Strong accessibility for hyperbolic groups

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We use an accessibility result of Delzant and Potyagailo to prove Swarup's Strong Accessibility Conjecture for Gromov hyperbolic groups with no 2–torsion. It follows that, if M is an irreducible, orientable, compact 3–manifold with hyperbolic fundamental group, then any hierarchy in which M is decomposed alternately along compressing disks and essential annuli is finite.

20E08, 20F65; 57M99, 57N35, 20F67

1 Introduction

The theory of group accessibility is made up of "accessibility results" and "strong accessibility results". Accessibility results show that a group can be decomposed as a graph of groups in a maximal way over a specific family of subgroups. Strong accessibility results show that a group has a finite hierarchy over a family of subgroups, ie a sequence of collections of graph of groups decompositions, beginning with a decomposition of the original group, and such that each later collection contains a decomposition of each of the vertex groups of the previous collection.

When decomposing over finite subgroups, these two notions are equivalent. In 1940, Gruško proved in [9] that finitely generated groups are accessible over the trivial group, ie, admit maximal free product decompositions. Wall coined the term "accessible" in this context in the early 1970's, and conjectured that every finitely generated group is accessible over finite subgroups (see Wall [18]). In 1985, Dunwoody proved in [7] that finitely presented groups are accessible over finite subgroups. (In fact, both Gruško and Dunwoody showed that any decomposition of a finitely generated or finitely presented group respectively over the appropriate family of subgroups has a maximal refinement over that family.) In [8], published in 1993, Dunwoody provided an example of a finitely generated group that is not accessible over finite subgroups.

As for the question of accessibility over more general families of subgroups, Bestvina and Feighn showed in [2] that, over any family of subgroups that are "small", any graph of groups decomposition of a finitely presented group can be refined to a maximal

Published: 5 September 2008 DOI: 10.2140/agt.2008.8.1459

one. (Any group that does not contain a copy of the free group on two generators, for example, is small.)

In [6], Delzant and Potyagailo proved a very general strong accessibility result. They showed that any finitely presented group without 2–torsion admits a finite hierarchy over any family of "elementary" subgroups (see Definition 7). In this paper, we use their work to prove the following strong accessibility result:

Theorem 22 *Let G be a hyperbolic group with no 2–torsion. Decompose G maximally as a graph of groups over finite subgroups, and then take the resulting vertex groups, and decompose those maximally as graphs of groups over two-ended subgroups. Now repeat this process on the new vertex groups and so on. Then this process must eventually terminate, with subgroups of G which are unsplittable over finite and two-ended subgroups.*

Swarup conjectured this result, without the hypothesis of G having no 2–torsion. In Bestvina's *Questions in geometric group theory* [1], this is referred to as Swarup's Strong Accessibility Conjecture.

This theorem is not a special case of the strong accessibility result of Delzant and Potyagailo for two reasons. Firstly, a hierarchy from [6] is over one fixed family of subgroups. For this result, however, we alternate between decomposing over finite subgroups and two-ended subgroups. Secondly, given a group and a family of elementary subgroups, [6] shows the existence of one finite hierarchy over the family. For Swarup's conjecture, it must instead be shown that any hierarchy as described is finite. By analyzing equivariant maps between G–trees, we are able to overcome these difficulties.

As a corollary to Theorem 22, we get the following result about finite hierarchies in 3–manifolds:

Theorem 25 *Let M be an irreducible, orientable, compact 3–manifold with hyperbolic fundamental group. The process of decomposing M along any maximal, disjoint collection of compressing disks, then decomposing the resulting manifolds along maximal, disjoint collections of essential annuli, then the resulting manifolds along compressing disks, then again along essential annuli and so on, must eventually terminate with a collection of manifolds, each of which has incompressible boundary and admits no essential annuli, or is a 3–ball.*

Acknowledgments I am grateful to my advisor, Peter Scott, for all his guidance and to the referee for many helpful comments. This research was supported in part by NSF grant DMS-0602191.

2 Preliminaries

To begin, we now review some basic notions and terminology that we will need. We shall first discuss what we will need from Bass–Serre theory.

Let a group G act simplicially on the left on a simplicial tree τ, and let the action be without inversions, ie, such that no element of G fixes an edge of τ, but swaps its vertices. Then we say that τ is a G*–tree*.

Associate to each vertex v_0 of $\Gamma = G\backslash\tau$ the stabilizer V of one of its preimages v under the projection map $\tau \to G\backslash\tau$, and associate to each edge e_0 the stabilizer E of one of its preimages as well. We shall call such V and E *vertex* and *edge groups*, respectively, and note that such groups associated to the vertices and edges of $G\backslash\tau$ are uniquely determined up to conjugacy.

To each pair (v_0, e_0) of a vertex v_0 in Γ and an oriented edge e_0 with terminal vertex v_0, associate an injective homomorphism from E to V induced by the inclusion of the stabilizer of a lift of e_0 into the stabilizer of a lift of v_0. Call Γ, together with all this data, a *graph of groups* structure for G, and denote the graph and data also by Γ. We shall also say that Γ is a *decomposition* of G.

If τ is a G–tree, G does not fix a point in τ, and $\Gamma = G\backslash\tau$ is finite, then we call Γ a *proper decomposition* of G. We shall say G acts *minimally* on a G–tree τ if τ contains no proper, G–invariant subtrees. Note that if G is finitely generated, acts minimally on τ, and τ is not a vertex, then $\Gamma = G\backslash\tau$ must be a finite graph, and hence is a proper decomposition of G.

Equivariant maps between G–trees will be very important in the proof of our main result, so we define the following:

Definition 1 Let a G*–map* be a simplicial, surjective, G–equivariant map between two G–trees that does not collapse any edge to a vertex.

We will now assume that τ and τ' are G–trees that are not vertices, and let $\Gamma = G\backslash\tau$ and $\Gamma' = G\backslash\tau'$.

Definition 2 If there is a simplicial, surjective, G–equivariant map $\tau' \to \tau$ (which may collapse edges to vertices), then we call the decomposition $\Gamma' = G\backslash\tau'$ a *refinement* of $\Gamma = G\backslash\tau$.

Definition 3 Let $\tau' \to \tau$ be as in Definition 2 and not be a simplicial homeomorphism, and assume that, for each edge e with vertices x and y of τ' which is collapsed to a vertex of τ, either x and y are in the same G–orbit, or $X = \mathrm{stab}(x)$ and $Y = \mathrm{stab}(y)$ properly contain $E = \mathrm{stab}(e)$. Then we call Γ' a *proper refinement* of Γ.

If all edge groups of a decomposition Γ of G are in some family $\mathscr{C}$ of subgroups of G, then we say that Γ is a decomposition of G *over* $\mathscr{C}$. Since the edge groups of Γ are determined only up to conjugacy, $\mathscr{C}$ should be closed under conjugacy. Note that if Γ is a decomposition of G over $\mathscr{C}$, and Γ' is a refinement of Γ such that the associated map $\tau' \to \tau$ does not collapse any edges to vertices, then Γ' is a decomposition of G over the elements of $\mathscr{C}$ and their subgroups.

A decomposition of G with one edge is a *splitting* of G, and a proper decomposition of G with one edge is a *proper splitting* of G. If there exist no proper splittings of G over a family $\mathscr{C}$ as above, then we say that G is *unsplittable over* $\mathscr{C}$.

Note that if G admits a decomposition Γ' over $\mathscr{C}$, arising from an action on a G–tree τ', then for any edge e of Γ' with edge group E, there is a splitting Γ of G associated to e, where Γ has one edge with edge group E, and Γ' is a refinement of Γ. To see this, let e be an edge in τ' with stabilizer E. Then, if $G\cdot\mathrm{int}(e)$ denotes the G–orbit of the interior of e in τ', let τ be the G–tree obtained by collapsing the components of $\tau' - G\cdot\mathrm{int}(e)$ to vertices, with the action of G induced from the action of G on τ'. Then we may take Γ to be $G\backslash\tau$. We observe that if G acts minimally on τ', then Γ must be a proper splitting.

Next, we define the notion of a compatible decomposition, which leads us to the idea of a hierarchy for a group.

Definition 4 If G has a decomposition Γ, and the vertex group of a vertex v of Γ admits a splitting, then we say that the splitting is *compatible* with the decomposition if there exists a refinement of Γ in which v is replaced with an edge that is associated to the splitting, as is described above. Equivalently, the splitting is compatible with the decomposition if a conjugate of each edge group of the edges incident to v is contained in a vertex group of the splitting.

Consider a group G, and a family $\mathscr{C}$ of subgroups of G which is closed under conjugacy.

Definition 5 A *hierarchy* for G over $\mathscr{C}$ is a sequence $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \dots$ of finite sets of conjugacy classes of subgroups of G, defined inductively as follows. The set $\mathcal{G}_0$ contains only (the conjugacy class of) G. If $i > 0$, then for any conjugacy class in $\mathcal{G}_{i-1}$, either $\mathcal{G}_i$ contains that conjugacy class, or $\mathcal{G}_i$ contains the conjugacy classes of the vertex groups of some proper decomposition of a representative of that class over $\mathscr{C}$. We require that at least one representative from $\mathcal{G}_{i-1}$ be decomposed.

Thus, if a hierarchy of G over $\mathscr{C}$ is finite, then its last set $\mathcal{G}_N$ contains only conjugacy classes of subgroups that are unsplittable over $\mathscr{C}$.

We note that the existence of a finite hierarchy over $\mathscr{C}$ does not, in general, imply the existence of any kind of maximal decomposition over $\mathscr{C}$, since the splittings of vertex groups need not be compatible with the decompositions producing those vertex groups.

Definition 6 If $\Gamma = G\backslash\tau$ is a finite decomposition of G over $\mathscr{C}$, and there exists no proper refinement of Γ over $\mathscr{C}$, then we say that Γ is a *maximal decomposition* of G over $\mathscr{C}$.

One could alternatively define a maximal decomposition of G over $\mathscr{C}$ to be a maximal collection of compatible splittings over $\mathscr{C}$. This is stronger than our definition, and often, such collections are infinite. For example, consider any group G that has an infinite descending chain of subgroups $G \supset C_0 \supset C_1 \supset C_2 \supset \ldots$ Then we have:

$$G = G *_{C_0} C_0 = G *_{C_0} C_0 *_{C_1} C_1 = G *_{C_0} C_0 *_{C_1} C_1 *_{C_2} C_2 = \ldots$$

Less trivially, consider the Baumslag–Solitar group $H = \mathrm{BS}(1,2) = \langle x, t : t^{-1}xt = x^2\rangle$. The normal closure of $\langle x\rangle$ in H is isomorphic to $\mathbb{Z}[1/2]$ under addition, by an isomorphism which takes x to 1, and $t^i x t^{-i}$ to $1/2^i$ for each i. Let A_i denote the infinite cyclic subgroup generated by $t^i x t^{-i}$, and let $K = H *_{A_0} H$. Note that $A_0 \subset A_1 \subset A_2 \subset \ldots$, and that K is finitely presented. We can refine the given decomposition of K as many times as we please, for we have that

$$K = H *_{A_0} (A_1 *_{A_1} H) = H *_{A_0} (A_1 *_{A_1} (A_2 *_{A_2} H)) = \ldots$$

with the splitting associated to each edge of any of these decompositions being proper. Thus both of the examples above have sequences of refinements that do not terminate.

That concludes our review of Bass–Serre theory. Now let G be a finitely generated group, with a finite generating set S such that $s \in S$ implies that $s^{-1} \in S$. The *Cayley graph* $\Gamma_G(S)$ of G with respect to S is a graph with vertex set equal to G, and the edges incident to each vertex g in bijection with the elements of S, with the edge corresponding to $s \in S$ connecting g to the vertex gs. By giving each edge in $\Gamma_G(S)$ length one, we may view the graph as a metric space. We remark that, if S and S' are two different finite generating sets for G, then $\Gamma_G(S)$ and $\Gamma_G(S')$ are quasi-isometric. (See, for instance, Bridson and Haefliger [4].)

The number of *ends* of a locally finite simplicial complex X, denoted $e(X)$, is defined to be the supremum over all finite subcomplexes K of X of the number of infinite components of $X - K$. The number of *ends* of a finitely generated group G, $e(G)$, is the number of ends of $\Gamma_G(S)$, where S is a finite generating set for G. This does not depend on the choice of S.

We recall that $e(G) = 2$ if and only if G has a finite index subgroup that is infinite cyclic. (See Scott and Wall [15].)

A finitely generated group G is said to be *hyperbolic* if there is some $\delta > 0$ and some finite generating set S for G such that, for any geodesic triangle in $\Gamma_G(S)$, each side of the triangle is contained in the union of the δ–neighborhoods of the other two sides. While the value of δ depends on our choice of a generating set, we note that hyperbolicity does not. For a proof of this fact, as well as an introduction to hyperbolic groups, we refer the reader to Bridson and Haefliger [4].

Two subgroups H and H' of a group G are said to be *commensurable* if their intersection is of finite index in both. The *commensurizer*, $\mathrm{Comm}_G(H)$, of H in G is the subgroup of elements g of G such that H and gHg^{-1} are commensurable. (We note that $\mathrm{Comm}_G(H)$ is called the "commensurator" of H in G by some authors.)

In [6], Delzant and Potyagailo prove the existence of a finite hierarchy for any finitely presented group with no 2–torsion over any family of "elementary" subgroups, which are defined as follows.

Definition 7 If G is a finitely presented group, and $\mathscr{C}$ a family of subgroups of G, then $\mathscr{C}$ is said to be *elementary* if the following conditions are satisfied:

(1) If $C \in \mathscr{C}$, then all subgroups and conjugates of C are in $\mathscr{C}$.

(2) Each infinite element of $\mathscr{C}$ is contained in a unique maximal subgroup in $\mathscr{C}$.

(3) Ascending unions of finite subgroups in $\mathscr{C}$ are contained in $\mathscr{C}$.

(4) If any $C \in \mathscr{C}$ acts on a tree, then C fixes a point in the tree, fixes a point in the boundary at infinity of the tree, or preserves but permutes two points in the boundary at infinity.

(5) If $C \in \mathscr{C}$ is an infinite, maximal element of $\mathscr{C}$ and $gCg^{-1} = C$, then $g \in C$.

We will be interested in applying the results of [6] to a pair $(G, \mathscr{C})$, when $\mathscr{C}$ is the set of all finite and two-ended subgroups of G. The following proposition is the reason we assume hyperbolicity in Theorem 22.

Proposition 8 *If G is a subgroup of a hyperbolic group, and $\mathscr{C}$ is the set of all finite and two-ended subgroups of G, then $\mathscr{C}$ is elementary.*

In order to show this, we must recall a few facts about hyperbolic groups. The first follows from Lemmas 1.16 and 1.17 of [12].

Lemma 9 *Any two-ended subgroup H of a hyperbolic group G is contained in a unique maximal two-ended subgroup, which is equal to the commensurizer $\mathrm{Comm}_G(H)$ of H in G.*

This implies the following:

Corollary 10 *If G is a subgroup of a hyperbolic group, and $H \subset G$ is two-ended, then H is contained in a unique maximal two-ended subgroup of G, which is equal to $\mathrm{Comm}_G(H)$.*

Proof Let G' be the hyperbolic group containing G, and let $H \subset G$ be two-ended. If $H' \subset G$ is two-ended and $H \subset H'$, then H' must commensurize H, ie $H' \subset \mathrm{Comm}_G(H)$. Note also that $H \subset \mathrm{Comm}_G(H) \subset \mathrm{Comm}_{G'}(H)$ and $\mathrm{Comm}_{G'}(H)$ is two-ended by Lemma 9, so $\mathrm{Comm}_G(H)$ is two-ended. Thus $\mathrm{Comm}_G(H)$ is the unique maximal two-ended subgroup containing H. □

The next fact is Theorem III.Γ.3.2 in [4]:

Theorem 11 *If G is a hyperbolic group, then G contains only finitely many conjugacy classes of finite subgroups.*

We can now give the proof of Proposition 8.

Proof of Proposition 8 Let G be a subgroup of a hyperbolic group, with $\mathscr{C}$ the collection of all finite and two-ended subgroups of G. Then property 1 of Definition 7 is immediate, and property 2 is shown in Corollary 10. Since G is contained in a hyperbolic group, property 3 follows from Theorem 11.

As for property 4, assume that a group C acts on a tree. If C is finite, then C must fix a point of the tree. If C is virtually $\mathbb{Z}$, then C must have an axis, so C preserves two points in the boundary of the tree. Thus $\mathscr{C}$ satisfies 4.

For property 5, let C be any maximal, infinite two-ended subgroup of G. It follows from Corollary 10 that $C = \mathrm{Comm}_G(C)$. If $N_G(H)$ denotes the normalizer of any subgroup H of G, then we always have that

$$H \subset N_G(H) \subset \mathrm{Comm}_G(H).$$

Thus $N_G(C) = C$, so 5 follows. □

We shall conclude with a discussion of folds between G–trees, which were introduced by Stallings [17]. Here, as well as in later arguments, we shall denote vertices and edges with lower case letters, and their stabilizers with the capitalizations of those letters.

Definition 12 A *fold* on a G–tree τ is a G–map that identifies two adjacent edges e and f of τ, identifies $g\cdot e$ with $g\cdot f$, for all $g\in G$, and makes no other identifications. If e and f meet at a vertex v, and are also incident to vertices x and y respectively, then the identification of e with f is such that x is identified with y.

The next result shows that any G–map can often be decomposed into a series of folds. It follows from the proposition in Section 2 of [2].

Proposition 13 *If ϕ is a surjective G–map from a G–tree τ' to a G–tree τ, $G\backslash\tau'$ is finite, and all the edge stabilizers of τ are finitely generated, then $\phi=\phi_n\circ\phi_{n-1}\circ\ldots\circ\phi_1$, for some collection of folds $\{\phi_i\}$.*

As described by Bestvina and Feighn in [2], folds can be broken up into several different types, depending on whether e and f are in the same G–orbit, and whether x, y and v are in the G–orbits of one another. We shall restrict to the case when neither x nor y are in the G–orbit of v. (By subdividing edges of our G–tree, we can always assume that any G–map ϕ as above is a composition of such folds.)

If $\phi\colon \tau'\to\tau$ is such a fold, then ϕ must be one of three types, which, following [2], we will call types IA, IIA, and IIIA. These types correspond to the following three cases: when no $g\in G$ takes x to y, when some $g\in G$ takes x to y and e to f, and when some $g\in G$ takes x to y, but does not take e to f.

Let π denote the projection map $\tau'\to\Gamma'=G\backslash\tau'$, and let $\Phi\colon\Gamma'\to\Gamma$ be the map induced from ϕ. Our figures below indicate how, in each case, Φ will alter $\pi(e\cup f)$. Since Φ cannot alter the underlying graph, or edge or vertex groups, of $\Gamma'-\pi(e\cup f)$, these must describe Φ completely.

When no $g\in G$ takes x to y, we will say that the fold is of type IA. In this case, $\pi(e\cup f)$ will change as indicated in Figure 1.

A fold of type IIA occurs when some $g\in G$ takes x to y and takes e to f, in which case we have that $g\in V$, the stabilizer of v. Here, the image under π of the segment $e\cup f$ is a single edge, and folding changes only the labeling of Γ'. See Figure 2.

Lastly, we have a fold of type IIIA when some $g\in G$ takes x to y and does not take e to f. Note that then g translates along an axis containing e and f. In Γ', we get what is shown in Figure 3.

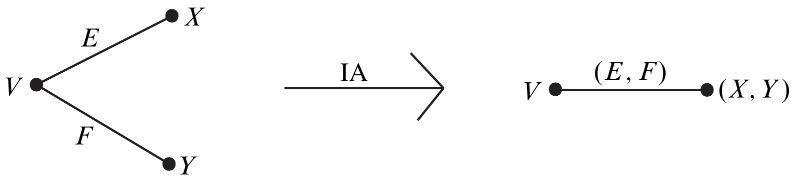

Figure 1: A fold of type IA, with vertices and edges labeled with their associated groups

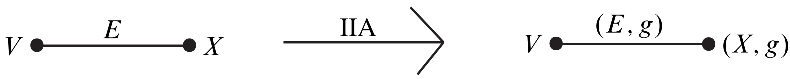

Figure 2: A fold of type IIA

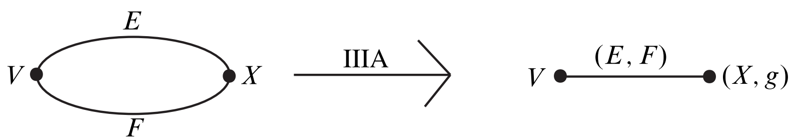

Figure 3: A fold of type IIIA

3 Strong accessibility

In this section, we shall prove Swarup's conjecture for hyperbolic groups with no 2–torsion. We shall first define the notion of complexity used by Delzant and Potyagailo in [6], and then carefully state their result.

Let G be a finitely presented group, let $\mathscr{C}$ be a family of elementary subgroups of G, and note that G is the fundamental group of a finite, two-dimensional simplicial orbihedron Π for which vertex stabilizers are in $\mathscr{C}$. (For example, G is the fundamental group of a finite, two-dimensional simplicial complex. In this case, vertex stabilizers are equal to the trivial group.) For any such Π, we define $T(\Pi)$ to be the number of faces of Π, and $b_1(\Pi)$ to be the first Betti number of the underlying space. Then we define the complexity of Π to be

$$c(\Pi) = (T(\Pi), b_1(\Pi)).$$

The complexity of G with respect to $\mathscr{C}$ is defined to be

$$c(G,\mathscr{C}) = c(G) = \min c(\Pi),$$

where the minimum is taken over all Π with vertex groups elements of $\mathscr{C}$ and $G = \pi_1^{\mathrm{orb}}(\Pi)$. Lexicographical ordering is used.

Proposition 3.4 of [6] shows that if $c(G) = (0,0)$, then G must be the fundamental group of a tree of groups (possibly just a vertex), with finite edge groups, and vertex groups in $\mathscr{C}$. We are taking G to be finitely presented, so we note that any such tree will be finite. A group is said to have a *dihedral* action on a tree if the group acts on the tree, has an axis, and elements of the group interchange the endpoints of the axis. Delzant and Potyagailo proved the following theorem:

Theorem 14 [6] *Let G be a finitely presented group, with $\mathscr{C}$ a family of elementary subgroups of G and $c(G,\mathscr{C}) > (0,0)$. Suppose G has a proper decomposition over $\mathscr{C}$, with τ the associated Bass–Serre tree, and suppose further that no $C \in \mathscr{C}$ has a dihedral action on τ.*

Then there is a proper decomposition of G over $\mathscr{C}$ with associated tree τ' such that there is a G–map $\tau' \to \tau$, and, if $\{G_v\}$ denotes the vertex groups of $G\backslash\tau'$, then $\sum T(G_v) \le T(G)$, and $\max_v c(G_v, \mathscr{C} \cap G_v) < c(G,\mathscr{C})$.

With $\mathscr{C}$ defined to be the finite and two-ended subgroups of a group G, as long as G has no 2–torsion, it follows that the action of any $C \in \mathscr{C}$ on any G–tree τ is not dihedral.

Remark 15 We will want to apply Proposition 13 to the map $\tau' \to \tau$. In order to do this, we need surjectivity. For the moment, we shall merely note that, if G acts on τ minimally, then $\tau' \to \tau$ must be surjective. If the action is not minimal, then τ' maps onto a G–tree contained in τ, and all of the edge and vertex groups of τ outside of this subtree are contained in $\mathscr{C}$.

We shall now present several lemmas, which will be used to bridge the gap between Theorem 14 and Swarup's Strong Accessiblity Conjecture for groups with no 2–torsion.

Lemma 16 *Let G be a finitely generated group, and $\mathscr{C}$ a family of subgroups of G which is closed under conjugation and subgroups. Suppose that $\phi\colon \tau' \to \tau$ is a surjective G–map between G–trees with all edge stabilizers in $\mathscr{C}$. Moreover, suppose ϕ is such that, for each edge e of τ', stab(e) is contained in $\mathrm{stab}(\phi(e))$ with finite index. Let $\Gamma' = G\backslash\tau'$, and $\Gamma = G\backslash\tau$, and suppose that Γ' is finite, and the edge groups of Γ are all finitely generated. Then if Γ' admits a proper refinement over $\mathscr{C}$, so does Γ, and the additional edge groups in the refinements are the same.*

From this, we immediately have the following:

Corollary 17 *If G, $\mathscr{C}$, Γ and Γ' are as in Lemma 16, and Γ is a maximal proper decomposition of G, then Γ' must be maximal as well.*

Proof of Lemma 16 We will start by showing that if ϕ is a fold, then a proper splitting of a vertex group of τ', which is compatible with Γ', induces a proper splitting of the image of the vertex group which is compatible with Γ, over the same edge group. It will then follow that a proper refinement of Γ' induces a proper refinement of Γ.

So assume that $\phi\colon \tau' \to \tau$ is a fold. We use our notation from above, so that ϕ identifies e to f and x to y, where e and f meet at the vertex $v \in \tau'$, and similarly, identifies $g \cdot e$ to $g \cdot f$ for each g in G. Let vertex $w \in \tau'$ be such that W, the stabilizer of w, admits a proper splitting over some $C \in \mathscr{C}$, which is compatible with Γ'. Thus there exists a tree $\overline{\tau'}$ and a G–equivariant map $\zeta'\colon \overline{\tau'} \to \tau'$ which merely collapses each edge in the orbit of c to a vertex in the orbit of w. We would like to find a tree $\overline{\tau}$ such that there is a similar collapsing map $\zeta\colon \overline{\tau} \to \tau$, a fold $\overline{\phi}$ taking $\overline{\tau'}$ to $\overline{\tau}$, and such that the following diagram commutes:

$$\begin{tikzcd} \overline{\tau'} \arrow[r, "\overline{\phi}"] \arrow[d, "\zeta'"'] & \overline{\tau} \arrow[d, "\zeta"] \\ \tau' \arrow[r, "\phi"] & \tau \end{tikzcd}$$

For our first case, assume that w is not in the G–orbit of v, nor of x nor y. Then we may define $\overline{\phi}$ to identify $\zeta'^{-1}(e)$ to $\zeta'^{-1}(f)$, and $\zeta'^{-1}(g \cdot e)$ to $\zeta'^{-1}(g \cdot f)$, for each g in G. The edge c, as well as the edges in the G–orbit of c, are untouched by such a fold, so the above diagram must commute. Also because no edge gets identified to c or any of its translates, and because the refinement $\overline{\Gamma'}$ of Γ' is proper, it follows that $\overline{\phi}$ induces a refinement of Γ which is proper.

Next, assume that w is in the G–orbit of x, and not of v. (w may be in the orbit of y.) Then we may again define $\overline{\phi}$ directly, taking that it identifies $\zeta'^{-1}(e)$ to $\zeta'^{-1}(f)$, and similarly for the G–orbits of e and f. Define the map $\overline{\phi}_*$ to take the stabilizer of any vertex or edge z in τ' to the stabilizer of $\overline{\phi}(z)$, and let a and b be the vertices of c. Recall that $A = \mathrm{stab}(a)$, and so on. Then in this case, $\overline{\phi}_*(C) = C$, while $A \subseteq \overline{\phi}_*(A)$ and $B \subseteq \overline{\phi}_*(B)$. It follows again that $\overline{\Gamma}$ is a proper refinement of Γ because $\overline{\Gamma'}$ is a proper refinement of Γ'. To see this, we note that if $C \hookrightarrow A$ and $C \hookrightarrow B$ are not isomorphisms, then neither are the new injections in $\overline{\tau}$. If instead $g \in V$ takes a to b, then g will take $\overline{\phi}(a)$ to $\overline{\phi}(b)$. Thus we have that $\overline{\Gamma}$ is a proper refinement of Γ.

It remains to consider the case in which w is in the G–orbit of v. Without loss of generality, we assume that $w = v$. By abuse of notation, we will denote $\zeta'^{-1}(e)$ by e, and $\zeta'^{-1}(f)$ by f. Suppose that e and f are adjacent in $\overline{\tau'}$, so both contain either a or b. Here again, we may simply define $\overline{\phi}$ to identify e to f, and extend equivariantly. Then $\overline{\phi_*}$ takes A, B and C to themselves, and if there is some $g \in V$ which takes a to b, then g must also take $\overline{\phi}(a)$ to $\overline{\phi}(b)$. Hence, this induced refinement $\overline{\Gamma}$ must be proper.

So for our last case, assume that $w = v$, and that e and f are not adjacent in $\overline{\tau'}$. Without loss of generality, take that e contains a and f contains b, ie $E \subseteq A$ and $F \subseteq B$. Here, we will use our hypothesis that E and F are of finite index in $\phi_*(E) = \phi_*(F)$ to show that either $E \subseteq C$ or $F \subseteq C$. If $E \subseteq C$, then we may alter $\overline{\tau'}$ by "sliding" e so that it is incident to b instead of a, and do the same with the G–orbit of e. If $F \subseteq C$, then we can slide f instead. By doing this, we are able to create a proper refinement of Γ' of the type discussed in the previous paragraph, and may refer now to that argument.

To show that this is possible, assume that neither E nor F is contained in C, and choose elements $g_E \in E - C$ and $g_F \in F - C$. Then the subset of $\overline{\tau'}$ which is fixed pointwise by g_E is a subtree of $\overline{\tau'}$ which is disjoint from the subtree of points fixed by g_F. Thus $g_E g_F$ acts by translation on an axis in $\overline{\tau'}$. Both E and F are contained in $\phi_*(E)$, hence so is $g_E g_F$, but because $g_E g_F$ has an axis, it is of infinite order, and no power $(g_E g_F)^n$ is contained in E or F, except when $n = 0$. This means that E and F must be of infinite index in $\phi_*(E)$, which is a contradiction. Thus either $E \subseteq C$ or $F \subseteq C$ as desired.

We have seen now that if $\phi\colon \Gamma' \to \Gamma$ is a fold, and if Γ' admits a proper refinement by a splitting over a subgroup C, then Γ must also admit a proper refinement by a splitting which is also over C. For general ϕ, Proposition 13 implies that ϕ is a composition of folds. If Γ' admits a proper refinement, then by what we have shown, the refinement pushes through each fold, giving a proper refinement of Γ, as desired. □

Next, we note the following fact, which we shall make use of with $n = 2$:

Lemma 18 *Let G be a finitely generated group, with a G–tree σ and associated decomposition Σ, identified with $G\backslash\sigma$. Let $V_1, \dots, V_n$ be stabilizers of vertices $v_1, \dots, v_n$ of σ, and let σ_0 be the smallest subtree of σ containing $\{v_1, \dots, v_n\}$. Then the orbit of σ_0 under $\langle V_1, \dots, V_n\rangle$ is connected, thus a subtree of σ.*

Proof Fix any $w \in \langle V_1, \dots, V_n\rangle$. It will suffice to show that $w \cdot \sigma_0$ is connected to σ_0 in $\langle V_1, \dots, V_n\rangle \cdot \sigma_0$.

We can write $w = w_1 w_2 \cdots w_{m-1} w_m$, where each w_i is contained in some V_{j_i}. Then $w_m \cdot \sigma_0$ intersects σ_0 at the vertex stabilized by V_{j_m}, the subtree $w_{m-1} \cdot (w_m \cdot \sigma_0 \cup \sigma_0) = (w_{m-1} w_m \cdot \sigma_0) \cup (w_{m-1} \cdot \sigma_0)$ intersects σ_0 at the vertex stabilized by $V_{j_{m-1}}$, the subtree $w_{m-2} \cdot ((w_{m-1} w_m \cdot \sigma_0) \cup (w_{m-1} \cdot \sigma_0) \cup \sigma_0) = (w_{m-2} w_{m-1} w_m \cdot \sigma_0) \cup (w_{m-2} w_{m-1} \cdot \sigma_0) \cup (w_{m-2} \cdot \sigma_0)$ intersects σ_0 at the vertex stabilized by $V_{j_{m-2}}$, and so on. Continuing in this manner, it follows that the translates $w \cdot \sigma_0 = w_1 w_2 \cdots w_m \cdot \sigma_0$, $w_1 w_2 \cdots w_{m-1} \cdot \sigma_0$, $\ldots$, $w_1 w_2 \cdot \sigma_0$, $w_1 \cdot \sigma_0$, σ_0 make a subtree, hence $w \cdot \sigma_0$ is connected to σ_0 in $\langle V_1, \ldots, V_n \rangle \cdot \sigma_0$. □

From this, it follows that if σ is a G–tree, and $v_1, \ldots, v_n$ are vertices of σ with respective stabilizers $V_1, \ldots, V_n \subset G$, then the $\langle V_1, \ldots, V_n \rangle$–orbit of the smallest subtree containing $\{v_1, \ldots, v_n\}$ is a $\langle V_1, \ldots, V_n \rangle$–tree.

We can now prove the following:

Lemma 19 *Let $\Gamma = G \backslash \tau$ be a maximal proper decomposition of a finitely presented group G over a family $\mathscr{C}$ which is closed under conjugation and subgroups. Let $\Gamma' = G \backslash \tau'$ be the decomposition from Theorem 14, with $\phi \colon \tau' \to \tau$ the associated G–map. Assume that, for each edge e of τ', stab(e) is contained in stab$(\phi(e))$ with finite index. Then, for each vertex group V of Γ, either V is a vertex group of Γ', or $V \in \mathscr{C}$.*

Proof By Remark 15, we can assume that $\phi \colon \tau' \to \tau$ from Theorem 14 is a surjection.

We may subdivide the edges of τ and τ' so that, for each edge of τ and τ', the vertices of that edge are in different G–orbits, yet still $\phi \colon \tau' \to \tau$ is a G–map. Again, from Proposition 13, ϕ is a composition of folds. Our subdivision of the edges of τ and τ' ensures that ϕ is, in fact, a composition of folds of types IA, IIA, and IIIA.

Assume first that ϕ is a fold of type IA, IIA, or IIIA. Using that Γ is maximal, we will show that, for any vertex group Z of Γ, either Z is isomorphic by the given injection to one of its edge groups, or Z is a vertex group of Γ', hence has smaller complexity than G. Thus for a composition of such folds, a vertex group of the target decomposition is either a vertex group of the source decomposition, or is in $\mathscr{C}$.

We employ our previous notation, so that ϕ is a fold which takes edge e of τ' to edge f, and vertex x to vertex y, with e and f sharing the additional vertex v. It is immediate that, for all vertices z' of τ', stab$(z') =$ stab$(\phi(z'))$ if z' is not in the G–orbit of x or y. Hence it suffices to show the above statement for $Z =$ stab$(\phi(x))$.

Consider the case in which ϕ is a fold of type IA. Recall that $\phi(x) = \phi(y)$ has stabilizer $Z = (X, Y)$, and consider the action of Z on τ'. Lemma 18 implies that this gives the

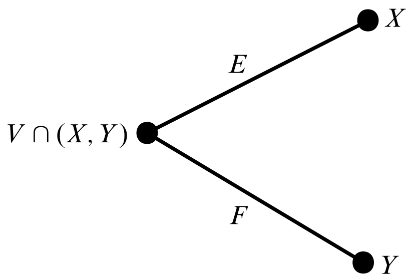

Figure 4: Decomposition of Z in the case when ϕ is a fold of type IA

decomposition of Z that is pictured in Figure 4. Thus $(X,Y)=X*_E(V\cap(X,Y))*_F Y$. If this decomposition gives a proper splitting of Z which is compatible with Γ, ie the edge stabilizer of any edge adjacent to $\phi(x)$ is contained in a vertex group of the splitting, then this splitting would induce a proper refinement of Γ. This would be a contradiction, however, because Γ is assumed to be maximal.

We claim first that the decomposition is compatible with Γ, hence either splitting from the decomposition is compatible with Γ. This follows because E and F are contained in V, so the stabilizer (E,F) of $\phi(e)$ is contained in $V\cap(X,Y)$, and any other edge incident to $\phi(x)$ is untouched by the fold, hence has stabilizer either contained in X or contained in Y.

Therefore this decomposition of Z must not give a proper splitting. The fact that $[X*_E(V\cap(X,Y))]*_F Y$ is not proper implies that either $Z=Y$ or $Y=F$. If $Z=Y$, then Z is a vertex group of τ'. Otherwise, $Y=F$. But also $X*_E[(V\cap(X,Y))*_F Y]$ is not a proper splitting, so either $Z=X$ or $X=E$. If $Z=X$, then, as before, Z is a vertex group of τ'. Otherwise, $Y=F$ and $X=E$, so $Z=(X,Y)=(E,F)$, and hence Z is an edge group of τ. Thus if ϕ is a fold of type IA, then either Z is isomorphic to a vertex group of Γ', or an edge group of Γ.

Consider next the case in which ϕ is a fold of type IIA. There is some $g\in G$ taking e to f, and fixing v, and $\phi(x)$ is stabilized by (X,g). The action of this subgroup on τ' gives the splitting of $Z=(X,g)$ that is in Figure 5, and hence $(X,g)=(V\cap(X,g))*_E X$. It is clear that $(E,g)\subset(V\cap(X,g))$, so if we show that any other edge group of Γ contained in (X,g) is contained in one of the new vertex groups, then the compatibility of this splitting of (X,g) with Γ will follow. But as above, since the

Figure 5: Decomposition of Z in the case when ϕ is a fold of type IIA

fold only affects the edge group labeled (E, g), then any other edge group incident to the vertex labeled (X, g) must have been contained in X.

We note that since $g \in V \cap (E, g)$, but $g \notin E$, this splitting induces a proper refinement of Γ unless $X = E$, in which case $(X, g) = (E, g)$. Thus if ϕ is of type IIA, $Z = (X, g)$ must be an edge group of Γ.

If ϕ is a fold of type IIIA, then there is some $g \in G$ taking x to y, but not taking e to f. Recall that $Z = \mathrm{stab}(\phi(x))$ is (X, g), and consider the action of (X, g) on τ'. The quotient by this action contains the decomposition of (X, g) given in Figure 6, thus $(X, g) = ((V \cap (X, g)) *_F X) *_E$, where this HNN extension is by g.

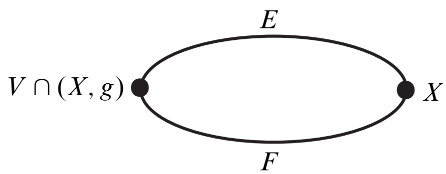

Figure 6: Decomposition of Z in the case when ϕ is a fold of type IIIA

A refinement by an HNN extension must always be proper, so it remains to show that this splitting induces a refinement of Γ, ie is compatible with the other splittings of Γ. To do this, we must show that the stabilizer of any edge incident to $\phi(x)$ is contained in $((V \cap (X, g)) *_F X)$. The argument for this is similar to the above: except for $\phi(e)$, any edge d incident to $\phi(x)$ is again untouched by the fold, hence has stabilizer equal to the stabilizer of $\phi^{-1}(d)$, which is contained in X, as $\phi^{-1}(d)$ is incident to x. $X \subset ((V \cap (X, g)) *_F X)$, so our splitting is compatible with the splitting over D.

Now recall that $\phi(e)$ is stabilized by (E, F). But both E and F stabilize v, hence are in V. Also, E and F, when conjugated by g, stabilize x, hence (E, F) is in (X, g). Thus $\mathrm{stab}(\phi(e)) = (E, F) \subset (V \cap (X, g)) \subset ((V \cap (X, g)) *_F X)$, so the given splitting of (X, g) is compatible with the other splittings of Γ. But this means that there is a proper refinement of Γ, a contradiction. Hence ϕ cannot be a fold of type IIIA.

We now address the situation in which $\phi = \phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_1$, where each ϕ_i is a fold of type IA, IIA, or IIIA. Let Γ_i denote the decomposition $G \backslash \phi_i \circ \phi_{i-1} \circ \ldots \circ \phi_1(\tau')$. Lemma 16, and the fact that Γ is maximal, imply that the decompositions Γ', Γ_1, Γ_2, $\ldots$, Γ_{n-1} are all maximal, and since Γ and Γ' are proper decompositions, each Γ_i is also proper. Thus, for each i, the vertex groups of Γ_i are edge groups of Γ_i, or are vertex groups of Γ_{i-1}. It follows that any vertex group of Γ is isomorphic to either a vertex group of Γ', or an edge group of some Γ_i, thus is in $\mathscr{C}$. (Note that our early subdivision of edges of Γ' only adds edge groups to the collection of vertex groups of Γ', hence does not affect this result.) □

Before proving Theorem 22, we shall need two additional facts. The first is a result from Scott and Swarup [14] about the existence of maximal decompositions over two-ended subgroups:

Theorem 20 *Let G be a one-ended, finitely presented group, and let Γ be a proper decomposition of G over two-ended subgroups. Then Γ admits a refinement Σ which is a maximal proper decomposition of G over two-ended subgroups.*

Proof Let τ be the G–tree corresponding to Γ. For any vertex v of valence two of Γ which is not the vertex of a circuit and has incident edges e and f such that $E = V = F$ by the given injections, collapse either e or f. Continue this process until no such vertices remain, and denote the resulting decomposition by $\overline{\Gamma}$. We have now removed enough redundancy from Γ to be able to apply Theorem 7.11 of [14], with corrected statement in [13], giving us that $\overline{\Gamma}$ has a maximal refinement $\overline{\Sigma}$.

We claim now that $\overline{\Sigma}$ induces a maximal refinement Σ of Γ, ie that we may put the collapsed edges back into $\overline{\Sigma}$ corresponding to their location in Γ. This can be done by merely subdividing each edge of $\overline{\Sigma}$ which corresponds to an edge e (respectively f) of Γ when, as in our notation above, the edge f (respectively e) was collapsed to a point. □

The second is the following:

Lemma 21 *If G is a finitely presented group, and Γ is a decomposition of G over finitely generated subgroups, then the vertex group(s) of Γ are also finitely presented.*

For a proof of this, we refer the reader to Lemma 1.1 in [3].

We can now prove Swarup's conjecture for hyperbolic groups with no 2–torsion:

Theorem 22 *Let G be a hyperbolic group with no 2–torsion. Decompose G maximally over finite subgroups, and then take the resulting vertex groups, and decompose those maximally over two-ended subgroups. Now repeat this process on the new vertex groups and so on. Then this process must eventually terminate, with subgroups of G which are unsplittable over finite and two-ended subgroups.*

Remark 23 We note that the proof below also goes through if G is a finitely presented subgroup of a hyperbolic group.

Proof First, we will note that the above process must terminate for any finitely generated group H such that $c(H) = (0, 0)$, with respect to the family of finite and two-ended subgroups of H. Recall that in this case, H is the fundamental group of a tree of groups with finite edge groups, and finite or two-ended vertex groups. If the tree consists of just one vertex, then H is finite or two-ended. When H is finite, then it is unsplittable over all subgroups and hence the process terminates. If H is two-ended, then H admits one nontrivial decomposition, which is over a finite subgroup and has finite vertex groups, thus the above process must also terminate.

More generally, let H be the fundamental group of a tree of groups as described above. The only vertex groups of the tree which admit any splittings are the two-ended groups. As noted above, each splits over a finite subgroup, and the resulting vertex groups are finite, hence unsplittable. Any collection of splittings of H over finite subgroups are compatible, hence we may combine any splittings of vertex groups of the tree with the splittings of H determined by the edges of the tree to get a decomposition of H over finite subgroups with vertex groups which are completely unsplittable. It follows that the process terminates for any H such that $c(H) = (0, 0)$.

Now we let G be any hyperbolic group. Then G must be finitely presented, thus, by [7], it has a maximal decomposition over finite subgroups. Choose such a decomposition (which must be finite), and let τ be the associated tree. Let $\mathscr{C}$ be the family of all finite and two-ended subgroups of G, and let τ' be the G–tree resulting from an application of Theorem 14. Note that since the map $\tau' \to \tau$ collapses no edges to vertices, stabilizers of edges of τ' are subgroups of stabilizers of edges of τ, hence the decomposition of G associated to τ' is over finite subgroups of G.

Thus Lemma 19, applied taking the family of elementary subgroups to be the collection of finite subgroups of G, implies that any vertex stabilizer V_1 of τ either is finite or is a vertex stabilizer of τ', hence is of smaller complexity (with respect to the family of finite and two-ended subgroups of V_1) than G. Certainly the process described above must terminate for finite groups, so we may assume that V_1 is not finite.

By Lemma 21, V_1 must be finitely presented. Let $\mathscr{C}_1$ be the collection of finite and two-ended subgroups of V_1, ie $\mathscr{C}_1 = \mathscr{C} \cap V_1$. By Proposition 8, $\mathscr{C}_1$ is elementary in V_1. Note that V_1 must have one end, so by Theorem 20, V_1 has a maximal decomposition over two-ended subgroups. By Remark 15, we can assume that this decomposition is finite. Let τ_1 be the corresponding V_1–tree, and τ_1' the tree from Theorem 14.

Since V_1 has one end, the edge groups of τ_1' are also two-ended, and thus any edge group of τ_1' is of finite index in the image edge group from the map $\tau_1' \to \tau_1$. Therefore, Lemma 19 gives us that if V_2 is a vertex group of τ_1, then V_2 is in $\mathscr{C}_1$ or has smaller complexity than V_1, with respect to the family $\mathscr{C}_2 = \mathscr{C}_1 \cap V_2$ of the finite and two-ended subgroups of V_2. We note that V_2 is finitely presented, and $\mathscr{C}_2$ is elementary in V_2.

If V_2 is in $\mathscr{C}_1$, then V_2 could admit one nontrivial maximal decomposition, which would be over a finite subgroup and would have finite vertex groups. Otherwise, we can repeat the arguments above, decomposing V_2 maximally over finite subgroups, decomposing the resulting vertex groups maximally over two-ended subgroups, etc. Complexity of the resulting groups continues to decrease, so we must eventually reach a collection of subgroups of G which are unsplittable over any finite or two-ended subgroups, as desired. □

4 Application to 3–manifolds

We will now use this result to get the hierarchy theorem for 3–manifolds stated earlier. First, we recall that a surface N in a 3–manifold M is said to be essential if N is properly embedded in M, 2–sided, π_1–injective into M, and is not properly homotopic into the boundary of M.

Lemma 24 *Let M be a compact, connected 3–manifold, and let $\mathcal{A} = \{A_i\}_{i \in I}$ be a nonempty, finite collection of disjoint, nonparallel, essential surfaces in M, such that $\{\pi_1(A_i)\}$ are contained in a family $\mathscr{C}$ of subgroups of $G = \pi_1(M)$ which is closed under subgroups and conjugation. Suppose further that $\mathcal{A}$ is maximal with respect to collections of disjoint, nonparallel essential surfaces of M with fundamental groups in $\mathscr{C}$. Let Γ be the decomposition of G which is dual to $\mathcal{A}$. Then Γ is a maximal proper decomposition of G over $\mathscr{C}$.*

Proof Assume for the contrapositive that Γ is not maximal. Then there exists some vertex group V of Γ which admits a proper splitting over some $C \in \mathscr{C}$ which is compatible with Γ. Let L be the graph of groups for such a splitting of V, and let p denote the midpoint of the edge of L. Let N denote the union of the component of $M - \mathcal{A}$ which corresponds to V with the surfaces A_i which correspond to the edge groups incident to V. We can define a map from N to L which is an isomorphism on π_1, with each A_i in N mapped to a vertex of L, and such that the map is transverse to p. Note that each component of the inverse image of p is a properly embedded, 2–sided surface in N. Furthermore, Stallings showed in [16] that we can homotope this map on N rel ∂N to a new map f such that the surfaces comprising $f^{-1}(p)$ are π_1–injective in M (see also Hempel [10]).

We may further assume that these components are not parallel to the boundary of N, because of the following. Let S denote a component of $f^{-1}(p)$ which is boundary parallel in N, and let R be the region made up of S and the component of $N - S$ through which S can be homotoped to ∂N, so R is homeomorphic to $S \times I$. Then we may homotope f to take R to p, and then to take a small neighborhood of R past p, so that p is not contained in $f(R)$. We may then homotope f to map the elements of $\mathcal{A} \cap R$ to the other vertex of L, so that still p is not in $f(R)$, and still f is an isomorphism on π_1. Note that, because L is the graph of groups of a proper splitting, and f is surjective on π_1, this process will never make $f^{-1}(p)$ empty.

We have arranged that the components of $f^{-1}(p)$ are essential in M. Because f is π_1–injective, the fundamental group of each component of $f^{-1}(p)$ is conjugate to a subgroup of C and so is in $\mathscr{C}$. Since f maps the A_i's to vertices of L, the surfaces $f^{-1}(p)$ are disjoint from $\mathcal{A}$. Also, as components of $f^{-1}(p)$ are not boundary parallel in N, they are not parallel to elements of $\mathcal{A}$. Hence $\mathcal{A}$ is not maximal. $\square$

We note that each component of $f^{-1}(p)$ induces a refinement of Γ. Suppose, in addition to the hypotheses on M in the above lemma, that M is irreducible. Then we can homotope f to remove any sphere components of $f^{-1}(p)$, so that any simply connected component of $f^{-1}(p)$ must be a compressing disk for M. Thus, a maximal collection of compressing disks in an irreducible, connected 3–manifold M induces a maximal proper decomposition of G over the trivial group.

It also follows that, if $\mathcal{A}$ is a maximal collection of annuli in M, and M is as in the above lemma, has incompressible boundary and is irreducible, then the graph of groups Γ corresponding to $\mathcal{A}$ must be maximal over the family generated by all infinite cyclic subgroups of $\pi_1(M)$.

Recall that, if M is orientable and irreducible and $\pi_1(M) = G$ is infinite, then G has no torsion (see Hempel [10]). Hence any essential surface in M with finite fundamental group must be simply connected, and any essential surface with two-ended fundamental group must be an annulus. We also note that it follows from the Geometrization Theorem proven by Perelman (see Morgan and Tian [11] and Cao and Zhu [5]) that M as above has a hyperbolic fundamental group if and only if M is hyperbolic and has no torus boundary components.

These observations, together with Theorem 22, imply the following theorem.

Theorem 25 *Let M be an irreducible, orientable, compact 3–manifold with hyperbolic fundamental group. The process of decomposing M along any maximal, disjoint collection of compressing disks, then decomposing the resulting manifolds along maximal, disjoint collections of essential annuli, then the resulting manifolds along compressing disks, then again along essential annuli and so on, must eventually terminate with a collection of manifolds, each of which has incompressible boundary and admits no essential annuli, or is a 3–ball.*

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Received: 17 August 2007 Revised: 12 December 2007