Pseudo-Anosov homeomorphisms and the lower central series of a surface group

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Let Γ_k be the lower central series of a surface group Γ of a compact surface S with one boundary component. A simple question to ponder is whether a mapping class of S can be determined to be pseudo-Anosov given only the data of its action on Γ/Γ_k for some k. In this paper, to each mapping class f which acts trivially on Γ/Γ_{k+1}, we associate an invariant $\Psi_k(f) \in \mathrm{End}(H_1(S,\mathbb{Z}))$ which is constructed from its action on Γ/Γ_{k+2}. We show that if the characteristic polynomial of $\Psi_k(f)$ is irreducible over $\mathbb{Z}$, then f must be pseudo-Anosov. Some explicit mapping classes are then shown to be pseudo-Anosov.

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1 Introduction

Denote by $\mathrm{Mod}(S)$ *the mapping class group* of a compact, oriented surface $S = S_{g,1}$ of genus $g \geq 2$ with one boundary component; ie, $\mathrm{Mod}(S)$ is the group of homeomorphisms of S fixing ∂S pointwise up to isotopies fixing ∂S pointwise. A basic question to contemplate is: what topological or dynamical data of a mapping class can be extracted from various kinds of algebraic data? The most complex kind of mapping class is a *pseudo-Anosov* mapping class, ie, a mapping class that has a representative homeomorphism which leaves invariant a pair of transverse measured foliations (see Farb and Margalit [7] for more information on pseudo-Anosov mapping classes). Thus, it is a natural question to ask if a given mapping class is pseudo-Anosov.

One kind of algebraic data is the action of a mapping class on $\Gamma := \pi_1(S,*)$ and its various quotients. Specifically, consider the sequence of k-step nilpotent quotients $N_k := \Gamma/\Gamma_{k+1}$ where $\{\Gamma_k\}$ is the lower central series of Γ defined inductively by:

$$\Gamma_1 = \Gamma \qquad \Gamma_k = [\Gamma, \Gamma_{k-1}] \text{ for } k > 1$$

Since elements of $\mathrm{Mod}(S)$ fix ∂S pointwise and we choose the basepoint $* \in \partial S$, we obtain a representation $\mathrm{Mod}(S) \to \mathrm{Aut}(\Gamma)$, and furthermore since each Γ_k is characteristic, we obtain a representation for each k:

$$\rho_k\colon \mathrm{Mod}(S) \to \mathrm{Aut}(\Gamma/\Gamma_{k+1})$$

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One natural question to ask is: given *only* the datum of $\rho_k(f)$ for $f \in \mathrm{Mod}(S)$, can we determine if the mapping class is pseudo-Anosov or not? If the mapping class is determined to be pseudo-Anosov, can we detect the dilatation? This paper is one step in answering the first question. (This paper does not address the second question. In Farb, Leininger and Margalit [6], it is shown that $\lim_{k\to\infty} \inf\{\text{dilatation of } f \mid f \in \ker(\rho_k)\} = \infty$.)

For $k \geq 1$, we define *the kth Torelli group* to be $\mathcal{I}_k(S) := \ker(\rho_k)$ (and so with our indexing, which is different from some other authors, the classical Torelli group is $\mathcal{I}_1(S)$). To each $f \in \mathcal{I}_k = \mathcal{I}_k(S)$, we will associate an invariant $\Psi_k(f) \in \mathrm{End}(H_1(S,\mathbb{Z}))$ which is constructed from $\rho_{k+1}(f)$ (see below or Section 5). We will prove the following.

Theorem 1.1 (Criterion for pseudo-Anosovs) *Let $f \in \mathcal{I}_k$. If the characteristic polynomial of $\Psi_k(f)$ is irreducible in $\mathbb{Z}[x]$, then f is pseudo-Anosov.*

Theorem 1.1 will follow immediately from the following theorem which we prove in Section 5. For the remainder of this paper, we let $H := H_1(S,\mathbb{Z})$.

Theorem 1.2 *Let $f \in \mathcal{I}_k$. If the characteristic polynomial $\chi(\Psi_k(f))$ of $\Psi_k(f) \in \mathrm{End}(H)$ has no (nontrivial) even degree or degree 1 factors over $\mathbb{Z}$, then f is pseudo-Anosov.*

Since Ψ_k uses only the data of $\rho_{k+1}(f)$ and $\ker(\rho_{k+1}) = \mathcal{I}_{k+1}$, we obtain the following corollary:

Corollary 1.3 *If $f \in \mathcal{I}_k$ satisfies the hypothesis of Theorem 1.2, then the whole coset $f\mathcal{I}_{k+1}$ is pseudo-Anosov.*

Note that the data of ρ_1 is not used in Theorem 1.2. Since

$$\Gamma/\Gamma_2 = H$$

the homomorphism ρ_1 is the standard representation into $\mathrm{Aut}(H)$ with image isomorphic to the integral symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$. It is not too difficult to find a criterion on $\rho_1(f)$ for f to be pseudo-Anosov, and in fact, Casson and Bleiler give such a criterion in [5, Lemma 5.1]. Casson and Bleiler show that if the characteristic polynomial, $\chi(\rho_1(f))$, is irreducible over $\mathbb{Z}$, has no roots of unity as eigenvalues, and is not equal to $g(t^n)$ for any $n > 1$ and $g \in \mathbb{Z}[x]$, then f is pseudo-Anosov.

The Casson–Bleiler criterion is well-known and has been around for many years. It is unfortunately unable to detect pseudo-Anosovs in any of the $\mathcal{I}_k$ simply because

$\mathcal{I}_k \subseteq \ker(\rho_1)$. (This is not to imply that the Casson–Bleiler criterion can detect all pseudo-Anosovs which act non-trivially on H; it cannot.) In this sense, this paper is an extension of the Casson–Bleiler criterion (for the case of a surface with one boundary component).

Remark 1.4 It is well-known that $\mathcal{I}_1$ has pseudo-Anosov elements thanks to criteria of Thurston [18], Penner [16], and others [2]. However, their methods of finding pseudo-Anosovs are all topological as opposed to algebraic in nature. Furthermore, their criteria require the specification of a particular mapping class and thus are not well-suited to dealing with the information of $\rho_k(f) \in \mathrm{Aut}(\Gamma_1/\Gamma_{k+1})$ which only specifies a coset of $\mathcal{I}_k$. Both Thurston's criterion and Penner's criterion require that a mapping class be described in terms of twists about two multi-curves. In [2], Bestvina–Handel describe an algorithm using train tracks that can determine whether any single mapping class is pseudo-Anosov or not. In fact, this algorithm has been implemented in a computer program by Peter Brinkmann [3].

Let us now outline the contents of the paper. In Section 2, we recall some basic properties of the series $\{\Gamma_k\}$. We then define for $f \in \mathcal{I}_k$ the invariant $\Psi_k(f) \in \mathrm{End}(H)$. ($\Psi_k(f)$ is in general non-trivial which might be rather suprising given that $\rho_1(f) \in \mathrm{Aut}(H)$ is necessarily trivial.) To define Ψ_k, we need two ingredients, the Johnson homomorphism τ and contractions:

$$\Phi_{2k}\colon \Gamma_{2k+1}/\Gamma_{2k+2} \to H$$

Defining Φ_{2k} requires a bit of work and is described in Section 4. In Section 3, we recall the definition of the Johnson homomorphism τ which we describe here as follows:

$$\tau\colon \mathcal{I}_k(S) \to \mathrm{Hom}\left(\bigoplus_{m=1}^{\infty} \Gamma_m/\Gamma_{m+1}, \bigoplus_{m=k+1}^{\infty} \Gamma_m/\Gamma_{m+1}\right)$$

We denote the image of f under τ as τ_f. By the definition (given in Section 3), $\tau_f(\Gamma_m/\Gamma_{m+1}) \subseteq \Gamma_{m+k}/\Gamma_{m+k+1}$. We define Ψ_k as follows:

$$\Psi_k(f) := \begin{cases} \Phi_k \circ (\tau_f|_H) & k \text{ even} \\ \Phi_{2k} \circ (\tau_f^2|_H) & k \text{ odd} \end{cases} \in \mathrm{End}(H)$$

Note that the map Ψ_k is a homomorphism for k even but not necessarily for k odd.

In Section 5, we prove Theorem 1.2. The general idea of the proof of Theorem 1.2 is to use the Nielsen–Thurston classification which states that a mapping class is pseudo-Anosov if and only if it is neither reducible nor of finite order. Recall that f is *reducible* if f fixes the isotopy class of an essential 1–dimensional submanifold where *essential* means that each component is neither null-homotopic nor homotopic to a boundary component. Since $\mathcal{I}_1$ is torsion-free, the classification reduces to: f is pseudo-Anosov if and only if it is irreducible. We then show that reducibility of f implies that $\chi(\Psi_k(f))$ has a linear or even degree factor by using the fact that a certain subgroup of $\pi_1(S)$ is invariant under $f_* \in \mathrm{Aut}(\pi_1(S))$. The proof of Theorem 1.2 will be outlined in more detail in Section 5.

For any particular $f \in \mathcal{I}_k$, the invariant $\Psi_k(f)$ is explicitly computable, provided one can compute τ_f. In Section 6, we show some mapping classes satisfy the hypothesis of Theorem 1.2 by computing $\Psi_k(f)$ directly. Nevertheless, at present the author has not found whole families of pseudo-Anosovs ranging over either g or k which satisfy the hypothesis of Theorem 1.2. Additionally, in Section 6 we compare Theorem 1.2 to the Thurston/Penner criteria.

Remark 1.5 We choose to work with a surface with a boundary component as opposed to a closed surface to simplify things technically. The fundamental group of a surface with boundary is a free group. As we shall see in Section 2, this will further imply that the Lie algebra associated to the $\{\Gamma_k\}$ is a free Lie algebra. While the author suspects that one may obtain a criterion for closed surfaces from this criterion, he has not done so at present.

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2 Basic facts about the lower central series

For the reader's convenience, we recall basic facts about central filtrations of a group. Suppose

$$G = G_1 \supset G_2 \supset G_3 \dots$$

is a filtration of G by normal subgroups. We call G a *central filtration* if $[G_k, G_l] \subseteq G_{k+l}$. We recount the following folklore result.

Theorem 2.1 *Let $\{G_i\}$ be a central filtration of G by normal subgroups. Then, the following hold:*

(1) *The function* $G_k \times G_l \to G_{k+l}$ *given by* $(x, y) \mapsto xyx^{-1}y^{-1}$ *induces a well-defined map*

$$G_k/G_{k+1} \times G_l/G_{l+1} \to G_{k+l}/G_{k+l+1}$$

(2) *Using the pairing from (1) as a bracket which we denote by* $[\ ,\ ]$*, we obtain a graded* $\mathbb{Z}$*–Lie algebra:*

$$L := \bigoplus_k G_k/G_{k+1}$$

For an explanation and proof see Sections 3.1 and 4.5 of Bass and Lubotzky [1]. Also, we recall for the reader that the lower central series is a central filtration (see [1, Section 4.4]).

The fundamental group of a surface with boundary is a free group. The Lie algebra associated to a free group's lower central series is a free Lie algebra [11, Theorem 5.12].

Theorem 2.2 *Let* G *be a free group with generators* $a_1, \dots, a_n$ *and lower central series* $G_1 \supset G_2 \supset \dots$*. Then the (graded)* $\mathbb{Z}$*–Lie algebra*

$$L := (\bigoplus_k G_k/G_{k+1},\ \ [\ ,\ ])$$

is a free $\mathbb{Z}$*–Lie algebra.* L *has as its generating set* $\{a_1, \dots, a_n\}$ *viewed as a subset of* G_1/G_2*.*

The definition of *free Lie algebra* is exactly what one expects: given a $\mathbb{Z}$–Lie algebra L' and elements $x_1, \dots, x_n \in L'$, there exists a unique Lie algebra homomorphism $h\colon L \to L'$ such that $h(a_i) = x_i$. The free Lie algebra in general is fairly complicated. Even computing the rank of G_k/G_{k+1} for arbitrary k is nontrivial. Thankfully, free Lie algebras embed in simpler Lie algebras.

A *free associative* $\mathbb{Z}$*–algebra* A with generators $b_1, \dots, b_n$ is a noncommutative ring with the universal property that given a $\mathbb{Z}$–algebra A' and elements $x_1, \dots, x_n \in A'$ there is a unique homomorphism $h\colon A \to A'$ such that $h(b_i) = x_i$. More concretely, A is (canonically isomorphic to) the noncommutative polynomial ring in n variables over $\mathbb{Z}$. However, viewing A as a polynomial ring is not particularly convenient for the purposes of this paper. If we let $M := \mathbb{Z}^n$, then A is isomorphic to the tensor algebra $\bigoplus_{k=0}^{\infty} M^{\otimes k}$ where $M^{\otimes 0} := \mathbb{Z}$. The algebra A has a canonical Lie bracket: $[x, y] := x \otimes y - y \otimes x$. Thus, we have a canonical Lie homomorphism $\mathcal{L} \to A$ defined by $a_i \mapsto b_i$. From Reutenauer [17, Corollary 0.3 and Theorem 0.5], we obtain the following.

Theorem 2.3 *If L is a free $\mathbb{Z}$–Lie algebra with generators $a_1, \ldots, a_n$ and A is a free associative algebra over $\mathbb{Z}$ with generators $b_1, \ldots, b_n$, then the canonical Lie homomorphism induced by $a_i \mapsto b_i$ is injective.*

Moreover, it is not hard to check that the map $L \to A$ respects the grading.

Now, let us apply Theorems 2.2 and 2.3 to the group $\Gamma := \pi_1(S)$ with (free) generators $a_1, \ldots, a_{2g}$. Let $\mathcal{L}$ be the graded Lie algebra associated to $\{\Gamma_k\}$. Let $\mathcal{A}$ be the tensor algebra $\bigoplus_{k=0}^{\infty} H^{\otimes k}$ where $H^{\otimes 0} := \mathbb{Z}$. Since $H \cong \mathbb{Z}^{2g}$, the algebra $\mathcal{A}$ is a free associative algebra. To simplify notation, let us define $\mathcal{L}_k := \Gamma_k / \Gamma_{k+1}$. Recall that $\mathcal{A} \cong \bigoplus_{k=0}^{\infty} M^{\otimes k}$ where $M = \mathbb{Z}^{2g}$. We have defined the a_i as elements of $\pi_1(S)$, but we can also consider the equivalence class of a_i in $\Gamma_1 / \Gamma_2 \subset \mathcal{L}$ or in $H = H^{\otimes 1} \subset \mathcal{A}$. Thus, we obtain a natural, injective map $\mathcal{L} \to \mathcal{A}$ defined by sending "a_i" to "a_i".

The mapping class group has a natural action on $\mathcal{L}$ by considering

$$\mathcal{L} = \bigoplus_{k=1}^{\infty} \Gamma_k / \Gamma_{k+1}$$

as a direct sum of representations $\mathrm{Mod}(S) \to \mathrm{Aut}(\Gamma_k / \Gamma_{k+1})$. We obtain an action on

$$\mathcal{A} = \bigoplus_{k=0}^{\infty} H^{\otimes k}$$

from the action on H. It is not hard to check that the map $\mathcal{L} \to \mathcal{A}$ respects this action. Since the $\mathrm{Mod}(S)$–action on $\mathcal{A}$ is induced by the action on H, it factors through to an $\mathrm{Sp}(2g, \mathbb{Z})$–action and so the $\mathrm{Mod}(S)$–action on $\mathcal{L}$ factors through $\mathrm{Sp}(2g, \mathbb{Z})$ also (This can also be proven directly.).

3 The Johnson Homomorphisms

All of the results in this section are the work of Johnson, Morita, Hain and others. Recall that

$$\mathcal{I}_k := \ker(\mathrm{Mod}(S) \to \mathrm{Aut}(\Gamma_1 / \Gamma_{k+1}))$$

and $H = H_1(S)$. A preliminary version of the Johnson homomorphism is a map:

$$\tau\colon \mathcal{I}_k \to \mathrm{Hom}(H, \Gamma_{k+1} / \Gamma_{k+2})$$

for each k. Note that the image of f under τ will be denoted τ_f as is standard. We define the preliminary version as follows. Let $f \in \mathcal{I}_k$. Since f_* acts trivially on

Γ_1/Γ_{k+1}, we obtain a well-defined map of sets:

$$\begin{aligned} t_f\colon \Gamma_1/\Gamma_{k+2} &\to \Gamma_{k+1}/\Gamma_{k+2} \\ x &\mapsto f_*(x)x^{-1} \end{aligned}$$

The following result is one part of [14, Proposition 2.3].

Proposition 3.1 (Johnson, Morita) *The set map $t_f\colon \Gamma_1/\Gamma_{k+2} \to \Gamma_{k+1}/\Gamma_{k+2}$ induces a well-defined homomorphism $H \to \Gamma_{k+1}/\Gamma_{k+2}$ which is τ_f. Moreover, τ is a homomorphism.*

Proof By the very definition of the lower central series, $\Gamma_{k+1}/\Gamma_{k+2}$ is in the center of Γ_1/Γ_{k+2}. Thus,

$$f_*(xy)(xy)^{-1} = f_*(xy)y^{-1}x^{-1} = f_*(x)(f_*(y)y^{-1})x^{-1} = f_*(x)x^{-1}(f_*(y)y^{-1})$$

and so t_f is in fact a homomorphism. As $\Gamma_{k+1}/\Gamma_{k+2}$ is abelian, this homomorphism factors through the abelianization of Γ_1/Γ_{k+1} which is $\Gamma_1/[\Gamma_1,\Gamma_1] = \Gamma_1/\Gamma_2 = H$. Hence, we obtain a homomorphism $H \to \Gamma_{k+1}/\Gamma_{k+2}$. Now, suppose we are given $f, g \in \mathcal{I}_k$. Then, we have

$$f_*(g_*(x))x^{-1} = f_*(g_*(x)x^{-1})f_*(x)x^{-1}$$
$$= (f_*(t_g(x))t_g(x)^{-1})t_g(x)f_*(x)x^{-1} = t_f(t_g(x))t_g(x)t_f(x)$$

Since $t_g(x) \in \Gamma_{k+1}/\Gamma_{k+2} \subseteq \ker t_f$, we find that $f_*(g_*(x))x^{-1} = t_g(x)t_f(x)$. □

Remark 3.2 In the above proof, we see that $\ker(t_f) \supset \Gamma_2/\Gamma_{k+2}$, and so for $x \in \Gamma_2/\Gamma_{k+2}$ we have

$$1 = t_f(x) = f_*(x)x^{-1} \Rightarrow f(x) = x$$

Thus f acts trivially on Γ_2/Γ_{k+2} and in particular on $\Gamma_{k+1}/\Gamma_{k+2}$. Looking at the short exact sequence

$$(1) \qquad 1 \to \Gamma_{k+1}/\Gamma_{k+2} \to \Gamma_1/\Gamma_{k+2} \to \Gamma_1/\Gamma_{k+1} \to 1$$

one might think that f must act trivially on Γ_1/Γ_{k+2} itself, but this is not the case. Elements in $(\Gamma_1/\Gamma_{k+2}) \setminus (\Gamma_2/\Gamma_{k+2})$ may be changed by elements in $\Gamma_{k+1}/\Gamma_{k+2}$ and this is precisely what τ_f measures.

In view of the remark, we see that τ_f retains the information of $f_* \in \mathrm{Aut}(\Gamma_1/\Gamma_{k+2})$. Furthermore, τ_f determines f_* as an element of $\mathrm{Aut}(\Gamma_1/\Gamma_{k+2})$ (assuming $f \in \mathcal{I}_k$). We simply note that $f_*(x) = \tau_f(\overline{x})x$ where $\overline{x}$ is the projection of x to H. Moreover, the following sequence is exact [14, Proposition 2.3]:

$$(2) \qquad 1 \to \mathrm{Hom}(H, \Gamma_{k+1}/\Gamma_{k+2}) \to \mathrm{Aut}(\Gamma_1/\Gamma_{k+2}) \to \mathrm{Aut}(\Gamma_1/\Gamma_{k+1})$$

Given $f \in \mathcal{I}_k$, one can similarly define a function:

$$\begin{aligned} \Gamma_m/\Gamma_{m+k+1} &\to \Gamma_{m+k}/\Gamma_{m+k+1} \\ x &\mapsto f_*(x)x^{-1} \end{aligned} \tag{3}$$

As before, this induces a well-defined homomorphism $\Gamma_m/\Gamma_{m+1} \to \Gamma_{m+k}/\Gamma_{m+k+1}$. (See [12, Lemma 3.2].)

Consider the free associative algebra $\mathcal{A}$ as defined in the previous section. Suppose one has chosen $2g$ elements $\{x_1, \dots, x_{2g}\} \subseteq \mathcal{A}$. From general theory about the free associative algebra, we know there is then a unique derivation $D\colon \mathcal{A} \to \mathcal{A}$ such that $D(a_i) = x_i$ where the a_i are generators of $\mathcal{A}$ (see [17, Lemma 0.7]). If $x_i \in \mathcal{L}$, then induction on the grading and the following computation show that $D(\mathcal{L}) \subseteq \mathcal{L}$ and that D is a derivation on $\mathcal{L}$:

$$D[y,z] = D(yz - zy) = (Dy)z + yDz - (Dz)y - zDy = [Dy,z] + [y,Dz]$$

Thus, given $f \in \mathcal{I}_k$, there is a unique derivation D_f of $\mathcal{A}$ which extends τ_f. It turns out that extending τ_f to all of $\mathcal{L}$ yields the same result regardless of whether one restricts D_f or uses (3). The following proposition follows more or less from [12, Lemma 2.3 and Proposition 2.5].

Proposition 3.3 (Morita) *For all $m \geq 1$, the map defined by (3) induces a homomorphism $\Gamma_m/\Gamma_{m+1} \to \Gamma_{m+k}/\Gamma_{m+k+1}$ and is equal to the map $D_f|_{\mathcal{L}_m}$.*

By abuse of notation, we will denote the extention to $\mathcal{L}$ by τ_f. The map τ has other nice algebraic properties. They are collected in the following theorem.

Theorem 3.4 (Morita) *Let τ be as defined above, a collection of homomorphisms $\mathcal{I}_k \to \mathrm{Der}(\mathcal{L})$, one for each k. Then, the following hold:*

(a) *The map $\tau\colon \mathcal{I}_k \to \mathrm{Der}(\mathcal{L})$ is a homomorphism with kernel $\mathcal{I}_{k+1}$. Hence, it induces a well-defined homomorphism $\mathcal{I}_k/\mathcal{I}_{k+1} \to \mathrm{Der}(\mathcal{L})$.*

(b) *The abelian group*

$$\bigoplus_{k=1}^{\infty} \mathcal{I}_k/\mathcal{I}_{k+1}$$

has a Lie algebra structure induced by:

$$\begin{aligned} \mathcal{I}_m \times \mathcal{I}_n &\to \mathcal{I}_{m+n} \\ (f,g) &\mapsto fgf^{-1}g^{-1} =: [f,g] \end{aligned}$$

(c) *The map* τ *induces a Lie algebra homomorphism:*

$$\bigoplus_{k=1}^{\infty} \mathcal{I}_k/\mathcal{I}_{k+1} \to \mathrm{Der}(\mathcal{L})$$

Furthermore, τ *respects the conjugation action of* $\mathrm{Mod}(S)$ *on* $\mathcal{I}_k$ *and* $\mathrm{Der}(\mathcal{L})$*.*

Sketch of proof This proof sketch will consist mainly of citations. For (a), recall that by Proposition 3.1, $\tau_{f\circ g}|_H = \tau_f|_H + \tau_g|_H$. Since the derivations $\tau_{f\circ g}$ and $\tau_f + \tau_g$ agree on generators, they must agree on all of $\mathcal{L}$. One deduces the kernel is $\mathcal{I}_{k+1}(S)$ from the exact sequence in (2). Part (b) is [13, Proposition 4.1]. Also, [13, Proposition 4.7] shows (in slightly different notation) that $\tau_{[f,g]}|_H = (\tau_f\tau_g - \tau_g\tau_f)|_H$. Since the two derivations $\tau_{[f,g]}$ and $\tau_f\tau_g - \tau_g\tau_f$ agree on H and since H generates $\mathcal{L}$, we must have equality. To show that the $\mathrm{Mod}(S)$ action is respected, we use the definition of τ_f given by (3). Suppose $g \in \mathrm{Mod}(S)$. In Γ_m/Γ_{m+k+1}, we have:

$$\begin{aligned}\tau_{gfg^{-1}}(x) &= g(f(g^{-1}(x)))x^{-1} = g(f(g(x))g^{-1}(x^{-1}))\\ &= g(f(g^{-1}(x))(g^{-1}(x))^{-1}) = g(\tau_f(g^{-1}(x)))\end{aligned}$$ □

Remark 3.5 *A priori*, it may seem that, for $f \in \mathcal{I}_k$, we are using the entire action of f_* on $\pi_1(S)$ since we use the action on Γ_m/Γ_{m+k+1} for all m. This would conflict with the characterization given in the introduction that we only use the data of $f_* \in \mathrm{Aut}(\Gamma_1/\Gamma_{k+2})$. However, since τ_f is a derivation on $\mathcal{L}$ which is generated by H, it is completely determined by $\tau_f|_H$ which is itself determined by $f_* \in \mathrm{Aut}(\Gamma_1/\Gamma_{k+2})$.

4 The contractions Φ_k

Our goal in this section is to find a contraction $\mathcal{L}_{k+1} \to \mathcal{L}_1$ respecting the Sp–action and thus the $\mathrm{Mod}(S)$–action by the results of Section 2. We remark that we want to respect the action so that $\chi(\Psi_k(f))$ will depend only on the conjugacy class of f and because the argument in Section 5 implicitly uses a change of coordinates. The following theorem simplifies this problem. Below, $\mathrm{Hom}_{\mathrm{Sp}}$ will denote the set of homomorphisms which respect the Sp action, and, for X an Sp–representation, X_{Sp} will indicate the space of vectors fixed by the Sp action. While I suspect the following may be known, I was not able to find it in the literature.

Theorem 4.1 *If* $f \in \mathrm{Hom}_{\mathrm{Sp}}(\mathcal{L}_{k+1}, \mathcal{L}_1)$*, then* $\exists\, n \in \mathbf{Z}$ *such that* nf *is the restriction of an element* $g \in \mathrm{Hom}_{\mathrm{Sp}}(\mathcal{A}_{k+1}, \mathcal{A}_1)$*, where* $\mathcal{A}_m$ *is the summand* $H^{\otimes m} \subset \mathcal{A}$*.*

Proof The theorem will follow if we can find a bilinear pairing on each $\mathcal{A}_{k+1}$ which is nondegenerate on both $\mathcal{A}_{k+1}$ and $\mathcal{L}_{k+1}$. Let $\{a_1, b_1, \dots, a_g, b_g\}$ be a symplectic basis of $H_1(S)$. The a_i and b_i also serve as a free generating set of $\mathcal{L}$ as a Lie algebra and of $\mathcal{A}$ as an associative algebra. We can easily define a pairing $\langle\, ,\rangle$ which is nondegenerate on $\mathcal{A}_{k+1}$. If $x = x_1 \otimes x_2 \cdots \otimes x_{k+1}$ and $y = y_1 \otimes y_2 \cdots \otimes y_{k+1}$, then set

$$\langle x, y\rangle := \langle x_1, y_1\rangle\langle x_2, y_2\rangle \dots \langle x_{k+1}, y_{k+1}\rangle$$

where $\langle x_i, y_i\rangle$ is the algebraic intersection pairing on H.

Now, let $\theta \in \mathrm{Aut}(\mathcal{A})$ be the algebra homomorphism defined by $\theta(a_i) = b_i$ and $\theta(b_i) = -a_i$. In particular, if $w = x_1 \otimes x_2 \cdots \otimes x_n$ then $\theta(w) = \theta(x_1) \otimes \theta(x_2) \otimes \cdots \otimes \theta(x_n)$. Let Y_k be the canonical basis of $H^{\otimes k}$ induced by the basis of H (ie, tensoring the a's and b's in every possible order). For two elements $y, y' \in Y_k$, one easily sees that $\langle y, y'\rangle \neq 0$ if and only if $y' = \pm\theta(y)$. Then, for $P = \sum_y c_y y$, we have $\langle P, \theta(P)\rangle > 0$, since all "cross terms" vanish and we are left with $\sum_y c_y^2 \langle y, \theta(P)\rangle$.

We now wish to show that $\langle\, ,\rangle$ is nondegenerate on the embedded copy of $\mathcal{L}_{k+1}$, but this is almost immediate. We only need that $P \in \mathcal{L}_{k+1}$ implies $\theta(P) \in \mathcal{L}_{k+1}$. Indeed, since $\mathcal{L}$ is the Lie subalgebra of $\mathcal{A}$ generated by $\{a_1, b_1, \dots, a_g, b_g\}$ and since θ preserves the Lie bracket and (up to sign) permutes the generators $\{a_1, b_1, \dots, a_g, b_g\}$, we see that $\theta(\mathcal{L}) = \mathcal{L}$.

Suppose $f \in \mathrm{Hom}_{\mathrm{Sp}}(\mathcal{L}_{k+1}, \mathcal{L}_1) \cong (\mathcal{L}_{k+1}^* \otimes \mathcal{L}_1)_{\mathrm{Sp}}$. Since $\mathcal{L}_{k+1}$ and $\mathcal{L}_{k+1}^*$ are finitely generated free $\mathbf{Z}$–modules, the pairing $\langle\, ,\rangle$ gives an embedding $\mathcal{L}_{k+1} \hookrightarrow \mathcal{L}_{k+1}^*$ whose image has finite index. Thus, there is some $n \in \mathbf{Z}$ such that nf is in the image of $(\mathcal{L}_{k+1} \otimes \mathcal{L}_1)_{\mathrm{Sp}}$, but we have:

$$(\mathcal{L}_{k+1} \otimes \mathcal{L}_1)_{\mathrm{Sp}} \hookrightarrow (\mathcal{A}_{k+1} \otimes \mathcal{A}_1)_{\mathrm{Sp}} \hookrightarrow (\mathcal{A}_{k+1}^* \otimes \mathcal{A}_1)_{\mathrm{Sp}} \cong \mathrm{Hom}_{\mathrm{Sp}}(\mathcal{A}_{k+1}, \mathcal{A}_1)$$

Thus, nf is the restriction of some $g \in \mathrm{Hom}_{\mathrm{Sp}}(\mathcal{A}_{k+1}, \mathcal{A}_1)$. □

Theorem 4.1 and its proof reduce our problem to finding tensors in $(\mathcal{A}_{k+1} \otimes \mathcal{A}_1)_{\mathrm{Sp}} \cong (H^{\otimes k+2})_{\mathrm{Sp}}$. Thus, if $k = 2n$ is even, we obtain such a tensor by taking the symplectic pairing $\omega_0 = \sum_i (a_i \otimes b_i - b_i \otimes a_i)$ and taking high tensor powers, ie, $\omega_0^{\otimes(n+1)}$. The element $\omega_0^{\otimes(n+1)}$ represents the contraction

$$x_1 \otimes x_2 \otimes \cdots \otimes x_{k+1} \mapsto \left(\prod_{j=1}^{n} \langle x_{2j-1}, x_{2j}\rangle\right) x_{k+1}.$$

This contraction is what we denote by Φ_k.

There is an obvious action of the permutation group $\mathfrak{S}_{2m}$ on $H^{\otimes 2m}$. Since $\mathrm{Sp}(2g,\mathbb{Z})$ acts diagonally on $H^{\otimes 2m}$, it is easy to see that for any $\sigma \in \mathfrak{S}_{2m}$, we have $\eta \in (H^{\otimes 2m})_{\mathrm{Sp}}$ if and only if $\sigma(\eta) \in (H^{\otimes 2m})_{\mathrm{Sp}}$. Thus, all the vectors $\sigma(\omega_0^{\otimes 2m})$ are Sp–invariant as well. For every $\sigma \in \mathfrak{S}_{2m}$, there is a corresponding σ' so that $\sigma(\omega_0^{2m})$ corresponds to the contraction:

$$x_1 \otimes x_2 \otimes \cdots \otimes x_{m-1} \mapsto \left(\prod_{j=1}^{n} \langle x_{\sigma'(2j-1)}, x_{\sigma'(2j)}\rangle\right) x_{\sigma'(m-1)}$$

Furthermore, it is a classical result of Weyl (see for example, [15, Section 4.1]) that $\{\sigma(\omega_0^{\otimes 2m})\}_{\sigma\in\mathfrak{S}_{2m}}$ is a generating set for $((H\otimes\mathbb{Q})^{\otimes 2m})_{\mathrm{Sp}(2g,\mathbb{Q})}$. By a result of Borel (see [4, Theorem 2.7 and Remark 2.8]), we can conclude that all $\mathrm{Sp}(2g,\mathbb{Z})$–fixed vectors are $\mathrm{Sp}(2g,\mathbb{Q})$–fixed vectors and thus $(H^{\otimes 2m})_{\mathrm{Sp}(2g,\mathbb{Z})} = \langle \sigma(\omega_0^{\otimes 2m}) \mid \sigma \in \mathfrak{S}_{2m}\rangle$.

5 Proof of Theorem 1.2

Recall from above that for each $k \geq 1$ we defined a map:

$$\begin{aligned} \Psi_k\colon \mathcal{I}_k &\to \mathrm{End}(H) \\ f &\mapsto \begin{cases} \Phi_k \circ (\tau_f|_H) & k \text{ even} \\ \Phi_{2k} \circ (\tau_f^2|_H) & k \text{ odd} \end{cases} \end{aligned}$$

Idea of proof of Theorem 1.2 Before diving into the proof, let us sketch the idea of the proof for one of the main cases. The goal is to show that if $f \in \mathcal{I}_k$ fixes the isotopy class of some curve γ, then $\Psi_k(f)$ has a non-trivial invariant subspace in H. Let us look at the case where f fixes some separating curve γ in S. If we were trying to prove Casson–Bleiler in this case, we would note that f leaves two subspaces of H invariant which correspond to the two invariant subsurfaces, call them S_1 and S_2 (See Figure 1.) Now suppose $f \in \mathcal{I}_2$ and the elements c_i (resp d_i) generate the image C of $H_1(S_1,\mathbb{Z})$ (resp. the image D of $H_1(S_2,\mathbb{Z})$) in H. Since $\hat{\iota}(c_i,d_j) = 0$ for all i,j and Φ_k is defined via $\hat{\iota}$, one might have the hope that $\tau_f(C) \subseteq \langle [c_i,[c_j,c_k]] \mid \text{all } i,j,k\rangle$ and $\tau_f(D) \subseteq \langle [d_i,[d_j,d_k]] \mid \text{all } i,j,k\rangle$ so as to obtain that $\Psi_2(f)(C) \subseteq C$ and $\Psi_2(f)(D) \subseteq D$.

The hope that $\tau_f(C) \subseteq \langle\{[c_i,[c_j,c_k]]\}\rangle$ and $\tau_f(D) \subseteq \langle\{[d_i,[d_j,d_k]]\}\rangle$ is, in fact, false, but the degree to which this naive hope fails can be controlled. What one can say is that (if we take S_2 as in Figure 1) $\pi_1(S_2)$ is an f–invariant subgroup of $\pi_1(S)$. Since $\pi_1(S_2)$ can be generated by lifts of the d_i and of γ and since $\gamma \in \langle [\tilde{c}_i,\tilde{c}_j] \mid \text{all } i,j\rangle$ for some lifts $\tilde{c}_i \in \pi_1(S)$ of the c_i, it turns out that $\tau_f(d_i) \in \langle\, [d_i,[d_j,d_k]], [d_i,[c_j,c_k]] \mid \text{all } i,j,k \,\rangle$. It is then not to hard to see that $\Phi_2(\tau_f(d_i)) \subseteq$

$\langle\{d_i\}\rangle = D$, and so $\Psi_2(f)$ has an invariant subspace D. The full proof of the separating case is a more general (and complete) version of this last argument.

We remark that the following proof of the main theorem remains valid if we replace Φ_k with any of the contractions induced by a $\sigma(\omega_0^{k+2})$ described in Section 4. In the following, all factorization and irreducibility is with respect to $\mathbb{Z}[x]$.

Proof of Theorem 1.2 Now let us prove the theorem. Let $f \in \mathcal{I}_k$. Recall that the Nielsen–Thurston classification and torsion-freeness of $\mathcal{I}_1 \supseteq \mathcal{I}_k$ imply that f is pseudo-Anosov if and only if f is irreducible. It is well-known that $\mathcal{I}_1$ is *pure*, meaning that if an isotopy class of 1–submanifold is fixed, then each component of the 1–submanifold is fixed (see Ivanov [9, Theorem 1.2]). Thus, the proof of Theorem 1.2 reduces to proving the following two claims.

Claim 1 Suppose f fixes an essential separating curve. Then, the characteristic polynomial of $\Psi_k(f)$ factors into two (nontrivial) even degree polynomials in $\mathbb{Z}[x]$.

Claim 2 Suppose f fixes a nonseparating curve. Then, $\Psi_k(f)$ has an eigenvector over $\mathbb{Z}$.

Before we begin the proofs of Claims 1 and 2, we state a theorem that will be used for both. (This is [17, Theorem 2.5])

Theorem 5.1 (Shirshov, Witt) *If $\mathcal{L}'$ is a subalgebra of a free Lie algebra $\mathcal{L}$ over a field, then $\mathcal{L}'$ is a free Lie algebra .*

Proof of Claim 1 Let γ be the (oriented) separating curve such that $f(\gamma) = \gamma$. Cutting along γ separates S into a $\Sigma_{g_1,1} =: S_1$ and a $\Sigma_{g_2,2} =: S_2$ where $g_1 + g_2 = g$. Let C (resp. D) be the image of $H_1(S_1, \mathbb{Z})$ (resp. $H_1(S_2, \mathbb{Z})$) in H. Since $f(S_i) = S_i$ (up to isotopy), one might hope that either $\Psi_k(f)(C) \subseteq C$ or $\Psi_k(f)(D) \subseteq D$. We will show that this actually holds for D.

We begin by defining a submodule of $\mathcal{L}$:

$$M := \bigoplus_m (\Lambda \cap \Gamma_m / \Lambda \cap \Gamma_{m+1})$$

where $\Lambda := \pi_1(S_2)$. Note that $M \cap \mathcal{L}_1 = D$. *Step 1* is to show that $\tau_f(M) \subseteq M$. *Step 2* is to show that M is a free Lie subalgebra and give generators of M as a Lie algebra. *Step 3* is to show, using the generators, that for any $x \in M$ we have $\Phi_n(x) \in D$. Then, it is clear from the definition of Ψ_k that for $d \in D$, we have $\Psi_k(f)(d) \in D$. Since D is an even rank subspace, that will complete the proof.

First, we need to set up some notation. Let $p_1 \in \partial S_1$ (resp. $p_2 \in \partial S_2 \cap \partial S$) be the basepoint of S_1 (resp S_2 and S). Let α be a path from p_2 to p_1, and let $\widetilde{\gamma} = \alpha\gamma\alpha^{-1} \in \pi_1(S_2)$. Let ι (resp. $\widehat{\iota}$) denote geometric (resp. algebraic) intersection number of unbased homotopy classes of closed curves. Choose $\{c_i'\}_{i=1}^{2g_1} \in \pi_1(S_1, p_1)$ and $\{d_i\}_{i=1}^{2(g_2)} \in \pi_1(S_1, p_2)$ with the following properties (see Figure 1):

(a) The set $\{c_i'\}_{i=1}^{2g_1}$ (resp. $\{\widetilde{\gamma}\} \cup \{d_i\}_{i=1}^{2(g_2)}$) generates $\pi_1(S_1, p_1)$ (resp. $\pi_1(S_2, p_2)$).

(b) For all m, n, we have $\iota(c_m', d_n) = \widehat{\iota}(c_m', d_n) = 0$. Furthermore,

$$\iota(c_m', c_n') = \begin{cases} 1 & \text{if } m = n + g_1 \text{ or } m = n - g_1 \\ 0 & otherwise \end{cases}$$
$$\iota(d_m, d_n) = \begin{cases} 1 & \text{if } m = n + g_2 \text{ or } m = n - g_2 \\ 0 & otherwise \end{cases}$$

and for $1 \le i \le g_1$ (resp. $1 \le i \le g_2$), we have $\widehat{\iota}(c_i', c_{i+g_1}') = 1$ (resp. $\widehat{\iota}(d_i, d_{i+(g_2)}) = 1$).

(c) As an element of $\pi_1(S_1, p_1)$, we have $\gamma = \prod_{i=1}^{g_1} [c_i', c_{i+g_1}']$.

In particular, the union $\{c_i'\}_{i=1}^{2g_1} \cup \{d_i\}_{i=1}^{2g_2}$ gives a symplectic basis in H. Now, let $c_i := \alpha c_i' \alpha^{-1}$. We have $\widetilde{\gamma} = \prod_{i=1}^{g_1} [c_i, c_{i+g_1}]$ and $\pi_1(S, p_2) = \langle \{c_i\}, \{d_i\} \rangle$. Furthermore, denote the inclusion map of S_2 by $j\colon S_2 \hookrightarrow S$. In the following, we will frequently view $d_i \in \mathcal{L}_1$ and $\widetilde{\gamma} \in \mathcal{L}_2$.

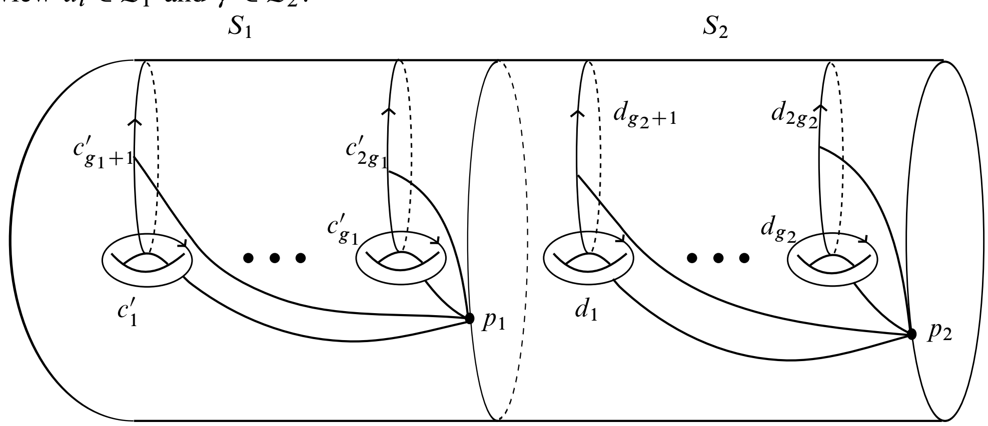

Figure 1

Step 1 First note that since S and S_2 share a base point, $\pi_1(S_2)$ gives a well-defined subgroup of $\pi_1(S) = \Gamma$ which is invariant under f_*. We remark that a similar statement is not true for S_1. Indeed, to embed $\pi_1(S_1)$ in $\pi_1(S)$ requires that we choose a path connecting base points (eg α); even after choosing a representative homeomorphism of f which fixes γ pointwise, this path is not necessarily preserved (up to homotopy rel endpoints).

Recall that one way of defining τ_f is to induce it from the map:

$$\begin{aligned} \Gamma_m &\to \Gamma_{m+k} \\ x &\mapsto f_*(x)x^{-1} \end{aligned}$$

Since $f_*(\Lambda) = \Lambda$, it is easy to see that

$$M = \bigoplus_m (\Lambda \cap \Gamma_m / \Lambda \cap \Gamma_{m+1})$$

is a τ_f–invariant submodule of $\mathcal{L}$.

Step 2 We wish to show M is a Lie subalgebra and find its generators. We will do this by showing that M is the Lie algebra homomorphic image of a Lie algebra N whose generators are easily found.

We first define a filtration of Λ which is a slight alteration of the lower central series. We let:

$$\begin{aligned} \Lambda_1 &:= \pi_1(S_2) \\ \Lambda_2 &:= \langle [\Lambda_1, \Lambda_1], \tilde{\gamma} \rangle \\ \Lambda_m &:= \langle [\Lambda_{m-n}, \Lambda_n] \rangle_{n=1}^{\lfloor \frac{m}{2} \rfloor} \quad \text{for } m \geq 3 \end{aligned}$$

By Theorem 2.1,

$$N := \bigoplus_n \Lambda_n / \Lambda_{n+1}$$

is a graded $\mathbb{Z}$–Lie algebra under the commutation bracket. Since $j_*(\Lambda_n) \subseteq \Gamma_n$, there is an induced Lie algebra homomorphism $N \to \mathcal{L}$. It is easy to check that, as a Lie algebra, N is generated by $\{d_i\}_{i=1}^{2g_2} \cup \{\tilde{\gamma}\}$ and so its image $M' := j_*(N)$ in $\mathcal{L}$ is also generated by $\{d_i\}_{i=1}^{2g_2} \cup \{\tilde{\gamma}\}$ (viewed in $\mathcal{L}$).

Proposition 5.2 *N maps isomorphically onto M'*

Proof of Proposition 5.2 We wish to use Theorem 5.1, but $\mathcal{L}$ is not an algebra over a field. As $\mathbb{Q}$ is a flat $\mathbb{Z}$–module, we have $M' \otimes \mathbb{Q} \hookrightarrow \mathcal{L} \otimes \mathbb{Q}$, and so $M'_{\mathbb{Q}} := M' \otimes \mathbb{Q}$ is a free Lie algebra generated by $\{d_i\}_{i=1}^{2(g_2)} \cup \{\tilde{\gamma}\}$, but it is not a priori clear that these generators are *free*. In the proof of Theorem 5.1 in [17], a recipe is given for finding free generators of a subalgebra, which we describe now.

For any subset $X \subseteq \mathcal{L} \otimes \mathbb{Q}$, let $\langle X \rangle$ denote the Lie subalgebra of $\mathcal{L} \otimes \mathbb{Q}$ generated by X. Let

$$E_n = M'_{\mathbb{Q}} \cap \left(\bigoplus_{i=1}^{n} \mathcal{L}_i \otimes \mathbb{Q} \right)$$

and let $E'_n = E_n \cap \langle E_{n-1} \rangle$. If we let $X_n :=$ a set of generators (as a $\mathbb{Q}$ vector space) for E_n mod E'_n, then $X = \bigcup_n X_n$ is a free generating set of $M'_{\mathbb{Q}}$.

We now show the afore-mentioned generators of $M'_{\mathbb{Q}}$ to be free. Clearly, we can set $X_1 := \{d_i\}_{i=1}^{2g_2}$. The only question is whether $\tilde{\gamma}$ is in the Lie algebra generated by X_1. Recall that $\tilde{\gamma} = \prod_{i=1}^{g_1} [c_i, c_{i+g_1}]$ and so in $\mathcal{L}_2$, we have $\tilde{\gamma} = \sum_{i=1}^{g_1} [c_i, c_{i+g_1}]$. As elements of H, the c_i and d_i freely generate $\mathcal{L} \otimes \mathbb{Q}$, so $\tilde{\gamma} \notin \langle X_1 \rangle$. Thus, we can set $X_2 = \{\tilde{\gamma}\}$, and so $\{d_i\}_{i=1}^{2g_2} \cup \{\tilde{\gamma}\}$ freely generates $M'_{\mathbb{Q}}$. But then clearly it *freely* generates M'.

Now, we can define an inverse Lie homomorphism $M' \to N$ by sending generators to generators, and so $N \to \mathcal{L}$ is injective. This proves Proposition 5.2. □

By the proposition, we have $\Lambda_n \setminus \Lambda_{n+1} \hookrightarrow \Gamma_n \setminus \Gamma_{n+1}$, but this implies that in fact $\Lambda_n = \Lambda \cap \Gamma_n$. Thus, $M = M'$.

Step 3 Recall that $C =$ image of $H_1(S_1)$ and $D =$ image of $H_1(S_2)$ in H; ie, $C = \langle \{c_i\}_{i=1}^{2g_1} \rangle$ and $D = \langle \{d_i\}_{i=1}^{2(g_2)} \rangle$. Suppose $x \in D$. Then, by Steps 1 and 2,

$$y := \begin{cases} \tau_f(x) & k \text{ even} \\ \tau_f^2(x) & k \text{ odd} \end{cases}$$

is an element of M. We can write $\tilde{\gamma}$ in $\mathcal{A}$ as $\sum_{i=1}^{2g_1} (c_i \otimes c_{i+g_1} - c_{i+g_1} \otimes c_i)$. Thus, M is contained in the subring generated by:

$$\left\{ \sum_{i=1}^{2g_1} (c_i \otimes c_{i+g_1} - c_{i+g_1} \otimes c_i) \right\} \cup \{d_i\}_{i=1}^{2(g_2)}$$

Consequently, we can write $y = \sum_m y_{m,1} \otimes \cdots \otimes y_{m,n}$ where an *even* number of the elements of $\{y_{m,1}, \ldots, y_{m,n}\}$ are in C and the rest are in D. Since $\hat{\imath}(c_i, d_j) = 0$ for all i, j, we have $\Phi_{n-1}(y_{m,1} \otimes \cdots \otimes y_{m,n}) \neq 0$ only if $y_{m,n} \in D$. Thus, $\Psi_k(f)(D) \subseteq D$, and we are done with Claim 1.

Proof of Claim 2 Let α be the nonseparating curve which is fixed by $f \in \mathcal{I}_k$. Let $\widehat{S}$ be the surface obtained by cutting along α, and $j\colon \widehat{S} \hookrightarrow S$ the canonical immersion.

Similar to the proof of Claim 1, we will show that $\Psi_k(f)(C) \subseteq C$ where $C :=$ image of $H_1(\widehat{S}, \mathbb{Z})$ in H. Analagous to the above, we let

$$M := \bigoplus_n \widehat{\Gamma} \cap \Gamma_n / \widehat{\Gamma} \cap \Gamma_{n+1}$$

where $\widehat{\Gamma} = \pi_1(\widehat{S})$. We go through the same 3 steps as in the proof of Claim 1:

- *Step 1* Show that $\tau_f(M) \subseteq M$.
- *Step 2* Show that M is a Lie subalgebra of $\mathcal{L}$ and find generators.
- *Step 3* Show that $\Phi_k(M \cap \mathcal{L}_{k+1}) \subseteq C$.

Let us first set up some notation. Let α_1 and α_2 be the boundary curves of $\widehat{S}$ such that $j_*(\alpha_1) = j_*(\alpha_2) = \alpha$. Choose based representatives a, a_1 and a_2 of α, α_1 and α_2 respectively as in Figure 2; in particular, $j_*(a_1) = a$. Also, let b be as depicted in Figure 2. Extend $\{a, b\}$ to a "standard" generating set $\{a, b\} \cup \{c_i\}_{i=1}^{2(g-1)}$; ie, the following hold:

(a) The set $\{a, b\} \cup \{c_i\}_{i=1}^{2(g-1)}$ gives a symplectic basis in homology.

(b) $\iota(a, b) = \widehat{\iota}(a, b) = 1$.

(c) All c_i can be homotoped to lie entirely inside the interior of $\widehat{S}$.

Letting a_1 and a_2 be as in Figure 2, one can easily check that $j_*(a_1 a_2^{-1}) = [a, b^{-1}]$.

Step 1 Choosing the same basepoint for $\widehat{S}$ and S, we have that $j_*\colon \pi_1(\widehat{S}) \to \pi_1(S)$ is injective and $\widehat{\Gamma} = \pi_1(\widehat{S})$ is invariant under f_*. Thus, we have

$$M := \bigoplus_n \widehat{\Gamma} \cap \Gamma_n / \widehat{\Gamma} \cap \Gamma_{n+1}$$

is a τ_f–invariant submodule of $\mathcal{L}$. It is also easy to see $M \cap \mathcal{L}_1 = C$.

Step 2 Just as in the proof of Claim 1, we choose a filtration of $\pi_1(\widehat{S})$ which is a slight alteration of the lower central series:

$$\widehat{\Gamma}_1 = \pi_1 \widehat{S}$$
$$\widehat{\Gamma}_2 = \langle [\widehat{\Gamma}_1, \widehat{\Gamma}_1], a_1 a_2^{-1} \rangle$$
$$\widehat{\Gamma}_n = \langle [\widehat{\Gamma}_{n-k}, \widehat{\Gamma}_k] \rangle_{k=1}^{\lfloor \frac{m}{2} \rfloor} \quad n \geq 3$$

By Theorem 2.1, we get a corresponding graded $\mathbb{Z}$–Lie algebra which we denote by $\widehat{M}$. Again, since $j_*(\widehat{\Gamma}_n) \subseteq \Gamma_n$, we get an induced Lie algebra homomorphism $\widehat{M} \to \mathcal{L}$. Note that $\widehat{M}$ is generated by $\{a_1\} \cup \{c_i\}_{i=1}^{2(g-1)} \in \widehat{M}_1$ and $a_1 a_2^{-1} \in \widehat{M}_2$. Since $a_1 a_2^{-1} \mapsto [a, b^{-1}]$, we have that $\{a, [a, b^{-1}]\} \cup \{c_i\}_{i=1}^{2(g-1)}$ generates $j_*(\widehat{M})$.

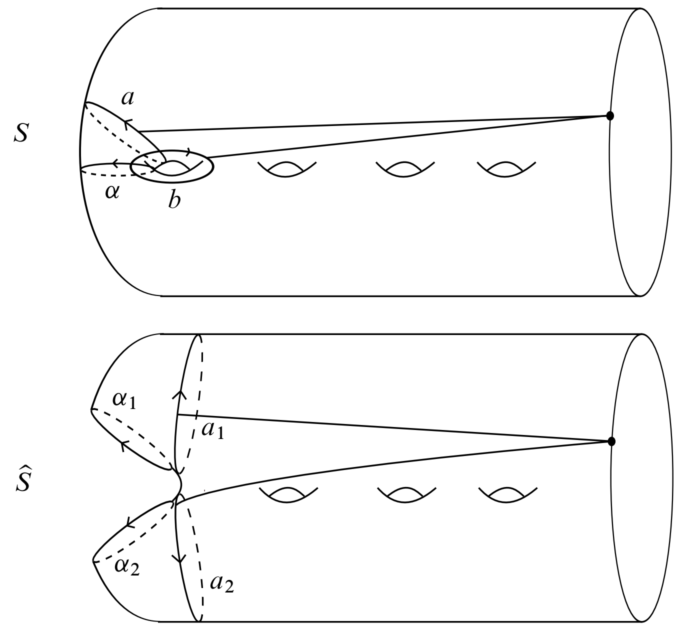

Figure 2

Proposition 5.3 *The Lie algebra* $\widehat{M}$ *maps isomorphically onto* $j_*(\widehat{M})$.

Proof of Proposition 5.3 Since the set $\{a,b\} \cup \{c_i\}_{i=1}^{2(g-1)}$ is a free generating set of $\mathcal{L}$, we have $[a,b^{-1}] \notin \langle a, \{c_i\}_{i=1}^{2(g-1)} \rangle$. Thus, by reasoning similar to that in the separating case, $\{a,[a,b^{-1}]\} \cup \{c_i\}_{i=1}^{2(g-1)}$ is a *free* generating set of $j_*(\widehat{M})$. We obtain an inverse Lie algebra map $j_*(\widehat{M}) \to \widehat{M}$ induced by

$$a \mapsto a_1, \qquad [a,b^{-1}] \mapsto a_1 a_2^{-1}, \qquad c_i \mapsto c_i$$ □

Since $\widehat{M}$ injects into $\mathcal{L}$, we have

$$\widehat{\Gamma}_m \setminus \widehat{\Gamma}_{m+1} \hookrightarrow \Gamma_m \setminus \Gamma_{m+1}$$

and so $\widehat{\Gamma}_m = \widehat{\Gamma} \cap \Gamma_m$. Thus, $j_*(\widehat{M}) = M$.

Step 3 Now, let $x \in C := \langle a, \{c_i\}_{i=1}^{2(g-1)} \rangle \subseteq H$. Then

$$y := \begin{cases} \tau_f(x) & k \text{ even} \\ \tau_f^2(x) & k \text{ odd} \end{cases}$$

is an element of M. As an element of $\mathcal{A}$, we may write $y = \sum_m y_{m,1} \otimes \cdots \otimes y_{m,n}$ where each $y_{m,r}$ is a multiple of one of a, b, c_i. Since $[a,b] = a \otimes b - b \otimes a$ as an element of $\mathcal{A}$ and $y \in M$, there are at least as many a terms as b terms in $y_{m,1}, \ldots, y_{m,n}$. Since b pairs nontrivially only with a in the set $\{a,b\} \cup \{c_i\}_{i=1}^{2(g-1)}$, we have $\Phi_{n-1}(y_{m,1} \otimes \cdots \otimes y_{m,n}) \neq 0$ only if $y_{m,n} \neq$ a multiple of b, in which case $\Phi_{n-1}((y_{m,1} \otimes \cdots \otimes y_{m,n}) \in C$. Thus, $\Psi_k(f)(C) \subseteq C$, and since C has rank $2g-1$, the characteristic polynomial of $\Psi_k(f)$ factors into a product of a degree 1 and degree $2g-1$ polynomial. □

6 Theorem 1.2 versus the Thurston–Penner criteria

In this section we will compare the criterion of Theorem 1.2 to the Thurston–Penner criteria. Since the Thurston–Penner criteria are topological and Theorem 1.2 is algebraic, one might expect that there is essentially no relation between the two. We will show this to be true in the following sense. There exist examples satisfying the Thurston or Penner criteria but not the hypothesis of Theorem 1.2 and examples satisfying both. As of the writing of this paper, it has not been proven that there are examples of pseudo-Anosovs which do not satisfy the Thurston–Penner criteria. However, we will give an example satisfying the hypothesis of Theorem 1.2 to which the Thurson–Penner criteria do not seem to apply directly.

Since we will be dealing with Dehn twists about separating curves, we first describe $\Psi_2(T_\gamma)$ where γ is one of the γ_i in Figure 3 and T_γ is the Dehn twist about γ. First let us set up a symplectic basis. Let $\{\alpha_i, \beta_i\}$ be the curves as depicted in Figure 3 with $a_i = [\alpha_i]$ and $b_i = [\beta_i]$ their homology classes. Our ordered basis of H throughout this section will be $\{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\}$.

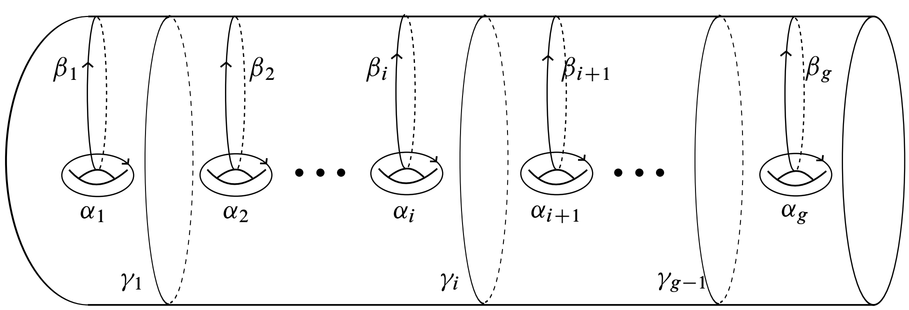

Figure 3

Lemma 6.1 *With* $\{a_i, b_i\}$ *and* $\{\gamma_i\}$ *as above, the element* $\Psi_2(T_{\gamma_i}) \in \mathrm{End}(H)$ *is the map defined by:*

$$a_j \;\mapsto\; \begin{cases} (2i+1)a_j & j \le i \\ 0 & j > i \end{cases}$$
$$b_j \;\mapsto\; \begin{cases} (2i+1)b_j & j \le i \\ 0 & j > i \end{cases}$$

Note that with the given indexing, i is the genus of γ_i.

Proof We can lift a_i, b_i, γ_i to $\tilde{a}_i, \tilde{b}_i, \tilde{\gamma}_i \in \pi_1(S)$ by connecting α_i, β_i and γ_i to the basepoint via paths. Furthermore, we can do it in such a way that $\tilde{\gamma}_i = \prod_{j=1}^{i} [\tilde{a}_j, \tilde{b}_j]$ in $\pi_1(S)$ and

$$T_{\gamma_i}(\tilde{a}_j) \;=\; \begin{cases} \tilde{\gamma}_i \tilde{a}_j \tilde{\gamma}_i^{-1} & j \le i \\ \tilde{a}_j & j > i \end{cases}$$
$$T_{\gamma_i}(\tilde{b}_j) \;=\; \begin{cases} \tilde{\gamma}_i \tilde{b}_j \tilde{\gamma}_i^{-1} & j \le i \\ \tilde{b}_j & j > i \end{cases}$$

Thus, for $j \le i$ and $f_i = T_{\gamma_i}$, we compute $f_i(\tilde{a}_j)\tilde{a}_j^{-1} = [\tilde{\gamma}_i, \tilde{a}_j]$ and

$$\tau_{f_i}(a_j) = \left[\sum_{k=1}^{i} [a_k, b_k], a_j\right] = \sum_{k=1}^{i} ((a_k \otimes b_k - b_k \otimes a_k) \otimes a_j - a_j \otimes (a_k \otimes b_k - b_k \otimes a_k))$$

For $j > i$, we easily see that $\tau_{f_i}(a_j) = 0$. Recall that $\Phi_2(c_1 \otimes c_2 \otimes c_3) = \hat{\imath}(c_1, c_2)c_3$. We then compute for $j \le i$ that $\Psi_2(f_i) = \Phi_2(\tau_{f_i}(a_j)) = (2i+1)a_j$. Clearly, $\Phi_2(\tau_{f_i}(a_j)) = 0$ for $j > i$. The computation for b_j is the same but with the the roles of a and b switched. □

Now let us consider T_γ where γ is an arbitrary separating curve not homotopic to the boundary. Recall that Ψ_k is $\mathrm{Mod}(S)$–equivariant (This follows from the $\mathrm{Mod}(S)$–equivariance of Φ_k and τ). The $\mathrm{Mod}(S)$–action on $\mathrm{End}(H)$ is as follows. If $\varphi \in \mathrm{Mod}(S)$ and $h \in \mathrm{End}(H)$, then

$$\varphi \cdot h = [\varphi] h [\varphi]^{-1}$$

where $[\varphi]$ denotes the projection of Φ to $\mathrm{Sp}(2g, \mathbb{Z})$. Thus, for $f \in \mathcal{I}_2$ and $\varphi \in \mathrm{Mod}(S)$, we find that $\Psi_k(\varphi f \varphi^{-1}) = [\varphi]\Psi_k(f)[\varphi]^{-1}$. Recall that if for a fixed g', two separating curves η_1 and η_2 both cut S into a $\Sigma_{g',1}$ and a $\Sigma_{g-g',2}$, then there is some $\varphi \in \mathrm{Mod}(S)$ such that $\varphi(\eta_1) = \eta_2$. Thus, $\Psi_2(T_\gamma)$ is of the form $\varphi \Psi_2(T_{\gamma_i})\varphi^{-1}$ for some i and some $\varphi \in \mathrm{Sp}(2g, \mathbb{Z})$. Similarly, if A is a multicurve of separating curves and T_A the multicurve twist, then

$$\Psi_2(T_A) = \varphi \Psi_2(\prod_{k=1}^{m} T_{\gamma_{i_k}})\varphi^{-1}$$

for some $\varphi \in \mathrm{Sp}(2g, \mathbb{Z})$ and some subset $\{\gamma_{i_k}\}$ of $\{\gamma_i\}$.

For the reader's convenience, we recall a few definitions and state a corollary to both the Thurston and Penner criteria. A *pants decomposition* is a maximal set of pairwise nonisotopic simple closed curves which are pairwise disjoint and not null-homotopic. For an $S_{g,b}$, a pants decomposition consists of $3g-3+2b$ curves and contains all the boundary curves. Recall that a simple closed curve γ is *essential* if it is neither homotopically trivial nor homotopic to a boundary component. We say that two curves η and ν *fill* a surface S if, for any essential simple closed curve γ, the curve γ either intersects η or ν nontrivially. We define the notion of filling for two multicurves similarly.

Corollary 6.2 (Thurston, Penner) *If two multicurves A and B fill a surface, then the product of multicurve twists $T_A T_B^{-1}$ is pseudo-Anosov.*

6.1 Negative results for Theorem 1.2

In this section we show that there is a pseudo-Anosov in $\mathcal{I}_2(S_{g,1})$ for each $g \geq 2$ which satisfies the Thurston–Penner criteria but not the hypothesis of Theorem 1.2. Let T_γ denote the twist about a simple closed curve γ.

Theorem 6.3 *For each $g \geq 2$, there exists two simple closed curves $\gamma_{g,1}$ and $\gamma_{g,2}$ filling $S = S_{g,1}$ such that $f_g := T_{\gamma_{g,1}} T_{\gamma_{g,2}}^{-1}$ does not satisfy the hypothesis of Theorem 1.2. However, by the Thurston–Penner criteria, we know f_g is pseudo-Anosov.*

Proof We break the proof into two cases. For $g = 2$, we will explicitly compute $\Psi_2(f_2)$. For $g \geq 3$, the main idea is to show that there is an f'_g such that f'_g is reducible and $\Psi_2(f'_g) = \Psi_2(f_g)$. Of course, then it is impossible for $\Psi_2(f_g)$ to satisfy the hypothesis of Theorem 1.2 since $\Psi_2(f'_g)$ does not.

We also need a consequence of [8, Lemma 2 of Expose 13] to construct the f_g. For the reader's convenience, we state the consequence.

Lemma 6.4 *Let S be a surface. Let γ be a simple closed curve on S and $P = \{\alpha_1, \ldots, \alpha_m\}$ a pants decomposition of S such that $\iota(\gamma, \alpha_i) \neq 0$ for all α_i that are not boundary components. Then, the curves γ and $T_P(\gamma)$ fill the surface.*

The case $g=2$ Let $\gamma_{2,1}$ and the η_i be as in Figure 4. Since the η_i are disjoint and $\{\eta_i\}$ is a 4 element set, $P=\{\eta_i\}$ is a pants decomposition. By Lemma 6.4, we know that $\gamma_{2,1}$ and $\gamma_{2,2}:=T_P(\gamma)$ fill S.

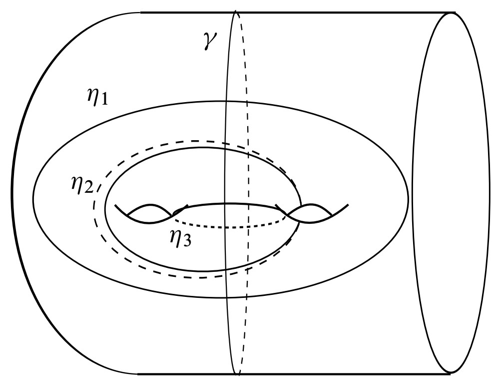

Figure 4

We now explicitly compute $\Psi_2(f_2)$ and see that its characteristic polynomial has degree 2 factors. Since Ψ_2 is a homomorphism and $\mathrm{Mod}(S)$–equivariant, we find that:

$$\begin{aligned}\Psi_2(T_{\gamma_{2,1}}T^{-1}_{T_P(\gamma_{2,1})}) &= \Psi_2(T_{\gamma_{2,1}})-[T_P]\circ\Psi_2(T_{\gamma_{2,1}})\circ[T_P]^{-1}\\ &= \Psi_2(T_{\gamma_{2,1}})-\\ &\quad [T_{\eta_1}][T_{\eta_2}][T_{\eta_3}][T_{\eta_4}]\Psi_2(T_{\gamma_{2,1}})[T_{\eta_4}]^{-1}[T_{\eta_3}]^{-1}[T_{\eta_2}]^{-1}[T_{\eta_1}]^{-1}\\ &= \Psi_2(T_{\gamma_{2,1}})-[T_{\eta_1}][T_{\eta_3}]\Psi_2(T_{\gamma_{2,1}})[T_{\eta_3}]^{-1}[T_{\eta_1}]^{-1}\end{aligned}$$

Note that since η_2 is separating, $[T_{\eta_2}]$ is trivial. For any simple closed curve β and $c\in H$, one can show that

$$[T_\beta](c)=c+\hat{\iota}([\beta],c)[\beta]$$

where $[\beta]$ is the homology class of β. We see that $[\eta_1]=a_1+a_2$ and $[\eta_3]=b_2-b_1$ and so one computes:

$$\Psi_2(T_{\gamma_{2,1}}T^{-1}_{T_P(\gamma_{2,1})})=3*\begin{pmatrix}-1&0&1&1\\0&-1&1&-1\\-1&-1&1&0\\-1&1&0&1\end{pmatrix}$$

The characteristic polynomial is computed to be $(9+x^2)^2$.

The case $g\geq 3$ First, we find a pair of filling curves using Lemma 6.4. We prove the case of $g=5$ with the generalization to the general case of $g\geq 3$ being clear. Let $\gamma_{5,1}$

be the curve γ_1 in Figure 3 and P the pants decomposition given by all the curves in Figure 6 and all the curves in Figure 5 except ν. One sees that $\gamma_{5,1}$ intersects every curve of P nontrivially. Thus, by Lemma 6.4, $\gamma_{5,1}$ and $\gamma_{5,2} := T_P(\gamma_{5,1})$ fill $S_{5,1}$.

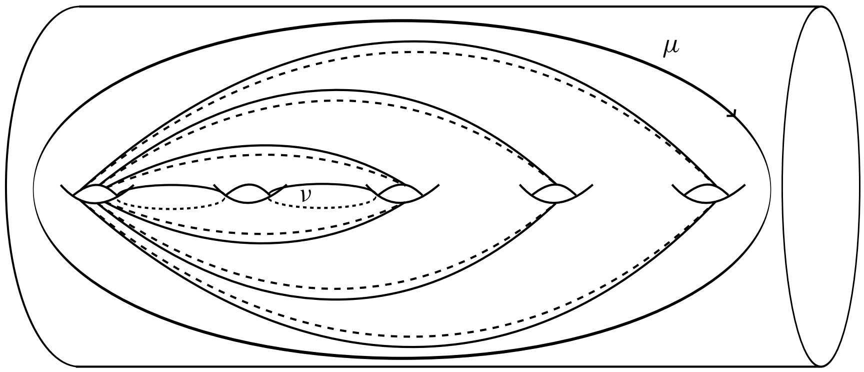

Figure 5

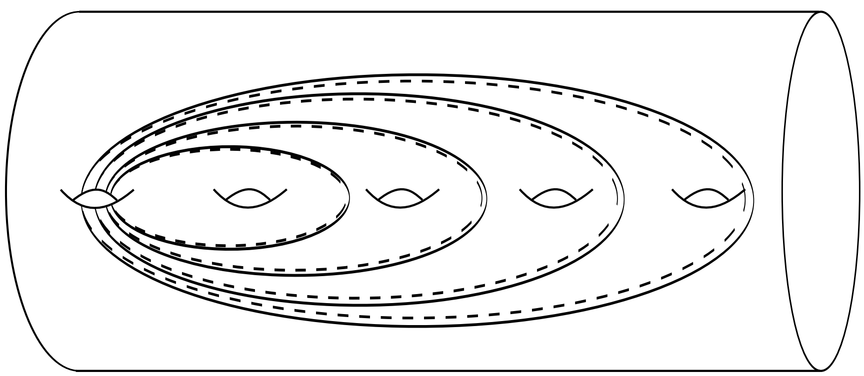

Figure 6

We next show that there is some $f_5' \in \mathcal{I}_2$ such that $\Psi_2(f_5') = \Psi_2(f_5)$ and f_5' is reducible. Let

$$P_{\text{nosep}} = \{\eta \in P \mid \eta \text{ is nonseparating}\}.$$

Since $[T_\eta] = Id$ for all η that are separating, we see that:

$$\begin{aligned} \Psi_2(f_5) &= \Psi_2(T_{\gamma_{5,1}} T^{-1}_{T_P(\gamma_{5,1})}) = \Psi_2(T_{\gamma_{5,1}}) - [T_P]\Psi_2(T_{\gamma_{5,1}})[T_P^{-1}] \\ &= \Psi_2(T_{\gamma_{5,1}}) - [T_{P_{\text{nosep}}}]\Psi_2(T_{\gamma_{5,1}})[T^{-1}_{P_{\text{nosep}}}] = \Psi_2(T_{\gamma_{5,1}} T^{-1}_{T_{P_{\text{nosep}}}(\gamma_{5,1})}) \end{aligned}$$

We let $f_5' = T_{\gamma_{5,1}} T^{-1}_{T_{P_{\text{nosep}}}(\gamma_{5,1})}$. Notice that the curve ν in Figure 5 intersects neither $\gamma_{5,1}$ nor $T_{P_{\text{nosep}}}(\gamma_{5,1})$, and so $f_5'(\nu) = \nu$. Thus, f_5' is reducible.

There is one more subtle point we must show to prove that in fact f_5 cannot be shown to be pseudo-Anosov using Theorem 1.2. We need to establish that $f_5 \in \mathcal{I}_2 \setminus \mathcal{I}_3$ since, if $f_5 \in \mathcal{I}_3$, we can then apply Ψ_3 and perhaps $\Psi_3(f_5)$ has irreducible characteristic polynomial. Since $\Psi_2(\mathcal{I}_3) = 0$, it is sufficient to prove $\Psi_2(f_5) \neq 0$. Suppose $\Psi_2(f_5) = 0$. Then we have that:

$$\Psi_2(T_{\gamma_{5,1}}) - [T_{P_{\text{nosep}}}]\Psi_2(T_{\gamma_{5,1}})[T^{-1}_{P_{\text{nosep}}}] = 0$$

In other words, the endomorphisms $[T_{P_{\text{nosep}}}]$ and $\Psi_2(T_{\gamma_{5,1}})$ commute. Letting $e_{i,j}$ be the elementary matrix with a 1 in the (i,j)th entry and 0's everywhere else, Lemma 6.1 implies $\Psi_2(T_{\gamma_{5,1}}) = e_{1,1} + e_{2,2}$ as a matrix in our chosen basis. Commuting with $e_{1,1} + e_{2,2}$ implies that the $(1,10)$th entry of $[T_{P_{\text{nosep}}}]$ is 0; ie, the a_1 coefficient of $[T_{P_{\text{nosep}}}](b_5)$ is 0. Since the only curve in Figure 5 that intersects β_5 nontrivially is μ, we have that

$$[T_{P_{\text{nosep}}}](b_5) = [T_\mu](b_5) = b_5 + \hat{\imath}(b_5, [\mu])[\mu] = b_5 - \sum_{i=1}^{5} a_i$$

which gives a contradiction. (For a proof of the second equality, see Farb and Margalit [7, Section 6.1.3].) □

6.2 Positive results for Theorem 1.2

In this section, we will exhibit two examples of mapping classes which satisfy the hypothesis of Theorem 1.2. Both examples were found through a computer search. We begin with an example satisfying both Theorem 1.2 and the Thurston–Penner criteria.

We first make some preliminary remarks. If A and B are multicurves and $T_A T_B^{-1}$ is pseudo-Anosov, then it is clear that $A \cup B$ fills S. Thus, if $T_A T_B^{-1}$ satisfies the hypothesis of Theorem 1.2, it immediately follows that the Thurston–Penner criteria imply that $T_A T_B^{-1}$ is pseudo-Anosov.

Now let us describe our example explicitly. Let $S = S_{5,1}$. We let $A = \{\gamma_1, \gamma_2, \gamma_3\}$ and $B' = \{\gamma_1, \gamma_2\}$ where the γ_i are the separating curves given in Figure 3. Let $h \in \mathrm{Mod}(S)$

be any mapping class such that its projection to $\mathrm{Sp}(2g,\mathbb{Z})$ is given by:

$$[h] = \begin{pmatrix} 2 & 0 & 1 & 1 & 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & -2 & 0 & 0 & -1 & 1 & -1 & 2 & -2 \\ 3 & 3 & 2 & -1 & 2 & 0 & 0 & 1 & 2 & -3 \\ 1 & -1 & 0 & 2 & 1 & 0 & 2 & 0 & 1 & 1 \\ 4 & 3 & 2 & -1 & 2 & 1 & 1 & 0 & 2 & -2 \\ 0 & -1 & 2 & 0 & 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 6 & 0 & 7 & 2 & 5 & 2 & 3 & 4 & 2 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $B = h(B')$. If we let $e_{i,j}$ be the elementary matrix with a 1 in the (i,j)th entry and 0's everywhere else, then using Lemma 6.1, we find

$$T_A = 15(e_{1,1} + e_{2,2}) + 12(e_{3,3} + e_{4,4}) + 7(e_{5,5} + e_{6,6})$$

and

$$T_{B'} = 8(e_{1,1} + e_{2,2}) + 5(e_{3,3} + e_{4,4}).$$

Putting this together, we compute (via Mathematica):

$$\begin{aligned} \Psi_2(T_A T_B^{-1}) &= \Psi(T_A) - [h]\Psi(T_{B'})[h]^{-1} \\ &= \begin{pmatrix} 42 & 0 & -6 & -33 & -26 & -33 & -25 & 11 & -5 & 26 \\ 0 & 42 & -44 & 14 & -8 & 30 & -116 & 18 & -16 & 24 \\ 14 & 33 & -28 & 0 & -14 & 24 & -89 & 14 & -19 & 38 \\ 44 & -6 & 0 & -28 & -28 & -36 & -22 & 8 & -2 & 20 \\ 30 & 33 & -36 & -24 & -22 & 0 & -89 & 22 & -19 & 46 \\ 8 & -26 & 28 & -14 & 0 & -22 & 68 & -10 & 8 & -8 \\ 18 & -11 & 8 & -14 & -10 & -22 & 13 & 0 & 3 & 2 \\ 116 & -25 & 22 & -89 & -68 & -89 & 0 & 13 & -10 & 68 \\ 24 & -26 & 20 & -38 & -8 & -46 & 68 & -2 & 8 & 0 \\ 16 & -5 & 2 & -19 & -8 & -19 & 10 & 3 & 0 & 8 \end{pmatrix} \end{aligned}$$

We compute (via Mathematica) the characteristic polynomial to be

$$(x^5 - 21x^4 + 107x^3 + 3837x^2 - 13500x + 151200)^2$$

and furthermore find that modulo 17 the polynomial

$$x^5 - 21x^4 + 107x^3 + 3837x^2 - 13500x + 151200$$

is irreducible, and hence irreducible over $\mathbb{Z}$. Thus, by Theorem 1.2, $T_A T_B^{-1}$ is pseudo-Anosov and we are done.

We now exhibit a mapping class $f \in \mathcal{I}_1(S_{4,1})$ for which there is no obvious way to apply the Thurston–Penner criteria. First, let us recall some facts about the Johnson homomorphism on $\mathcal{I}_1$. There is the following sequence of canonical embeddings and isomorphisms:

$$\Lambda^3 H \hookrightarrow \Lambda^2 H \otimes H \cong (\Gamma_2/\Gamma_3) \otimes H \cong \mathrm{Hom}(H, \Gamma_2/\Gamma_3)$$

[10, Theorem 1] implies:

$$\tau(\mathcal{I}_1/\mathcal{I}_2) = \mathrm{image}(\Lambda^3 H) \subseteq \mathrm{Hom}(H, \Gamma_2/\Gamma_3)$$

We define a *bounding pair* to be a pair of nonisotopic disjoint curves whose union separates the surface. The *bounding pair map* associated to an ordered bounding pair (η, γ) is the product of Dehn twists $T_\eta T_\gamma^{-1}$. Let $h = T_{\beta_i} T_{\beta_i'}^{-1}$ be the bounding pair map for β_i and β_i' as given in Figure 7.

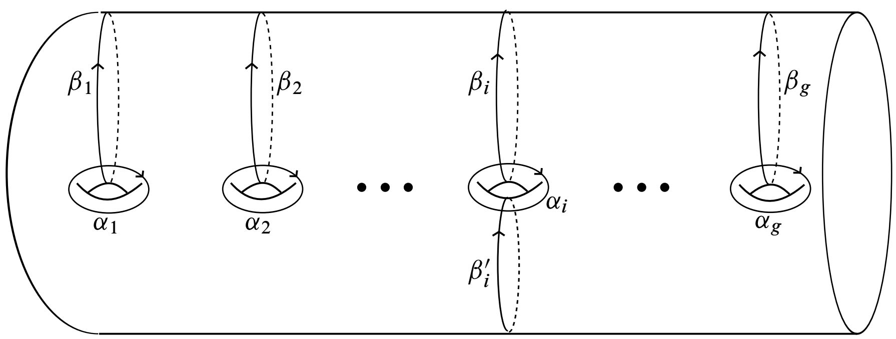

Figure 7

In [10, Lemma 4B], Johnson computes that

$$\tau_h = \left(\sum_{j=1}^{i-1} a_j \wedge b_j\right) \wedge b_i \tag{4}$$

Now, let us describe the example. Let

$$\begin{aligned} y = (a_4 + b_2 + b_3) \wedge a_1 \wedge b_1 + (a_3 + b_4) \wedge a_2 \wedge b_2 \\ + (a_1 + a_2 + b_1) \wedge a_3 \wedge b_3 + (a_1 + a_2) \wedge a_4 \wedge b_4 \in \Lambda^2 H. \end{aligned}$$

From the previous paragraph, we know there exists $f \in \mathcal{I}$ such that $\tau_f = y$ which we construct now. Consider the bounding pairs illustrated in Figure 8 . Let f be the product of bounding pair maps about these bounding pairs. Since τ is a homomorphism to an abelian group, τ_f is the same regardless of how the bounding pair maps are composed. Using (4), one computes that $\tau_f = y$.

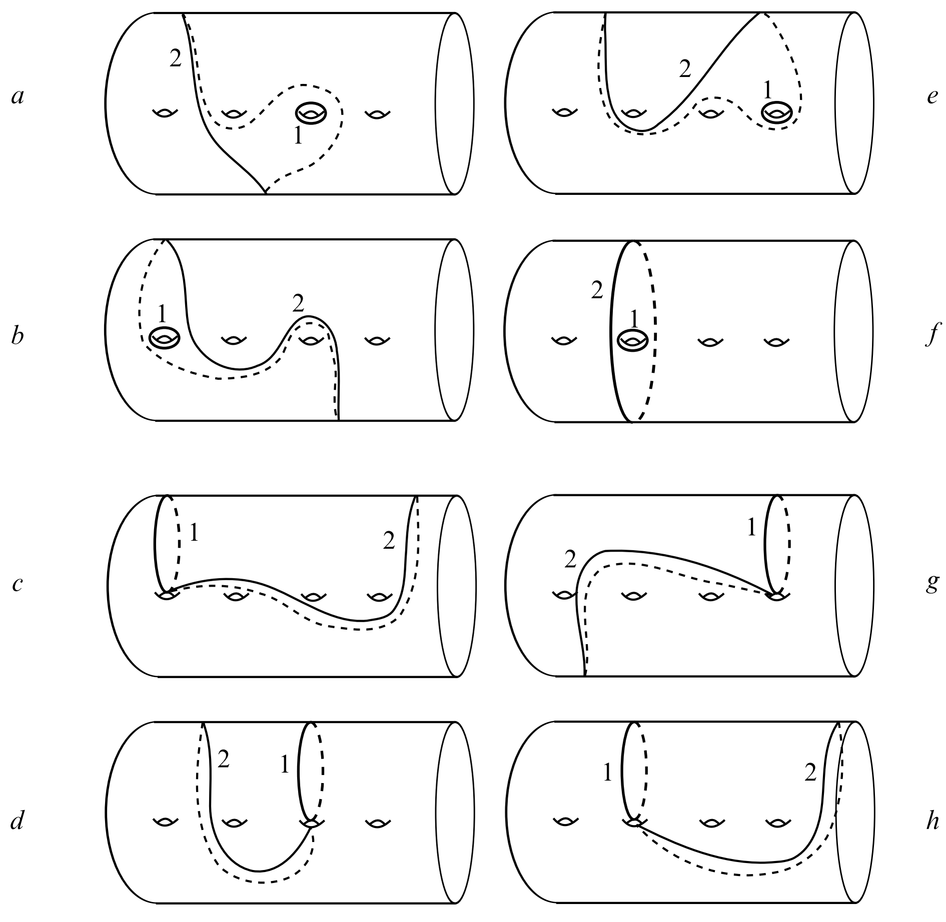

Figure 8: The product of the bounding pair maps indicated in a–h yields y

Via computation (with Mathematica), we find that with respect to the symplectic basis $\{a_1, b_1, \ldots, a_4, b_4\}$:

$$\Psi_1(f) = \begin{pmatrix} -6 & -2 & 2 & 0 & 2 & 2 & -2 & 0 \\ 4 & 2 & 2 & -2 & -2 & 2 & 2 & -2 \\ 4 & -2 & 2 & 0 & -2 & 2 & 2 & 0 \\ -2 & 4 & -2 & 0 & 0 & 2 & 0 & 2 \\ -4 & -4 & 2 & 4 & -2 & 4 & 0 & -2 \\ -4 & -4 & 0 & 6 & 2 & 2 & -2 & 0 \\ -2 & 4 & -2 & 2 & 2 & 2 & 2 & 2 \\ 4 & -2 & -2 & -4 & -4 & 2 & 4 & 0 \end{pmatrix}$$

The characteristic polynomial of $\Psi_1(f)/2$ is

$$\chi(\Psi_1(f)/2) = x^8 - 8x^6 + 26x^5 - 18x^4 - 76x^3 + 241x^2 - 558x + 553.$$

This polynomial is found to be irreducible mod 11 via Mathematica and is hence irreducible. By Theorem 1.2, f is pseudo-Anosov. Note that curves c_2, d_2, and g_2 in Figure 8 all pairwise intersect, and so the criteria of Thurston and Penner do not seem to apply directly to f.

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