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On smoothable surgery for 4–manifolds

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Under certain homological hypotheses on a compact 4–manifold, we prove exactness of the topological surgery sequence at the stably smoothable normal invariants. The main examples are the class of finite connected sums of 4–manifolds with certain product geometries. Most of these compact manifolds have non-vanishing second mod 2 homology and have fundamental groups of exponential growth, which are not known to be tractable by Freedman–Quinn topological surgery. Necessarily, the $*$–construction of certain non-smoothable homotopy equivalences requires surgery on topologically embedded 2–spheres and is not attacked here by transversality and cobordism.

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1 Introduction

1.1 Objectives

The main theorem of this paper is a limited form of the surgery exact sequence for compact 4–manifolds (Theorem 4.1). Corollaries include exactness at the smooth normal invariants of the 4–torus T^4 (Example 4.4) and the real projective 4–space $\mathbb{RP}^4$ (Corollary 4.7). C T C Wall proved an even more limited form of the surgery exact sequence [34, Theorem 16.6] and remarked that his techniques do not apply to T^4 and $\mathbb{RP}^4$. Although our new hypotheses depend on the L–theory assembly map, we provide a remedy along essentially the same lines.

1.2 Results

Let $(X, \partial X)$ be a based, compact, connected, topological 4–manifold with fundamental group $\pi = \pi_1(X)$ and orientation character $\omega = w_1(\tau_X)\colon \pi \to \mathbb{Z}^\times$. The reader is referred to Section 1.4 for an explanation of surgical language.

If X has a preferred smooth structure, consider the following surgery sequence.

$$\mathcal{S}^s_{\mathrm{DIFF}}(X) \xrightarrow{\eta} \mathcal{N}_{\mathrm{DIFF}}(X) \xrightarrow{\sigma_*} L^h_4(\mathbb{Z}[\pi]^\omega) \tag{1–1}$$

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Otherwise, let TOP0 refer to manifolds with the same smoothing invariant as X.

$$\mathcal{S}^s_{\mathrm{TOP0}}(X) \xrightarrow{\eta} \mathcal{N}_{\mathrm{TOP0}}(X) \xrightarrow{\sigma_*} L^h_4(\mathbb{Z}[\pi]^\omega) \tag{1–2}$$

The first examples consists of orientable 4–manifolds X with torsion-free, infinite fundamental groups, mostly of exponential growth. These include the 4–torus T^4 and connected sums of certain aspherical 4–manifolds of non-positive curvature.

Corollary 4.3 *Let π be a free product of groups of the form*

$$\pi = \bigstar_{i=1}^n \Lambda_i$$

for some $n > 0$, where each Λ_i is a torsion-free lattice in either $\mathrm{Isom}(\mathbb{E}^{m_i})$ *or* $\mathrm{Isom}(\mathbb{H}^{m_i})$ *or* $\mathrm{Isom}(\mathbb{CH}^{m_i})$ *for some $m_i > 0$. Suppose the orientation character ω is trivial. Then the surgery sequences* (1–1) *and* (1–2) *are exact.*

The second examples consist of a generalization X of non-aspherical, orientable, simply-connected 4–manifolds. These include the outcome of smooth surgery on the core circle of the mapping torus of an orientation-preserving self-diffeomorphism of a 3–dimensional lens space $L(p,q)$. The fundamental groups have torsion.

Corollary 4.6 *Let π be a free product of groups of the form*

$$\pi = \bigstar_{i=1}^n O_i$$

for some $n > 0$, where each O_i is an odd-torsion group. (Necessarily ω is trivial.) Then the surgery sequences (1–1) *and* (1–2) *are exact.*

The third examples consist of non-aspherical, non-orientable 4–manifolds X whose connected summands are non-orientable with fundamental group of order two. These include the real projective 4–space $\mathbb{RP}^4$.

Corollary 4.7 *Suppose X is a* DIFF *4–manifold of the form*

$$X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2)$$

for some $n > 0$ and $r \geq 0$, and each summand X_i is either $S^2 \times \mathbb{RP}^2$ or $S^2 \rtimes \mathbb{RP}^2$ or $\#_{S^1} n(\mathbb{RP}^4)$ for some $1 \leq n \leq 4$. Then the surgery sequences (1–1) *and* (1–2) *are exact.*

The fourth examples consist of orientable 4–manifolds X whose connected summands are mostly aspherical 3–manifold bundles over the circle. The important non-aspherical examples include $\# n(S^3 \times S^1)$ with free fundamental group. The aspherical examples are composed of fibers of a specific type of Haken 3–manifolds.

Corollary 4.8 *Suppose X is a* TOP *4–manifold of the form*

$$X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2)$$

for some $n > 0$ and $r \geq 0$, and each summand X_i is the total space of a fiber bundle

$$H_i \longrightarrow X_i \longrightarrow S^1.$$

Here, we suppose H_i is a compact, connected 3–manifold such that:

(1) *H_i is S^3 or D^3, or*

(2) *H_i is irreducible with non-zero first Betti number.*

Moreover, if H_i is non-orientable, we assume that the quotient group $H_1(H_i;\mathbb{Z})_{(\alpha_i)_*}$ of coinvariants is 2–torsionfree, where $\alpha_i \colon H_i \to H_i$ is the monodromy homeomorphism. Then the surgery sequence (1–2) *is exact.*

Finally, the fifth examples consist of possibly non-orientable 4–manifolds X with torsion-free fundamental group. The connected summands are surface bundles over surfaces, most of which are aspherical with fundamental groups of exponential growth. The aspherical, non-orientable examples of subexponential growth include simple torus bundles $T^2 \rtimes Kl$ over the Klein bottle, excluded from Corollary 4.3.

Corollary 4.9 *Suppose X is a* TOP *4–manifold of the form*

$$X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2)$$

for some $n > 0$ and $r \geq 0$, and each summand X_i is the total space of a fiber bundle

$$\Sigma_i^f \longrightarrow X_i \longrightarrow \Sigma_i^b.$$

Here, we suppose the fiber and base are compact, connected 2–manifolds, $\Sigma_i^f \neq \mathbb{RP}^2$, and Σ_i^b has positive genus. Moreover, if X_i is non-orientable, we assume that the fiber Σ_i^f is orientable and that the monodromy action of $\pi_1(\Sigma_i^b)$ of the base preserves any orientation on the fiber. Then the surgery sequence (1–2) *is exact.*

1.3 Techniques

Our methods employ various bits of geometric topology: topological transversality in all dimensions (see Freedman–Quinn [9]), and the analysis of smooth normal invariants of the Novikov pinching trick, which is used to construct homotopy self-equivalences of

4–manifolds (see Cochran–Habegger [6] and Wall [34]). Our hypotheses are algebraic-topological in nature and come from the surgery characteristic class formulas of Sullivan–Wall [34] and from the assembly map components of Taylor–Williams [30], as well as control of π_2 in non-orientable cases.

Jonathan Hillman has successfully employed these now standard techniques to classify 4–manifolds, up to s–cobordism, in the homotopy type of certain surface bundles over surfaces (see [12, Section 2] and [13, Chapter 6]). Along the same lines, our abundant families of 4–manifold examples also have fundamental groups of exponential growth, and so, too, are currently inaccessible by topological surgery (see Freedman–Quinn [9], Freedman–Teichner [10] and Krushkal–Quinn [18]).

The reader should be aware that the topological transversality used in Section 4 produces 5–dimensional TOP normal bordisms $W \to X \times \Delta^1$ which may not be smoothable, although the boundary $\partial W = \partial_- W \cup \partial_+ W$ is smoothable. In particular, W may not admit a TOP handlebody structure relative to $\partial_- W$. Hence W may not be the trace of surgeries on topologically embedded 2–spheres in X. Therefore, in general, W cannot be produced by Freedman–Quinn surgery theory, which has been developed only for a certain class of fundamental groups $\pi_1(X)$ of subexponential growth. In this way, topological cobordism is superior to surgery.

1.4 Language

For any group π, we shall write $\mathrm{Wh}_0(\pi) := \widetilde{K}_0(\mathbb{Z}[\pi])$ for the *projective class group* and $\mathrm{Wh}_1(\pi) := \widetilde{K}_1(\mathbb{Z}[\pi])/\langle\pi\rangle$ for the *Whitehead group*.

Let CAT be either the manifold category TOP or PL = DIFF in dimensions < 7. Suppose $(X, \partial X)$ is a based, compact, connected CAT 4–manifold. Let us briefly introduce some basic notation used throughout this paper. The fundamental group $\pi = \pi_1(X)$ depends on a choice of basepoint; a basepoint is essential if X is non-orientable. The *orientation character* $\omega = w_1(X)\colon \pi \to \mathbb{Z}^\times$ is a homomorphism that assigns $+1$ or -1 to a loop $\lambda\colon S^1 \to X$ if the pullback bundle $\lambda^*(\tau_X)$ is orientable or non-orientable. Recall that any finitely presented group π and arbitrary orientation character ω can be realized on some closed, smooth 4–manifold X by a straightforward surgical construction. A choice of generator $[X] \in H_4(X, \partial X; \mathbb{Z}^\omega)$ is called a *twisted orientation class*.

Let us introduce the terms in the surgery sequence investigated in Section 4. The *simple structure set* $\mathcal{S}^s_{\mathrm{CAT}}(X)$ consists of CAT s–bordism classes in $\mathbb{R}^\infty$ of simple homotopy equivalences $h\colon Y \to X$ such that $\partial h\colon \partial Y \to \partial X$ is the identity. Here, *simple* means that the torsion of the acyclic $\mathbb{Z}[\pi]$–module complex $\mathrm{Cone}(\widetilde{h})$ is zero, for some preferred finite homotopy CW-structures on Y and X. Indeed, any compact topological manifold $(X, \partial X)$ has a canonical simple homotopy type, obtained by cleanly embedding X into euclidean space (see Kirby–Siebenmann [15, Theorem III.4.1]). Therefore, for any abelian group A and $n > 1$, pulling back the inverse of the Hurewicz isomorphism induces a bijection from $[X/\partial X, K(A,n)]_0$ to $H^n(X, \partial X; A)$. This identification shall be used implicitly throughout the paper.

Denote G_n as the topological monoid of homotopy self-equivalences $S^{n-1} \to S^{n-1}$, and $\mathrm{G} := \mathrm{colim}_n \, \mathrm{G}_n$ as the direct limit of $\{ \mathrm{G}_n \to \mathrm{G}_{n+1} \}$. The *normal invariant set* $\mathcal{N}_{\mathrm{CAT}}(X) \cong [X/\partial X, \mathrm{G}/\mathrm{CAT}]_0$ consists of CAT normal bordism classes in $\mathbb{R}^\infty$ of degree one, CAT normal maps $f \colon M \to X$ such that $\partial f \colon \partial X \to \partial X$ is the identity; we suppress the normal data and define $X/\varnothing = X \sqcup \mathrm{pt}$. Denote $\widehat{f} \colon X/\partial X \to \mathrm{G}/\mathrm{CAT}$ as the associated homotopy class of based maps. Indeed, transversality in the TOP category holds for all dimensions and codimensions (see Kirby–Siebenmann [15] and Freedman–Quinn [9]). The normal invariants map $\eta \colon \mathcal{S}^s_{\mathrm{CAT}}(X) \to \mathcal{N}_{\mathrm{CAT}}(X)$ is a forgetful map. The *surgery obstruction group* $L^h_4(\mathbb{Z}[\pi]^\omega)$ consists of Witt classes of nonsingular quadratic forms over the group ring $\mathbb{Z}[\pi]$ with involution $(g \mapsto \omega(g) g^{-1})$. The surgery obstruction map $\sigma^h_* \colon \mathcal{N}_{\mathrm{CAT}}(X) \to L^h_4(\mathbb{Z}[\pi]^\omega)$ vanishes on the image of η. The basepoint of the former two sets is the identity map $\mathbf{1}_X \colon X \to X$, and the basepoint of the latter set is the Witt class 0.

1.5 Invariants

The unique homotopy class of classifying maps $u \colon X \to B\pi$ of the universal cover induces homomorphisms

$$u_0 \colon H_0(X; \mathbb{Z}^\omega) \longrightarrow H_0(\pi; \mathbb{Z}^\omega)$$
$$u_2 \colon H_2(X; \mathbb{Z}_2) \longrightarrow H_2(\pi; \mathbb{Z}_2).$$

Next, recall that the manifold X has a second Wu class

$$v_2(X) \in H^2(X; \mathbb{Z}_2) = \mathrm{Hom}(H_2(X; \mathbb{Z}_2), \mathbb{Z}_2)$$

defined for all $a \in H^2(X, \partial X; \mathbb{Z}_2)$ by $\langle v_2(X), a \cap [X] \rangle = \langle a \cup a, [X] \rangle$. This unoriented cobordism characteristic class is uniquely determined from the Stiefel–Whitney classes of the tangent microbundle τ_X by the formula

$$v_2(X) = w_1(X) \cup w_1(X) + w_2(X).$$

Observe that $v_2(X)$ vanishes if X is a TOP Spin-manifold.

Finally, let us introduce the relevant surgery characteristic classes. Observe

$$H_0(\pi;\mathbb{Z}^\omega) = \mathbb{Z}/\langle \omega(g)-1 \mid g\in\pi\rangle = \begin{cases}\mathbb{Z} & \text{if } \omega = 1\\ \mathbb{Z}_2 & \text{if } \omega\neq 1.\end{cases} \tag{1–3}$$

The 0th component of the 2–local assembly map A_π (see Taylor–Williams [30]) has an integral lift

$$I_0\colon H_0(\pi;\mathbb{Z}^\omega) \longrightarrow L_4^h(\mathbb{Z}[\pi]^\omega).$$

The image $I_0(1)$ equals the Witt class of the E_8 quadratic form (see Davis [7, Remark 3.7]). The 2nd component of the 2–local assembly map A_π [30] has an integral lift

$$\kappa_2\colon H_2(\pi;\mathbb{Z}_2) \longrightarrow L_4^h(\mathbb{Z}[\pi]^\omega).$$

Let $f\colon M\to X$ be a degree one, TOP normal map. According to René Thom [31], every homology class in $H_2(X,\partial X;\mathbb{Z}_2)$ is represented by $g_*[\Sigma]$ for some compact, possibly non-orientable surface Σ and TOP immersion $g\colon(\Sigma,\partial\Sigma)\to(X,\partial X)$. The codimension two Kervaire–Arf invariant

$$\mathrm{kerv}(f)\colon H_2(X,\partial X;\mathbb{Z}_2)\longrightarrow \mathbb{Z}_2$$

assigns to each two-dimensional homology class $g_*[\Sigma]$ the Arf invariant of the degree one, normal map $g^*(f)\colon f^*(\Sigma)\to\Sigma$. The element $\mathrm{kerv}(f)\in H^2(X,\partial X;\mathbb{Z}_2)$ is invariant under TOP normal bordism of f; it may not vanish for homotopy equivalences. If M and X are oriented, then there is a signature invariant

$$\mathrm{sign}(f) := (\mathrm{sign}(M)-\mathrm{sign}(X))/8 \in H_0(X;\mathbb{Z}),$$

which does vanish for homotopy equivalences. For any compact topological manifold X, the Kirby–Siebenmann invariant $\mathrm{ks}(X)\in H^4(X,\partial X;\mathbb{Z}_2)$ is the sole obstruction to the existence of a DIFF structure on $X\times\mathbb{R}$ or equivalently on $X\#r(S^2\times S^2)$ for some $r\geq 0$. Furthermore, the image of $\mathrm{ks}(X)\cap[X]$ in $\mathbb{Z}_2$ under the augmentation map $X\to\mathrm{pt}$ is an invariant of unoriented TOP cobordism (see Freedman–Quinn [9, Section 10.2B]). Define

$$\mathrm{ks}(f) := f_*(\mathrm{ks}(M)\cap[M]) - (\mathrm{ks}(X)\cap[X]) \in H_0(X;\mathbb{Z}_2).$$

In Section 4, we shall use Sullivan's surgery characteristic class formulas as geometrically identified in dimension four by J F Davis [7, Proposition 3.6]:

$$\widehat{f}^*(k_2)\cap[X] = \mathrm{kerv}(f)\cap[X]\in H_2(X;\mathbb{Z}_2) \tag{1–4}$$

$$\widehat{f}^*(\ell_4)\cap[X] = \begin{cases}\mathrm{sign}(f)\in H_0(X;\mathbb{Z}) & \text{if } \omega=1\\ \mathrm{ks}(f)+(\mathrm{kerv}(f)^2\cap[X])\in H_0(X;\mathbb{Z}_2) & \text{if } \omega\neq 1.\end{cases} \tag{1–5}$$

Herein is used the 5th stage Postnikov tower (see Kirby–Siebenmann [15] and Wall [34])

$$k_2 + \ell_4 \colon \mathrm{G/TOP}^{[5]} \xrightarrow{\simeq} K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}, 4).$$

The two expressions in (1–5) agree modulo two [15, Annex 3, Theorem 15.1]:

$$\mathrm{ks}(f) = \big(\mathrm{red}_2(\widehat{f})^*(\ell_4) - (\widehat{f})^*(k_2)^2\big) \cap [X] \in H_0(X; \mathbb{Z}_2). \tag{1–6}$$

2 Smoothing normal bordisms

Let $(X, \partial X)$ be a based, compact, connected, DIFF 4–manifold. We start with group-theoretic criteria on the existence and uniqueness of smoothing the topological normal bordisms relative ∂X from the identity map on X to itself.

Proposition 2.1 *With respect to the Whitney sum H–space structures on the* CAT *normal invariants, there are exact sequences of abelian groups:*

$$0 \longrightarrow \mathrm{Tor}_1(H_0(\pi; \mathbb{Z}^\omega), \mathbb{Z}_2) \longrightarrow \mathcal{N}_{\mathrm{DIFF}}(X) \xrightarrow{\mathrm{red}_{\mathrm{TOP}}} \mathcal{N}_{\mathrm{TOP}}(X) \xrightarrow{\mathrm{ks}} H_0(\pi; \mathbb{Z}_2) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Tor}_1(H_1(\pi; \mathbb{Z}^\omega), \mathbb{Z}_2) \longrightarrow \mathcal{N}_{\mathrm{DIFF}}(X \times \Delta^1) \xrightarrow{\mathrm{red}_{\mathrm{TOP}}} \mathcal{N}_{\mathrm{TOP}}(X \times \Delta^1) \xrightarrow{\mathrm{ks}} H_1(\pi; \mathbb{Z}^\omega) \otimes \mathbb{Z}_2 \longrightarrow 0.$$

Proof Since X is a CAT manifold, by CAT transversality and Cerf's result that PL/O is 6–connected (see Kirby–Siebenmann [15] and Freedman–Quinn [9]), we can identify the based sets

$$\begin{aligned} \mathcal{N}_{\mathrm{DIFF}}(X) &= [X/\partial X, \mathrm{G/PL}]_0 \\ \mathcal{N}_{\mathrm{TOP}}(X) &= [X/\partial X, \mathrm{G/TOP}]_0 \\ \mathcal{N}_{\mathrm{DIFF}}(X \times \Delta^1) &= [S^1 \wedge (X/\partial X), \mathrm{G/PL}]_0 \\ \mathcal{N}_{\mathrm{TOP}}(X \times \Delta^1) &= [S^1 \wedge (X/\partial X), \mathrm{G/TOP}]_0. \end{aligned}$$

Furthermore, each right-hand set is an abelian group with respect to the H–space structure on G/CAT given by Whitney sum of CAT microbundles.

For any based space Z with the homotopy type of a CW-complex, there is the Siebenmann–Morita exact sequence of abelian groups [15, Annex 3, Theorem 15.1]:

$$0 \longrightarrow \mathrm{Cok}\big(\mathrm{red}_2^{(3)}\big) \longrightarrow [Z, \mathrm{G/PL}]_0 \xrightarrow{\mathrm{red}_{\mathrm{TOP}}} [Z, \mathrm{G/TOP}]_0 \xrightarrow{\mathrm{ks}} \mathrm{Im}\big(\mathrm{red}_2^{(4)} + \mathrm{Sq}^2\big) \longrightarrow 0.$$

Here, the stable cohomology operations

$$\mathrm{red}_2^{(n)}\colon H^n(Z;\mathbb{Z}) \longrightarrow H^n(Z;\mathbb{Z}_2) \qquad \text{and} \qquad \mathrm{Sq}^2\colon H^2(Z;\mathbb{Z}_2) \longrightarrow H^4(Z;\mathbb{Z}_2)$$

are reduction modulo two and the second Steenrod square. The homomorphism ks is given by the formula $\mathrm{ks}(a,b) = \mathrm{red}_2^{(4)}(a) - \mathrm{Sq}^2(b)$, as stated in (1–6), which follows from Sullivan's determination (3–1) below.

Suppose $Z = X/\partial X$. By Poincaré duality and the universal coefficient sequence, there are isomorphisms

$$\begin{aligned}\mathrm{Cok}\big(\mathrm{red}_2^{(3)}\big) &\cong \mathrm{Cok}\big(\mathrm{red}_2\colon H_1(X;\mathbb{Z}^\omega) \to H_1(X;\mathbb{Z}_2)\big) \cong \mathrm{Tor}_1(H_0(X;\mathbb{Z}^\omega),\mathbb{Z}_2)\\ \mathrm{Im}\big(\mathrm{red}_2^{(4)}\big) &\cong \mathrm{Im}\big(\mathrm{red}_2\colon H_0(X;\mathbb{Z}^\omega) \to H_0(X;\mathbb{Z}_2)\big) = H_0(X;\mathbb{Z}_2).\end{aligned}$$

Therefore we obtain the exact sequence for the normal invariants of X.

Suppose $Z = S^1 \wedge (X/\partial X)$. By the suspension isomorphism Σ, Poincaré duality, and the universal coefficient sequence, there are isomorphisms

$$\begin{aligned}\mathrm{Cok}\big(\mathrm{red}_2^{(3)}\big) &\cong \mathrm{Cok}\big(\mathrm{red}_2^{(2)}\colon H^2(X,\partial X;\mathbb{Z}) \to H^2(X,\partial X;\mathbb{Z}_2)\big)\\ &\cong \mathrm{Cok}\big(\mathrm{red}_2\colon H_2(X;\mathbb{Z}^\omega) \to H_2(X;\mathbb{Z}_2)\big) \cong \mathrm{Tor}_1(H_1(X;\mathbb{Z}^\omega),\mathbb{Z}_2).\end{aligned}$$

Note, since the cohomology operations $\mathrm{red}_2^{(4)}$ and Sq^2 are stable, that

$$(\Sigma^{-1}\circ \mathrm{ks})(\Sigma a, \Sigma b) = \mathrm{red}_2^{(3)}(a) - \mathrm{Sq}^2(b) = \mathrm{red}_2^{(3)}(a)$$

for all $a \in H^3(X,\partial X;\mathbb{Z})$ and $b \in H^1(X,\partial X;\mathbb{Z}_2)$. Then, by Poincaré duality and the universal coefficient sequence, we have

$$\begin{aligned}\mathrm{Im}(\mathrm{ks}) &\cong \mathrm{Im}\big(\mathrm{red}_2^{(3)}\colon H^3(X,\partial X;\mathbb{Z}) \to H^3(X,\partial X;\mathbb{Z}_2)\big)\\ &\cong \mathrm{Im}\big(\mathrm{red}_2\colon H_1(X;\mathbb{Z}^\omega) \to H_1(X;\mathbb{Z}_2)\big) = H_1(\pi;\mathbb{Z}^\omega)\otimes\mathbb{Z}_2.\end{aligned}$$

Therefore we obtain the exact sequence for the normal invariants of $X \times \Delta^1$. □

3 Homotopy self-equivalences

Recall Sullivan's determination (see Madsen–Milgram [20] and Wall [34])

(3–1) $$k_2 + 2\ell_4\colon \mathrm{G/PL}^{[5]} \xrightarrow{\simeq} K(\mathbb{Z}_2,2) \times_{\delta(\mathrm{Sq}^2)} K(\mathbb{Z},4).$$

The homomorphism $\delta\colon H^4(K(\mathbb{Z}_2,2);\mathbb{Z}_2) \to H^5(K(\mathbb{Z}_2,2);\mathbb{Z})$ is the Bockstein associated to the coefficient exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$, and the element Sq^2 is the 2nd Steenrod square. The cohomology classes k_2 and $2\ell_4$ map to a generator of the base $K(\mathbb{Z}_2, 2)$ and of the fiber $K(\mathbb{Z}, 4)$. Moreover, the cohomology class $2\ell_4$ of G/PL is the pullback of the cohomology class ℓ_4 of G/TOP under the forgetful map $\mathrm{red}_{\mathrm{TOP}}\colon \mathrm{G/PL} \to \mathrm{G/TOP}$. The above homotopy equivalence gives

$$\mathrm{red}_2(2\ell_4) = (k_2)^2 \in H^4(\mathrm{G/PL}; \mathbb{Z}_2);$$

compare [20, Theorem 4.32, Footnote]. There exists a symmetric L–theory twisted orientation class $[X]_{\mathbf{L}^\cdot} \in H_4(X, \partial X; \mathbf{L}^{\cdot\omega})$ fitting into a commutative diagram (Figure 3.1), due to Sullivan–Wall [34, Theorem 13B.3] and Quinn–Ranicki [24, Theorem 18.5].

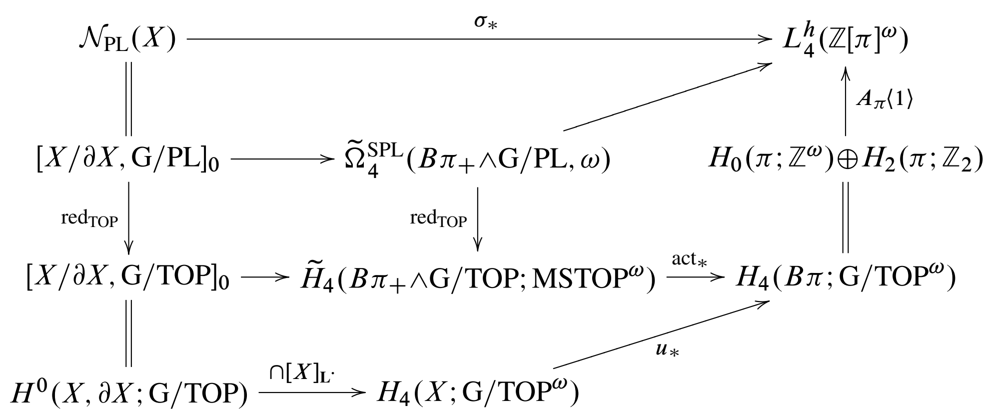

Figure 3.1: Factorization of the smoothable surgery obstruction

Here, the identification $\mathcal{N}_{\mathrm{PL}}(X) = [X/\partial X, \mathrm{G/PL}]_0$ only makes sense if $\mathrm{ks}(X) = 0$. It follows that the image $\hat{\sigma}(g) \in H_4(\pi; \mathrm{G/TOP}^\omega)$, through the scalar product act[1], of a normal invariant $g\colon X/\partial X \to \mathrm{G/PL}$ consists of two characteristic classes:

$$\hat{\sigma}(g) = u_0\big(g^*(2\ell_4) \cap [X]\big) \oplus u_2\big(g^*(k_2) \cap [X]\big),$$

which are determined by the TOP manifold-theoretic invariants in Section 1.5.

We caution the reader that $\ell_4 \notin H^4(\mathrm{G/PL}; \mathbb{Z})$; the notation $2\ell_4$ is purely formal.

Definition 3.1 Let $(X, \partial X)$ be any based, compact, connected TOP 4–manifold. Define the *stably smoothable* subsets

$$\mathcal{N}_{\mathrm{TOP}0}(X) := \{\, f \in \mathcal{N}_{\mathrm{TOP}}(X) \mid \mathrm{ks}(f) = 0 \,\}$$
$$\mathcal{S}^s_{\mathrm{TOP}0}(X) := \big\{\, h \in \mathcal{S}^s_{\mathrm{TOP}}(X) \mid \mathrm{ks}(h) = 0 \,\big\}.$$

[1] $\mathbf{L}.\langle 1\rangle = \mathrm{G/TOP}$ is a module spectrum over the ring spectrum $\mathbf{L}^\cdot = \mathrm{MSTOP}$ via Brown representation. Refer to [24, Remark B9] and [34, Theorem 9.8] on the level of homotopy groups or to Sullivan's original method of proof in his thesis [29].

Recall X has fundamental group π and orientation character ω.

Hypothesis 3.2 *Let X be orientable. Suppose that the homomorphism*

$$\kappa_2\colon H_2(\pi;\mathbb{Z}_2) \longrightarrow L_4^h(\mathbb{Z}[\pi]^\omega)$$

is injective on the subgroup $u_2(\operatorname{Ker} v_2(X))$.

Hypothesis 3.3 *Let X be non-orientable such that π contains an orientation-reversing element of finite order, and if* CAT = DIFF, *then suppose that orientation-reversing element has order two. Suppose that κ_2 is injective on all $H_2(\pi;\mathbb{Z}_2)$, and suppose that* $\operatorname{Ker}(u_2) \subseteq \operatorname{Ker}(v_2)$.

Hypothesis 3.4 *Let X be non-orientable such that there exists an epimorphism $\pi^\omega \to \mathbb{Z}^-$. Suppose that κ_2 is injective on the subgroup $u_2(\operatorname{Ker} v_2(X))$.*

Proposition 3.5 *Let $f\colon M \to X$ be a degree one, normal map of compact, connected* TOP *4–manifolds such that $\partial f = \mathbf{1}_{\partial X}$. Suppose Hypothesis 3.2 or 3.3 or 3.4. If $\sigma_*(f) = 0$ and $\operatorname{ks}(f) = 0$, then f is* TOP *normally bordant to a homotopy self-equivalence $h\colon X \to X$ relative to ∂X.*

Proof Since $\operatorname{ks}(f) = 0$, there is a (formal) based map $g\colon X/\partial X \to \mathrm{G/PL}$ such that

$$\mathrm{red}_{\mathrm{TOP}} \circ g = \widehat{f}\colon X/\partial X \to \mathrm{G/TOP}.$$

So g has vanishing surgery obstruction:

$$0 = \sigma_*(f) = (I_0 + \kappa_2)(\widehat{\sigma}(g)) \in L_4^h(\mathbb{Z}[\pi]^\omega).$$

Case 1 Suppose X is orientable; that is, $\omega = 1$. Then the inclusion $1^+ \to \pi^\omega$ is retractive and induces a split monomorphism $L_4^h(\mathbb{Z}[1]) \to L_4^h(\mathbb{Z}[\pi])$ with cokernel defined as $\widetilde{L}_4^h(\mathbb{Z}[\pi])$. So the above sum of maps is direct:

$$0 = (I_0 \oplus \kappa_2)(\widehat{\sigma}(g)) \in L_4^h(\mathbb{Z}[1]) \oplus \widetilde{L}_4^h(\mathbb{Z}[\pi]).$$

Then both the signature and the square of the Kervaire–Arf invariant vanish (1–6):

$$0 = g^*(2\ell_4) = g^*(k_2)^2.$$

So $(\widehat{f})^*(k_2) \cap [X] \in \operatorname{Ker} v_2(X)$. Therefore, since κ_2 is injective on the subgroup $u_2(\operatorname{Ker} v_2(X))$, we have $(\widehat{f})^*(k_2) \cap [X] \in \operatorname{Ker}(u_2)$. So, by the Hopf exact sequence

$$\pi_2(X) \otimes \mathbb{Z}_2 \xrightarrow{\mathrm{Hur}} H_2(X;\mathbb{Z}_2) \xrightarrow{u_2} H_2(\pi;\mathbb{Z}_2) \longrightarrow 0,$$

there exists $\alpha \in \pi_2(X)$ such that our homology class is spherical:

$$(\widehat{f})^*(k_2) \cap [X] = (\mathrm{red}_2 \circ \mathrm{Hur})(\alpha).$$

Case 2 Suppose X is non-orientable; that is, $\omega \neq 1$. Let $x \in \pi$ be an orientation-reversing element: $\omega(x) = -1$. First, consider the case that x has finite order. By taking an odd order power, we may assume that x has order 2^N for some $N > 0$. Then the map induced by $1^+ \to \pi^\omega$ has a factorization through $(C_{2^N})^-$:

$$L_4^h(\mathbb{Z}[1]) \longrightarrow L_4^h(\mathbb{Z}[C_{2^N}]^-) \xrightarrow{x_*} L_4^h(\mathbb{Z}[\pi]^\omega).$$

The abelian group in the middle is zero by Wall [33, Theorem 3.4.5, Remark]. Then, since I_0: $H_0(\pi;\mathbb{Z}^\omega) \to L_4^h(\mathbb{Z}[\pi]^\omega)$ factors through $L_4^h(\mathbb{Z}[1]) \cong \mathbb{Z}$, generated by the Witt class $[E_8]$, we must have $I_0 = 0$. So

$$0 = \sigma_*(f) = \kappa_2(\widehat{\sigma}(f)) \in L_4^h(\mathbb{Z}[\pi]^\omega),$$

and since κ_2 is injective on all $H_2(\pi;\mathbb{Z}_2)$, we have

$$0 = \widehat{\sigma}(f) = u_2\big((\widehat{f})^*(k_2) \cap [X]\big).$$

Then $(\widehat{f})^*(k_2) \cap [X] \in \mathrm{Ker}(u_2) \subseteq \mathrm{Ker}(v_2)$ by hypothesis, and the class is spherical.

Next, consider the case there are no orientation-reversing elements of finite order. Then, by hypothesis, there is an epimorphism p: $\pi^\omega \to \mathbb{Z}^-$, which is split by a monomorphism with image generated by some orientation-reversing infinite cyclic element $y \in \pi$. Define $\overline{L}_4^h(\mathbb{Z}[\pi]^\omega)$ as the kernel of p_*. Then y_* induces a direct sum decomposition

$$L_4^h(\mathbb{Z}[\pi]^\omega) = L_4^h(\mathbb{Z}[\mathbb{Z}]^-) \oplus \overline{L}_4^h(\mathbb{Z}[\pi]^\omega).$$

The abelian group of the non-orientable Laurent extension in the middle is isomorphic to $\mathbb{Z}_2$, generated by the Witt class $[E_8]$, according to the quadratic version of Milgram–Ranicki [21, Theorem 4.1] with orientation $u = -1$. Then the map I_0: $H_0(\pi;\mathbb{Z}^\omega) \to L_4^h(\mathbb{Z}[\pi]^\omega)$ factors through the summand $L_4^h(\mathbb{Z}[\mathbb{Z}]^-)$ by an isomorphism; functorially, κ_2 has zero projection onto that factor. So the sum of maps is direct, similar to the oriented case:

$$0 = (I_0 \oplus \kappa_2)(\widehat{\sigma}(g)) \in L_4^h(\mathbb{Z}[\mathbb{Z}]^-) \oplus \overline{L}_4^h(\mathbb{Z}[\pi]^\omega).$$

A similar argument, using the smooth normal invariant g, shows that

$$0 = (\widehat{f})^*(\ell_4) = (\widehat{f})^*(k_2)^2.$$

Hence $(\widehat{f})^*(k_2) \cap [X] \in \mathrm{Ker}\, v_2(X)$. Since κ_2 is injective on $u_2(\mathrm{Ker}\, v_2(X))$, we also have $(\widehat{f})^*(k_2) \cap [X] \in \mathrm{Ker}(u_2)$, thus the class is spherical.

General case Let us return to the general case of X without any condition on orientability. For any $\alpha \in \pi_2(X)$, there is a homotopy operation, called the Novikov pinch map, defined by the homotopy self-equivalence

$$h\colon X \xrightarrow{\text{pinch}} X \vee S^4 \xrightarrow{\mathbf{1}_X \vee \Sigma\eta} X \vee S^3 \xrightarrow{\mathbf{1}_X \vee \eta} X \vee S^2 \xrightarrow{\mathbf{1}_X \vee \alpha} X.$$

Here, $\eta\colon S^3 \to S^2$ and $\Sigma\eta\colon S^4 \to S^3$ are the complex Hopf map and its suspension that generate the stable homotopy groups π_1^s and π_2^s.

For the normal invariant of the self-equivalence $h\colon X \to X$ associated to our particular α, there is a formula in the simply-connected case due to Cochran and Habegger [6, Theorem 5.1] and generalized to the non-simply connected case by Kirby and Taylor [16, Theorem 18, Remarks]:

$$\begin{aligned}(\widehat{h})^*(k_2) &= \big(1 + \langle v_2(X), (\widehat{f})^*(k_2) \cap [X]\rangle\big) \cdot (\widehat{f})^*(k_2) = (\widehat{f})^*(k_2)\\ (\widehat{h})^*(\ell_4) &= 0 = (\widehat{f})^*(\ell_4).\end{aligned}$$

Here, we have used $(\widehat{f})^*(k_2) \cap [X] \in \operatorname{Ker} v_2(X)$ and, if X is non-orientable, $\mathrm{ks}(f) = 0$ in (1–5). Therefore $f\colon M \to X$ is TOP normally bordant to the homotopy self-equivalence $h\colon X \to X$ relative to the identity $\partial X \to \partial X$ on the boundary. □

4 Smoothable surgery for 4–manifolds

Terry Wall asked if the smooth surgery sequence is exact at the normal invariants for the 4–torus T^4 and real projective 4–space $\mathbb{RP}^4$; see the remark after [34, Theorem 16.6]. The latter case of $\mathbb{RP}^4$ was affirmed implicitly in the work of Cappell and Shaneson [4]. The main theorem of this section affirms the former case of T^4 and extends their circle sum technique for $\mathbb{RP}^4$ to a broader class of non-orientable 4–manifolds, using the assembly map and smoothing theory.

Theorem 4.1 *Let $(X, \partial X)$ be a based, compact, connected,* CAT *4–manifold with fundamental group $\pi = \pi_1(X)$ and orientation character $\omega = w_1(X)\colon \pi \to \mathbb{Z}^\times$.*

(1) *Suppose Hypothesis 3.2 or 3.3. Then the surgery sequence of based sets is exact at the smooth normal invariants:*

(4–1) $$\mathcal{S}^s_{\mathrm{DIFF}}(X) \xrightarrow{\eta} \mathcal{N}_{\mathrm{DIFF}}(X) \xrightarrow{\sigma_*} L^h_4(\mathbb{Z}[\pi]^\omega).$$

(2) *Suppose Hypothesis 3.2 or 3.3 or 3.4. Then the surgery sequence of based sets is exact at the stably smoothable normal invariants:*

(4–2) $$\mathcal{S}^s_{\mathrm{TOP0}}(X) \xrightarrow{\eta} \mathcal{N}_{\mathrm{TOP0}}(X) \xrightarrow{\sigma_*} L^h_4(\mathbb{Z}[\pi]^\omega).$$

The above theorem generalizes a statement of Wall [34, Theorem 16.6] proven correctly by Cochran and Habegger [6] for closed, oriented DIFF 4–manifolds.

Corollary 4.2 (Wall) *Suppose the orientation character ω is trivial and the group homology vanishes: $H_2(\pi;\mathbb{Z}_2)=0$. Then the surgery sequence* (4–1) *is exact.*

A fundamental result from geometric group theory is that any torsion-free, finitely presented group Γ is of the form $\Gamma = \bigstar_{i=1}^n \Gamma_i$ for some $n \geq 0$, where each Γ_i is either $\mathbb{Z}$ or a one-ended, finitely presented group. Geometric examples of such Γ_i are torsion-free lattices of any rank. The Borel/Novikov Conjecture (that is, the Integral Novikov Conjecture) would imply that κ_2 is injective for all finitely generated, torsion-free groups π and all ω (see Davis [7]). At the moment, we have:

Corollary 4.3 *Let π be a free product of groups of the form*

$$\pi = \bigstar_{i=1}^n \Lambda_i$$

for some $n>0$, where each Λ_i is a torsion-free lattice in either $\mathrm{Isom}(\mathbb{E}^{m_i})$ *or* $\mathrm{Isom}(\mathbb{H}^{m_i})$ *or* $\mathrm{Isom}(\mathbb{CH}^{m_i})$ *for some $m_i>0$. Suppose the orientation character ω is trivial. Then the surgery sequences* (4–1) *and* (4–2) *are exact.*

Example 4.4 Besides stabilization with connected summands of $S^2\times S^2$, the preceding corollary includes the orientable manifolds $X=T^4=\prod 4(S^1)$ and $X=\#n(S^1\times S^3)$ and $X=\#n(T^2\times S^2)$ for all $n>0$. Also included are the compact, connected, orientable 4–manifolds X whose interiors $X-\partial X$ admit a complete hyperbolic metric. In addition, the corollary applies to the total space of any orientable fiber bundle $S^2\to X\to\Sigma$ for some compact, connected, orientable 2–manifold Σ of positive genus.

Remark 4.5 Many surgical theorems on TOP 4–manifolds require π to have subexponential growth (see Freedman–Quinn [10] and Krushkal–Quinn [18]) in order to find topologically embedded Whitney discs. Currently, the Topological Surgery Conjecture remains open for the more general class of discrete, amenable groups. In our case, observe that all crystallographic groups $1\to\mathbb{Z}^m\to\pi\to finite\to 1$ have subexponential growth for all $m>0$. On the other hand, observe that all torsion-free lattices π in $\mathrm{Isom}(\mathbb{H}^m)$ and all free groups $\pi=F_n$ have exponential growth if and only if $m,n>1$.

Indeed, taking all $\mathbb{E}^{m_i}=\mathbb{R}$, we obtain the finite-rank free groups $\pi=F_n$. Thus we partially strengthen a theorem of Krushkal and Lee [17] if X is a compact, connected, oriented TOP 4–manifold with fundamental group F_n. They only required X to be a finite Poincaré complex of dimension 4 ($\partial X = \varnothing$) but insisted that the intersection form over $\mathbb{Z}[\pi]$ of their degree one, TOP normal maps $f\colon M \to X$ be tensored up from the simply-connected case $\mathbb{Z}[1]$. Now, our shortcoming is that exactness is not proven at $\mathcal{N}_{\mathrm{TOP}}(X)$. This is because self-equivalences do not represent the homotopy equivalences with $\mathrm{ks} \neq 0$, such as the well-known non-smoothable homotopy equivalences $*\mathbb{CP}^2 \to \mathbb{CP}^2$ and $*\mathbb{RP}^4 \to \mathbb{RP}^4$.

Consider examples of infinite groups with odd torsion and trivial orientation. The original case $n = 1$ below was observed by S Cappell [3, Theorem 5]. Observe that the free products below have exponential growth if and only if $n > 1$.

Corollary 4.6 *Let π be a free product of groups of the form*

$$\pi = \bigstar_{i=1}^{n} O_i$$

for some $n > 0$, where each O_i is an odd-torsion group. (Necessarily ω is trivial.) Then the surgery sequences (4–1) *and* (4–2) *are exact.*

Consider non-orientable 4–manifolds X whose fundamental group $\pi = \bigstar n(C_2)$ is infinite and has 2–torsion. We denote $S^2 \rtimes \mathbb{RP}^2$ the total space of the 2–sphere bundle classified by the unique homotopy class of non-nullhomotopic map $\mathbb{RP}^2 \to BSO(3)$. This total space was denoted as the sphere bundle $S(\gamma \oplus \gamma \oplus \mathbb{R})$ in the classification of Hambleton, Kreck and Teichner [11], where γ is the canonical line bundle over $\mathbb{RP}^2$.

Corollary 4.7 *Suppose X is a* DIFF *4–manifold of the form*

$$X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2)$$

for some $n > 0$ and $r \geq 0$, and each summand X_i is either $S^2 \times \mathbb{RP}^2$ or $S^2 \rtimes \mathbb{RP}^2$ or $\#_{S^1} n(\mathbb{RP}^4)$ for some $1 \leq n \leq 4$. Then the surgery sequences (4–1) *and* (4–2) *are exact.*

In symplectic topology, the circle sum $M \#_{S^1} N$ is defined as $(M - \mathring{E}) \cup_{\partial E} (N - \mathring{E})$, where E is the total space of a 3–plane bundle over S^1 with given embeddings in the 4–manifolds M and N. The preceding corollary takes circle sums along the order-two generator $\mathbb{RP}^1$ of $\pi_1(\mathbb{RP}^4)$; the normal sphere bundle $\partial E = S^2 \rtimes \mathbb{RP}^1$ is non-orientable. Observe that all the free products π in Corollary 4.7 have exponential growth if and only if $n > 2$.

Next, consider non-orientable 4–manifolds X whose fundamental groups π are infinite and torsion-free. Interesting examples have π in Waldhausen's class Cl of groups with vanishing Whitehead groups $\mathrm{Wh}_*(\pi)$ [32, Section 19], such as Haken 3–manifold bundles over the circle.

Corollary 4.8 *Suppose X is a* TOP *4–manifold of the form*

$$X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2)$$

for some $n > 0$ and $r \geq 0$, and each summand X_i is the total space of a fiber bundle

$$H_i \longrightarrow X_i \longrightarrow S^1.$$

Here, we suppose H_i is a compact, connected 3–manifold such that:

(1) *H_i is S^3 or D^3, or*

(2) *H_i is irreducible with non-zero first Betti number.*

Moreover, if H_i is non-orientable, we assume that the quotient group $H_1(H_i;\mathbb{Z})_{(\alpha_i)_*}$ of coinvariants is 2–torsionfree, where $\alpha_i\colon H_i \to H_i$ is the monodromy homeomorphism. Then the surgery sequence (4–2) *is exact.*

Finally, consider certain surface bundles over surfaces, which have fundamental group in the same class Cl. Let $Kl = \mathbb{RP}^2 \# \mathbb{RP}^2$ be the Klein bottle, whose fundamental group $\pi^\omega = \mathbb{Z}^+ \rtimes \mathbb{Z}^-$ has the indicated orientation. Observe that any non-orientable, compact surface of positive genus admits a collapse map onto Kl.

Corollary 4.9 *Suppose X is a* TOP *4–manifold of the form*

$$X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2)$$

for some $n > 0$ and $r \geq 0$, and each summand X_i is the total space of a fiber bundle

$$\Sigma_i^f \longrightarrow X_i \longrightarrow \Sigma_i^b.$$

Here, we suppose the fiber and base are compact, connected 2–manifolds, $\Sigma_i^f \neq \mathbb{RP}^2$, and Σ_i^b has positive genus. Moreover, if X_i is non-orientable, we assume that the fiber Σ_i^f is orientable and that the monodromy action of $\pi_1(\Sigma_i^b)$ of the base preserves any orientation on the fiber. Then the surgery sequence (4–2) *is exact.*

4.1 Proofs in the orientable case

Proof of Theorem 4.1 for orientable X Suppose X satisfies Hypothesis 3.2. Let $f\colon M \to X$ be a degree one, normal map of compact, connected, oriented TOP 4–manifolds such that: $\partial f = \mathbf{1}_{\partial X}$ on the boundary, f has vanishing surgery obstruction $\sigma_*(f) = 0$, and f has vanishing Kirby–Siebenmann stable PL triangulation obstruction $\mathrm{ks}(f) = 0$.

Then, by Proposition 3.5, f is TOP normally bordant to a homotopy self-equivalence $h\colon X \to X$ relative to ∂X. Thus exactness is proven at $\mathcal{N}_{\mathrm{TOP0}}(X)$.

Note, since X is orientable (1–3), that

$$\mathrm{Tor}_1(H_0(\pi;\mathbb{Z}^\omega),\mathbb{Z}_2) = \mathrm{Tor}_1(\mathbb{Z},\mathbb{Z}_2) = 0.$$

Then, by Proposition 2.1, $\mathrm{red}_{\mathrm{TOP}}$ induces an isomorphism from $\mathcal{N}_{\mathrm{DIFF}}(X)$ to $\mathcal{N}_{\mathrm{TOP0}}(X)$. Thus exactness is proven at $\mathcal{N}_{\mathrm{DIFF}}(X)$. □

Proof of Corollary 4.2 The result follows immediately from Theorem 4.1, since

$$\kappa_2\colon H_2(\pi;\mathbb{Z}_2) = 0 \longrightarrow L_4^h(\mathbb{Z}[\pi])$$

is automatically injective. □

Proof of Corollary 4.3 By Theorem 4.1, it suffices to show κ_2 is injective by induction on n. Suppose $n=0$. Then it is automatically injective:

$$\kappa_2\colon H_2(1;\mathbb{Z}_2) = 0 \longrightarrow L_4^h(\mathbb{Z}[1]) = \mathbb{Z}.$$

Let Λ be a torsion-free lattice in either $\mathrm{Isom}(\mathbb{E}^m)$ or $\mathrm{Isom}(\mathbb{H}^m)$ or $\mathrm{Isom}(\mathbb{CH}^m)$. Since isometric quotients of the homogeneous[2] spaces $\mathbb{E}^m$ or $\mathbb{H}^m$ or $\mathbb{CH}^m$ have uniformly bounded curvature matrix (hence A–regular), by Farrell–Jones [8, Proposition 0.10], the connective (integral) assembly map is split injective:

$$A_\Lambda\langle 1\rangle\colon H_4(\Lambda;\mathrm{G/TOP}) = H_0(\Lambda;\mathbb{Z}) \oplus H_2(\Lambda;\mathbb{Z}_2) \longrightarrow L_4^h(\mathbb{Z}[\Lambda]).$$

The decomposition of the domain follows from the Atiyah–Hirzebruch spectral sequence for the connective spectrum $\mathrm{G/TOP} = \mathbf{L}.\langle 1\rangle$. Therefore the integral lift of the 2–local component is injective:

$$\kappa_2\colon H_2(\Lambda;\mathbb{Z}_2) \longrightarrow L_4^h(\mathbb{Z}[\Lambda]).$$

Suppose for some $n>0$ that κ_2 is injective for $\pi_n = \bigstar_{i=1}^{n-1}\Lambda_i$. Let Λ_n be a torsion-free lattice in either $\mathrm{Isom}(\mathbb{E}^{m_n})$ or $\mathrm{Isom}(\mathbb{H}^{m_n})$. Write $\pi := \pi_n * \Lambda_n$. By the Mayer–Vietoris sequence in K–theory (see Waldhausen [32]), and since

$$\mathrm{Wh}_1(1) = \mathrm{Wh}_0(1) = 0 = \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[1];\mathbb{Z}[\pi_n - 1],\mathbb{Z}[\Lambda_n - 1]),$$

note $\mathrm{Wh}_1(\pi) = \mathrm{Wh}_1(\pi_n) \oplus \mathrm{Wh}_1(\Lambda_n)$. Also, since the trivial group 1 is square-root closed in the torsion-free groups π and Λ_n, we have

$$\mathrm{UNil}_4^h(\mathbb{Z}[1];\mathbb{Z}[\pi_n - 1],\mathbb{Z}[\Lambda_n - 1]) = 0$$

[2] *Homogeneous*: the full group of isometries acts transitively on the riemannian manifold.

by a corollary of Cappell [2, Corollary 4], which was proven in [3, Lemmas II.7, 8, 9]. So

$$L_4^h(\mathbb{Z}[\pi]) = \widetilde{L}_4^h(\mathbb{Z}[\pi_n]) \oplus L_4^h(\mathbb{Z}[1]) \oplus \widetilde{L}_4^h(\mathbb{Z}[\Lambda_n])$$
$$H_2(\pi;\mathbb{Z}_2) = H_2(\pi_n;\mathbb{Z}_2) \oplus H_2(1;\mathbb{Z}_2) \oplus H_2(\Lambda_n;\mathbb{Z}_2)$$

by the Mayer–Vietoris sequences in L–theory [2, Theorem 5(ii)] and group homology (see Brown [1, Section VII.9]). Therefore, since κ_2 factors through the summand $\widetilde{L}_4^h(\mathbb{Z}[-])$, we conclude that

$$\kappa_2 = \begin{pmatrix} \kappa_2 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_2 \end{pmatrix} \colon H_2(\pi;\mathbb{Z}_2) \longrightarrow L_4^h(\mathbb{Z}[\pi])$$

is injective. The corollary is proven for n factors, thus completing the induction. □

Proof of Corollary 4.6 Since each O_i is odd-torsion, a transfer argument [1] shows that $H_2(O_i;\mathbb{Z}_2)=0$. Then, by the Mayer–Vietoris sequence in group homology [1, Section VII.9] and induction, we conclude $H_2(\bigstar_{i=1}^n O_i;\mathbb{Z}_2) = 0$. Therefore κ_2 is automatically injective. □

4.2 Proofs in the non-orientable case

Proof of Theorem 4.1 for non-orientable X Suppose X satisfies Hypothesis 3.3. Let $f\colon M\to X$ be a degree one, TOP normal map such that $\sigma_*(f)=0$ and $\mathrm{ks}(f)=0$. Then, by Proposition 3.5, f is TOP normally bordant to a homotopy self-equivalence $h\colon X\to X$ relative to ∂X. Thus exactness is proven at $\mathcal{N}_{\mathrm{TOP}0}(X)$.

Further suppose the non-orientable 4–manifold X is smooth. Since in this case we assume that π has an orientation-reversing element of order two, by Cappell and Shaneson [4, Theorem 3.1], there exists a closed DIFF 4–manifold X_1 and a simple homotopy equivalence $h_1\colon X_1\to X$ such that $\eta_{\mathrm{DIFF}}(h_1)\neq 0$ and $\eta_{\mathrm{TOP}}(h_1)=0$. The above argument of Proof 4.1 in the orientable case shows for non-orientable X that the kernel of $\mathcal{N}_{\mathrm{DIFF}}(X)\to\mathcal{N}_{\mathrm{TOP}}(X)$ is cyclic of order two. Note

$$\eta_{\mathrm{DIFF}}(h\circ h_1) = \eta_{\mathrm{DIFF}}(h) + h_*\,\eta_{\mathrm{DIFF}}(h_1) \neq \eta_{\mathrm{DIFF}}(h)$$
$$\eta_{\mathrm{TOP}}(h\circ h_1) = \eta_{\mathrm{TOP}}(h) + h_*\,\eta_{\mathrm{TOP}}(h_1) = \eta_{\mathrm{TOP}}(h)$$

by the surgery sum formula given in Proposition 4.10 and Ranicki [23, Proposition 4.3]. Therefore f is DIFF normally bordant to either the simple homotopy equivalence $h\colon X\to X$ or $h\circ h_1\colon X_1\to X$ relative to ∂X. Thus exactness is proven at $\mathcal{N}_{\mathrm{DIFF}}(X)$.

Suppose X satisfies Hypothesis 3.4. Let $f\colon M\to X$ be a degree one, TOP normal map such that $\sigma_*(f)=0$ and $\mathrm{ks}(f)=0$. Then, by Proposition 3.5, f is TOP normally bordant to a homotopy self-equivalence $h\colon X \to X$ relative to ∂X. Thus exactness is proven at $\mathcal{N}_{\mathrm{TOP}0}(X)$. □

The following formula generalizes an analogous result of J. Shaneson [28, Proposition 2.2], which was stated in the smooth case.

Proposition 4.10 *Suppose M, N, X are compact* PL *manifolds. Let $f\colon M \to N$ be a degree one,* PL *normal map such that $\partial f\colon \partial M \to \partial N$ is the identity map. Let $h\colon N \to X$ be a homotopy equivalence such that $\partial h\colon \partial N \to \partial X$ is the identity. Then there is a sum formula for* PL *normal invariants:*

$$\widehat{h \circ f} = \eta(h) + h_*(\widehat{f}) \in [X/\partial X, \mathrm{G/PL}]_0.$$

Proof Any element of the abelian group $[X/\partial X, \mathrm{G/PL}]_0$ is the stable equivalence class of a pair (ξ, t), where ξ is a PL fiber bundle over $X/\partial X$ with fiber $(\mathbb{R}^n, 0)$ for some n, and $t\colon \xi \to \varepsilon^n = (X/\partial X) \times (\mathbb{R}^n, 0)$ is a fiber homotopy equivalence of the absolute fiber $\mathbb{R}^n - \{0\} \simeq S^{n-1}$. The abelian group structure on $[X/\partial X, \mathrm{G/PL}]_0$ is the π_0 of the Whitney sum H–space structure on the Δ–set $\mathrm{Map}_0(X/\partial X, G/\mathrm{PL})$ defined rigorously in Rourke [25, Proposition 2.3].

Let ν_M be the PL normal $(\mathbb{R}^n, 0)$–bundle of the unique isotopy class of embedding $M \hookrightarrow S^{n+\dim(M)}$, where $n > \dim(M) + 1$. For a certain stable fiber homotopy trivialization s induced by the embedding of M and the normal map f, the normal invariant of the degree one, PL normal map $(f, \xi)\colon M \to N$ is defined by

$$\widehat{(f, \xi)} = (\xi - \nu_N, s).$$

For the homotopy equivalence $h\colon N \to X$ with homotopy inverse $\overline{h}\colon X \to N$ and any PL bundle χ over N, define the pushforward bundle $h_*(\chi) := (\overline{h})^*(\chi)$ over X. Let r be the stable fiber homotopy trivialization associated to the degree one, PL normal map $(h, h_*(\nu_N))$. Then note

$$\begin{aligned}\widehat{h \circ f} &= (h_*(\xi) - \nu_X, r + h_*(s))\\ &= (h_*(\nu_N) - \nu_X, r) + (h_*(\xi) - h_*(\nu_N), h_*(s)) = \eta(h) + h_*(\widehat{f}).\end{aligned}$$

Here, the addition is the Whitney sum of stable PL bundles with fiber $(\mathbb{R}^n, 0)$ equipped with stable fiber homotopy trivializations. □

Proposition 4.11 (López de Medrano) *The following map is an isomorphism:*

$$\kappa_2\colon H_2(C_2; \mathbb{Z}_2) \longrightarrow L^h_4(\mathbb{Z}[C_2]^-).$$

Note that the source and target of κ_2 are isomorphic to $\mathbb{Z}_2$ (see Wall [34, Theorem 13A.1]).

Proof Observe that the connective assembly map

$$A_\pi\langle 1\rangle\colon\ H_\oplus(\pi;\mathrm{G/TOP}^\omega) \longrightarrow L^h_\oplus(\mathbb{Z}[\pi]^\omega)$$

is a homomorphism of $L^*(\mathbb{Z})$–modules. Then, by action of the symmetric complex $\sigma^*(\mathbb{CP}^2) \in L^4(\mathbb{Z})$, there is a commutative diagram (Figure 4.1) where the vertical

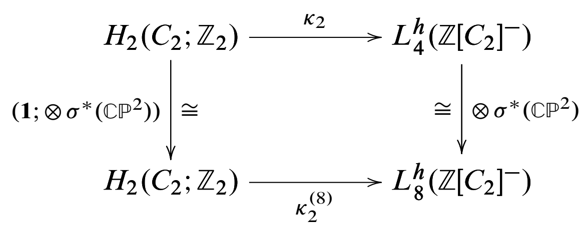

Figure 4.1: Periodicity for κ_2

maps are isomorphisms by decorated periodicity (see Sullivan [29]). So it is equivalent to show that $\kappa_2^{(8)}$ is non-trivial.

Consider the commutative diagram of Figure 4.2. The PL surgery obstruction map σ_*

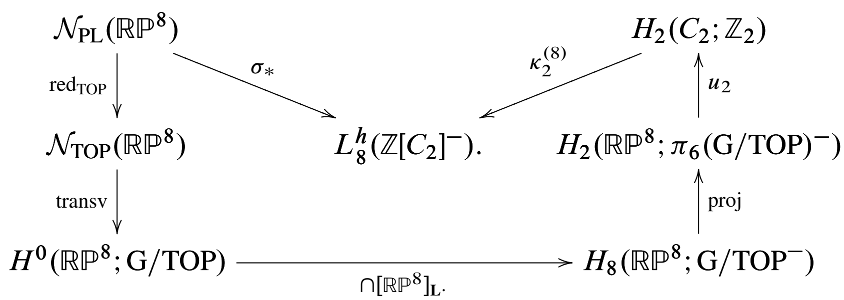

Figure 4.2: Calculation of $\kappa_2^{(8)}$

for $\mathbb{RP}^8$ was shown to be non-trivial in López de Medrano [19, Theorem IV.3.3] and given by a codimension two Kervaire–Arf invariant. So the map $\kappa_2^{(8)}$ is non-trivial. Therefore κ_2 is an isomorphism. □

Proof of Corollary 4.7 We proceed by induction on the number $n > 0$ of free C_2 factors in π to show that $\kappa_2\colon H_2(\pi;\mathbb{Z}_2) \to L^h_4(\mathbb{Z}[\pi]^\omega)$ is injective.

Suppose $n = 1$. Then

$$\kappa_2\colon H_2(C_2;\mathbb{Z}_2) \longrightarrow L^h_4(\mathbb{Z}[C_2]^-)$$

is an isomorphism by Proposition 4.11.

Suppose the inductive hypothesis is true for $n > 0$. Write

$$\pi_n := \bigstar(n-1)(C_2), \qquad \pi^\omega = (\pi_n)^{\omega_n} * (C_2)^-.$$

By the Mayer–Vietoris sequence in group homology (see Brown [1, Section VII.9]), we have

$$H_2(\pi;\mathbb{Z}_2) = H_2(\pi_n;\mathbb{Z}_2) \oplus H_2(C_2;\mathbb{Z}_2).$$

By the Mayer–Vietoris sequence in L^h_*–theory (see Cappell [2, Theorem 5(ii)]), using the Mayer–Vietoris sequence in K–theory (see Waldhausen [32]) for h–decorations, we have

$$L^h_4(\mathbb{Z}[\pi]^\omega) = L^h_4(\mathbb{Z}[\pi_n]^{\omega_n}) \oplus L^h_4(\mathbb{Z}[C_2]^-) \oplus \mathrm{UNil}^h_4(\mathbb{Z};\mathbb{Z}[\pi_n - 1]^{\omega_n}, \mathbb{Z}^-).$$

Since κ_2 is natural in groups with orientation character, we have

$$\kappa_2 = \begin{pmatrix} \kappa_2 & 0 & 0 \\ 0 & \kappa_2 & 0 \end{pmatrix} \colon H_2(\pi;\mathbb{Z}_2) \longrightarrow L^h_4(\mathbb{Z}[\pi]^\omega).$$

Therefore, by induction, we obtain that κ_2 is injective for the free product π^ω.

Let $i > 0$. If $X_i = S^2 \times \mathbb{RP}^2$ or $X_i = S^2 \rtimes \mathbb{RP}^2$, then a Leray–Serre spectral sequence argument shows that

$$\mathrm{Ker}(u_2) = \mathbb{Z}_2[S^2] \subseteq \mathbb{Z}_2[S^2] \oplus \mathbb{Z}_2[\mathbb{RP}^2] = \mathrm{Ker}(v_2).$$

If $X_i = P_j \#_{S^1} P_k$, then a Mayer–Vietoris and Poincaré duality argument shows that

$$\mathrm{Ker}(u_2) = \mathbb{Z}_2([\mathbb{RP}^2_j] + [\mathbb{RP}^2_k]) = \mathrm{Ker}(v_2).$$

Hence Theorem 4.1 applies for both sets of X_i. □

4.3 Proofs in both cases of orientability

Proof of Corollary 4.8 Write Λ_i as the fundamental group and ω_i as the orientation character of X_i. Then the connective assembly map

$$A_{\Lambda_i}\langle 1\rangle\colon H_4(B\Lambda_i;\mathrm{G/TOP}^{\omega_i}) \longrightarrow L^h_4(\mathbb{Z}[\Lambda_i]^{\omega_i})$$

is an isomorphism, as follows. Note $\Lambda_i = \pi_1(X_i) = \pi_1(H_i) \rtimes \mathbb{Z}$.

Suppose H_i has type (1). Then, by Wall [34, Theorem 13A.8], the map $A_{\Lambda_i}\langle 1\rangle$ is an isomorphism in dimension 4, given by signature (mod 2 if $\omega_i \neq 1$).

Suppose H_i has type (2). Then, by Roushon [27, Theorem 1.1(1)] if ∂H_i is non-empty and by Roushon [26, Theorem 1.2] if ∂H_i is empty, the connective assembly map $A_{\pi_1(H_i)}\langle 1\rangle$ is an isomorphism in dimensions 4 and 5. Since $\pi_1(H_i)$, hence Λ_i, is a member of Waldhausen's class Cl [32, Proposition 19.5(6,8)], we obtain $\mathrm{Wh}_*(\Lambda_i) = 0$ by [32, Proposition 19.3]. So, by the Ranicki–Shaneson sequence in L^h_*–theory [22, Theorem 5.2], and by the five-lemma, we obtain that the connective assembly map $A_{\Lambda_i}\langle 1\rangle$ is an isomorphism in dimension 4.

Therefore, for both types, the integral lift κ_2 of the 2–local component of $A_{\Lambda_i}\langle 1\rangle$ is injective. So, by the inductive Mayer–Vietoris argument of Corollary 4.3, we conclude that κ_2 is injective for the free product $\pi = \bigstar_{i=1}^{n} \Lambda_i$.

If X is orientable, then X satisfies Hypothesis 3.2. Otherwise, suppose X is non-orientable. Then consider all X_i which are non-orientable. If H_i is orientable, then the monodromy homeomorphism $\alpha_i\colon H_i \to H_i$ must reverse orientation. So there is a lift $\pi_1(X_i) \to \pi_1(S^1) \xrightarrow{1} \mathbb{Z}$ of the orientation character. Otherwise, if H_i is non-orientable, then $H_1(X_i) = H_1(H_i)_{(\alpha_i)_*} \times \mathbb{Z}$ by the Wang sequence and is 2–torsionfree by hypothesis. So there is a lift $\pi_1(X_i) \to H_1(X_i) \to \mathbb{Z}$ of the orientation character. Hence there is an epimorphism $(\Lambda_i)^{\omega_i} \to \mathbb{Z}^-$. Thus there is an epimorphism $\pi^\omega \to \mathbb{Z}^-$. So X satisfies Hypothesis 3.4. Therefore, in both cases of orientability of X, Theorem 4.1 is applicable. □

Proof of Corollary 4.9 Write Λ_i as the fundamental group and ω_i as the orientation character of X_i.

Suppose $\Sigma_i^f = S^2$. Since $\pi_1(X_i) = \pi_1(\Sigma_i^b)$ is the fundamental group of an aspherical, compact surface, by the proof of a result of J Hillman [12, Lemma 8], the connective assembly map $A_{\Lambda_i}\langle 1\rangle$ is an isomorphism in dimension 4.

Suppose $\Sigma_i^f \neq S^2$. Since Σ_i^f and Σ_i^b are aspherical, X_i is aspherical. By a result of J Hillman [12, Lemma 6], the connective assembly map $A_{\Lambda_i}\langle 1\rangle$ is an isomorphism in dimension 4.

Indeed, in both cases, the Mayer–Vietoris argument extends to fiber bundles where the surfaces are aspherical, compact, and connected, which are possibly non-orientable and with non-empty boundary (see Cavicchioli, Hegenbarth and Spaggiari [5, Theorem 2.4] for details).

Then the integral lift of the 2–local component of $A_{\lambda_i}\langle 1\rangle$ is injective:

$$\kappa_2\colon H_2(\Lambda_i; \mathbb{Z}_2) \longrightarrow L^h_4(\mathbb{Z}[\Lambda_i]^{\omega_i}).$$

So, by the Mayer–Vietoris argument of Corollary 4.3, we conclude that κ_2 is an injective for the free product $\pi = \bigstar_{i=1}^{n} \Lambda_i$.

If X is orientable, then X satisfies Hypothesis 3.2. Otherwise, suppose X is non-orientable. Then consider all X_i which are non-orientable. By hypothesis, the fiber Σ_i^f is orientable and the monodromy action of $\pi_1(\Sigma_i^b)$ on $H_2(\Sigma_i^f;\mathbb{Z})$, induced by the bundle $\Sigma_i^f \to X_i \to \Sigma_i^b$, is trivial. We must have that the surface Σ_i^b is non-orientable. So, since Σ_i^b is the connected sum of a compact orientable surface and non-zero copies of Klein bottles Kl, by collapsing to any Kl–summand, there is a lift $\pi_1(X_i) \to \pi_1(\Sigma_i^b) \to \pi_1(Kl) \to \mathbb{Z}$ of the orientation character. Hence there is an epimorphism $(\Lambda_i)^{\omega_i} \to \mathbb{Z}^-$. Thus there is an epimorphism $\pi^\omega \to \mathbb{Z}^-$. So X satisfies Hypothesis 3.4. Therefore, in both cases of orientability of X, Theorem 4.1 is applicable. □

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