Saddle tangencies and the distance of Heegaard splittings

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We give another proof of a theorem of Scharlemann and Tomova and of a theorem of Hartshorn. The two theorems together say the following. Let M be a compact orientable irreducible 3–manifold and P a Heegaard surface of M. Suppose Q is either an incompressible surface or a strongly irreducible Heegaard surface in M. Then either the Hempel distance $d(P) \leq 2\text{genus}(Q)$ or P is isotopic to Q. This theorem can be naturally extended to bicompressible but weakly incompressible surfaces.

57N10; 57M50

1 Introduction

Let P be a closed orientable surface of genus at least 2. The curve complex of P, introduced by Harvey [6], is the complex whose vertices are the isotopy classes of essential simple closed curves in P, and $k+1$ vertices determine a k–simplex if they are represented by pairwise disjoint curves. We denote the curve complex of P by $\mathcal{C}(P)$. For any two vertices in $\mathcal{C}(P)$, the distance $d(x, y)$ is the minimal number of 1–simplices in a simplicial path jointing x to y. To simplify notation, unless necessary, we do not distinguish a vertex in $\mathcal{C}(P)$ from a simple closed curve in P representing this vertex.

Let M be a compact orientable irreducible 3–manifold and P an embedded connected separating surface in M with $\text{genus}(P) \geq 2$. Let U and V be the closure of the two components of $M - P$. We may view $\partial U = \partial V = P$. As in Scharlemann–Tomova [14], we say P is *bicompressible* if P is compressible in both U and V. Let $\mathcal{U}$ and $\mathcal{V}$ be the set of vertices in $\mathcal{C}(P)$ represented by curves bounding compressing disks in U and V respectively. The distance $d(P)$ is defined to be the distance between $\mathcal{U}$ and $\mathcal{V}$ in the curve complex $\mathcal{C}(P)$. If P is a Heegaard surface, then $d(P)$ is the distance defined by Hempel [7]. We say P is *strongly irreducible* or following the definition in [14], say P is *weakly incompressible* if $d(P) \geq 2$, ie every compressing disk in U intersects every compressing disk in V.

Let Q be another closed orientable surface embedded in M. Let $g(Q)$ be the genus of Q. A theorem of Hartshorn [5] says that if Q is incompressible and P is a strongly

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irreducible Heegaard surface, then $d(P) \leq 2g(Q)$. In [14], Scharlemann and Tomova showed that if both P and Q are connected, separating, bicompressible and strongly irreducible, then either $d(P) \leq 2g(Q)$ or P and Q are well-separated or P and Q are isotopic. In particular, if both P and Q are strongly irreducible Heegaard surfaces, either P and Q are isotopic or $d(P) \leq 2g(Q)$.

Combining Hartshorn's theorem and the theorem of Scharlemann and Tomova, we have the following Theorem.

Theorem 1.1 *Suppose M is a compact orientable irreducible 3–manifold and P is a separating bicompressible and strongly irreducible (or weakly incompressible) surface in M. Let Q be an embedded closed orientable surface in M and suppose Q is either incompressible or separating, bicompressible but strongly irreducible. Then either*

(1) $d(P) \leq 2g(Q)$, *or*

(2) *after isotopy, $P_t \cap Q = \varnothing$ for all t, where P_t ($t \in [0,1]$) is a level surface in a sweep-out for P, see Section 2 for definition, or*

(3) *P and Q are isotopic.*

Remark The statement of Theorem 1.1 is basically the same as the main theorem of [14]. If Q is separating, bicompressible but strongly irreducible and $P_t \cap Q = \varnothing$ for all $t \in [0,1]$, then it is easy to see that P and Q are well-separated. Note that part (3) of the theorem never happens if Q is incompressible.

In this paper, we give another proof of Theorem 1.1. Some arguments were originally used in a different proof of the main theorem by the author [9]. The motivation for this paper is a conjecture in [9] which generalizes both the main theorem of [9] and the theorem of Scharlemann and Tomova. We hope this proof and the techniques in [9; 10] can lead to a solution of this conjecture. Some arguments in the proof are similar to those in [1; 14].

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2 Saddle tangencies

Notation Throughout this paper, we denote the interior of X by $\text{int}(X)$ for any space X.

Let P be a bicompressible surface and let U and V be the closure of the two components of $M - P$ as above. Let P^U and P^V be the possibly disconnected surfaces obtained by maximally compressing P in U and V respectively. Since M is irreducible, after capping off 2–sphere components by 3–balls, we may assume P^U and P^V do not contain 2–sphere components. Moreover, we may also assume $P^U \subset \mathrm{int}(U)$ and $P^V \subset \mathrm{int}(V)$. Since P is strongly irreducible, as in Casson–Gordon [3], P^U and P^V are incompressible in M. Furthermore, $P^U \cup P^V$ bounds a submanifold M_P of M and P is a strongly irreducible Heegaard surface of M_P. Note that if U is a handlebody, then $P^U = \varnothing$. If P is a Heegaard surface of M, then we may view $M_P = M$.

The surface P cuts M_P into a pair of compression bodies $U \cap M_P$ and $V \cap M_P$. There are a pair of properly embedded graphs $G^U \subset U \cap M_P$ and $G^V \subset V \cap M_P$ which are the spines of the two compression bodies. The endpoints of the graphs G^U and G^V lie in P^U and P^V respectively. Let $\Sigma_U = P^U \cup G^U$ and $\Sigma_V = P^V \cup G^V$, then $M_P - (\Sigma_U \cup \Sigma_V)$ is homeomorphic to $P \times (0,1)$. Throughout this paper, Σ_U and Σ_V are fixed.

We consider a sweepout $H\colon P \times (I, \partial I) \to (M_P, \Sigma_U \cup \Sigma_V)$, see [11], where $I = [0,1]$ and $H|_{P\times(0,1)}$ is an embedding. We denote $H(P \times \{x\})$ by P_x for any $x \in I$. We may assume $P_0 = \Sigma_U$, $P_1 = \Sigma_V$ and each P_x $(i \neq 0,1)$ is isotopic to P. To simplify notation, we will not distinguish $H(P \times (0,1))$ from $P \times (0,1)$.

Let $\pi\colon P \times I \to P$ be the projection. To simplify notation, we do not distinguish between an essential simple closed curve γ in P_x and the vertex represented by $\pi(\gamma)$ in the curve complex $\mathcal{C}(P)$.

Definition 2.1 Let Q be a properly embedded compact surface in M. We say Q is in *regular position with respect to* $P \times I$ if

(1) $Q \cap G^U$ and $Q \cap G^V$ consist of finitely many points and Q is transverse to $P^U \cup P^V$ and

(2) Q is transverse to each P_x, $x \in (0,1)$, except for finitely many critical levels $t_1, \dots, t_n \in (0,1)$ and

(3) at each critical level t_i, Q is transverse to P_{t_i} except for a single saddle or center tangency.

If $x \in (0,1)$ is not one of the t_i's, then we say x or P_x is a regular level. Clearly every embedded surface Q can be isotoped into a regular position.

Definition 2.2 We say Q is *irreducible with respect to* $P \times I$ if

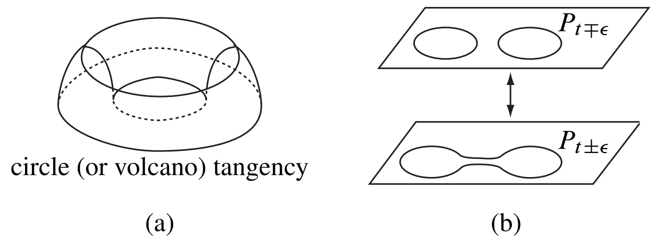

Figure 2.1

(1) Q is in regular position with respect to $P \times I$ and

(2) at each regular level P_x, if a component γ of $Q \cap P_x$ is trivial in P_x, then γ is also trivial in Q.

In this section, we assume Q is irreducible with respect to the sweepout $P \times I$. We first perform some isotopy on Q to eliminate center tangencies and trivial intersection curves. Lemma 2.1 can be viewed as a special case of a theorem of Thurston [15] and [4, Theorem 7.1].

Lemma 2.1 *Let Q be an embedded surface in M and suppose Q is irreducible with respect to the sweepout $P \times I$. Then, one can perform an isotopy on Q so that*

(1) $Q \cap (G^U \cup G^V)$ *consists of finitely many points, Q is transverse to $P^U \cup P^V$, and $Q \cap (P^U \cup P^V)$ consists of curves essential in Q;*

(2) *Q is transverse to each P_x, $x \in (0,1)$, except for finitely many critical levels $t_1, \dots, t_n \in (0,1)$;*

(3) *at each critical level t_i, Q is transverse to P_{t_i} except for a saddle or circle tangency, as shown in Figure 2.1(a);*

(4) *at each regular level x, every component of $Q \cap P_x$ is an essential curve in P_x.*

Proof Since $P^U \cup P^V$ is incompressible in M and M is irreducible, after some standard isotopy we may assume condition (1) in the lemma holds.

Note that the intersection of Q with $P \times I$ yields a (singular) foliation of $Q \cap M_P$ with each leaf a component of $Q \cap P_x$ for some $x \in I$. A singular point in the foliation is either a point in $Q \cap (G^U \cup G^V)$ or a saddle or center tangency.

Let x be a regular level and suppose a component γ of $Q \cap P_x$ is trivial in P_x. Suppose γ is innermost in P_x, ie the disk bounded by γ in P_x does not contain any other intersection curve with Q. Since Q is irreducible with respect to $P \times I$, γ bounds a disk D_γ in Q. If the induced foliation on D_γ contains more than one singular point, since γ is trivial in P_x, we can construct a disk $D' \subset P \times (x-\epsilon, x+\epsilon)$ for some small ϵ such that

(1) $\partial D' = \gamma$,

(2) the induced foliation of $D' \cap (P \times I)$ consists of parallel circles except for a singular point corresponding to a center tangency,

(3) $(Q - D_\gamma) \cup D'$ is embedded in M and irreducible with respect to $P \times I$.

Since M is irreducible, $(Q - D_\gamma) \cup D'$ is isotopic to Q. Moreover, the induced foliation on $(Q - D_\gamma) \cup D'$ has fewer singular points. So after finitely many such operations, we may assume that for any regular level x and for any component γ of $Q \cap P_x$ that is trivial in P_x, the disk bounded by γ in Q lies in M_P and is transverse to $P \times (0,1)$ except for a single center tangency.

Let t be a critical level and suppose $Q \cap P_t$ contains a saddle tangency. Let ϵ be a sufficiently small number. So the component of $Q \cap (P \times [t-\epsilon, t+\epsilon])$ that contains the saddle tangent point is a pair of pants F. Figure 2.1(b) is a picture of the curves changing from $F \cap P_{t-\epsilon}$ to $F \cap P_{t+\epsilon}$.

We first claim that at most one component of ∂F is trivial in the corresponding level surface $P_{t\pm\epsilon}$. Let γ_1, γ_2 and γ_3 be the 3 components of ∂F and suppose γ_1 and γ_2 are both trivial in the corresponding level surfaces. Then by the change of $F \cap P_x$ near the saddle tangency as shown in Figure 2.1(b), γ_3 must also be trivial in the corresponding level surface $P_{t\pm\epsilon}$. Since Q is irreducible with respect to $P \times I$, γ_1 and γ_2 bound disks D_1 and D_2 in Q respectively. By the assumption above, the disk D_i does not contain any saddle tangency and hence $F \cap D_i = \gamma_i$, $i = 1, 2$. Thus $F \cup D_1 \cup D_2$ is a disk in Q bounded by γ_3 and $F \cup D_1 \cup D_2$ contains a saddle tangent point. This contradicts the assumption above. Thus at most one component of ∂F is trivial in $P_{t\pm\epsilon}$.

Let F and γ_i be as above. Suppose γ_1 and γ_2 lie in $P_{t-\epsilon}$ and γ_3 lies in $P_{t+\epsilon}$. If γ_1 is trivial in $P_{t-\epsilon}$ and let D_1 be the disk in Q bounded by γ_1, then $F \cap D_1 = \gamma_1$ as above and $F \cup D_1$ is an annulus in Q bounded by $\gamma_2 \cup \gamma_3$. Since D_1 is isotopic to a disk in $P_{t-\epsilon}$, we can first push D_1 into $P \times [t-\epsilon, t+\epsilon]$, then as shown in Figure 2.2(a), we may perform another isotopy on Q canceling the center tangency in D_1 and the saddle tangency in F. If γ_3 is trivial in $P_{t+\epsilon}$, by the assumption above, both γ_1 and γ_2 are essential in $P_{t-\epsilon}$. Hence γ_1 and γ_2 must be parallel in $P_{t-\epsilon}$. Let D_3 be the disk in Q bounded by γ_3. As above, $F \cap D_3 = \gamma_3$ and $F \cup D_3$ is an annulus in Q bounded by $\gamma_1 \cup \gamma_2$. Since D_3 is isotopic to the disk in $P_{t+\epsilon}$ bounded by γ_3,

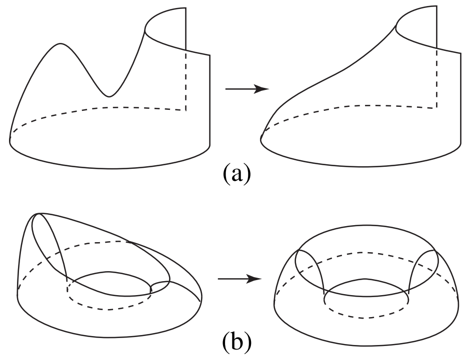

Figure 2.2

we can first push the annulus $F \cup D_3$ into a ∂–parallel annulus in $P \times [t-\epsilon, t+\epsilon]$. Then an isotopy as shown in Figure 2.2(b) can cancel the center tangency in D_3 and the saddle tangency in F, changing $F \cup D_3$ into an annulus with a circle (or volcano) tangency. Note that the circle tangency is an essential curve in the corresponding level surface P_x.

Note that condition (1) of the lemma implies that for a small ϵ, $Q \cap P_\epsilon$ and $Q \cap P_{1-\epsilon}$ consist of essential curves in P_ϵ and $P_{1-\epsilon}$ respectively. Since Q is not a 2–sphere, a curve of $Q \cap P_x$ that is trivial in P_x will eventually meet and cancel with a saddle tangency. Thus after a finite number of isotopies as above, we can eliminate all the curves of $Q \cap P_x$ that are trivial in P_x, and get a surface Q satisfying all the conditions in the lemma. □

Note that a circle tangency does not create any singularity in the foliation of $Q \cap M_P$ induced from $P \times I$. Thus, if Q satisfies the conditions in Lemma 2.1, a singular point in the foliation of $Q \cap M_P$ corresponds to either a saddle tangency or a point in $Q \cap (G^U \cup G^V)$. It is possible that Q does not intersect $M_P = P \times I$, ie $P_t \times Q = \varnothing$ for all t, after isotopy.

Lemma 2.2 *Let P and Q be as above and assume Q satisfies the conditions in Lemma 2.1. Suppose $Q \cap \Sigma_U \neq \varnothing$ and $Q \cap \Sigma_V \neq \varnothing$. Then the distance $d(P) = d(\mathcal{U}, \mathcal{V}) \leq 2g(Q)$.*

Proof Since Q is connected and P is separating, $Q \cap \Sigma_U \neq \varnothing$ and $Q \cap \Sigma_V \neq \varnothing$ imply $Q \cap P_t \neq \varnothing$ for every t.

Claim 1 Let t be a critical level and ϵ a sufficiently small number. Let σ and w be any components of $Q \cap P_{t-\epsilon}$ and $Q \cap P_{t+\epsilon}$ respectively. Then $d(\sigma, w) \leq 1$.

Proof of Claim 1 The claim is obvious if P_t contains a circle tangency. So we suppose P_t contains a saddle tangency. Let F be the component of $Q \cap (P \times [t-\epsilon, t+\epsilon])$ that contains the saddle tangency. Then F is a pair of pants and all other components of $Q \cap (P \times [t-\epsilon, t+\epsilon])$ are essential vertical annuli in $P \times [t-\epsilon, t+\epsilon]$. If σ is a boundary curve of a vertical annulus, then σ is isotopic to a component of $Q \cap P_{t+\epsilon}$ and hence $d(\sigma, w) \leq 1$ for any curve w in $Q \cap P_{t+\epsilon}$. If neither σ nor w is a boundary curve of a vertical annulus, then σ and w are components of ∂F and $d(\sigma, w) = 1$ as shown in Figure 2.1(b). □

Let $s_0 < \cdots < s_n$ be a collection of regular levels such that $s_0 = \delta$, $s_n = 1-\delta$ for a small δ and there is exactly one saddle or circle tangency in each $P \times (s_i, s_{i+1})$. Let $\Gamma_i = Q \cap P_{s_i}$ for each i.

Recall that $P_0 = \Sigma_U = P^U \cup G^U$ and $P_1 = \Sigma_V = P^V \cup G^V$. Since $s_0 = \delta$ for a small δ, we may assume $d(\mathcal{U}, \Gamma_0)$ is either 0 or 1, and if $d(\mathcal{U}, \Gamma_0) = 1$ then $d(\mathcal{U}, \sigma) = 1$ for any component σ of Γ_0. Similarly, $d(\mathcal{V}, \Gamma_n)$ is either 0 or 1, and if $d(\mathcal{V}, \Gamma_n) = 1$ then $d(\mathcal{V}, w) = 1$ for any component w of Γ_n.

Suppose $d(\mathcal{U}, \mathcal{V}) > 2g(Q)$ and hence $d(\mathcal{U}, \mathcal{V}) > 2$. Let k be the smallest integer such that $d(\mathcal{U}, \Gamma_k) \neq 0$ and l the largest integer such that $d(\Gamma_l, \mathcal{V}) \neq 0$. Since $d(\mathcal{U}, \Gamma_0) \leq 1$ and $d(\mathcal{V}, \Gamma_n) \leq 1$, by Claim 1 above, $d(\mathcal{U}, \Gamma_k) = d(\Gamma_l, \mathcal{V}) = 1$ and $k \leq l$. Without loss of generality, we assume $k < l$. Next we show that every curve in Γ_k is essential in Q. Suppose a curve γ in Γ_k is trivial in Q and let D be the disk bounded by γ in Q. Since P^U and P^V are incompressible, we may assume $D \subset M_P$. Since P is a strongly irreducible Heegaard surface of M_P, by the no-nesting lemma of Scharlemann [12, Lemma 2.2], γ must bound a disk in one of the two compression bodies, ie either $\gamma \in \mathcal{U}$ or $\gamma \in \mathcal{V}$. However, $\gamma \in \mathcal{U}$ contradicts $d(\mathcal{U}, \Gamma_k) \neq 0$, and $\gamma \in \mathcal{V}$ contradicts $d(\mathcal{U}, \mathcal{V}) > 2$. Thus every curve in Γ_k must be essential in Q. Similarly every curve in Γ_l is also essential in Q.

Let $Q' = Q \cap (P \times [s_k, s_l])$, and let U' and V' be the two components of $M - P \times (s_k, s_l)$ containing G^U and G^V respectively, $F_U = Q \cap U'$ and $F_V = Q \cap V'$. Since Γ_k and Γ_l are essential in Q, F_U, Q' and F_V are essential subsurfaces of $Q = F_U \cup Q' \cup F_V$.

Claim 2 Let σ_k be any component of Γ_k, then $d(\sigma_k, \mathcal{U}) \leq 1$.

Proof of Claim 2 By the definition of k and the argument above, Claim 2 holds if $k=0$. If $k>0$, then $d(\mathcal{U},\Gamma_{k-1})=0$ and $d(\mathcal{U},\Gamma_k)=1$. Let w be a component of Γ_{k-1} that represents a vertex in $\mathcal{U}$. By Claim 1, for any component σ_k of Γ_k, $d(\sigma_k,\mathcal{U})\le d(\sigma_k,w)\le 1$. □

Claim 3 There is a component σ_k of Γ_k and a component σ_l of Γ_l such that $d(\sigma_k,\sigma_l)\le -\chi(Q')$.

Proof of Claim 3 Let $t_1<\cdots<t_N$ be the levels in (s_k,s_l) that contain the saddle tangencies. For a sufficiently small ϵ, $P\times[t_i+\epsilon,t_{i+1}-\epsilon]$ contains no saddle tangency for each i (to simplify notation we set $t_0+\epsilon=s_k$ and $t_{N+1}-\epsilon=s_l$). So by the conditions in Lemma 2.1, $Q\cap(P\times[t_i+\epsilon,t_{i+1}-\epsilon])$ consists of annuli for each $i=0,\dots,N$. If $Q\cap(P\times[t_i+\epsilon,t_{i+1}-\epsilon])$ consists of ∂–parallel annuli, then $Q\cap P_t=\varnothing$ for some t after isotopy, a contradiction to our assumption at the beginning. Thus an annulus component A_i of $Q\cap(P\times[t_i+\epsilon,t_{i+1}-\epsilon])$ is vertical. We choose γ_i to be a meridian circle in A_i for each i and assume $\sigma_k=\gamma_0=A_0\cap P_{s_k}\subset\Gamma_k$ and $\sigma_l=\gamma_N=A_N\cap P_{s_l}\subset\Gamma_l$. Since each A_i is vertical, γ_i is parallel to a component of $Q\cap P_{t_{i+1}-\epsilon}$. Similarly γ_{i+1} is parallel to a component of $Q\cap P_{t_{i+1}+\epsilon}$. By Claim 1, $d(\gamma_i,\gamma_{i+1})\le 1$ and hence $d(\sigma_k,\sigma_l)=d(\gamma_0,\gamma_N)\le N$. Moreover, since the only singular points in the induced foliation of Q' are the saddle tangencies, by a standard index argument, $-\chi(Q')=N$ and hence $d(\sigma_k,\sigma_l)\le-\chi(Q')$. □

Since Q', F_U and F_V are essential subsurfaces of Q, $\chi(Q')\ge\chi(Q)$. By Claim 2, $d(\sigma_k,\mathcal{U})\le 1$ and similarly $d(\sigma_l,\mathcal{V})\le 1$. Therefore, $d(\mathcal{U},\mathcal{V})\le d(\mathcal{U},\sigma_k)+d(\sigma_k,\sigma_l)+d(\sigma_l,\mathcal{V})\le 1-\chi(Q')+1\le 2-\chi(Q)=2g(Q)$. □

Lemma 2.2 implies that if $d(\mathcal{U},\mathcal{V})$ is large, then not every Q can be put into a position satisfying all the hypotheses of Lemma 2.2.

Corollary 2.1 *Let P and Q be as in Theorem 1.1. Then Theorem 1.1 holds if Q is incompressible.*

Proof If Q is incompressible, then Q can be isotoped to be irreducible with respect to $P\times I$. Moreover, if $Q\cap\Sigma_U=\varnothing$, then since Q is incompressible, Q can be isotoped out of the compression body $M_P-N(\Sigma_U)$. Hence $Q\cap M_P=\varnothing$ after isotopy and part (2) of Theorem 1.1 holds. Now Corollary 2.1 follows from Lemma 2.1 and Lemma 2.2. □

3 The graphics of sweepouts

In this section, we suppose Q is separating, bicompressible and strongly irreducible.

Let X and Y be the closure of the 2 components of $M - Q$. Let Q^X and Q^Y be the possibly disconnected surfaces obtained by maximal compressing Q in X and Y respectively and capping off 2–sphere components by 3–balls. Similar to the argument on P^U and P^V above, we may assume $Q^X \subset \mathrm{int}(X)$ and $Q^Y \subset \mathrm{int}(Y)$ are incompressible in M. Furthermore, $Q^X \cup Q^Y$ bounds a submanifold M_Q of M and Q is a strongly irreducible Heegaard surface of M_Q. If X is a handlebody, then $Q^X = \varnothing$. If Q is a Heegaard surface of M, we may view $M_Q = M$.

As in Section 2, the surface Q cuts M_Q into a pair of compression bodies $X \cap M_Q$ and $Y \cap M_Q$. Let graphs $G^X \subset X \cap M_Q$ and $G^Y \subset Y \cap M_Q$ be the spines of the two compression bodies and let $\Sigma_X = Q^X \cup G^X$ and $\Sigma_Y = Q^Y \cup G^Y$. Then $M_Q - (\Sigma_X \cup \Sigma_Y)$ is homeomorphic to $Q \times (0,1)$.

Now we consider the two sweepouts $H\colon P \times (I, \partial I) \to (M_P, \Sigma_U \cup \Sigma_V)$ and $H'\colon Q \times (I, \partial I) \to (M_Q, \Sigma_X \cup \Sigma_Y)$. Let $P_t = H(P \times \{t\})$ and $Q_t = H'(Q \times \{t\})$, $t \in I$. We may assume $Q_0 = \Sigma_X$, $Q_1 = \Sigma_Y$ and Q_x is isotopic to Q for each $x \in (0,1)$.

The graphic Λ of the sweepouts, defined in [11], is the set of points $(s,t) \in (0,1) \times (0,1)$ such that P_s is not transverse to Q_t. We briefly describe the graphic below and refer to [11] for more details. As in [11], Cerf theory implies that after some isotopy, we may assume that Λ is a graph in $(0,1) \times (0,1)$ whose edges are the set of points (s,t) for which P_s is transverse to Q_t except for a single saddle or center tangency. There are two types of vertices in Λ, birth-and-death vertices and crossing vertices, as shown in Figure 3.1(a). Moreover, each arc $(0,1) \times \{x\}$ contains at most one vertex, $x \in (0,1)$. The complement of Λ, $(0,1) \times (0,1) - \Lambda$, is a finite collection of regions. Note that for every (s,t) in $(0,1) \times (0,1) - \Lambda$, P_s is transverse to Q_t, and for any two points (s,t) and (s',t') in the same region, $P_s \cap Q_t$ and $P_{s'} \cap Q_{t'}$ have the same intersection pattern.

Let $(s,t) \in (0,1) \times (0,1) - \Lambda$. Suppose there are disks or annuli $D_P \subset P_s$ and $D_Q \subset Q_t$ with $D_P \cap D_Q = \partial D_P = \partial D_Q \subset P_s \cap Q_t$. Suppose D_P is parallel to D_Q (fixing $\partial D_P = \partial D_Q$) in M and suppose $D_P \cup D_Q$ bounds a 3–ball or solid torus E. Moreover, suppose $Q_t \cap E = D_Q$. Then we can perform an isotopy on Q_t by pushing D_Q across E and remove the intersection $\partial D_P = \partial D_Q$. This isotopy is the same as the operation that changes Q_t to $(Q_t - D_Q) \cup D_P$ and then perturbs the resulting surface. We call such an isotopy a *trivial isotopy* on Q_t at P_s. We may view a trivial isotopy on Q_t as associated with the disk or annulus $D_Q \subset Q_t$. Suppose we are to perform another trivial isotopy on Q_t at $P_{s'}$ and let $D'_Q \subset Q_t$ be the disk or annulus

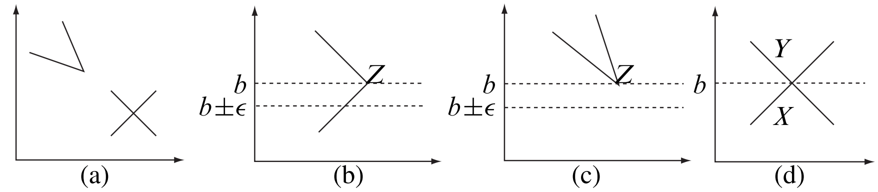

Figure 3.1

in the isotopy as above. Then D_Q and D'_Q are either disjoint or nested in Q_t. Thus either the two trivial isotopies are disjoint or we can view one isotopy as a middle step of the other.

Labelling For any Q_t, we use X_t (resp. Y_t) to denote the component of $M - Q_t$ that contains Σ_X (resp. Σ_Y). We label a region, ie a component of $(0,1)\times(0,1)-\Lambda$, X (resp. Y) if for a point (s,t) in the region, either (1) there is a component of $P_s \cap Q_t$ that is trivial in P_s but bounds an essential disk in X_t (resp. Y_t), or (2) Σ_U or Σ_V lies in Y_t (resp. X_t) after some *trivial isotopies* on Q_t at finitely many regular levels P_x. We label $t \in (0,1)$ X (resp. Y) if the horizontal line segment $(0,1)\times\{t\}$ intersects a region labelled X (resp. Y). Note that since a trivial isotopy does not increase $|\Sigma_U \cap Q_t|$ or $|\Sigma_V \cap Q_t|$, if t is not labelled, $Q_t \cap \Sigma_U \neq \varnothing$ and $Q_t \cap \Sigma_V \neq \varnothing$ after any trivial isotopies.

Lemma 3.1 *Either Theorem 1.1 holds or for a sufficiently small $\delta > 0$, δ is labelled X and $1-\delta$ is labelled Y.*

Proof For a sufficiently small $\delta > 0$, $H'(Q \times [0,\delta])$ is a small neighborhood of $\Sigma_X = Q^X \cup G^X$. If $P_s \cap G^X \neq \varnothing$ for some s, then by definition, δ is labelled X for a sufficiently small δ. Suppose δ is not labelled X, then the graph G^X must be disjoint from $M_P = H(P \times I)$. Moreover, if $Q^X \cap P_t = \varnothing$ for some t after isotopy, since Q^X is incompressible, we can isotope Q^X out of the two compression bodies $M_P - P_t$. Hence, $Q_\delta \cap M_P = \varnothing$ after isotopy and part (2) of Theorem 1.1 holds. If $Q^X \cap P_t \neq \varnothing$ for all t, since Q^X is incompressible, by Corollary 2.1, $d(P) \leq 2g(Q^X) \leq 2g(Q)$ and Theorem 1.1 follows. The proof for $1-\delta$ is similar. □

Lemma 3.2 *Either Theorem 1.1 holds or no $t \in (0,1)$ is labelled both X and Y.*

Proof We first remark that if $\Sigma_U \subset Y_t$ then one cannot move Σ_U to X_t by a trivial isotopy, since if this happens, then Σ_U must lie in E, where E is the 3–ball or solid torus in the trivial isotopy described above. However, since $g(P) \geq 2$ and P is strongly irreducible, Σ_U cannot lie in a 3–ball or solid torus by [3]. So by our labelling, if t is labelled both X and Y, then one can always find $s_1 \neq s_2$ such that P_{s_1} and P_{s_2} are transverse to Q_t and one of the following three cases occurs.

Case 1 A component of $P_{s_1} \cap Q_t$ contains a curve bounding an essential disk D_X in X_t and a component of $P_{s_2} \cap Q_t$ contains a curve bounding an essential disk D_Y in Y_t. In this case, since $s_1 \neq s_2$, $\partial D_X \cap \partial D_Y = \varnothing$ in Q_t, which contradicts the assumption that Q is strongly irreducible.

Case 2 After trivial isotopies, $\Sigma_U \subset Y_t$ and $\Sigma_V \subset X_t$. This means that $Q_t \subset P \times (0,1) \subset M_P$ and Q_t separates Σ_U and Σ_V in M_P. The proof for this case is similar to that of [14, Lemma 2.3]. If Q_t is incompressible in $P \times (0,1)$, then Q_t is isotopic to P and Theorem 1.1 holds. If Q_t is compressible on both sides in $P \times (0,1)$, similar to the construction of M_Q earlier, by maximally compressing Q_t in $P \times (0,1)$ on both sides and capping off 2–sphere components, we obtain a submanifold M'_Q of $P \times (0,1)$ such that Q_t is a strongly irreducible Heegaard surface of M'_Q. Moreover, by [3], $\partial M'_Q$ is incompressible in $P \times (0,1)$. So each component of $\partial M'_Q$ is parallel to P and M'_Q must be a product of P and an interval. Thus we can view Q_t as a strongly irreducible Heegaard surface of a product $P \times [0,1]$. By Scharlemann–Thompson [13], either Q_t is isotopic to P or Q_t cuts $P \times [0,1]$ into a handlebody and a compression body. In the later case, both Σ_U and Σ_V lie in Y_t (or both in X_t), a contradiction. If Q_t is compressible on only one side, say the X_t side. Then after maximally compressing Q_t in $P \times (0,1)$ on the X_t side, one obtains an incompressible surface Q' in $P \times (0,1)$ (note that $Q' \neq \varnothing$ as $\Sigma_V \subset X_t$). Thus Q' is incompressible in $P \times (0,1)$ and must be parallel to P. Moreover, since Q_t is connected and separating, Q' is a single parallel copy of P. So Q_t and Q' bound a compression body W in $P \times (0,1)$, and Q_t is bicompressible in the submanifold $Y_t \cup W$ of M. Since Q_t is strongly irreducible, Casson-Gordon [3] implies that Q' is incompressible in $Y_t \cup W$. However, since Q' is parallel to P, this contradicts the assumption that P is compressible on both sides.

Case 3 After trivial isotopies, $\Sigma_U \subset Y_t$ and a component of $P_{s_1} \cap Q_t$ contains a curve γ that is trivial in P_{s_1} and bounds an essential disk D in Y_t. Note that if a component of $P_{s_1} \cap Q_t$ also bounds an essential disk in X_t, then this contradicts that Q is strongly irreducible as in case (1). Thus, after some isotopy on Q_t, we may assume that γ is innermost in P_{s_1} and the disk D bounded by γ in P_{s_1} is an essential disk in Y_t. Since $\Sigma_U \subset Y_t$ and $D \subset Y_t - \Sigma_U$, by maximally compressing Q_t in $Y_t - \Sigma_U$ and capping off 2–sphere components, we obtained a (possibly disconnected) surface Q_t^Y. Note that $Q_t^Y \neq \varnothing$ because Σ_U is not contained in a 3–ball. Since Q_t

is strongly irreducible, by [3], Q_t^Y is incompressible in $M - \Sigma_U$. Note that if P is a Heegaard surface of a closed manifold M, this is already a contradiction since Q_t^Y lies in the handlebody $M - N(\Sigma_U)$ and cannot be incompressible. Q_t^Y cuts Y_t into H_1 and H_2, where H_2 is the compression body bounded by Q_t and Q_t^Y. Since the compressions on Q_t are disjoint from Σ_U and since Σ_U does not lie in a 3–ball, $\Sigma_U \cap H_2 = \varnothing$. Hence $\Sigma_U \subset H_1$. Since Q_t^Y is incompressible in $M - \Sigma_U$, we can push Q_t^Y out of the compression body $M_P - N(\Sigma_U)$ or equivalently push $M_P - N(\Sigma_U)$ into H_1. So we can isotope M_P into H_1. In particular, $Q_t \cap M_P = \varnothing$ after isotopy and part (2) of Theorem 1.1 holds. □

Lemma 3.3 *If $t \in (0,1)$ has no label and $(0,1) \times \{t\}$ contains no vertex of Λ, then Q_t is irreducible with respect to $P \times I$ and Theorem 1.1 holds.*

Proof Since $(0,1) \times \{t\}$ contains no vertex of Λ, Q_t is in regular position with respect to $P \times I$. For any $(s,t) \notin \Lambda$, suppose a curve γ in $P_s \cap Q_t$ is trivial in P_s. If γ is an essential curve in Q_t, by assuming γ to be an innermost such curve, the disk bounded by γ in P_s can be isotoped to be an essential disk in either X_t or Y_t. Since $t \in (0,1)$ has no label, γ must be trivial in Q_t. Thus by definition, Q_t is irreducible with respect to $P \times I$. So after isotopy we may assume Q satisfies the conditions in Lemma 2.1. Moreover, since t has no label, $Q_t \cap \Sigma_U \neq \varnothing$ and $Q_t \cap \Sigma_V \neq \varnothing$ after the isotopy in the proof of Lemma 2.1. So Theorem 1.1 follows from Lemma 2.2. □

Suppose Theorem 1.1 is not true. Then by Lemma 3.1, for a small δ, δ is labelled X and $1-\delta$ is labelled Y. As t changes from δ to $1-\delta$, the label changes from X to Y. So by Lemma 3.2 and Lemma 3.3, there must be a number $b \in (0,1)$ such that

(1) $(0,1) \times \{b\}$ contains a vertex of Λ and

(2) b has no label and

(3) $b-\epsilon$ is labelled X and $b+\epsilon$ is labelled Y for sufficiently small $\epsilon > 0$.

Let $Z = (a,b)$ be the vertex of Λ in $(0,1) \times \{b\}$. If Z is a birth-and-death vertex, then since no region that intersects $(0,1) \times \{b\}$ is labelled, as shown in Figure 3.1(b) and (c), after perturbing $(0,1) \times \{b\}$ a little, we can find a line segment $(0,1) \times \{b \pm \epsilon\}$ that does not intersect any labelled region, a contradiction to our assumption above. Therefore, $Z = (a,b)$ must be a crossing vertex. Figure 3.1(d) is a picture of Z.

Since $Z = (a,b)$ is a crossing vertex, as explained in [11] (see Kobayashi–Saeki [8, Figure 2.6]), P_a is transverse to Q_b except for two saddle tangencies. Since b is not labelled, for any $s \neq a$ in $(0,1)$, either (1) $P_s \cap Q_b$ contains a single center or saddle tangency or (2) P_s is transverse to Q_b and if a component of $P_s \cap Q_b$ is trivial in P_s then it is also trivial in Q_b. Moreover, after trivial isotopies, $Q_b \cap \Sigma_U \neq \varnothing$ and $Q_b \cap \Sigma_V \neq \varnothing$. Since P is separating and Q is connected, this implies that $Q_b \cap P_s \neq \varnothing$ for all $s \in I$.

Now we consider $Q_b \cap (P \times [a-\epsilon, a+\epsilon])$ for a small ϵ. Let F be the union of the components of $Q_b \cap (P \times [a-\epsilon, a+\epsilon])$ that contain the two saddle tangencies. Thus F is either the union of two disjoint pairs of pants or a connected surface with $\chi(F) = -2$. All other components of $Q_b \cap (P \times [a-\epsilon, a+\epsilon])$, denoted by $A_1, \dots, A_m$, are vertical annuli in $P \times [a-\epsilon, a+\epsilon]$.

Next we consider the case that a component of $Q_b \cap P_{a\pm\epsilon}$ is trivial in $P_{a\pm\epsilon}$. If a component γ of ∂A_i, $i = 1, \dots, m$, is trivial and innermost in $P_{a\pm\epsilon}$, then by our assumption, γ bounds a disk D_γ in Q_b. We can perform a trivial isotopy on Q_b by pushing the disk $D_\gamma \cup A_i$ away from $P \times [a-\epsilon, a+\epsilon]$. Thus, after a finite number of such operations, we may assume the boundary of every annular component A_i is essential in $P_{a\pm\epsilon}$.

Suppose a component γ of ∂F is an innermost trivial curve in $P_{a\pm\epsilon}$. So γ bounds a disk D_γ in Q_b. If D_γ contains a component of F, then as in the proof of Lemma 2.1, after replacing D_γ by a disk which is transverse to every P_x except for a single center tangency, we get a surface isotopic to Q_b and has at most one saddle tangency in $P \times [a-\epsilon, a+\epsilon]$. This means that after the isotopy, Q_b is irreducible with respect to $P \times I$ and Theorem 1.1 follows from Lemma 2.2 and Lemma 3.3. So we may assume that $D_\gamma \cap F = \gamma$ for any component γ of ∂F that is trivial in $P_{a\pm\epsilon}$.

Let $\widehat{F}$ be the union of F and all the disks D_γ in Q_b bounded by ∂F as above. We may push all such disks D_γ into $P \times (a-\epsilon, a+\epsilon)$ and isotope $\widehat{F}$ into a surface properly embedded in $P \times [a-\epsilon, a+\epsilon]$. By the construction, $\partial \widehat{F}$ is essential in $P_{a\pm\epsilon}$. So $\widehat{F}$ has no disk component. If $\widehat{F}$ consists of annuli, then since $\partial \widehat{F}$ is essential in $P_{a\pm\epsilon}$, each annulus is either vertical or ∂–parallel in $P \times [a-\epsilon, a+\epsilon]$. Thus, after some isotopy, Q_b becomes irreducible with respect to $P \times I$ and Theorem 1.1 follows from Lemma 2.2 and Lemma 3.3. So we may assume $\chi(\widehat{F})$ is either -2 or -1, ie at most one component of ∂F is trivial in $P_{a\pm\epsilon}$.

Suppose $\chi(\widehat{F}) = -1$. If $\widehat{F}$ is a once-punctured torus, then $\widehat{F}$ must be incompressible in $P \times [a-\epsilon, a+\epsilon]$. Otherwise a compression on $\widehat{F}$ yields a disk, contradicting that $\partial \widehat{F}$ is essential in $P_{a\pm\epsilon}$. As $\widehat{F}$ is properly embedded in the product $P \times [a-\epsilon, a+\epsilon]$, $\widehat{F}$ must be ∂–compressible. A ∂–compression on $\widehat{F}$ yields an incompressible annulus with both boundary circles in $P_{a-\epsilon}$ (or $P_{a+\epsilon}$). So the resulting annulus is ∂–parallel. Since $\widehat{F}$ is incompressible, this implies that $\widehat{F}$ itself is ∂–parallel. Hence we can perform an isotopy on $\widehat{F}$ so that Q_b becomes irreducible with respect to $P \times I$.

Similarly, if $\widehat{F}$ is a pair of pants, then $\widehat{F}$ must be incompressible but ∂–compressible. So a ∂–compression on $\widehat{F}$ yields one or two incompressible annuli. This implies that either $\widehat{F}$ is ∂–parallel or we can perform an isotopy on $\widehat{F}$ so that $\widehat{F}$ is transverse to each P_x except for a single saddle tangency. In either case, we can isotope $\widehat{F}$ so that Q_b becomes irreducible with respect to $P \times I$ and Theorem 1.1 follows from Lemma 2.2 and Lemma 3.3.

Therefore, we may assume $\chi(\widehat{F}) = -2$. Hence $F = \widehat{F}$ and every component of ∂F is essential in $P_{a\pm\epsilon}$.

Since b is not labelled and since every component of ∂F above is essential in $P_{a\pm\epsilon}$, at each regular level $x \in (0,1)$, if a component of $P_x \cap Q_b$ is trivial in P_x, then it must also be trivial in Q_b. Thus, we can apply Lemma 2.1 on $Q_b \cap (P \times ([0, a-\epsilon] \cup [a+\epsilon, 1]))$. So after some isotopies, Q_b satisfies all the conditions in Lemma 2.1 except for the level P_a where $P_a \cap Q_b$ contains 2 saddle tangencies. Moreover, since b is not labelled, $Q_b \cap \Sigma_U \neq \varnothing$ and $Q_b \cap \Sigma_V \neq \varnothing$. Hence $Q_b \cap P_s \neq \varnothing$ for every s.

Claim A Let σ and w be any components of $Q_b \cap P_{a-\epsilon}$ and $Q_b \cap P_{a+\epsilon}$ respectively. Then $d(\sigma, w) \leq 2 = -\chi(F) = -\chi(Q_b \cap (P \times [a-\epsilon, a+\epsilon]))$.

Proof of Claim A If σ is a boundary curve of a vertical annulus component of $Q_b \cap (P \times [a-\epsilon, a+\epsilon])$, then σ is isotopic to a component of $Q \cap P_{a+\epsilon}$ and hence $d(\sigma, w) \leq 1$ for any curve w in $Q \cap P_{a+\epsilon}$. So we may assume neither σ nor w is a boundary curve of a vertical annulus. Thus σ and w are both components of ∂F.

Let Ω be the union of the components of $P_a \cap Q_b$ that contain the 2 saddle tangent points. So Ω is a possibly disconnected graph with 2 vertices of valence 4. Let $N(\Omega)$ be a regular neighborhood of Ω in P_a and let $\pi\colon P \times I \to P_a$ be the projection, then $\pi(\partial F) \subset N(\Omega)$ after isotopy. Since P has genus at least 2, there must be an essential curve α in P_a disjoint from $N(\Omega)$. So $d(\sigma, w) \leq d(\sigma, \alpha) + d(\alpha, w) \leq 2 = -\chi(F)$. □

Now Theorem 1.1 follows from the argument in the proof of Lemma 2.2. As in the proof of Lemma 2.2, let $s_0 < \cdots < s_n$ be a collection of regular levels such that $s_0 = \delta$, $s_n = 1 - \delta$ for a small δ and there is exactly one critical level in each $P \times (s_i, s_{i+1})$. Let $\Gamma_i = Q \cap P_{s_i}$ for each i.

Since we assume Q is bicompressible in this section and since M is irreducible, if Q is a torus, then M must be a lens space and P and Q must be isotopic Heegaard surfaces of the lens space (see Bonahon–Otal [2]). So we may assume $g(Q) \geq 2$.

Suppose $d(\mathcal{U}, \mathcal{V}) > g(Q)$. Since $g(Q) \geq 2$, we have $d(\mathcal{U}, \mathcal{V}) > 4$. Let k be the smallest integer such that $d(\mathcal{U}, \Gamma_k) \neq 0$ and l the largest integer such that $d(\Gamma_l, \mathcal{V}) \neq 0$. By

Claim A above and Claim 1 in the proof of Lemma 2.2, $d(\mathcal{U}, \Gamma_k)$ and $d(\Gamma_l, \mathcal{V})$ are either 1 or 2 and $k \le l$. Without loss of generality, we assume $k < l$.

Similar to the proof of Lemma 2.2, Γ_k and Γ_l must be essential in Q_b. Let $Q' = Q_b \cap (P \times [s_k, s_l])$, and let U' and V' be the two components of $M - P \times (s_k, s_l)$ containing G^U and G^V respectively, $F_U = Q_b \cap U'$ and $F_V = Q_b \cap V'$. Since Γ_k and Γ_l are essential in Q_b, F_U, Q' and F_V are essential subsurfaces of $Q_b = F_U \cup Q' \cup F_V$.

Claim B Let σ_k be any component of Γ_k, then $d(\sigma_k, \mathcal{U}) \le 1 - \chi(F_U)$.

Proof of Claim B If a component A of F_U is a ∂–parallel annulus in U', then we may first isotope A into $P \times (s_k - \epsilon, s_k]$. Then we isotope A so that A is transverse to each P_x except for a circle tangency. Since ∂F_U is essential in P_{s_k}, after the isotopy, Q_b still satisfies the conditions in Lemma 2.1 except at the level P_a as above. Now we push A out of U'. After the isotopy, we still have $d(\mathcal{U}, \Gamma_k) \neq 0$. If k is no longer the smallest number so that $d(\mathcal{U}, \Gamma_k) \neq 0$ after the isotopy, then we can find a new k and proceed as above. Eventually F_U does not contain any ∂–parallel annulus after some isotopies. We can view these isotopies as trivial isotopies, so by our assumptions above, $Q_b \cap \Sigma_U \neq \varnothing$ after the isotopies.

We first show that $d(\sigma_k, \mathcal{U}) \le 2$. As in the proof of Lemma 2.2, $d(\sigma_k, \mathcal{U}) \le 1$ if $k = 0$. So we may assume $k > 0$. By the definition of k, $d(\mathcal{U}, \Gamma_{k-1}) = 0$. Thus there is a component w of Γ_{k-1} representing a vertex in $\mathcal{U}$. By Claim A above and the Claim 1 in the proof of Lemma 2.2, $d(\sigma_k, w) \le 2$ and hence $d(\sigma_k, \mathcal{U}) \le 2$.

Since F_U is an essential subsurface of Q_b, $\chi(F_U) \le 0$. Since $d(\sigma_k, \mathcal{U}) \le 2$ and $\chi(F_U) \le 0$, to prove the claim, we only need to consider the case that $\chi(F_U) = 0$. Suppose $\chi(F_U) = 0$. Since $d(\mathcal{U}, \Gamma_k) \neq 0$, F_U consists of incompressible annuli in U'. Let A be the component of F_U that contains σ_k. If A is also ∂–incompressible, then A can be isotoped away from any compressing disk of U' and hence $d(\sigma_k, \mathcal{U}) \le 1 = 1 - \chi(F_U)$. If A is ∂–compressible, then since F_U contains no ∂–parallel annulus, a ∂–compression on A yields a compressing disk of U' disjoint from A. Thus, $d(\sigma_k, \mathcal{U}) \le 1 = 1 - \chi(F_U)$ in any case. □

Similar to Claim B, for any component σ_l of Γ_l, $d(\mathcal{V}, \sigma_l) \le 1 - \chi(F_V)$. Although $P_a \cap Q_b$ contains 2 saddle tangencies, by Claim A and our assumptions on Q_b, Claim 3 in the proof of Lemma 2.2 also holds in this case, ie there is a component σ_k of Γ_k and a component σ_l of Γ_l such that $d(\sigma_k, \sigma_l) \le -\chi(Q')$.

Since Q', F_U and F_V are essential subsurfaces of Q_b, $d(\mathcal{U}, \mathcal{V}) \le d(\mathcal{U}, \sigma_k) + d(\sigma_k, \sigma_l) + d(\sigma_l, \mathcal{V}) \le 1 - \chi(F_U) - \chi(Q') + 1 - \chi(F_V) = 2 - \chi(Q) = 2g(Q)$. Thus Theorem 1.1 is proved. □

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