

Cohomology of the space of commuting n -tuples in a compact Lie group

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Consider the space $\text{Hom}(\mathbb{Z}^n, G)$ of pairwise commuting n -tuples of elements in a compact Lie group G . This forms a real algebraic variety, which is generally singular. In this paper, we construct a desingularization of the generic component of $\text{Hom}(\mathbb{Z}^n, G)$, which allows us to derive formulas for its ordinary and equivariant cohomology in terms of the Lie algebra of a maximal torus in G and the action of the Weyl group. This is an application of a general theorem concerning G -spaces for which every element is fixed by a maximal torus.

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1 Introduction

Moduli spaces of flat bundles are important in physics, where they form critical level sets of Lagrangians for a number of important quantum field theories, such as the Yang–Mills and Chern–Simons theories. In this paper, we consider the moduli space of flat G -bundles over a compact torus, $(S^1)^n$, where G is a compact Lie group. The fundamental group of $(S^1)^n$ is isomorphic to \mathbb{Z}^n , and the holonomy map allows us to identify the moduli space of flat bundles with $\text{Hom}(\mathbb{Z}^n, G)/G$, the space of homomorphisms of the fundamental group into G , modulo conjugation. When $n=2$, $\text{Hom}(\mathbb{Z}^2, G)/G$ is isomorphic to the moduli space of semistable holomorphic $G_{\mathbb{C}}$ bundles over an elliptic curve, where $G_{\mathbb{C}}$ denotes the complexification of G . The G -space $\text{Hom}(\mathbb{Z}^n, G)$ is the moduli space of based flat connections and will be the principal object of study in this paper.

Because a homomorphism is determined by where it sends generators, we may also identify $\text{Hom}(\mathbb{Z}^n, G)$ with $\{(g_1, \dots, g_n) \in G^n \mid g_i g_j = g_j g_i \ \forall i, j\}$, the space of pairwise commuting n -tuples in G . G carries a unique structure as a real algebraic group and $\text{Hom}(\mathbb{Z}^n, G)$ inherits the structure of a compact affine real algebraic variety. G acts regularly on $\text{Hom}(\mathbb{Z}^n, G)$ by conjugation, and the quotient $\text{Hom}(\mathbb{Z}^n, G)/G$ is a space with orbifold singularities.

A mistaken assumption about the nature of these spaces led to an undercount of the number of vacuum states in supersymmetric Yang–Mills theory over spatial $(S^1)^3$ by

Witten in [15]. The source of the error was identified in Witten [16], and motivated a systematic study of the spaces $\text{Hom}(\mathbb{Z}^n, G)/G$ in Borel–Friedman–Morgan [4] and Kac–Smilga [12], with particular focus on the case $n = 3$. In both these papers, $\text{Hom}(\mathbb{Z}^n, G)/G$ is shown to be describable in terms of the combinatorics of root systems. One striking result to emerge is that $\text{Hom}(\mathbb{Z}^n, G)/G$ need not be connected, even when G is simply connected.

The study of the space $\text{Hom}(\mathbb{Z}^n, G)$ was initiated more recently in Adem–Cohen [1] in the broader context of Lie subgroups of $GL_n(\mathbb{C})$. Their study was motivated by connections to orbifolds and pure braid groups. Their method is to consider the filtration of $\text{Hom}(\mathbb{Z}^n, G)$ by subspaces $S_n(j, G) := \{(g_1, \dots, g_n) \mid \text{at least } j \text{ entries equal } 1_G\}$, where 1_G denotes the identity in G . They show that after suspending once, $\text{Hom}(\mathbb{Z}^n, G)$ decomposes up to homotopy equivalence as a wedge of spaces:

$$(1) \quad \Sigma(\text{Hom}(\mathbb{Z}^n, G)) \sim \bigvee_{1 \leq k \leq n} \Sigma \left(\bigvee_{\binom{n}{k}} \text{Hom}(\mathbb{Z}^k, G)/S_k(1, G) \right)$$

They proceed to compute homology groups explicitly for $\text{Hom}(\mathbb{Z}^2, SU(2))$, and $\text{Hom}(\mathbb{Z}^3, SU(2))$, though this method has so far not led to general formulas for cohomology. See Torres–Giese–Sjerve [14] for a study of the case $G = SO(3)$, including a description of connected components, fundamental groups and \mathbb{Z}_2 homology.

Let $R_{n,G}$ denote the connected component of $\text{Hom}(\mathbb{Z}^n, G)$ containing the n -tuple $(1_G, \dots, 1_G)$. $R_{n,G}$ can be characterized as those n -tuples whose entries lie in a common maximal torus. If G is connected and $g \in G$ is generic (that is, it has centralizer a maximal torus), then any commuting n -tuple containing g lies in $R_{n,G}$, so we will call $R_{n,G}$ the generic component of $\text{Hom}(\mathbb{Z}^n, G)$. In many interesting cases (see Theorem 4.1), $\text{Hom}(\mathbb{Z}^n, G)$ is connected and so $R_{n,G} = \text{Hom}(\mathbb{Z}^n, G)$.

As motivation to focus on this component, we remark that the space $R_{2n,G}$ is isomorphic to the moduli space of based flat G -bundles over a genus n Riemann surface, whose holonomy is reducible to a maximal torus, and that $R_{n,G}$ is also closely related to the corresponding space for the closed nonorientable surface $\mathbb{R}P^{\#(n+1)}$. In upcoming work by the author [3] the results of this paper will be used in the computation of the cohomology of moduli spaces of flat $SU(2)$ -bundles over nonorientable surfaces.

In this paper, we construct a desingularization of $R_{n,G}$, and present a nice formula for its cohomology ring over fields with characteristic relatively prime to the order of the Weyl group of G .

Our formula is a generalization of an old formula (see Greub–Halperin–Vanstone [9]) for the cohomology of $G = \text{Hom}(\mathbb{Z}, G)$,

$$(2) \quad H(G) \cong H(G/T \times T)^W$$

where T is a maximal torus in G and $W = N(T)/T$ is the Weyl group. This formula is a consequence of the observation that the map $\phi: G/T \times T \rightarrow G$, which sends $([g], t)$ to $\phi([g], t) = g t g^{-1}$ is “almost” a covering map with covering transformation group W in a sense we make precise in Section 2. Basically, this means that W acts freely on $G/T \times T$ leaving ϕ invariant and that $\phi^{-1}(g)/W$ is cohomologically equivalent to a point for all $g \in G$.

For example, when $G = SU(2)$, $G/T \times T$ is isomorphic to $S^2 \times S^1$ and $\phi^{-1}(g)$ is two points unless $g = \pm 1_G$ when $\phi^{-1}(g) \cong S^2$. The quotient by $W = \mathbb{Z}_2$ is $\mathbb{R}P^2$ over these exceptional points, which is cohomologically trivial for coefficient fields of characteristic not equal to 2. Indeed, the space $S^2 \times_{\mathbb{Z}_2} S^1$ is just the real blow up of $SU(2)$ over the points $\pm 1_G$.

For general n , the formula is

$$(3) \quad H(R_{n,G}) \cong H(G/T \times T^n)^W$$

with the same restriction on the coefficient field. The method of proof is to generalize this construction of an “almost” covering map to wider class of G -spaces that includes $R_{n,G}$. The main condition for this construction to work is that every stabilizer must contain a maximal torus of G . As a bonus we also get a formula for the equivariant cohomology,

$$(4) \quad H_G(R_{n,G}) \cong H_T(T^n)^W.$$

In the case $G = SU(2)$ and $n \geq 2$, $R_{n,G}$ has 2^n isolated singularities at the points $(\pm 1_G)^n$. Locally these singularities are cones over $S^2 \times_{\mathbb{Z}_2} S^{n-1}$, where \mathbb{Z}_2 acts antipodally on each factor, so these singularities look like the total space of the vector bundle $S^2 \times_{\mathbb{Z}_2} \mathbb{R}^n \rightarrow S^2/\mathbb{Z}_2 = \mathbb{R}P^2$, quotiented by the zero section. $G/T \times T^n \cong S^2 \times (S^1)^n$, and its quotient $S^2 \times_{\mathbb{Z}_2} (S^1)^n$ is a smooth manifold and can be identified as the blow up of $R_{n,G}$ by these zero sections.

Some of the advantages of our approach over the approach in Adem–Cohen [1] are that we obtain general formulas for the cohomology of $R_{n,G}$, and these formulas actually describe $H(R_{n,G})$ as a ring, not just as a group. One of the disadvantages is that while we do get some negative results about torsion ($H(R_{n,G})$ has no p -torsion for primes p that do not divide the order of the Weyl group), the remaining torsion is not accessible by these methods.

The layout of this paper is as follows:

In [Section 2](#), we define the notion of a cohomological principal bundle. When the structure group is finite, these are the “almost” covering maps described above, which have the important property that the cohomology ring of the base is isomorphic to the invariant subring of the cohomology of the total space. I got the idea for this from Cappell–Lee–Miller [\[6\]](#) where a construction similar to the one in this paper is used in the study of the moduli space of flat $SU(2)$ connections on a Riemann surface. Cohomological principal bundles also emerge in a different way in the study of $SU(2)$ connections on nonorientable surfaces in [\[3\]](#) and seem to be a recurring phenomenon in such problems.

In [Section 3](#), we prove a general result ([Theorem 3.3](#)) concerning the cohomology of spaces admitting G actions for which every point is fixed by a maximal torus.

In [Section 4](#), we apply the theorems proved in [Section 3](#) to the case of commuting n -tuples, obtaining formulas for the ordinary and equivariant cohomology of $R_{n,G}$.

In [Section 5](#), we explore some consequences of these formulas by working out a more explicit description in the case $G = SU(2)$ and computing Poincaré polynomials for $G = SU(2)$, $SU(3)$ and $SU(4)$.

Also included are two appendices reviewing some results that are applied in the paper. [Appendix A](#) is an overview of equivariant cohomology from the Borel model perspective, and in [Appendix B](#) we work out formulas for $H(T^n)$ and $H(G/T)$ as modules of the Weyl group.

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2 Cohomological principal bundles

Let $f: X \rightarrow Y$ be a continuous map between topological spaces X and Y , and let Γ be a topological group acting freely on the right of X , such that $X \rightarrow X/\Gamma$ is a principal bundle.

Definition 1 We say $(f: X \rightarrow Y, \Gamma)$ is a *cohomological principal bundle* for the cohomology theory H if:

- (i) f is a closed surjection

(ii) f descends through the quotient to a map h

$$(5) \quad \begin{array}{ccc} X & & \\ \downarrow \pi & \searrow f & \\ X/\Gamma & \xrightarrow{h} & Y \end{array}$$

(iii) $H(h^{-1}(y)) \cong H(pt)$ for all $y \in Y$

Let $H(X, F)$ denote sheaf cohomology of the constant sheaf F_X , where F is a field (in all applications we have in mind, sheaf cohomology is isomorphic to singular cohomology). To prove [Proposition 2.3](#) we will require the following two standard results (see Bredon [\[5, 11.7 and 19.2\]](#)):

Theorem 2.1 (Vietoris–Begle mapping theorem) *Let $h: Z \rightarrow Y$ be a closed surjection, where Z is a paracompact Hausdorff space. Suppose that for all $y \in Y$, $H(f^{-1}(y), F) \cong H(pt, F)$. Then*

$$(6) \quad f^*: H(Y, F) \rightarrow H(Z, F)$$

is an isomorphism.

Theorem 2.2 *Let X be a topological space, let Γ be a finite group acting on X and let $\pi: X \rightarrow X/\Gamma$ denote the quotient map onto the orbit space X/Γ . If F is a field satisfying $\gcd(\text{char}(F), \#\Gamma) = 1$, then*

$$(7) \quad \pi^*: H(X/\Gamma, F) \rightarrow H(X, F)^\Gamma$$

is an isomorphism, where $H(X, F)^\Gamma$ denotes the ring of Γ invariants.

Combining [Theorem 2.1](#) and [Theorem 2.2](#), we get:

Proposition 2.3 *Let Γ be a finite group of order N , let X be a paracompact Hausdorff space and let $(f: X \rightarrow Y, \Gamma)$ be a cohomological principal bundle for $H(\cdot, F)$, where $\gcd(\text{char}(F), N) = 1$. Then f^* induces an isomorphism*

$$(8) \quad f^*: H(Y, F) \cong H(X, F)^\Gamma$$

where $H(X, F)^\Gamma$ denotes the ring of Γ invariants.

Proof Since X is a paracompact Hausdorff space, X/Γ is as well. The induced map $h: X/\Gamma \rightarrow Y$ satisfies the conditions of the Vietoris mapping theorem, so

$$(9) \quad h^*: H(Y, F) \cong H(X/\Gamma, F).$$

On the other hand, by [Theorem 2.2](#), $\pi^*: H(X/\Gamma, F) \cong H(X, F)^\Gamma$. Since $f^* = \pi^* \circ h^*$ this completes the proof. \square

Though [Proposition 2.3](#) suffices for our purposes, it will more convenient to use the following corollary.

Corollary 2.4 *Let Γ be a compact Lie group with N connected components, and let Γ_0 be the identity component. Let $(f: X \rightarrow Y, \Gamma)$ be a cohomological principal bundle for $H(\cdot, F)$, where $\gcd(\text{char}(F), N) = 1$ and X is a paracompact Hausdorff space. Then $H(Y, F) \cong H(X/\Gamma_0, F)^{\Gamma/\Gamma_0}$.*

Proof f descends to a map $g: X/\Gamma_0 \rightarrow Y$ and the residual action of Γ/Γ_0 acts on X/Γ_0 making the pair into a cohomological covering map. Since $X \rightarrow X/\Gamma_0$ is a principal bundle, it is a closed map and thus X/Γ_0 is paracompact Hausdorff (see Engelking [8] 5.1). The result then follows from [Proposition 2.3](#). \square

3 Cohomology of G -spaces with stabilizers containing a maximal torus

Let G be a connected compact group, T a maximal torus of G and X a space on which G acts. If every point $x \in X$ is fixed under this action by some maximal torus of G , then because all maximal tori in G are conjugate, every G orbit must intersect the T fixed point set X^T . It follows that the map

$$(10) \quad \phi: G \times X^T \rightarrow X, \phi((g, x)) = g \cdot x$$

is surjective. G acts on $G \times X^T$ by $g \cdot (h, x) = (gh, x)$, and ϕ is equivariant for this action. The normalizer of T in G , denoted $N(T) = N_G(T)$, acts freely on $G \times X^T$ from the right by

$$(11) \quad (g, x) \cdot n = (gn, n^{-1} \cdot x)$$

leaving ϕ invariant and commuting with the G action. We will show in [Theorem 3.3](#) that under very mild conditions the pair $(\phi: G \times X^T \rightarrow X, N(T))$ is a cohomological principal bundle. We begin with a couple of lemmas.

Lemma 3.1 *Let G act on X from the left and let $x \in X^T$. Then $g \cdot x \in X^T$ if and only if $g \in N(T)G_x^0$, where G_x^0 is the identity component of the stabilizer G_x .*

Proof If $g \cdot x \in X^T$, then $g^{-1}tg \cdot x = x$ for all $t \in T$, so

$$(12) \quad g^{-1}Tg \subset G_x.$$

Since T is maximal in G , it is also maximal in G_x , so for some $h \in G_x^0$, $h^{-1}g^{-1}Tgh = T$, and thus $g \in N(T)G_x^0$. The other direction is clear. \square

Let $W_G := N_G(T)/T$ denote the Weyl group of G .

Lemma 3.2 Let $(\phi: G \times X^T \rightarrow X, N(T))$ be defined as above. For every $x \in X$, $H(\phi^{-1}(x)/N(T), F) \cong H(pt, F)$, for F satisfying $\gcd(\text{char}(F), \#W_G) = 1$.

Proof We may assume by equivariance that $x \in X^T$. Then

$$(13) \quad \phi^{-1}(x) = \{(g, y) | g \cdot y = x\} = \{(g, y) | y \in X^T, g^{-1}x = y\} \cong G_x^0 N(T)$$

where this last isomorphism follows from the preceding lemma. It follows that

$$(14) \quad \phi^{-1}(x)/N(T) \cong G_x^0 N(T)/N(T) \cong G_x^0/N_{G_x^0}(T).$$

Now since $\#W_{G_x^0}$ divides $\#W_G$, we deduce from [Proposition A.4](#) that

$$(15) \quad H(G_x^0/N_{G_x^0}(T), F) \cong H(pt, F)$$

completing the proof. \square

Theorem 3.3 Let G be a compact, connected Lie group with maximal torus T , acting on a paracompact Hausdorff space X . Suppose that for every point $x \in X$, G_x contains a maximal torus of G . Then $(\phi: G \times X^T \rightarrow X, N(T))$ is a cohomological principal bundle for $H(\cdot, F)$, where $\gcd(\text{char}(F), \#W_G) = 1$. In particular

$$(16) \quad H(X, F) \cong H(G/T \times X^T, F)^{W_G}.$$

Proof First note that since X^T is a closed subset of X it inherits a paracompact Hausdorff topology. The only conditions in [Definition 1](#) that are not immediate are (iii), which follows from [Lemma 3.2](#), and closedness of the map ϕ . But ϕ is a restriction of the action map $G \times X \rightarrow X$ which is easily shown to be closed, so ϕ is also closed.

The final assertion follows from [Corollary 2.4](#)

\square

It is helpful to consider two extreme cases of this theorem.

Example 1 Suppose that G acts trivially on X . Then $X = X^T$ and $G \times_{N(T)} X^T = G/N(T) \times X$. By [Proposition A.4](#), we know that $G/N(T)$ has trivial cohomology, so by the Kunnet theorem $H(X, F) \cong H(G/N(T) \times X, F) \cong H(G/T \times X^T, F)^{W_G}$.

Example 2 Suppose that G acts so that G_x is a maximal torus for every $x \in X$. By [Lemma 3.1](#) we deduce that each orbit must intersect X^T precisely $\#W_G$ times. It follows easily that $G/T \times X^T \rightarrow X$ is a covering space map with deck transformation group W_G , and thus $H(G/T \times X^T, F)^{W_G} \cong H(X, F)$.

Corollary 3.4 *Let G act on X satisfying the hypotheses of [Theorem 3.3](#). Then $\dim H(X, \mathbb{C}) = \dim H(X^T, \mathbb{C})$.*

Proof It is a general property proven using characters, that if V is a finite dimensional representation of a finite group Γ , then $\dim(\mathbb{C}\Gamma \otimes V)^\Gamma = \dim V$. Because $H(G/T, \mathbb{C}) \cong \mathbb{C}W$ as a left W representation (see [Proposition B.1](#)) it follows that

$$(17) \quad \dim H(X, \mathbb{C}) = \dim(H(G/T, \mathbb{C}) \otimes H(X^T, \mathbb{C}))^W = \dim H(X^T, \mathbb{C}).$$

This completes the proof □

When X is a smooth compact manifold or more generally when the set of infinitesimal stabilizers of the action is finite, [Corollary 3.4](#) combined with [Proposition A.3](#) implies that the action is T equivariantly formal (see [Appendix A](#) for a review of equivariant cohomology). We adopt the convention that equivariant cohomology is assumed to be taken with complex coefficients unless otherwise stated, that is, $H_G(\cdot) = H_G(\cdot, \mathbb{C})$.

Now if we have a G equivariant map $\rho: Y \rightarrow X$ between G -spaces Y and X , we obtain a map between homotopy quotients $\rho_G: Y_G \rightarrow X_G$. If ρ induces an isomorphism $H(X, \mathbb{C}) \cong H(Y, \mathbb{C})$, we see by considering the Serre spectral sequences of the standard fibrations of X_G and Y_G over BG that $\rho_G^*: H_G(X) \cong H_G(Y)$.

Theorem 3.5 *Let G act on X satisfying the hypotheses of [Theorem 3.3](#). Then $H_G(X) \cong H_T(X^T)^W = (H(X^T) \otimes H_T)^W$.*

Proof The G -equivariant map $\phi: G/T \times_W X^T \rightarrow X$ induces an isomorphism in \mathbb{C} cohomology, so

$$(18) \quad H_G(X) \cong H_G(G/T \times_W X^T) \cong H_G(G/T \times X^T)^W.$$

It follows from a well known formula in equivariant cohomology (equation [\(28\)](#)) that $H_G(G/T \times X^T) \cong H_T(X^T)$, so we deduce that

$$(19) \quad H_G(X) \cong H_T(X^T)^W.$$

□

Remark 1 This theorem can also be proven by showing directly that $(EG \times X^T \rightarrow X_G, N(T))$ is a cohomological principal bundle.

Remark 2 By the familiar identity $H_G(X) \cong H_T(X)^W$, [Theorem 3.5](#) shows that the localization map $i^*: H_T(X) \rightarrow H_T(X^T)$ restricts to an isomorphism between the Weyl invariant subrings.

Just to clarify this theorem, we work out carefully how W acts on $H_T(X^T)$. If E is the total space of a universal G -bundle, then $(G/T \times X^T)_G = E \times_G (G/T \times X^T)$. If $n \in N(T)$ represents an element of W , and (e, g, x) represents an element of $(G/T \times X^T)_G$, then $[(e, g, x)] \cdot [n] = [(e, gn, n^{-1} \cdot x)]$. Thus in terms of the identification with $E \times_T X^T$ the action looks like $[(e, x)] \cdot [n] = [(e \cdot n, n^{-1} \cdot x)]$. If we turn the right representation of W on $H(X^T)$ into a left representation in the usual way, the action of W on $H_T(X^T)$ is just the tensor product of the representations on $H(X^T)$ and H_T .

4 Cohomology of $R_{n,G}$

Let G be a connected, compact Lie group and let $R_{n,G}$ be the identity component of $\{(g_1, \dots, g_n) \in G^n \mid g_i g_j = g_j g_i \ \forall i, j\} \cong \text{Hom}(\mathbb{Z}^n, G)$, topologized as a subspace of G^n . The following theorem, paraphrased from Kac–Smilga [\[12\]](#), shows that in many cases $R_{n,G}$ is in fact the only component of $\text{Hom}(\mathbb{Z}^n, G)$.

Theorem 4.1 *Let G be a compact, simple Lie group. The space $\text{Hom}(\mathbb{Z}^n, G)$ is connected if and only if any of the following conditions are met:*

- (i) $n = 1$ and G is connected.
- (ii) $n = 2$ and G is 1-connected.
- (iii) $n \geq 3$ and $G = SU(m)$ or $Sp(m)$ for $m \geq 1$.

The following lemma, which is a consequence of classification of components of $\text{Hom}(\mathbb{Z}^n, G)$ in [\[12\]](#) and Borel–Friedman–Morgan [\[4\]](#), gives a concrete description of $R_{n,G}$.

Lemma 4.2 *For a compact Lie group G , every commuting n -tuple in $R_{n,G}$ lies in a maximal torus of G .*

It follows that every point in $R_{n,G}$ is fixed by a maximal torus under the conjugation action.

Our main result is:

Theorem 4.3 Let G be a connected, compact Lie group and let T be a maximal torus in G . The pair $(\phi: G \times T^n \rightarrow R_{n,G}, N(T))$ forms a cohomological principal bundle for cohomology over fields F of characteristic relatively prime to $\#W$, where W is the Weyl group. In particular, $H(R_{n,G}, F) \cong H(G/T \times T^n, F)^W$.

Proof This is a straightforward application of [Theorem 3.3](#). □

We deduce using [Theorem 3.5](#):

Corollary 4.4 $H_G(R_{n,G}) \cong H_T(T^n)^W$.

Remark 3 It may be shown (see Schwartz [\[13\]](#)), that the quotient $G \times_{N(T)} T^n$ is a nonsingular real algebraic variety. The induced map $h: G \times_{N(T)} T^n \rightarrow R_{n,G}$ is surjective, regular and induces birational equivalence. Thus $h: G \times_{N(T)} T^n \rightarrow R_{n,G}$ is a resolution of singularities for $R_{n,G}$.

Remark 4 G acts on $G \times_{N(T)} T^n = G/T \times_W T^n$ by left multiplication on the G factor, making h into a G -equivariant map. h descends to an isomorphism between $G \backslash (G/T \times_W T^n) = T^n/W$ and $R_{n,G}/G$.

Since G/T has a CW-structure with only even dimensional Schubert cells, $H(G/T, \mathbb{Z})$ is torsion free, and $H(T^n, \mathbb{Z})$ is also torsion free. If $(\text{char}(F), \#W) = 1$, it follows by [Lemma 4.6](#) that $\dim H^k(R_{n,G}, F) = \dim H^k(R_{n,G}, \mathbb{C})$. The universal coefficient theorem now applies to prove:

Corollary 4.5 $\text{Tor}(\mathbb{Z}_p, H^*(R_{n,G}, \mathbb{Z})) = 0$, for primes p satisfying $(p, \#W) = 1$.

Lemma 4.6 Let Γ be a finite group acting linearly on a free, finitely generated \mathbb{Z} -module $M \cong \mathbb{Z}^n$. Given any field F , there is an induced linear action on $M_F = M \otimes F$. If $\gcd(\text{char}(F), \#\Gamma) = 1$, then $\dim_F((M_F)^\Gamma) = \text{rank}_{\mathbb{Z}}(M^\Gamma)$

Proof Because M^Γ is a saturated sublattice of M (in the sense that $\alpha m \in M^\Gamma$ for nonzero $\alpha \in \mathbb{Z}$ implies $m \in M$), it follows that any basis of M^Γ extends to a basis of M .

Let $\psi: M \rightarrow M \otimes F$ send m to $m \otimes 1$ and let $V := \text{span}_F\{\psi(M^\Gamma)\}$. Then $\dim_F(V) = \text{rank}_{\mathbb{Z}} M^\Gamma$ and $V \subset (M_F)^\Gamma$. In fact, we will show $V = (M_F)^\Gamma$, thus proving the theorem.

Because $\#\Gamma$ is invertible in F , we can define a projection operator $P: M_F \rightarrow M_F^\Gamma$ by

$$(20) \quad P(x) = \frac{1}{\#\Gamma} \sum_{g \in \Gamma} gx.$$

The image of P is spanned by vectors $\sum_{g \in \Gamma} g\psi(m) = \psi(\sum_{g \in \Gamma} gm)$, which all lie in V which proves that $V = (M_F)^\Gamma$ completing the proof. \square

In view of [Corollary 4.5](#), little will be lost by focusing on the characteristic zero case from now on.

The first homology group $H_1(T^n, \mathbb{R})$ is canonically isomorphic to the Lie algebra, $Lie(T^n) = \mathfrak{t}^n$. It follows that $H^*(T^n) \cong \bigwedge \mathfrak{t}^{*n}$ with its usual grading. It is also well known that the equivariant cohomology of a point $H_T(pt, \mathbb{R})$ is canonically isomorphic to the symmetric algebra $S\mathfrak{t}^*$, with grading $deg(S^i \mathfrak{t}) = 2i$ (see [Appendix A](#) for more details). Combining this with [Theorem 4.3](#) and [Corollary 4.4](#) results in the simple formulas

$$(21) \quad H_G(R_{n,G}, \mathbb{R}) \cong \left(\bigwedge \mathfrak{t}^{*n} \otimes S\mathfrak{t}^* \right)^W$$

$$(22) \quad H(R_{n,G}, \mathbb{R}) \cong \left(\bigwedge \mathfrak{t}^{*n} \otimes S\mathfrak{t}^* \right)^W / \langle S\mathfrak{t}^{*W+} \rangle$$

where $\langle S\mathfrak{t}^{*W+} \rangle$ denotes the ideal generated by the image of the ring of positive degree elements in $S\mathfrak{t}^{*W} \cong H_G$.

5 Examples

Throughout this section we will always assume cohomology to be taken with coefficients in \mathbb{C} .

5.1 $G = SU(2)$

We begin with the case $G = SU(2)$. In this case both $H(R_{n,G})$ and $H_G(R_{n,G})$ admit simple, explicit descriptions. Here, \mathfrak{t} is one dimensional so that $\mathfrak{t}^{*n} \otimes \mathbb{C} \cong \mathbb{C}^n$ and $W \cong \mathbb{Z}_2$ acts by multiplication by $\{\pm 1\}$. $G/T \cong \mathbb{C}P^1$ and W acts as an orientation reversing involution. Letting y denote a generator of $H^2(G/T)$, we have

$$(23) \quad H(G/T \times T^n) \cong \bigwedge \mathbb{C}^n \otimes \mathbb{C}[y] / \langle y^2=0 \rangle$$

where y has degree 2 and \mathbb{C}^n has degree 1. Identifying $H(R_{n,SU(2)})$ with the Weyl invariant subring gives

$$(24) \quad H(R_{n,SU(2)}) \cong \left(\bigoplus_{i \text{ even}} \bigwedge^i \mathbb{C}^n \right) \oplus \left(\bigoplus_{i \text{ odd}} \bigwedge^i \mathbb{C}^n \otimes y \right).$$

We can easily compute Betti numbers to be $\dim H^d(R_{n,SU(2)}) = \binom{n}{d}$ if d is even, and $\binom{n}{d-2}$ if d is odd.

Similarly, the $SU(2)$ equivariant cohomology of $R_{n,SU(2)}$ is isomorphic to

$$(25) \quad H_G(R_{n,SU(2)}) \cong \bigoplus_{i+j \equiv 0 \pmod{2}} \left(\bigwedge^i \mathbb{C}^n \right) \otimes y^j \subseteq \bigwedge \mathbb{C}^n \otimes \mathbb{C}[y]$$

where the generators of the exterior algebra have order 1 and that of the polynomial algebra has order 2.

5.2 Poincaré polynomials

The computation of the Poincaré polynomial of $R_{n,G}$ can be organized using polynomials with character coefficients. If a group W acts on a graded vector space $\bigoplus V_i$ with character χ^i on V_i , then the Poincaré polynomial of the representation is $\sum_i \chi^i t^i$. Direct sums and tensor products of graded representations pass to sums and products of their Poincaré polynomials. The ordinary Poincaré polynomial of the invariant part of the representation can be extracted by taking the inner product of the coefficients with the character of the trivial representation.

For example, when $G = SU(2)$, the character table for $W = S_2$ is

	(1)	(12)
χ_1	1	1
χ_2	1	-1

and the formula for the Poincaré polynomial of $R_{n,SU(2)}$ is

$$\begin{aligned} P_t(H(R_{n,SU(2)})) &= \langle \chi_1, (\chi_1 + \chi_2 t)^n (\chi_1 + \chi_2 t^2) \rangle \\ &= \frac{1}{2} [(1+t)^n (1+t^2) + (1-t)^n (1-t^2)] \\ &= \frac{1}{2} \sum_i \binom{n}{i} [(1+(-1)^i)t^i + (1-(-1)^i)t^{i+2}] \end{aligned}$$

which concurs with our earlier calculation.

For $G = SU(3)$, the character table for $W = S_3$ is

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

and the Poincaré polynomial is

$$P_t(H(R_n, SU(3))) = \langle \chi_1, (\chi_1 + \chi_3 t + \chi_2 t^2)^n (\chi_1 + \chi_3 t^2 + \chi_3 t^4 + \chi_2 t^6) \rangle$$

$$= \frac{1}{6} [(1+t+t^2)^n (1+t^2+t^4+t^6) + 3(1-t^2)^n (1-t^6) + 2(1-t+t^2)^n (1-t^2-t^4+t^6)]$$

For $G = SU(4)$, the character table for $W = S_4$ is

	(1)	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	1	-1

and the Poincaré polynomial of $R_n, SU(4)$ is

$$P_t(H(R_n, SU(4))) = \langle \chi_2, (\chi_1 + \chi_4 t + \chi_5 t^2 + \chi_2 t^3)^n$$

$$(\chi_1 + \chi_4 t^2 + (\chi_3 + \chi_4) t^4 + (\chi_4 + \chi_5) t^6 + (\chi_3 + \chi_5) t^8 + \chi_5 t^{10} + \chi_2 t^{12}) \rangle.$$

Appendix A Equivariant cohomology

In this section, we briefly review equivariant cohomology from the Borel model perspective. Our main sources are Atiyah–Bott [2] and Hsiang [11].

Let G be a compact, connected Lie group and X a topological space with continuous left G action. We define the equivariant cohomology $H_G(X)$ to be the singular cohomology over \mathbb{C} of the space X_G obtained from a universal G -space EG by the mixing construction:

$$(26) \quad X_G := EG \times_G X$$

In (26) G acts on EG from the right, and $EG \times_G X$ denotes the quotient of $EG \times X$ by the relation $(e, gx) \sim (eg, x)$. X_G projects naturally onto the classifying space $BG = EG/G$ forming a fibre bundle with fibre X . We will call this the standard fibre bundle.

We adopt the notation $H_G := H(BG, \mathbb{C})$. The standard fibre bundle $\pi: X_G \rightarrow BG$ induces a natural map $\pi^*: H_G \rightarrow H_G(X)$, making $H_G(X)$ into an H_G -module.

The Serre spectral sequence of the standard fibration is frequently used to compute $H_G(X)$. If this spectral sequence collapses at the E_2 page, then $H_G(X) \cong H(X) \otimes_{\mathbb{C}} H_G$ as H_G -modules and we say that the action is formal. When the action is formal, the Leray–Hirsch theorem implies that

$$(27) \quad H_G(X)/\langle H_G^+ \rangle \cong H(X)$$

where $\langle H_G^+ \rangle$ denotes the ideal in $H_G(X)$ generated by the image of the positive degree ideal in H_G under π^* . We will use the following simple criterion for formality.

Proposition A.1 *Let G be a compact, connected Lie group, and X a space such that $H^i(X) = 0$ for i odd. Then any action of G on X is formal.*

Proof According to [2], $H_G^i = 0$ for odd i , so that the Serre spectral sequence of the standard fibration satisfies $E_2^{p,q} = 0$ unless p and q are even. Thus all subsequent differentials must be zero and the spectral sequence collapses at E_2 . \square

Let $T = (S^1)^m$ be a torus. Let X be a smooth compact manifold on which T acts smoothly, or more generally let the set of infinitesimal stabilizers of the action be finite. The fixed point set X^T includes into X by the map i , inducing a map $i^*: H_T(X) \rightarrow H_T(X^T)$. The localization theorem [2] states:

Theorem A.2 *The kernel and cokernel of i^* are torsion H_T -modules.*

Combining the localization theorem with the Serre spectral sequence of the standard fibration gives (see Guillemin–Sternberg [10]):

Proposition A.3 *With conditions as in Theorem A.2, the action of T on X is formal if and only if $\dim H(X^T) = \dim H(X)$.*

When G is a connected, compact Lie group with maximal torus T , any G -space X becomes a T -space by restricting the action. EG also forms a model for ET under the restricted action, and there is a natural map $\rho: X_T \rightarrow X_G$, which is a fibre bundle with fibre G/T . In fact, there is a natural identification

$$(28) \quad X_T \cong (X \times G/T)_G$$

where G acts by left multiplication on the homogeneous space G/T , and G acts on the product $X \times G/T$ via the diagonal action. Under this identification, the fibre bundle is just the projection map $(X \times G/T)_G \rightarrow X_G$.

The homotopy quotient X_T retains a natural action of the Weyl group $W = N(T)/T$ and we obtain an isomorphism

$$(29) \quad \rho^*: H_G(X) \cong H_T(X)^W.$$

In fact by [Proposition A.4](#), $(\rho: X_T \rightarrow X_G, W)$ is a cohomological principal bundle, so this formula follows from [Proposition 2.3](#).

Proposition A.4 *Let G be a compact, connected Lie group with maximal torus T . Then $H(G/N_G(T), F) \cong H(pt, F)$ for $\gcd(\text{char}(F), \#W_G) = 1$.*

Proof Since G/T has a CW structure with $\#W_G$ cells (the Schubert cells) all in even degrees,

$$(30) \quad H(G/N_G(T), F) = H(G/T, F)^{W_G}$$

has Euler characteristic 1, and is zero in odd degrees, and so must be trivial. \square

As a graded ring,

$$(31) \quad H_G \cong (St_{\mathbb{C}}^*)^W$$

the Weyl invariant part of the symmetric algebra over $\mathfrak{t}_{\mathbb{C}}^*$, where $\mathfrak{t}_{\mathbb{C}}$ is the complexified Lie algebra of a maximal torus in G and we set the degree of $\mathfrak{t}_{\mathbb{C}}^* = 2$. This is a polynomial algebra with even degree generators. For instance, when $G = U(N)$ the generators are the universal Chern classes $\{c_1, c_2, \dots, c_N\}$.

Appendix B W -module structure of $H(T^n)$ and $H(G/T)$

Let G be a compact, connected Lie group and T be a maximal torus in G . We assume throughout this section that $H(\cdot)$ is singular cohomology over \mathbb{C} .

Let $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$ be the complexification of the Lie algebra of T . It is immediately verified by the reader using invariant de Rham forms that

$$(32) \quad H(T^n) \cong \left(\bigwedge \mathfrak{t}_{\mathbb{C}}^* \right)^{\otimes n}$$

where the degree of $\mathfrak{t}_{\mathbb{C}}^*$ is 1. The Weyl group $W = N(T)/T$ acts on T^n by conjugation, inducing a W -module structure on $H(T^n)$ which is the same one induced by the usual action on $\mathfrak{t}_{\mathbb{C}}^*$.

The W -module structure of $H(G/T)$ is somewhat more complicated. We will content ourselves here with determining it as an ungraded representation.

Proposition B.1 As an (ungraded) left W -module, $H(G/T) \cong \mathbb{C}W$, the group ring.

Proof The set $N(T)/T = W$ will come up in this proof both as a group, and as a subset of G/T . To avoid confusion, we use W when we regard it as a group and $N(T)/T$ when we regard it a subset.

G , and hence T acts on G/T by multiplication on the left. G/T possesses a CW structure of even dimensional Schubert cells, so by [Proposition A.1](#) we know that the T action is formal and that

$$(33) \quad H_T(G/T) \cong H(G/T) \otimes H_T$$

as H_T -modules. More precisely, the natural map $\phi: H_T(G/T) \rightarrow H(G/T)$ is surjective. A linear section $s: H(G/T) \rightarrow H_T(G/T)$ which respects the grading determines an isomorphism $\tilde{s}: H(G/T) \otimes H_T \rightarrow H_T(G/T)$ by

$$(34) \quad \tilde{s}(x, \alpha) = s(x)\alpha.$$

The fixed point set of the T action on G/T is $N(T)/T$. Because $H_T(G/T)$ is a free H_T -module, the localization map $i^*: H_T(G/T) \rightarrow H_T(N(T)/T)$ is an inclusion whose cokernel is a torsion H_T -module. Since T acts trivially on $N(T)/T$,

$$(35) \quad H_T(N(T)/T) \cong H(N(T)/T) \otimes H_T$$

canonically as rings. Now W acts on the right of G/T , preserving $N(T)/T$, and commuting with the T action, so it induces an action on $H_T(G/T)$ and $H_T(N(T)/T)$ which is equivariant for i^* . By choosing the section s to be W -equivariant also, we find that in terms of the equations (33) and (35), W acts trivially on the H_T factor. Let Q_T denote the quotient field of H_T . Tensoring equations (33) and (35) by Q_T gives an isomorphism of W representations:

$$(36) \quad H(G/T) \otimes_{\mathbb{C}} Q_T \cong H(N(T)/T) \otimes_{\mathbb{C}} Q_T$$

But this proves the theorem, because Q_T is a field extension of \mathbb{C} . □

For an alternative proof, see Chriss–Ginzburg [7] chapter 6.

Theorem B.2 As graded rings and W modules, $H(G/T) \cong H_T / \langle H_T^{W+} \rangle$

Proof $H(G/T)$ is trivial in odd degrees, so by equation (27) and [Proposition A.1](#) we know

$$(37) \quad H(G/T) \cong H_G(G/T) / \langle H_G^+ \rangle$$

On the other hand, by equation (28) we know $H_G(G/T) \cong H_T$, and by equation (29) that $H_G^+ \cong H_T^{W+}$, which verifies the formula. □

We point out a consequence of this theorem.

Corollary B.3 $St_{\mathbb{C}}^* \cong \mathbb{C}W \otimes (St_{\mathbb{C}}^*)^W$ as an ungraded W representation, under the usual left action.

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