On a conjecture of Gottlieb

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We give a counterexample to a conjecture of D H Gottlieb and prove a strengthened version of it.

The conjecture says that a map from a finite CW–complex X to an aspherical CW–complex Y with non-zero Euler characteristic can have non-trivial degree (suitably defined) only if the centralizer of the image of the fundamental group of X is trivial.

As a corollary we show that in the above situation all components of non-zero degree maps in the space of maps from X to Y are contractible.

We use L^2–Betti numbers and homological algebra over von Neumann algebras to prove the modified conjecture.

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1 A version of Gottlieb's conjecture

Let X and Y be finite CW–complexes. In [3; 4], Gottlieb defines a notion of degree of a continuous map $f\colon X \to Y$ as follows. Let $f_*\colon H_*(X,\mathbb{Z}) \to H_*(Y,\mathbb{Z})$ be the induced map in reduced integral homology. The degree $\deg(f)$ of f is the least integer $n \in \mathbb{N}$, such that there exists a group homomorphism $\tau\colon H_*(Y,\mathbb{Z}) \to H_*(X,\mathbb{Z})$ which satisfies $f_* \circ \tau = n \cdot \mathrm{id}$. He makes the following conjecture (compare Gottlieb [4]).

Conjecture 1 (Gottlieb) *Let (Y,y) be a finite aspherical CW–complex which is not acyclic. Let $f\colon (X,x) \to (Y,y)$ be a continuous map with $\deg(f) \neq 0$. If $\chi(Y) \neq 0$, then the centralizer of $f_*(\pi_1(X,x))$ in $\pi_1(Y,y)$ is trivial.*

In this note we give a counterexample to this form of the conjecture (see Example 12) and prove a version with a stronger hypothesis, see Theorem 4. Let us rephrase one important consequence of *non-vanishing degree* in the case of mappings between closed oriented manifolds, so that it is applicable in a more general setting.

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Definition 2 Let $f\colon (X,x) \to (Y,y)$ be a continuous map. We say that f is a *superposition*, if for any $\mathbb{Q}\pi_1(Y,y)$–module L, the induced map

$$f_*\colon H_*^{\pi_1(X,x)}(\widetilde{X}; f^*L) \to H_*^{\pi_1(Y,y)}(\widetilde{Y}; L)$$

is surjective.

We will see in Theorem 8, that a map of non-vanishing degree between closed oriented manifolds, or more generally between oriented Poincaré duality complexes, is a superposition. Moreover, an equivariant version of the Becker–Gottlieb transfer gives plenty of examples of maps between CW–complexes which are not Poincaré complexes.

The problem with Gottlieb's definition of degree seems to be that it takes only untwisted coefficients into account. Lead by Gottlieb, one can therefore define a stronger version of degree as follows.

Definition 3 Let $f\colon X \to Y$ be a map between finite CW–complexes. Its *twisted rational degree* $\deg_{tw,\mathbb{Q}}(f)$ is 1 if f is a superposition, and is 0 otherwise.

Its *twisted degree* $\deg_{tw}(f)$ is the least positive integer $n \in \mathbb{N}$ such that for each $\mathbb{Z}\pi_1(Y)$–module L there is a group homomorphism $\tau_L\colon H_*(Y,L) \to H_*(X, f^*L)$ such that $f_* \circ \tau_L = n \cdot \mathrm{id}$, or 0 if no such integer exists.

Clearly, a map of non-zero twisted degree is a superposition, so that the next result shows that Gottlieb's conjecture is correct if one requires that the *twisted* degree is non-zero.

Our main result is the following theorem.

Theorem 4 *Let (Y,y) be a finite aspherical CW–complex. Let $f\colon (X,x) \to (Y,y)$ be a continuous superposition. If $\chi(Y) \neq 0$, then the centralizer of $f_*(\pi_1(X,x))$ in $\pi_1(Y,y)$ is trivial.*

Assuming Theorem 4, we can show some corollaries which generalize results from Gottlieb [4]. Let f be a continuous map from X to Y. We denote by $\mathrm{map}(X,Y,f)$ the space of continuous maps from X to Y which are homotopic to f.

Corollary 5 *Let Y be a finite aspherical CW–complex. Let $f\colon X \to Y$ be a continuous superposition. If $\chi(Y) \neq 0$ and Y is aspherical, then the mapping space* $\mathrm{map}(X,Y,f)$ *is contractible.*

Proof If Y is aspherical, then $\mathrm{map}(X, Y, f)$ is also aspherical because of the following reasoning: we have to extend a given map from S^n to $\mathrm{map}(X, Y, f)$ to D^{n+1}. By the exponential law, this means to extend a map from $X \times S^n$ to $X \times D^{n+1}$. The latter space is obtained from the former by attaching cells of dimension $n+1$ or higher. If $n \geq 2$, because $\pi_k(Y) = 0$ for $k \geq 2$, we can extend the map cell by cell as required.

Gottlieb showed in [2] that $\pi_1(\mathrm{map}(X, Y, f), f)$ is naturally isomorphic to the centralizer of $f_*(\pi_1(X, x))$ in $\pi_1(Y, y)$. Hence the claim follows from Theorem 4. $\square$

Corollary 6 *Let (Y, y) be a finite aspherical CW–complex with $\chi(Y) \neq 0$. Every subgroup of finite index in $\pi_1(Y, y)$ has trivial centralizer.*

Proof Let G be a finite index subgroup of $\pi_1(Y, y)$. The induced map $f\colon BG \to Y$ is a superposition. Hence, Theorem 4 implies the claim. $\square$

Proof of Theorem 4 Because $Y = B\pi_1(Y)$ is finite dimensional, $\pi_1(Y)$ is torsion free. Therefore every non-trivial subgroup is infinite. Let us assume that the centralizer of $f_*(\pi_1(X, x))$ in $\pi_1(Y, y)$ is infinite.

If π is a discrete group, let $L\pi$ be its group von Neumann algebra. If $\chi(Y)$ is not zero, then the equivariant L^2–homology

$$H_*^{\pi_1(Y,y)}(\widetilde{Y}; L\pi_1(Y, y))$$

cannot be zero-dimensional in all degrees. Indeed,

$$0 \neq \chi(Y) = \sum_{k=0}^{\infty} (-1)^k \beta_k^{(2)}(Y),$$

by Atiyah's L^2–index theorem (see Lück [7, Theorem 6.80]). Here $\beta_k^{(2)}(Y)$ denotes the k-th L^2–Betti number

$$\beta_k^{(2)}(Y) = \dim_{L\pi_1(Y,y)} H_k^{\pi_1(Y,y)}(\widetilde{Y}; L\pi_1(Y, y)).$$

By assumption, the map $f\colon (X, x) \to (Y, y)$ is a superposition, so it induces a surjection

$$H_k^{\pi_1(X,x)}(\widetilde{X}; f^* L\pi_1(Y, y)) \to H_k^{\pi_1(Y,y)}(\widetilde{Y}; L\pi_1(Y, y)),$$

for every $k \in \mathbb{N}$. However, since Y is aspherical, for every subgroup G of $\pi_1(Y, y)$ which contains $f_*(\pi_1(X, x))$, this map can be factorized through

$$H_k^G(EG; \mathrm{res}_G^{\pi_1(Y,y)} L\pi_1(Y, y)) = H_k^G(EG; LG) \otimes_{LG} L\pi_1(Y, y).$$

Here, we used that $L\pi_1(Y,y)$ is flat as LG–module, compare [7, Theorem 6.29]. By the same theorem, the $L\pi_1(Y,y)$–dimension of the right hand side is equal to $\beta_k^{(2)}(G)$. To derive a contradiction, it suffices to construct a subgroup as above which has only vanishing L^2–Betti numbers.

If the centralizer of $f_*(\pi_1(X,x))$ intersects non-trivially with $\pi_1(X,x)$, then the intersection is infinite, since $\pi_1(Y,y)$ is torsion-free. In this case $f_*(\pi_1(X,x))$ has an infinite center and all its L^2–Betti numbers are zero by [7, Theorem 7.2]. If the intersection is trivial, we may pick a non-torsion element which centralizes $f_*(\pi_1(X,x))$. Together with $f_*(\pi_1(X,x))$, it generates a copy of $f_*(\pi_1(X,x))\times\mathbb{Z}$, which has trivial L^2–Betti numbers by the Künneth Theorem for L^2–Betti numbers, see [7, Theorem 6.54(4)]. Hence, we arrive at a contradiction.

Since $\pi_1(Y,y)$ is torsion-free, we conclude that the centralizer of $f_*(\pi_1(X,x))$ is trivial. This finishes the proof. □

Remark 7 Note that, because the classifying space of $\mathrm{im}(f_*)\subset\pi_1(Y)$ is in general not a finite CW–complex, we have to use the generalization of L^2–Betti numbers to arbitrary CW–complexes of Lück as developed in [5; 6].

The next theorem gives examples of maps which are superpositions.

Theorem 8 *The following classes of maps are superpositions:*

(1) *retractions,*

(2) *continuous maps between oriented closed manifolds, or more generally Poincaré duality spaces, which have non-vanishing degree and*

(3) *continuous maps $f\colon (X,x)\to(Y,y)$, whose homotopy fiber has the homotopy type of a finite CW–complex and non-vanishing Euler characteristic.*

Proof The statement about retractions follows from functoriality; for the second statement one uses the transfer given by Poincaré duality, and for the third the Becker–Gottlieb transfer (with twisted coefficients). □

Remark 9 The proofs of the results presented so far show that the assumptions can be weakened as follows:

- $\chi(Y)\neq 0$ can be replaced by the assumption that Y has at least one non-vanishing L^2–Betti number and
- the map f being a superposition can be replaced by the assumption that f induces a surjective homomorphism in L^2–cohomology.

That surjectivity in L^2–cohomology is true for inclusions of finite index subgroups and therefore Corollary 6 holds under the weaker assumptions follows eg from Schick [8].

Remark 10 One should observe that Gottlieb's theorem, stating that the center of an aspherical finite CW–complex with non-trivial Euler characteristic is trivial, has been generalized considerably. Its strongest version now reads that such a group does not contain an infinite amenable normal subgroup.

Our main application states that the centralizer of an image group is trivial; again we expect a generalization similar to the one about normal amenable subgroups. However, the correct notion of "amenable centralizer" still has to be developed.

2 A counterexample to a strong form of the conjecture

We finish this note by giving the desired counterexample to Gottlieb's Conjecture. The tools in the construction are the techniques from the work of Baumslag, Dyer and Heller, see [1].

Theorem 11 (Baumslag–Dyer–Heller) *There exists a finite aspherical and acyclic CW–complex $(D,*)$ whose fundamental group $\pi_1(D,*)$ contains a copy of $\mathbb{Z}$.*

Example 12 *There exists a finite aspherical CW–complex Y with $\chi(Y)=2$ and a continuous map $f\colon \mathbb{T}^2\to Y$ which is of degree one (taking Gottlieb's definition) and injective on the fundamental group. In particular, the centralizer of $f_*(\pi_1(\mathbb{T}^2,*))=\mathbb{Z}^2$ is infinite.*

Construction of the Example Let D be as in Theorem 11. Starting with $\mathbb{T}^2$, we glue in two copies of D, along a circle in D representing the generator of $\mathbb{Z}\subset\pi_1(D,*)$, and along each of the generators of the fundamental group of $\mathbb{T}^2$, to obtain the new space Y. Let $f\colon \mathbb{T}^2\to Y$ be the natural map, induced by the glueing process.

This is exactly the type of construction of Baumslag-Dyer-Heller; since we glue along inclusions on the level of fundamental groups, the resulting space Y is aspheric and the map $f_*\colon \mathbb{Z}^2\to\pi_1(Y,y)$ is injective. On the other hand, a look at the Mayer–Vietoris sequence shows that the map from $\mathbb{T}^2$ to Y is an isomorphism in second integral homology, whereas $H_1(Y,\mathbb{Z})=0$. $\square$

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