient isotopy) *Suppose that S and T are infinite genus Heegaard surfaces for M with the same partition of ∂M. Consider the following condition:*

($*$) *For each i, $S \cap W_i$ has infinite genus if and only if $T \cap W_i$ is of infinite genus.*

Then ($*$) holds if and only if S and T are properly ambient isotopic.

Proof of Proposition 6.7 The proof of Lemma 6.1 can be adapted to show that if S and T are properly ambient isotopic then ($*$) holds.

Suppose, then, that S and T satisfy $(*)$. We desire to show that S and T are properly ambient isotopic. Using Proposition 6.6, we will be able to enlarge C to a compact set C' such that (after performing proper ambient isotopies of S and T) C' has the following properties:

(1) $\mathrm{cl}(M-C')$ is homeomorphic to $\cup X_i \times \mathbb{R}_+$

(2) $S\cap C' = T\cap C'$

(3) $V_S \cap \mathrm{fr}\, C' = V_T \cap \mathrm{fr}\, C'$ and each of these consists of a single disc on each component of $\mathrm{fr}\, C'$.

(4) For each $W_i' = \mathrm{cl}(M-C)\cap W_i$ where S and T are of infinite genus, the surfaces $S\cap W_i'$ and $T\cap W_i'$ are relative Heegaard surfaces for W_i'.

(5) For each W_i' where S and T are not of infinite genus, the surfaces $S\cap W_i'$ and $T\cap W_i'$ are of genus zero.

The way to achieve this is to take an exhaustion $\{K_i\}$ for M which is well-placed on S such that in each component of $\mathrm{cl}(M-K_1)$ S and T are either both of infinite genus or both of genus zero. Then use the fact that S and T are approximately isotopic to isotope them so that $S\cap K_1 = T\cap K_1$. Let $C' = K_1$. If a certain X_i is not a 2–sphere, Corollary 6.14 guarantees that a further proper ambient isotopy of S and T can be performed which is supported on a neighborhood of $W_i' = \mathrm{cl}(M-C')\cap W_i$ so that after the isotopy $S\cap C'$ still equals $T\cap C'$ but we now have property (4) in addition to property (3) for that W_i'. In the case when $X_i = S^2$, S and T automatically give relative Heegaard splittings of W_i' as $\partial_-(U\cap W_i')$ and $\partial_-(V\cap W_i')$ can be taken to be the discs $U\cap X_i$ and $V\cap X_i$ respectively.

For each W_i' in which S and T are of infinite genus, Theorem 6.4 guarantees $S\cap W_i'$ and $T\cap W_i'$ are infinitely stabilized. Since W_i' is 1–ended, Theorem 6.3 guarantees that the absolute Heegaard splittings of W_i' induced by $S\cap W_i'$ and $T\cap W_i'$ are equivalent by a proper ambient isotopy in W_i'. By the Marionette Lemma, $S\cap W_i'$ and $T\cap W_i'$ are properly ambient isotopic within W_i'. For each W_i' where S and T are of genus zero, the fact that S and T are properly ambient isotopic in W_i' follows from Proposition 6.5.

Since in each component of $\mathrm{cl}(M-C')$ there is a proper ambient isotopy of S and T in that component so that they coincide, and since S and T already coincide in C' there is a proper ambient isotopy of M taking T to S. □

Proposition 6.8 (No 2–sphere boundary components) *If no X_i is a 2–sphere then any two Heegaard splittings of M with the same partition of ∂M are equivalent up to proper ambient isotopy.*

Proof of Proposition 6.8 By Theorem 6.4, S and T are end-stabilized. Theorem 6.3 then implies that they are properly ambient isotopic. □

Appendix A Infinitely stabilized Heegaard splittings

The goal of this section is to give a detailed proof the following theorem which is due, essentially, to Frohman and Meeks. Our methods are the same but we elaborate in order to fix the error mentioned in the introduction. We refer the reader to earlier sections for the definitions of the terms used here.

Theorem A.1 *Let M be a non-compact orientable 3–manifold with compact boundary not containing any 2–sphere components. Suppose that $M = U_S \cup_S V_S$ and $M = U_T \cup_T V_T$ are two Heegaard splittings of M with the same partition of ∂M. If both S and T are infinitely stablized then they are approximately isotopic. If both S and T are end-stabilized then they are properly ambient isotopic.*

In [13], Frohman and Meeks introduce a technique which they call "stealing handles from infinity". This method provides a proper isotopy of an infinitely stabilized splitting so that for any compact submanifold K, $S \cap K$ is stabilized an arbitrary number of times.

Proposition A.2 (Frohman and Meeks [13, Proposition 2.1]) *Suppose that $M = U \cup_S V$ is an infinitely stabilized Heegaard splitting of M. Let C be a submanifold of M which is adapted to S. Then for any given $n \in \mathbb{N}$ there is a proper ambient isotopy of S so that $S \cap C$ has been stabilized at least n times.*

Sketch of Proof Since S is infinitely stabilized, we can find n disjoint stabilizing balls for S in the complement of C. We may then use paths in the surface S to isotope these balls along S into C. □

Definition An exhaustion $\{K_i\}$ is *perfectly adapted* to S if it is adapted to S and, additionally, each $\mathrm{cl}(K_{i+1} - K_i)$ is adapted to S. (See Section 5.1.) Note that a subsequence of a perfectly adapted sequence is perfectly adapted.

A useful corollary of Proposition A.2 is:

Corollary A.3 *Suppose that $U_S \cup_S V_S$ and $U_T \cup_T V_T$ are two end-stabilized splittings of M with the same partition of ∂M. Suppose there is an exhaustion $\{K_i\}$ for M with the following properties:*

(i) $\partial M \subset K_1$

(ii) $V_S \cap \operatorname{fr} K_i$ *and* $V_T \cap \operatorname{fr} K_i$ *consist of discs for all* i.

(iii) $V_S \cap \operatorname{fr} K_i = V_T \cap \operatorname{fr} K_i$ *for all* i.

(iv) $\{K_i\}$ *is perfectly adapted to both* S *and* T.

Then S *and* T *are equivalent up to proper ambient isotopy.*

Proof By the Reidemeister-Singer theorem and the Marionette Lemma, after finitely many stabilizations of $S \cap K_1$ and $T \cap K_1$ there is an ambient isotopy of K_1 so that $S \cap K_1 = T \cap K_1$. Since both S and T are end-stabilized, these stabilizations can be achieved by stealing handles from infinity. Thus, we may assume that $S \cap K_1 = T \cap K_1$. By the assumption that $\{K_i\}$ is perfectly adapted to both S and T, the intersections of $U_S \cup_S V_S$ and $U_T \cup_T V_T$ with any compact component L of $\operatorname{cl}(M - K_1)$ give a relative Heegaard splittings of L. By stealing more handles from infinity and passing them through K_1 we may stabilize $S \cap L$ and $T \cap L$ enough times so that after performing an ambient isotopy of L, S and T coincide in $K_1 \cup L$. We may do this for each compact component of $\operatorname{cl}(M - K_1)$. Since there are only finitely many such components, we have constructed proper ambient isotopies of S and T so that they coincide on K_1 and each compact component of $\operatorname{cl}(M - K_1)$. We proceed by induction.

Suppose that we have performed proper ambient isotopies of M so that $S \cap K_{n-1} = T \cap K_{n-1}$ and S and T coincide on each compact component of $\operatorname{cl}(M - K_{n-1})$. We will show that there are proper ambient isotopies of S and T which are fixed on K_{n-1} so that after the isotopies S and T coincide on K_n and each compact component of $\operatorname{cl}(M - K_n)$. This will show that the composition of the isotopies of S converges to a proper ambient isotopy of S and the composition of the isotopies of T converges to a proper ambient isotopy of T. Thus, we will have shown that there are proper ambient isotopies of S and T which make them coincide with a third Heegaard surface for M. Hence, S and T are properly ambient isotopic.

Let L be a component of $\operatorname{cl}(K_n - K_{n-1})$. By hypothesis, both S and T are relative Heegaard surfaces for L. If every non-compact component of $\operatorname{cl}(M - L)$ contains K_{n-1} then L is contained in a compact component of $\operatorname{cl}(M - K_{n-1})$ and so $S \cap L = T \cap L$.

We may, thus, suppose that there is a non-compact component of $\operatorname{cl}(M - L)$ which does not contain K_{n-1}. The surfaces S and T are both end-stabilized and so we may steal handles from that non-compact component of $\operatorname{cl}(M - L)$ in order to stabilize $S \cap L$ and $T \cap L$ enough times so that they are ambient isotopic in L. Since, S and

T already coincide on $\operatorname{fr} K_{n-1}$ we may take the ambient isotopy to be the identity on $\operatorname{fr} K_{n-1} \cap L$. Thus, there is a proper ambient isotopy of S and a proper ambient isotopy of T, each fixed on K_{n-1} so that after the isotopies $S \cap K_n = T \cap K_n$.

Now suppose that L' is a compact component of $\operatorname{cl}(M - K_n)$. As before, S and T both give relative Heegaard splittings of L'. If $S \cap L' \neq T \cap L'$ then L' is not contained in a compact component of $\operatorname{cl}(M - K_{n-1})$. As in each component of $\operatorname{cl}(K_n - K_{n-1})$ S and T are connected surfaces, this implies that there are paths in S and T from a non-compact component of $\operatorname{cl}(M - K_n)$ to L' which do not intersect K_{n-1}. Thus, we may stabilize $S \cap L'$ and $T \cap L'$ as much as we wish by stealing handles from infinity via paths that do not intersect K_{n-1}. Now isotope in L' so that the splittings coincide. We have, therefore, constructed proper ambient isotopies of S and T which are fixed on K_{n-1} such that after performing the isotopies $S \cap K_n$ equals $T \cap K_n$ and S and T also coincide on each compact component of $\operatorname{cl}(M - K_n)$. Thus, S and T are properly ambient isotopic in M. □

To show that two end-stabilized splittings of M with the same partition of ∂M are properly ambient isotopic, we will show that there is an exhaustion for M satisfying the requirements of Corollary A.3. The first task is to show that if S and T have perfectly adapted exhaustions then there is a perfectly adapted sequence of M adapted to both S and T simultaneously.

Lemma A.4 (Frohman and Meeks [13, Proposition 2.3]) *Suppose that K_1 and K_2 are two submanifolds of M such that K_1, K_2, and $\operatorname{cl}(K_2 - K_1)$ are adapted to S. Suppose that L_1 and L_2 are two submanifolds of M such that L_1, L_2 and $\operatorname{cl}(L_2 - L_1)$ are adapted to T. Assume also that $K_1 \subset L_1 \subset K_2 \subset L_2$ where each inclusion is into the interior of the succeeding submanifold.*

Then after stabilizing and isotoping S in $\operatorname{cl}(K_2 - K_1)$ and stabilizing and isotoping T in $\operatorname{cl}(L_2 - L_1)$ there is a submanifold J_1 of M adapted to both S and T so that $V_S \cap \operatorname{fr} J_1$ equals $V_T \cap \operatorname{fr} J_1$ and these intersections consist of discs.

Proof Push the frontier of K_2 slightly into K_2 to form a surface $F \subset K_2$. Let M_1 be the submanifold bounded by $\operatorname{fr} K_2$ and F. (M_1 is, of course, homeomorphic to $\operatorname{fr} K_2 \times I$.) Let M_2 be the submanifold bounded by F and $\operatorname{fr} K_1$. Let N_1 be the submanifold with boundary $\operatorname{fr} L_2 \cup F$ and let N_2 be the submanifold with boundary $F \cup \operatorname{fr} L_1$. Let $J_1 = K_1 \cup M_2$. Take Heegaard splittings of M_1, M_2, N_1 and N_2 with Heegaard surfaces S_1, S_2, T_1 and T_2 respectively. We should choose these splittings so that all the boundary components of each submanifold are contained in the same compressionbody of the splitting.

We can use the Heegaard surfaces S_1 and S_2 to form a Heegaard surface $\overline{S}$ for $\mathrm{cl}(K_2 - K_1)$. To do this, note that there are surfaces S_1' and S_2' in M_1 and M_2 which are subsurfaces of S_1 and S_2 except at a finite number of open discs which are parallel to $F = M_1 \cap M_2$. The surfaces S_1' and S_2' cobound a product region $S_2' \times I$. The surface F may be assumed to be $S_2' \times \{\frac{1}{2}\}$. Take a disc $D \subset S_2' \cap S_2$ so that in the product region $S_2' \times I$ the tube $D \times I$ is disjoint from $\mathrm{cl}(S_1' - S_1)$. The Heegard surface $\overline{S}$ for $\mathrm{cl}(K_2 - K_1)$ is formed by taking $(S_1 \cup S_2 \cup D \times I) - \mathrm{int}(D \times I)$. We say that $\overline{S}$ is formed by *tubing together* S_1 and S_2. This process is different from the amalgamation of Heegaard splittings. Similarly, we may form a Heegaard surface $\overline{T}$ for $\mathrm{cl}(L_2 - L_1)$ by tubing together T_1 and T_2. Since in both constructions the tube intersects F in a single disc, we may arrange that $\overline{S} \cap F = \overline{T} \cap F$ and that these intersections are a single inessential loop on F. Finally, using the product region in the compressionbodies containing $\mathrm{fr}(K_2 - K_1)$ we may use vertical tubes to extend $\overline{S}$ to be a relative Heegaard splitting for $\mathrm{cl}(K_2 - K_1)$ which coincides with S on $\mathrm{fr}(K_2 - K_1)$. Similarly, extend $\overline{T}$ to be a relative Heegaard splitting for $\mathrm{cl}(L_2 - L_1)$ which coincides with T on $\mathrm{fr}(L_2 - L_1)$. We call the Heegaard splittings given by $\overline{S}$ and $\overline{T}$ the *model splittings*.

The Reidemeister-Singer theorem and the Marionette Lemma imply that by stabilizing S and $\overline{S}$ enough in $\mathrm{cl}(K_2 - K_1)$ we may perform an ambient isotopy of $\mathrm{cl}(K_2 - K_1)$ which brings $S \cap \mathrm{cl}(K_2 - K_1)$ to $\overline{S}$. Similarly, we may stabilize $T \cap \mathrm{cl}(L_2 - L_1)$ and $\overline{T}$ enough times so that there is an ambient isotopy of $\mathrm{cl}(L_2 - L_1)$ which brings $T \cap \mathrm{cl}(L_2 - L - 1)$ to $\overline{T}$. Since $\overline{S}$ and $\overline{T}$ coincide on $\mathrm{fr}\, J_1 = F$ we have now arranged that J_1 is a submanifold adapted to both S and T and that $S \cap \mathrm{fr}\, J_1 = T \cap \mathrm{fr}\, J_1$ and these intersections consists of a single inessential loop on each component of $\mathrm{fr}\, J_1$. □

Corollary A.5 *Suppose that $\{K_i\}$ is an exhaustion perfectly adapted to S and that $\{L_i\}$ is an exhaustion perfectly adapted to T. Assume that, for all i, $K_i \subset L_i \subset K_{i+1}$. Then after stabilizing S and T in each component of $\mathrm{cl}(K_{i+1} - K_i)$ and $\mathrm{cl}(L_{i+1} - L_i)$ respectively we may properly isotope S and T so that there is an exhaustion $\{J_i\}$ which is perfectly adapted to both S and T and is such that $S \cap \mathrm{fr}\, J_i = T \cap \mathrm{fr}\, J_i$ and the intersection consists of a single inessential loop on each component of $\mathrm{fr}\, J_i$.*

Proof Construct J_1 as in the proposition. Assuming that we have constructed J_{n-1} we will demonstrate how to construct J_n. Build J_n as in the proposition, letting K_{n+1}, K_n, L_{n+1}, L_n play the roles of K_2, K_1, L_2 and L_1. Choose model splittings for each component of $\mathrm{cl}(K_{n+1} - K_n)$ and $\mathrm{cl}(L_{n+1} - L_n)$ which coincide with the model splittings of $\mathrm{cl}(K_n - K_{n-1})$ and $\mathrm{cl}(L_n - L_{n-1})$ on $\mathrm{fr}\, K_n$ and $\mathrm{fr}\, L_n$ respectively. Stabilize the model splittings enough times so that after stabilizing $S \cap \mathrm{cl}(K_{n+1} - K_n)$

and $T \cap \mathrm{cl}(L_{n+1} - L_n)$ we may perform ambient isotopies of $S \cap \mathrm{cl}(K_{n+1} - K_n)$ and $T \cap \mathrm{cl}(L_{n+1} - L_n)$ so that they coincide with the model splittings. These isotopies are supported off K_{n-1} and L_{n-1} respectively. Note that, by the construction of the model splittings, $\mathrm{cl}(J_n - J_{n-1})$ is adapted to both S and T (after performing the isotopies).

We thus obtain an exhaustion $\{J_i\}$ for M. The final remarks of the previous paragraph show that there are proper ambient isotopies of S and T so that $\{J_i\}$ is perfectly adapted to both Heegaard surfaces. □

Remark So far we have shown that if S and T are end-stabilized splittings and if there are exhaustions perfectly adapted to each of them then (after stealing handles from infinity and performing other proper ambient isotopies of S and T) there is an exhaustion which is perfectly adapted to both of them at the same time and furthermore S and T coincide on the frontiers of the exhausting submanifolds. Corollary A.3 then shows that S and T are properly ambient isotopic. It thus remains to show that there is a perfectly adapted exhaustion adapted to any given end-stabilized splitting. The following lemmas show how we can achieve this. The first one fixes the misstatement in [13, Proposition 2.2] mentioned in the introduction.

Lemma A.6 *Let $M = U \cup_S V$ be an absolute Heegaard splitting of the non-compact 3–manifold M and let $\{K_i\}$ be an exhaustion for M adapted to S. Assume that, for each i, $V \cap \mathrm{fr}\, K_i$ consists of discs and that the sequence $\{K_i\}$ has the outer collar property with respect to U. Then after stabilizing $S \cap \mathrm{cl}(K_n - K_{n-1})$, for each $n \geq 3$, a finite number of times, there is a proper ambient isotopy of $S \cap K_n$ with the following properties:*

(i) *The isotopy is fixed on $K_{n-2} \cup \mathrm{cl}(M - K_n)$.*

(ii) *$S \cap K_{n-1}$ is the same before and after the isotopy.*

(iii) *After the isotopy, S is a relative Heegaard surface for $\mathrm{cl}(K_n - K_{n-1})$.*

The proof is similar to the proof of Theorem 6.4. The reader is referred to Section 6.2 for the definitions and properties of edge-slides.

Proof Let N be a component of $\mathrm{cl}(K_n - K_{n-1})$. Let $F_2 = \mathrm{fr}\, K_n \cap N$ and $F_1 = \mathrm{fr}\, K_{n-1} \cap N$. Since $\{K_i\}$ has the outer collar property, there are discs $\delta_1 \subset (U \cap K_{n-1})$ with boundary on S so that $\sigma(U \cap K_{n-1}; \delta_1)$ contains a product region $P_1^U = (F_1 \cap U) \times I \subset U \cap \mathrm{cl}(K_{n-1} - K_{n-2})$ with $F_1 \cap U = (F_1 \cap U) \times \{0\}$. Let $(F_1' \cap U)$ signify $(F_1 \cap U) \times \{1\}$; it is a subsurface of S except at the remnants of the discs δ_1. Similarly, there are discs $\delta_2 \subset U \cap K_n$ with boundary on S so that $\sigma(U \cap K_n; \delta_1)$ contains a product region $P_2^U = (F_2 \cap U) \times I \subset U \cap \text{cl}(K_n - K_{n-1})$ with $(F_2 \cap U) = (F_2 \cap U) \times \{0\}$. Let $(F_2' \cap U)$ signify $(F_2 \cap U) \times \{1\}$; it is a subsurface of S except at the remnants of the discs δ_2. The boundaries of the surfaces $F_1' \cap U$ and $F_2' \cap U$ are simple closed curves on S which bound discs in V. Let F_1' and F_2' be the surfaces $F_1' \cap U$ and $F_2' \cap U$ together with discs in V bound by $\partial F_1' \cap U$ and $\partial F_2' \cap U$. Let P_1 and P_2 be the product regions bounded by $F_1' \cup F_1$ and $F_2' \cup F_2$ respectively. P_1^U and P_2^U are the product regions which are the intersections of P_1 with U and P_2 with U. Let $N' = N \cup P_1$.

Choose a spine for V which intersects each disc of $\delta_1 \cup \delta_2$ exactly once. We may assume that the spine intersects P_1 and P_2 in vertical arcs. Let Σ be the intersection of this spine with N'. Corollary 3.5 shows that $U \cap \text{cl}(K_n - K_{n-1})$ and $V \cap \text{cl}(K_n - K_{n-1})$ are compressionbodies. Since there are not closed components of $\partial_- \text{cl}(N' - \eta(\partial N' \cup \Sigma))$, $\text{cl}(N' - \eta(\partial N' \cup \Sigma))$ is a handlebody and so Σ is a reduced spine for N'. (Recall that $\{K_i\}$ is adapted to S and so $U \cap K_{n-1}$ is correctly embedded in $U \cap K_n$. This is needed to apply Corollary 3.5.)

We now construct a model splitting of N'. Let $X \cup_W Y$ be any relative Heegaard splitting of N with $Y \cap \text{fr}\, N = V \cap \text{fr}\, N$. Let Σ' be a reduced spine for Y. We may assume that $\Sigma' \cap P_2$ consists of vertical arcs. Using the product region P_1 we may extend Σ' to be a graph in N' whose intersection with P_1 consists of vertical arcs. $\Sigma' \cap \text{cl}(N' - P_1)$ is a reduced spine for $\text{cl}(N' - P_2)$.

The Reidemeister-Singer theorem and the Marionette Lemma imply that by stabilizing the Heegaard splittings of $N'' = \text{cl}(N' - P_2)$ induced by $\Sigma \cap N''$ and $\Sigma' \cap N''$ they become isotopic. Perform the necessary stabilizations in such a way that the graphs $\Sigma \cap N''$ and $\Sigma' \cap N''$ still intersect P_1 in vertical arcs. Edge-slides of reduced spines are equivalent to isotopies of the Heegaard surfaces, so there is a sequence of edge-slides which takes (the now stabilized) $\Sigma \cap N''$ to $\Sigma' \cap N''$. These edge-slides may involve sliding edges of $\Sigma \cap N''$ over other edges or over the surfaces $F_1' \cup F_2'$.

These edge-slides define an ambient isotopy of $S \cap N'$ which is fixed off a regular neighborhood of $\text{cl}(N' - P_2)$. In particular, the isotopy is fixed on $K_{n-2} \cup \text{cl}(M - K_n)$. After the isotopy, $S \cap K_{n-1}$ is exactly the same as it was before. Now, however, $S \cap \text{cl}(K_n - K_{n-1})$ is a relative Heegaard surface for N since the model surface was. □

Lemma A.7 *Suppose that $M = U \cup_S V$ is an end-stabilized absolute Heegaard splitting of M. Then there is an exhaustion $\{L_i\}$ which is perfectly adapted to S.*

Proof By Section 4.3 and Corollary 4.3, there is an exhaustion $\{K_i\}$ which is adapted to S, has the outer collar property, and is such that $V \cap \text{fr}\, K_i$ consists of discs for all i. Recall that, since S is end-stabilized, any time we need to stabilize some $S \cap \mathrm{cl}(K_i - K_j)$ we may do so by a proper ambient isotopy of S in such a way that K_j is fixed throughout the isotopy. This means the isotopies needed to make each $\mathrm{cl}(K_i - K_j)$ of arbitrarily high genus can be achieved by a single proper ambient isotopy of S in M.

For each $\mathrm{cl}(K_{3i+1} - K_{3i})$, steal handles from infinity and perform the isotopy of $S \cap \mathrm{cl}(K_{3i+1} - K_{3i})$ needed in order to make $\mathrm{cl}(K_{3i+1} - K_{3i})$ adapted to S. Since each of these isotopies is fixed on K_{3i-2} their union is a proper ambient isotopy of S. Let $L_i = K_{3i}$ for each i. We claim that $\{L_i\}$ is perfectly adapted to S.

It is, of course, adapted to S as each K_i is adapted to S before and after the isotopy. We need to show that after this isotopy $\mathrm{cl}(K_{3i} - K_{3i-3})$ is adapted to S for $i \geq 2$. To see this, note that since V intersects each $\mathrm{fr}\, K_{3i}$ in discs $V \cap \mathrm{cl}(K_{3i} - K_{3i-3})$ is a relative compressionbody with preferred surface $S \cap \mathrm{cl}(K_{3i} - K_{3i-3})$ for each i. To see that $U \cap \mathrm{cl}(K_{3i} - K_{3i-3})$ is a relative compressionbody with preferred surface $S \cap \mathrm{cl}(K_{3i} - K_{3i-3})$ note first that $\{L_i\}$ has the outer collar property. Furthermore, after the isotopy, there are discs $(\delta_1, \partial\delta_1) \subset (U \cap \mathrm{cl}(K_{3i-2} - K_{3i-3}), S \cap \mathrm{cl}(K_{3i-2} - K_{3i-3}))$ which cut off a product region $(U \cap \mathrm{fr}\, K_{3i-3}) \times I$ contained in $U \cap (\mathrm{cl}(K_{3i-2} - K_{3i-3})) \subset U \cap \mathrm{cl}(K_{3i} - K_{3i-3})$. Hence $\{L_i\}$ has both the inner and outer collar properties. It is easy to see that $\{L_i\}$ is perfectly adapted to S (cf Section 4.1). □

Proof of Theorem A.1 Suppose, first, that $U_S \cup_S V_S$ and $U_T \cup_T V_T$ are two absolute infinitely stabilized Heegaard splittings of M with the same partition of ∂M. To show that they are approximately isotopic we will show that given any compact set C there are proper ambient isotopies of S and of T so that after the isotopies, S and T coincide on C. By Section 4.3 and Corollary 4.3, there are exhaustions $\{K_i\}$ and $\{L_i\}$ adapted to S and T respectively which have the outer collar property and are such that $V_S \cap \mathrm{fr}\, K_i$ and $V_T \cap \mathrm{fr}\, L_i$ consist of discs. Take subsequences so that $C \subset K_1 \subset L_1 \subset K_2 \subset L_2$. By Lemma A.6 we may steal handles from infinity for both S and T and then perform further proper ambient isotopies so that K_1, K_2 and $\mathrm{cl}(K_2 - K_1)$ are adapted to S and L_1, L_2, and $\mathrm{cl}(L_2 - L_1)$ are adapted to T. By Lemma A.4 we may steal more handles from infinity and perform more ambient isotopies of S and T so that there is a submanifold J_1 containing K_1 which is adapted to both S and T. By stealing more handles from infinity, we may stabilize $S \cap J_1$ and $T \cap J_1$ enough times so that they are ambient (in J_1) isotopic (Reidemeister-Singer theorem and the Marionette Lemma). Isotope S and T so that they coincide on J_1. They then also coincide on C and so they are approximately isotopic.

Now suppose that S and T are end-stabilized. By Lemma A.7 there are exhaustions $\{K_i\}$ and $\{L_i\}$ perfectly adapted to S and T respectively. By Corollary A.5, we may perform proper ambient isotopies of S and T so that there is an exhaustion $\{J_i\}$ perfectly adapted to both of them and is such that, for each i, $V_S \cap \operatorname{fr} J_i = V_T \cap \operatorname{fr} J_i$ and the intersections consist of discs. By Corollary A.3, S and T are properly ambient isotopic. ◻

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