Concordance of $\mathbb{Z}_p \times \mathbb{Z}_p$ actions on S^4

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We prove that locally linear, orientation-preserving actions of $G = \mathbb{Z}_p \times \mathbb{Z}_p$ on S^4 are concordant if and only if a $\mathbb{Z}_2$–valued surgery obstruction vanishes, and discuss constructions and examples.

57S25; 57S17

1 Introduction

The main result of [7] was that if M is a simply-connected four-manifold admitting an effective, homologically trivial, locally linear action by $G = \mathbb{Z}_p \times \mathbb{Z}_p$, where p is prime, then[1] M is equivariantly homeomorphic to a connected sum of standard actions on copies of $\pm\mathbb{C}P^2$ and $S^2 \times S^2$ with a possibly non-standard action on S^4. In this note we further examine these non-standard actions on the sphere. We describe some constructions arising from counterexamples to the generalized Smith Conjecture and then consider the classification of actions up to concordance. An analysis of singular sets and quotient spaces, combined with an application of the results of Cappell and Shaneson [2], allows us to prove:

Theorem 1.1 *A locally linear orientation-preserving action of $\mathbb{Z}_p \times \mathbb{Z}_p$ on S^4 is topologically concordant to a linear action if and only if a certain Kervaire–Arf invariant $c \in \mathbb{Z}_2$ vanishes. Thus after normalizing rotation numbers, there are at most two concordance classes of $\mathbb{Z}_p \times \mathbb{Z}_p$ actions on S^4.*

The invariant c is well-behaved with respect to two connected-sum constructions. We finish with a discussion of the existence of actions on which c is nontrivial, which is still an open question.

[1] with one exception: pseudofree actions on $\mathbb{Z}_3 \times \mathbb{Z}_3$ on homotopy $\mathbb{C}P^2$ s.

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2 Warmup: $\mathbb{Z}_p$ actions on S^4

Smith Theory (see Smith [15] and Bredon [1]) shows that when a cyclic group of prime order p acts on a sphere, the fixed-point set is again a $\mathbb{Z}_p$–homology sphere.

The classical Smith Conjecture (whose solution, involving the work of many mathematicians, was published in 1984, see Bass and Morgan [10]) goes further, stating the fixed-point set of a tame cyclic group action on S^3 is either empty, or an *unknotted* S^1. But the natural generalization of the Smith conjecture to higher dimensions is false: Giffen [4], Gordon [5], and Sumners [16] all constructed counterexamples in spheres of dimension 4 and higher. Along with [7], these counterexamples motivate our study.

Suppose $\langle g\rangle \cong \mathbb{Z}_p$ acts on $X = S^4$, locally linearly and preserving orientation. $\mathrm{Fix}(g)$ is then a $\mathbb{Z}_p$–homology sphere of dimension 0 or 2; since $\mathrm{Fix}(g)$ is also a submanifold, it must be either S^0 or S^2. Both are possible, but let us assume $\mathrm{Fix}(g) \cong S^2$. By local linearity, the quotient map $X \to X^* = X/\langle g\rangle$ is a branched covering, and the quotient space is a manifold with the image $\mathrm{Fix}(g)^*$ of the fixed set embedded as a locally flat submanifold. According to Freedman and Quinn [3, 9.3A] (see also Quinn [12]), $\mathrm{Fix}(g)^*$ has a normal bundle N^*. The lift of N^* back to X is an equivariant tubular neighborhood $N(\mathrm{Fix}(g))$ of $\mathrm{Fix}(g)$. Then $X \backslash N(\mathrm{Fix}(g))$ has the same boundary, and by Alexander duality, the same integral homology, as $S^1 \times B^3$.

Lemma 2.1 *$X^* = X/\langle g\rangle$ is homeomorphic to S^4.*

Proof First note that near a fixed point, a complex coordinate chart may be chosen so that the quotient map takes the form $(z, w) \mapsto (z, w^p)$. Thus the quotient space is topologically a manifold.

Let $X^* = X/\langle g\rangle$. Then $\pi_1(X - N(\mathrm{Fix}(g)))$ is a normal subgroup of index p in $\pi_1(X^* \backslash (N(\mathrm{Fix}(g))/\langle g\rangle))$. Filling in $N^* = N(\mathrm{Fix}(g))/\langle g\rangle$ kills a meridian μ of $\mathrm{Fix}(g)$. This in particular kills μ^p. But μ^p normally generates $\pi_1(X - N(\mathrm{Fix}(g)))$, and each μ^i represents one of its p cosets. It follows that X^* is simply connected.

Finally, the transfer map $\mu_*\colon H_2(X^*; \mathbb{Q}) \to H_2(X; \mathbb{Q})^{\mathbb{Z}_p} = 0$ is an isomorphism, so X^* is a rational, and hence integral, homology sphere. It follows from the four-dimensional topological Poincaré conjecture that $X^* \cong S^4$. □

Definition We will say actions ψ_0 and ψ_1 of a group G on a manifold M are *concordant* if there is a locally linear action Ψ of G on $M \times I$ such that $\Psi|_{M\times\{i\}} = \psi_i$, and such that for each $g \in G$, $\mathrm{Fix}(g, M \times I) \cong \mathrm{Fix}(g, M) \times I$.

Remarks (1) This stratified notion of concordance is the most natural one in our context, but for other purposes one might instead consider the *a priori* weaker notion of unstratified concordance. For example, Bredon [1, 1.7] describes involutions of S^5 whose fixed-point sets are lens spaces, and puncturing such an example twice along its fixed set yields a concordance from a standard $\mathbb{Z}_2$ action on S^4 to itself whose fixed stratum is not a cylinder.

(2) Rotation numbers are carried along tubular neighborhoods of the singular strata, so it follows immediately that concordant actions must have the same rotation numbers.

We begin with an observation which is essentially an equivariant version of a theorem of Kervaire, and sketch a proof as motivation for what follows. A more careful version of the argument is given following its generalization to $\mathbb{Z}_p \times \mathbb{Z}_p$ below. Kervaire worked in the smooth category, but his result holds in the case of topologically locally flat 2–spheres in S^4.

Theorem 2.2 *Every locally linear $\mathbb{Z}_p$ action on S^4 which fixes a 2–sphere is topologically concordant to a linear action.*

Sketch of proof Consider the quotient space pair $(X^*, Y^*) = (S^4, \mathrm{Fix}(g))/\langle g \rangle$. According to Kervaire [6], all even-dimensional knots are slice, so the pair (X^*, Y^*) bounds a ball pair (B^5, B^3).

Cappell and Shaneson's homology surgery groups [2] measure the obstruction to surgering $(B^5 - N(B^3))$ rel boundary to make it $\mathbb{Z}[\mathbb{Z}_p]$–homology equivalent to a standard pair $(B^5 - N(B^3_{\mathrm{std}}))$. The obstruction groups vanish in this case, so the $\mathbb{Z}_p$–cover

$$\widetilde{B^5 - N(B^3)}$$

may be assumed to be a homology $S^1 \times B^4$ with fundamental group $\mathbb{Z}$. Filling in a thickened B^3 makes the covering simply connected, and hence a topological ball. Removing a small (B^5, B^3) pair with a standard action along the fixed-point set yields the desired concordance. □

3 Examples of rank two group actions

Henceforth, let $G = \mathbb{Z}_p \times \mathbb{Z}_p$.

The *standard linear* G action is defined to be the restriction to the unit sphere in $\mathbb{R}^5$ of

$$g \mapsto \begin{pmatrix} \cos(2\pi/p) & -\sin(2\pi/p) & 0 & 0 & 0 \\ \sin(2\pi/p) & \cos(2\pi/p) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad h \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos(2\pi/p) & -\sin(2\pi/p) & 0 \\ 0 & 0 & \sin(2\pi/p) & \cos(2\pi/p) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

An easy exercise in representation theory shows that any orthogonal action (preserving orientation if $p=2$) is equivalent to the standard action after normalizing by precomposition with an automorphism of G.

In general, given a locally linear, homologically trivial G action on $X = S^4$, Smith Theory arguments show that $\mathrm{Fix}(G)$ will consist of two points, x_1 and x_2. As the G–action is linear on $T_{x_1}X$, we can choose generators g and h of G which fix transverse 2–spheres, and so that after the two-spheres are oriented, g and h have rotation numbers $(1,0)$ and $(0,1)$ at x_1. We refer to the spheres fixed by g and h as Y_g and Y_h, respectively. Of course, $Y_g \cap Y_h = \{x_1, x_2\}$, while $Y_g \cup Y_h$, the entire singular set of the action, will be denoted Σ.

Hence we assume for convenience from now on that generators g, h for G are chosen to have rotation numbers $(1,0)$ and $(0,1)$, respectively, in a neighborhood of a fixed point x_1.

3.1 A construction using counterexamples to the generalized Smith conjecture

Giffen [4] produced the first known examples of $\mathbb{Z}_p$ actions on S^4 with knotted two-spheres as fixed point set. He showed that, when $k \geq 2$ and $p = \pm 1 \pmod{k}$, then the p–fold branched cover of any k–twist spun classical knot is a homotopy four-sphere, and diffeomorphic to S^4 when p is odd.

Gordon [5] re-cast Giffen's construction without the branched covering using a sort of equivariant Dehn surgery on the twist-spun knot. Gordon also gave another construction: If (S^4, K) is any knot in S^4, and $g_p \colon S^4 \to S^4$ is a standard rotation action on S^4 fixing S^2, then the equivariant connected sum of p copies of K admits a $\mathbb{Z}_p$ action (with unknotted fixed set). Removing a neighborhood of pK and re-gluing it differently results in an action with knotted fixed set.

Finally, Sumners [16] gave what is perhaps the simplest construction: Start with a standard rotation action on B^5 fixing B^3, glue on orbits of 1–handles, and then glue in orbits of 2–handles which cancel the 1–handles geometrically in ∂B^5, but not even homotopically in $\partial(B^5 \backslash B^3)$. The resulting action on ∂B^5 has the nice additional property that a concordance to a standard linear action is obvious (and obviously smooth).

The standard (or any other) action of $G = \mathbb{Z}_p \times \mathbb{Z}_p$ on S^4 can be modified via two distinct equivariant connected sum operations. First, given actions of G on two copies X, X' of S^4, choose fixed points x_1 and x'_1 with equivariant four-ball neighborhoods B and B'. In the equivariant connected sum $(X \backslash \operatorname{Int}(B)) \#_{\partial B} (-X' \backslash \operatorname{Int}(B'))$, the fundamental group of the complement of the singular set is the product $(\pi_1(X \backslash \Sigma)) *_{\mathbb{Z}\times\mathbb{Z}} (\pi_1(X' \backslash \Sigma'))$, where the amalgamating subgroup is generated by meridians of S_g and S_h.

Second, a G–action can be combined with p copies of a $\mathbb{Z}_p$ action:

Given an action of $\mathbb{Z}_p \times \mathbb{Z}_p$ on a four-sphere X, choose $y \in X^h \backslash X^G$ with a $\langle h \rangle$–equivariant four-ball neighborhood B. Then $\langle g \rangle\, B$ is a closed G–subspace of X, and $\partial(X \backslash \operatorname{Int}(\langle g \rangle\, B))$ consists of p copies of a standard linear action by $\langle h \rangle$ on S^3, permuted cyclically by g.

Now suppose X' is a four-sphere equipped with a locally linear action of $\mathbb{Z}_p = \langle h \rangle$ with rotation numbers $(0, 1)$ along its (two-sphere) fixed set. Choose a point $y' \in (X')^h$, and a corresponding equivariant ball neighborhood B' Then the equivariant connected sum $(X \backslash \operatorname{Int}(\langle g \rangle\, B)) \#_{\langle g \rangle \times \partial B} (\langle g \rangle \times (-X' \backslash \operatorname{Int}(B'))$ has fundamental group

$$\pi_1(X - \Sigma) *_{\mathbb{Z}} \pi_1(X' - S_h) *_{\mathbb{Z}} \pi_1(g(X' \backslash S_h)) *_{\mathbb{Z}} \ldots *_{\mathbb{Z}} \pi_1(g^{p-1}(X' \backslash S_h)).$$

In both constructions, the amalgamating subgroups map injectively into all factors, and it follows that the factor groups inject into $\pi_1(S^4 - \Sigma)$ (see Rolfsen [14, 4.7]).

Hence if any of the individual $\mathbb{Z}_p$ actions have nonabelian knot groups (as is the case in the counterexamples to the four-dimensional Smith Conjecture described above), the resulting action will be non-linear.

4 General analysis

We prove Theorem 1.1 by pinpointing the obstructions to generalizing the argument of Theorem 2.2.

Let $X = S^4$. We seek to analyze a given action via its quotient space, or more precisely, the triple $(X^*, Y_g^*, Y_h^*) = (X/G, Y_g/G, Y_h/G)$. As in the case of Theorem 2.2, X^* is a simply-connected homology four-sphere, hence homeomorphic to S^4. Each of Y_g^* and Y_h^* is itself a (topological) two-sphere, over which Y_g and Y_h are p–fold cyclic branched covers. The original action can then be recovered from the quotient as the $\mathbb{Z}_p \times \mathbb{Z}_p$ branched cover[2] of X^* over the knotted spheres Y_g^* and Y_h^*. In this way, classification of G–actions is expressed in terms of knot theory.

As a model, let us denote the quotient of S^4 by the standard action as $X^{\mathrm{s},*}$, and choose closed tubular neighborhoods $N(Y_g^{\mathrm{s},*})$ and $N(Y_h^{\mathrm{s},*})$ of $Y_g^{\mathrm{s},*}$ and $Y_h^{\mathrm{s},*}$ in $X^{\mathrm{s},*}$. Each is a trivial D^2–bundle over S^2, and their union is a neighborhood $N(\Sigma^{\mathrm{s},*})$ of the singular set $\Sigma^{\mathrm{s},*} \subset X^*$, formed by plumbing the bundles together at x_1^* and x_2^*.

The closure of $X^{\mathrm{s},*} \backslash N(\Sigma^{\mathrm{s},*})$ is homeomorphic to $D^2 \times T^2$, with boundary $S^1 \times T^2 \cong T^3$. The D^2 factor is bounded by a curve γ^{std} running from x_1^* to x_2^* along a "longitude" line of (a thickened) $Y_g^{\mathrm{s},*}$, then returning to x_1^* along $Y_h^{\mathrm{s},*}$; finally, $D^2 \times T^2$ has "corners" along $\{x_1^*, x_2^*\} \times T^2$.

Our goal is to construct a cobordism rel boundary W from $\overline{X^* \backslash N(\Sigma^*)}$ to $\overline{X^{\mathrm{s},*} \backslash N(\Sigma^{\mathrm{s},*})}$ and then modify it so that $W \cup (N(\Sigma^{\mathrm{s},*}) \times I) \cong S^4 \times I$, and so that its iterated branched cover is also a cylinder.

It follows from Freedman and Quinn [3, 9.3] that each of Y_g^* and Y_h^* has a topological normal bundle, so as before we can form $N(\Sigma^*)$. Alexander duality shows that $X \backslash N(\Sigma^*)$ has the homology of $D^2 \times T^2$. Let μ_g and μ_h be small meridional loops in $\partial(X^* \backslash N(\Sigma^*))$ around Y_g^* and Y_h^*, respectively, and let γ be a path running from x_1^* to x_2^* along Y_g^*, then back to x_1 along Y_h^*. Fix a homeomorphism f from $\partial(\overline{X^* \backslash N(\Sigma^*)})$ to $\partial(\overline{X^{\mathrm{s},*} \backslash N(\Sigma^{\mathrm{s},*})})$ sending μ_g, μ_h, and γ to their standard counterparts.

The space $D^2 \times T^2$ is a $K(\mathbb{Z} \times \mathbb{Z}, 1)$, so the only potential obstruction to extending f over all of $X^* \backslash N(\Sigma^*)$ lies in $H^2(\overline{X^* \backslash N(\Sigma^*)}, \partial(\overline{X^* \backslash N(\Sigma^*)}); \mathbb{Z} \times \mathbb{Z})$. This group is generated by the classes $\delta[\gamma_i^*]$, where γ_1^* is a cocycle evaluating to $(1,0)$ on γ, and γ_2^* is a cocycle evaluating to $(0,1)$. As $f \circ \gamma = \gamma^{\mathrm{std}}$ bounds a disk in $D^2 \times T^2$, the obstruction to extending f vanishes. Since f is a homeomorphism on $\partial(\overline{X^* \backslash N(\Sigma^*)})$, it is a degree one map relative to $\partial(\overline{X^* \backslash N(\Sigma^*)})$. Finally, since $X^* \cong S^4$, the stablized tangent bundle τ of $X^* \backslash N(\Sigma^*)$ admits an obvious trivialization. Since $f^*(\varepsilon)$ and τ are individually trivial, trivializations can be combined to yield a stable trivialization F of $\tau \oplus f^*\varepsilon$. The data $((X^* \backslash N(\Sigma^*), \partial), f, \varepsilon, F)$ together define a degree one normal map.

[2]This can be viewed as an iterated branched cover, first by $\langle g \rangle$, and then by $\langle h \rangle$, or vice versa. The iterated branched cover is well-defined because $\mathbb{Z}_p \times \mathbb{Z}_p$ is abelian.

According to Freedman [3], surgery "works" for four-manifolds with fundamental group $\mathbb{Z}\times\mathbb{Z}$, so there is an exact sequence[3]

$$L_5(\mathbb{Z}\times\mathbb{Z}) \to S_{\mathrm{TOP}}(D^2\times T^2,\partial) \to NM_{\mathrm{TOP}}(D^2\times T^2,\partial) \xrightarrow{\theta} L_4(\mathbb{Z}\times\mathbb{Z}).$$

$S_{\mathrm{TOP}}(D^2\times T^2,\partial)$ is a structure set measuring the difference between homotopy $D^2\times T^2$ manifolds with boundary T^3, and topological ones. The extension of the classification of "fake tori" [3, 11.5] to dimension four shows that $|S_{\mathrm{TOP}}(D^2\times T^2,\partial)| = 1$.

So f, together with the pullback $f^*\varepsilon$ of the trivial bundle, defines a surgery obstruction which vanishes if and only if f is normally cobordant rel boundary to a homeomorphism.

According to Wall [18, 13A.8, 13B.8], $L_4(\mathbb{Z}\times\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}_2$, with generators for the factors given by the signature and a codimension two Kervaire–Arf invariant, respectively: $\theta(f) = (\sigma(f), c(f))$. Here, σ is the difference in signatures between $(\overline{X^*\backslash N(\Sigma^*)},\partial)$ and $(T^2\times D^2,\partial)$. Alexander duality shows that the two spaces have isomorphic cohomology, so the signature component of the obstruction vanishes.

The second component, c is more interesting. Composing the projection $p\colon D^2\times T^2 \to T^2 \cong K(\mathbb{Z}\times\mathbb{Z},1)$ with f realizes the obstruction as the element of $L_2(1)$ defined by the stable normal bundle of the transverse preimage of a regular value of pf. The submanifold $(pf)^{-1}(*)$ of $(\overline{X^*\backslash N(\Sigma^*)})$ is exactly the preimage via f of a meridional disk, the first factor of $D^2\times T^2$. And then $c(f)$ is the Kervaire invariant of the associated normal map.

This potential obstruction to null-concordance of our group action has the following equivalent interpretations:

(1) The surgery obstruction (rel boundary), of the normal map $\overline{X^*\backslash N(\Sigma^*)} \to D^2\times T^2$.

(2) The codimension two surgery obstruction of the restricted normal map $f^{-1}(D^2\times x_0) \to D^2\times x_0$, where x_0 is a regular value of a map to T^2

(3) The element of the spin bordism group $\Omega_2^{\mathrm{spin}} \cong \mathbb{Z}_2$ defined by identifying $-(D^2\times x_0)$ to $f^{-1}(D^2\times x_0)$ and their normal bundles along their common boundary, using F.

[3] Since $Wh(\mathbb{Z}\times\mathbb{Z}) = 0$, the distinction between homotopy and simple homotopy is not relevant to the structure set or the the surgery groups.

We return to this invariant later. Let us suppose now that it vanishes. Then the map f is normally cobordant to a homeomorphism. Let (W_0^*, f, F) be a normal cobordism.

The cobordism $f\colon W_0^* \to D^2 \times T^2$ easily extends to a normal map $W_0^* \to D^2 \times T^2 \times I$, which we continue to denote by f. Let $\mathcal{F}\colon \mathbb{Z}[\mathbb{Z}\times\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}_p \times \mathbb{Z}_p]$ be the homomorphism of group rings corresponding to the covering $X \setminus N(\Sigma) \to X^* \setminus N(\Sigma^*)$. The homology surgery groups of Cappell and Shaneson [2] measure the obstruction to surgering f to be a $\mathbb{Z}[\mathbb{Z}_p \times \mathbb{Z}_p]$–homology equivalence. The relevant obstruction group $\Gamma_5^h(\mathcal{F})$ injects into $L_5(\mathbb{Z}_p \times \mathbb{Z}_p)$ (see [2, pages 285, 295–296]). The latter group vanishes for all primes p (Wall [17] is a convenient reference), so W_0^* is normally bordant rel boundary to a new bordism W_1^* which is $\mathbb{Z}[\mathbb{Z}_p \times \mathbb{Z}_p]$–homology equivalent to a cylinder. According to [2, 2.1], f may be assumed to be 2–connected. The $\mathbb{Z}_p \times \mathbb{Z}_p$–cover W_1 of W_1^* therefore has the following properties:

(1) $\tilde{f}\colon W_1 \to (X - N(\Sigma^s)) \times I$ induces an isomorphism on integral homology.

(2) $\partial W_1 = (X - N(\Sigma)) \cup (T^3 \times I) \cup (X^s - N(\Sigma^s))$.

(3) $\pi_1(W_1) \cong \mathbb{Z} \times \mathbb{Z}$.

(4) W_1 is equipped with a $\mathbb{Z}_p \times \mathbb{Z}_p$ action.

Equivariantly gluing in a copy of $N(\Sigma) \times I$ results in a simply-connected five-manifold homotopy equivalent, and hence homeomorphic, to $S^4 \times I$, yielding the desired concordance.

4.1 Proof of Theorem 2.2

The above argument simplifies in the case $G = \langle g \rangle = \mathbb{Z}_p$ of Theorem 2.2. Recall that we assume $\mathbb{Z}_p$ has two-dimensional fixed point set, and that a generator g of $\mathbb{Z}_p$ has been chosen with rotation numbers $(1,0)$. Let $X = S^4$, $X^* = X/\langle g \rangle$, and let $Y_g^* \subset X^*$ be the image of $\mathrm{Fix}(g)$. Then X can be viewed as a cyclic branched cover of X^* with branch set Y_g^*. If $X^{s,*}$ denotes the quotient of S^4 by the standard linear action, and $N(Y_g^*)$, a closed tubular neighborhood of Y_g^*, then the closure of $X^{s^*} \setminus N(Y_g^{s,*})$ is homeomorphic to $D^3 \times S^1$.

As before, we construct a degree one normal map $((X^* \setminus N(Y_g^*), \partial), f, \varepsilon, F)$ from $X^* \setminus N(Y_g^*)$ to $X^{s,*} \setminus N(Y_g^{s,*})$. It defines a surgery obstruction in $L_4(\mathbb{Z})$. But $L_4(\mathbb{Z})$ is an infinite cyclic group generated by the signature difference σ alone; there is no codimension two obstruction. And $\sigma(f) = 0$ because f^* is a cohomology isomorphism. Hence a normal cobordism (W_0^*, f, F) exists, and $f\colon W_0^* \to D^3 \times S^1$ extends to a normal map $W_0^* \to D^3 \times S^1 \times I$.

If $\mathcal{F}\colon \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}_p]$ is the homomorphism of group rings corresponding to the covering $X \backslash N(\mathrm{Fix}(g)) \to X^* \backslash N(Y_g^*)$, then as before, $\Gamma_5^h(\mathcal{F}) = 0$, and the construction proceeds with no obstructions.

5 Behavior of the obstruction

Are there examples of $\mathbb{Z}_p \times \mathbb{Z}_p$–actions which are not concordant to linear actions? We do not yet have a definitive answer to this question, but as we shall see, the obstruction c vanishes for known constructions of $\mathbb{Z}_p \times \mathbb{Z}_p$–actions.

Example Montesinos [8; 9] considered "twins": pairs of two-spheres in S^4 which intersect transversely in two points. Any such pair of twins has a regular neighborhood homeomorphic to a pair of trivial D^2 bundles over S^2 plumbed together at their north and south poles. As before, the obstruction to extending a homeomorphism on the boundaries of these regular neighborhoods to a map of the twin complements to $D^2 \times T^2$ vanishes, so our definition of the Arf invariant in fact extends to the complement of any pair of "twin" two-spheres.

Pairs of twins arise naturally in the spinning of classical knots: One sphere is formed by the spun knot, and the other is the boundary of the ball B^3, held fixed during the spinning. A map $f\colon B^3 \backslash B^1 \to S^1$ defining a Seifert surface relative to the boundary of the spinning ball extends to

$$f \times \mathrm{id}\colon (B^3 \backslash B^1) \times S^1 \to S^1 \times S^1$$

so that the same Seifert surface used to calculate the Arf invariant of the classical knot can also be used to calculate c (see Robertello [13]). So, for example, the pair of twins (Spun trefoil, ∂B^3) realizes the nontrivial invariant $c = 1 \in (\mathbb{Z}_2, +)$. Note that the branched covers of S^4 over the spun trefoil are $S^1 \times \mathbb{Z}_p$–manifolds, but by the solution of the classical Smith conjecture, they are never four-spheres.

Proposition 5.1 *The invariant c is additive under connected sum of $\mathbb{Z}_p \times \mathbb{Z}_p$ actions at a fixed point, and is unchanged when a given $\mathbb{Z}_p \times \mathbb{Z}_p$ action is modified by equivariant connected sum with p copies of a $\mathbb{Z}_p$ action.*

Proof First consider the case of two actions joined at a fixed point. Note that c is calculated by extending a standard map $f\colon \partial N(\Sigma^*) \to T^2$ to $X^* \backslash N(\Sigma^*)$, then examining the Kervaire–Arf invariant on the normal bundle of $f^{-1}(*)$. Because the map and the action are standard on $\partial N(\Sigma^*)$, the maps (f_1 and f_2, say) may be assumed to agree where their domains overlap, so that the new surface used to compute

c is the boundary connected sum $f_1^{-1}(*)\natural f_2^{-1}(*)$. As in the case of Seifert surfaces for classical knots, the Arf invariant is additive under this operation.

Now consider the case of a G–action summed with a $\langle g\rangle$–orbit of $\langle h\rangle$–actions. We claim that an appropriate map $f\colon S^4\backslash\Sigma\to T^2$ may be modified to yield a new map for the revised action by changing f only near the sum points. Since we may simply choose a point $y\in T^2$ away from the image of these sum points to calculate c, it will follow immediately that c is unchanged.

As before, the actions may be assumed standard on an equivariant regular neighborhood of the singular set. A map $f\colon X\backslash N(\Sigma)\to T^2$ which is convenient for the calculation of c is obtained as follows: as the quotient $(X/\langle g\rangle, Y_g/\langle g\rangle)$ is again a knotted sphere pair, the complement $(X/\langle g\rangle)\backslash(Y_g/\langle g\rangle)$ admits a map f_g to S^1 inducing an isomorphism on H_1. Then f_g lifts to an equivariant map $\widetilde{f_g}\colon X\backslash Y_g\to\widetilde{S}^1\cong S^1$. A similar construction applies to $X\backslash Y_h$. We define $f=(\widetilde{f_g}\times\widetilde{f_h})|_{X\backslash\Sigma}\colon X\backslash\Sigma\to S^1\times S^1$.

Let x be a point on $Y_h\subset X$, away from Y_g, such that $f(x)\in T^2$ is a regular value for f. As the G–action is standard near Σ, a neighborhood of x may be parametrized in the form $B^2\times(-\delta,\delta)\times(-\epsilon,\epsilon)$, so that $h\cdot(re^{i\theta},t_1,t_2)=(re^{i(\theta+\frac{2\pi}{p})},t_1,t_2)$, and so that, near $f(x)\in T^2$, $f(re^{i\theta},t_1,t_2)=(e^{i\theta},t_2)$.

Meanwhile, let X' be a sphere with an $\langle h\rangle$–action fixing Y'_h, and let $f'_h\colon X'\backslash Y_h\to S^1$ be an equivariant Seifert map. Choose a point $x'\in Y'_h$; near x', choose coordinates so that $f'_h(re^{i\theta},t_1,t_2)=e^{i\theta}$, and extend f'_h to $S^1\times(-\epsilon,\epsilon)$ in the obvious way, so that p copies of X' may be glued G–equivariantly to $\langle g\rangle\times\{x\}$ in S^4. □

6 Concluding remarks and questions

It follows from the additivity of the Arf invariant that the constructions of $\mathbb{Z}_p\times\mathbb{Z}_p$ actions outlined in Section 3 yield actions which are concordant to linear actions. Modifications to these actions to change the Arf invariant would need to be sufficiently global as to change $f^{-1}(x)$ for every $x\in T^2$.

It is not clear whether such "highly nonlinear" actions exist, but it is interesting to compare the classical case of $\mathbb{Z}_2$ actions in dimension three. Murasugi [11] proved that for a knot K, $\mathrm{Arf}(K)=0\iff\Delta_K(-1)\equiv\pm1\pmod 8$. As $|\Delta_K(-1)|$ is the order of the first homology of the two-fold branched cover of K, $\mathrm{Arf}(K)$ is therefore an obstruction to a knot having a homology sphere as a double branched cover, and hence also an obstruction (which does not require the full strength of the classical Smith Conjecture) to the existence of a $\mathbb{Z}_2$–action fixing K.

A simplified, essentially homological proof of this fact might generalize to yield an obstruction for $\mathbb{Z}_2 \times \mathbb{Z}_2$ actions on S^4. The realization question for odd primes seems harder. We hope to consider both in future work.

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