Knot Floer homology and Seifert surfaces

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Let K be a knot in S^3 of genus g and let $n > 0$. We show that if $\operatorname{rk}\widehat{HFK}(K,g) < 2^{n+1}$ (where $\widehat{HFK}$ denotes knot Floer homology), in particular if K is an alternating knot such that the leading coefficient a_g of its Alexander polynomial satisfies $|a_g| < 2^{n+1}$, then K has at most n pairwise disjoint nonisotopic genus g Seifert surfaces. For $n = 1$ this implies that K has a unique minimal genus Seifert surface up to isotopy.

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1 Introduction and preliminaries

If S_1 and S_2 are Seifert surfaces of a knot $K \subset S^3$ then S_1 and S_2 are said to be equivalent if $S_1 \cap X(K)$ and $S_2 \cap X(K)$ are ambient isotopic in the knot exterior $X(K) = S^3 \setminus N(K)$, where $N(K)$ is a regular neighborhood of K. In [4] Kakimizu assigned a simplicial complex $MS(K)$ to every knot K in S^3 as follows.

Definition 1.1 $MS(K)$ is a simplicial complex whose vertices are the equivalence classes of the minimal genus Seifert surfaces of K. The equivalence classes $\sigma_0, \dots, \sigma_n$ span an n–simplex if and only if for each $0 \le i \le n$ there is a representative S_i of σ_i such that the surfaces $S_0, \dots, S_n$ are pairwise disjoint.

Scharlemann and Thompson [10] showed that the complex $MS(K)$ is always connected. In other words, if S and T are minimal genus Seifert surfaces for a knot K then there is a sequence $S = S_1, S_2, \dots, S_k = T$ of minimal genus Seifert surfaces such that $S_i \cap S_{i+1} = \varnothing$ for $0 \le i \le k-1$.

The main goal of this short note is to show that for a genus g knot K and for $n > 0$ the condition $\operatorname{rk}\widehat{HFK}(K,g) < 2^{n+1}$ implies $\dim MS(K) < n$, consequently for $n = 1$ the knot K has a unique Seifert surface up to equivalence. This condition involves the use of knot Floer homology introduced by Ozsváth and Szabó in [8] and independently by Rasmussen in [9]. However, when K is alternating then this condition is equivalent to $|a_g| < 2^{n+1}$, where a_g is the leading coefficient of the Alexander polynomial of K. The alternating case is already a new result whose statement doesn't involve knot Floer

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homology. On the other hand, the proof of this particular case seems to need sutured Floer homology techniques, which is a generalization of knot Floer homology that was introduced by the author in [2].

The above statement does not hold for $n = 0$ since every knot has at least one minimal genus Seifert surface. However, it was shown by Ni [6] and the author [3] that $\text{rk}\,\widehat{HFK}(K, g) < 2$ implies that the knot K is fibred, and hence $MS(K)$ is a single point.

To a knot K in S^3 and every $j \in \mathbb{Z}$ knot Floer homology assigns a graded abelian group $\widehat{HFK}(K, j)$ whose Euler characteristic is the coefficient a_j of the Alexander polynomial $\Delta_K(t)$. Ozsváth and Szabó [7] have shown that if K is alternating then $\widehat{HFK}(K, j)$ is nonzero in at most one grading, thus $\text{rk}\,\widehat{HFK}(K, j) = |a_j|$.

Next we are going to review some necessary definitions and results from the theory of sutured manifolds and sutured Floer homology. Sutured manifolds were introduced by Gabai in [1].

Definition 1.2 A *sutured manifold* (M, γ) is a compact oriented 3–manifold M with boundary together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. Furthermore, the interior of each component of $A(\gamma)$ contains a *suture*, ie, a homologically nontrivial oriented simple closed curve. We denote the union of the sutures by $s(\gamma)$.

Finally every component of $R(\gamma) = \partial M \setminus \text{Int}(\gamma)$ is oriented. Define $R_+(\gamma)$ (or $R_-(\gamma)$) to be those components of $\partial M \setminus \text{Int}(\gamma)$ whose normal vectors point out of (into) M. The orientation on $R(\gamma)$ must be coherent with respect to $s(\gamma)$, ie, if δ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then δ must represent the same homology class in $H_1(\gamma)$ as some suture.

A sutured manifold is called *taut* if $R(\gamma)$ is incompressible and Thurston norm minimizing in $H_2(M, \gamma)$.

The following definition was introduced in [2].

Definition 1.3 A sutured manifold (M, γ) is called *balanced* if M has no closed components, $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$, and the map $\pi_0(A(\gamma)) \to \pi_0(\partial M)$ is surjective.

Example 1.4 If R is a Seifert surface of a knot K in S^3 then we can associate to it a balanced sutured manifold $S^3(R) = (M, \gamma)$ such that $M = S^3 \setminus (R \times I)$ and $\gamma = K \times I$. Observe that $R_-(\gamma) = R \times \{0\}$ and $R_+(\gamma) = R \times \{1\}$. Furthermore, $S^3(R)$ is taut if and only if R is of minimal genus.

Sutured Floer homology is an invariant of balanced sutured manifolds defined by the author in [2], and is a common generalization of the invariants $\widehat{HF}$ and $\widehat{HFK}$. It assigns an abelian group $SFH(M,\gamma)$ to each balanced sutured manifold (M,γ). The following theorem is a special case of [3, Theorem 1.5].

Theorem 1.5 *Let K be a genus g knot in S^3 and suppose that R is a minimal genus Seifert surface for K. Then*

$$SFH(S^3(R)) \cong \widehat{HFK}(K,g).$$

A sutured manifold (M,γ) is called a product if it is homeomorphic to $(\Sigma \times I, \partial\Sigma \times I)$, where Σ is an oriented surface with boundary. If (M,γ) is a product, $SFH(M,\gamma) \cong \mathbb{Z}$. Let us recall [3, Theorem 1.4] and [3, Theorem 9.3].

Theorem 1.6 *If (M,γ) is a taut balanced sutured manifold then* $\mathrm{rk}\, SFH(M,\gamma) \geq 1$. *Furthermore, if (M,γ) is not a product then* $\mathrm{rk}\, SFH(M,\gamma) \geq 2$.

Definition 1.7 Let (M,γ) be a balanced sutured manifold. An oriented surface $S \subset M$ is called a *horizontal surface* if S is open, $\partial S = s(\gamma)$ in an oriented sense; moreover, $[S] = [R_+(\gamma)]$ in $H_2(M,\gamma)$, and $\chi(S) = \chi(R_+(\gamma))$.

A horizontal surface S defines a horizontal decomposition

$$(M,\gamma) \rightsquigarrow^S (M_-,\gamma_-) \coprod (M_+,\gamma_+)$$

as follows. Let $M_\pm$ be the union of the components of $M \setminus \mathrm{Int}(N(S))$ that intersect $R_\pm(\gamma)$. Similarly, let $\gamma_\pm$ be the union of the components of $\gamma \setminus \mathrm{Int}(N(S))$ that intersect $R_\pm(\gamma)$.

The following proposition is a special case of [3, Proposition 8.6].

Proposition 1.8 *Suppose that (M,γ) is a taut balanced sutured manifold and let S be a horizontal surface in it. Then*

$$\mathrm{rk}\, SFH(M,\gamma) = \mathrm{rk}\, SFH(M_-,\gamma_-) \cdot \mathrm{rk}\, SFH(M_+,\gamma_+).$$

The following definition can be found for example in [6].

Definition 1.9 A balanced sutured manifold (M,γ) is called *horizontally prime* if every horizontal surface S in (M,γ) is isotopic to either $R_+(\gamma)$ or $R_-(\gamma)$ rel γ.

2 The results

Theorem 2.1 *Let (M,γ) be a taut balanced sutured manifold such that both $R_+(\gamma)$ and $R_-(\gamma)$ are connected. Suppose that there is a sequence of pairwise disjoint nonisotopic connected horizontal surfaces $R_-(\gamma) = S_0, S_1, \dots, S_n = R_+(\gamma)$. Then*

$$\operatorname{rk} SFH(M,\gamma) \geq 2^n.$$

Proof We prove the theorem using induction on n. If $n = 1$ then (M,γ) is not a product since $R_-(\gamma)$ and $R_+(\gamma)$ are nonisotopic. Thus Theorem 1.6 implies that $\operatorname{rk} SFH(M,\gamma) \geq 2$.

Now suppose that the theorem is true for $n-1$. Since each S_k is connected we can suppose without loss of generality that S_1 separates S_i and S_0 for every $i \geq 2$. Let (M_-,γ_-) and (M_+,γ_+) be the sutured manifolds obtained after horizontally decomposing (M,γ) along S_1. Note that both (M_-,γ_-) and (M_+,γ_+) are taut. As S_0 and S_1 are nonisotopic, (M_-,γ_-) is not a product so as before $\operatorname{rk} SFH(M_-,\gamma_-) \geq 2$. Applying the induction hypothesis to (M_+,γ_+) and to the surfaces $R_-(\gamma_+), S_2, \dots, S_n = R_+(\gamma_+)$ we get that $\operatorname{rk} SFH(M_+,\gamma_+) \geq 2^{n-1}$. So using Proposition 1.8 we see that $\operatorname{rk} SFH(M,\gamma) \geq 2^n$. □

Corollary 2.2 *If (M,γ) is a taut balanced sutured manifold and $\operatorname{rk} SFH(M,\gamma) < 4$ then (M,γ) is horizontally prime. More generally, if $n > 0$ and $\operatorname{rk} SFH(M,\gamma) < 2^{n+1}$ then (M,γ) can be cut into horizontally prime pieces by less than n horizontal decompositions.*

Proof Suppose that $\operatorname{rk} SFH(M,\gamma) < 2^{n+1}$. If (M,γ) is not horizontally prime then there is a surface S_1 in (M,γ) which is not isotopic to $R_\pm(\gamma)$. Decomposing (M,γ) along S_1 we get two sutured manifolds (M_-,γ_-) and (M_+,γ_+). If they are not both horizontally prime then repeat the above process with a nonprime piece and obtain a horizontal surface S_2, etc. This process has to end in less than n steps according to Theorem 2.1. □

Theorem 2.3 *Let K be a knot in S^3 of genus g and let $n > 0$. If $\operatorname{rk} \widehat{HFK}(K,g) < 2^{n+1}$ then K has at most n pairwise disjoint nonisotopic genus g Seifert surfaces, in other words, $\dim MS(K) < n$. If $n = 1$ then K has a unique Seifert surface up to equivalence.*

Proof Suppose that $R, S_1, \dots, S_n$ are pairwise disjoint nonisotopic Seifert surfaces for K. According to Theorem 1.5 we have $\widehat{HFK}(K,g) \cong SFH(S^3(R))$. Let $S^3(R) =$

(M,γ). If $R_+(\gamma)$ and $R_-(\gamma)$ were isotopic then (M,γ) would be a product and S_1 and R would be equivalent. So the surfaces $R_-(\gamma) = S_0, S_1, \dots, S_n, S_{n+1} = R_+(\gamma)$ satisfy the conditions of Theorem 2.1, thus $\operatorname{rk} SFH(S^3(R)) \ge 2^{n+1}$, a contradiction.

In particular, if $n = 1$ then $\dim MS(K) = 0$. But according to [10] the complex $MS(K)$ is connected, so it consists of a single point. □

Corollary 2.4 *Suppose that K is an alternating knot in S^3 of genus g and let $n > 0$. If the leading coefficient a_g of its Alexander polynomial satisfies $|a_g| < 2^{n+1}$ then* $\dim MS(K) < n$. *If $|a_g| < 4$ then K has a unique Seifert surface up to equivalence.*

Proof This follows from Theorem 2.3 and the fact that for alternating knots the equality $\operatorname{rk} \widehat{HFK}(K,g) = |a_g|$ holds. □

Remark In [5] Kakimizu classified the minimal genus Seifert surfaces of all the prime knots with at most 10 crossings. The $n = 1$ case of Corollary 2.4 is sharp since the knot 7_4 is alternating, the leading coefficient of its Alexander polynomial is 4, and has 2 inequivalent minimal genus Seifert surfaces. On the other hand, the Alexander polynomial of the alternating knot 9_2 is also 4, but has a unique minimal genus Seifert surface up to equivalence.

Also note that [3, Theorem 1.7] implies that if the leading coefficient a_g of the Alexander polynomial of an alternating knot K satisfies $|a_g| < 4$ then the knot exterior $X(K)$ admits a depth ≤ 2 taut foliation transversal to $\partial X(K)$. Indeed, for alternating knots $g = g(K)$ and $|a_g| = \operatorname{rk} \widehat{HFK}(K,g) \ne 0$, so the conditions of [3, Thorem 1.7] are satisfied.

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