A class function on the mapping class group of an orientable surface and the Meyer cocycle

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In this paper we define a $\mathbf{QP}^1$–valued class function on the mapping class group $\mathcal{M}_{g,2}$ of a surface $\Sigma_{g,2}$ of genus g with two boundary components. Let E be a $\Sigma_{g,2}$–bundle over a pair of pants P. Gluing to E the product of an annulus and P along the boundaries of each fiber, we obtain a closed surface bundle over P. We have another closed surface bundle by gluing to E the product of P and two disks.

The sign of our class function cobounds the 2–cocycle on $\mathcal{M}_{g,2}$ defined by the difference of the signature of these two surface bundles over P.

57N13, 55R40; 57M07

1 Introduction

Let $\Sigma_{g,r}$ be a compact oriented surface of genus g with r boundary components. The mapping class group $\mathcal{M}_{g,r}$ is $\pi_0\mathrm{Diff}_+(\Sigma_{g,r}, \partial\Sigma_{g,r})$ where $\mathrm{Diff}_+(\Sigma_{g,r}, \partial\Sigma_{g,r})$ is the group of orientation preserving diffeomorphisms of $\Sigma_{g,r}$ which restrict to the identity on the boundary $\partial\Sigma_{g,r}$. We simply denote $\Sigma_g := \Sigma_{g,0}$ and $\mathcal{M}_g := \mathcal{M}_{g,0}$. Harer [4] proved that

$$H^2(\mathcal{M}_{g,r}; \mathbf{Z}) \cong \mathbf{Z} \quad g \geq 3,\ r \geq 0,$$

see also Korkmaz and Stipsicz [8].

Meyer [9] defined a cocycle $\tau_g \in Z^2(\mathcal{M}_g; \mathbf{Z})$ $(g \geq 0)$ called the Meyer cocycle which represents four times generator of the second cohomology class when $g \geq 3$. Let D_1, D_2, and D_3 be mutually disjoint disks in S^2, and $\mathrm{Int}\, D_i$ the interior of D_i for $i = 1, 2, 3$. We denote by $P := S^2 - \amalg_{i=1}^3 \mathrm{Int}\, D_i$ the pair of pants, and $\alpha, \beta, \gamma \in \pi_1(P)$ be the homotopy classes as shown in Figure 1. We consider a $\Sigma_{g,r}$–bundle $E_{g,r}^{\varphi,\psi}$ on the pair of pants P which has monodromies φ, ψ, $(\psi\varphi)^{-1} \in \mathcal{M}_{g,r}$ along α, β, $\gamma \in \pi_1(P)$. The diffeomorphism type of $E_{g,r}^{\varphi,\psi}$ does not depend on the choice of representatives in the mapping classes φ and ψ. Since $E_{g,r}^{\varphi,\psi}$ is the oriented fiber

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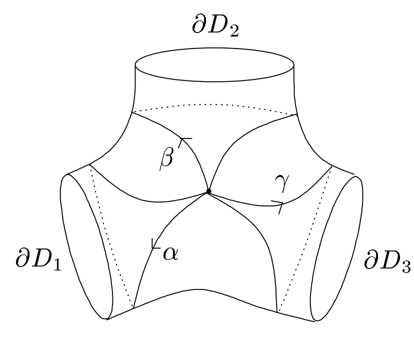

Figure 1

bundle, it has the canonical orientation comes from that of $\Sigma_{g,r}$ and P. The Meyer cocycle is defined by

$$\begin{array}{rcccl} \tau_g\colon \mathcal{M}_g & \times & \mathcal{M}_g & \to & \mathbf{Z}, \\ (\varphi & , & \psi) & \mapsto & \operatorname{Sign} E_g^{\varphi,\psi} \end{array}$$

where $\operatorname{Sign} E_g^{\varphi,\psi}$ is the signature of the compact oriented 4–manifold $E_g^{\varphi,\psi}$. For $k > 0$, it is known as Novikov additivity that when two compact oriented $4k$–manifolds are glued by an orientation reversing diffeomorphism of their boundaries, the signature of their union is the sum of their signature. When a pants decomposition of a closed oriented 2–manifold is given, the signature of a Σ_g–bundle on the 2–manifold is the sum of the signature of the Σ_g–bundles restricted to each pair of pants. Therefore, it is important to study the Meyer cocycle to calculate the signature of compact 4–manifolds. For $g = 1, 2$ the Meyer cocycle τ_g is a coboundary, and the cobounding function of this cocycle is calculated by several authors, for instance, Meyer [9], Atiyah [1], Kasagawa [6] and Iida [5]. The Meyer cocycle is not a coboundary if genus $g \geq 3$, but the cocycle can be a coboundary when it is restricted to some subgroups. For example, on the subgroup called the hyperelliptic mapping class group, the cobounding function is calculated by Endo [2] and Morifuji [11].

Let I be the unit interval $[0,1] \subset \mathbf{R}$. By sewing a pair of disks onto the surface $\Sigma_{g,2}$ along the boundary, we have Σ_g. For $h \in \operatorname{Diff}_+(\Sigma_{g,2}, \partial\Sigma_{g,2})$, if we extend h by the identity on the pair of disks, we have a self-diffeomorphism of Σ_g. We denote it by $h \cup id_{\amalg_{i=1}^2 D^2}$. By sewing an annulus $S^1 \times I$ onto the surface $\Sigma_{g,2}$ along the boundary, we have Σ_{g+1}. In the same way, if we extend $h \in \operatorname{Diff}_+(\Sigma_{g,2}, \partial\Sigma_{g,2})$ by the identity on the annulus, we have a self-diffeomorphism $h \cup id_{S^1 \times I}$.

Define the induced homomorphism on the mapping class group by

$$\begin{array}{rccc} \theta\colon & \mathcal{M}_{g,2} & \to & \mathcal{M}_g \\ & [h] & \mapsto & [h \cup id_{\amalg_{i=1}^2 D^2}] \end{array}$$

and

$$\begin{array}{rccc} \eta\colon & \mathcal{M}_{g,2} & \to & \mathcal{M}_{g+1,0}. \\ & [h] & \mapsto & [h \cup id_{S^1 \times I}]. \end{array}$$

Harer [3; 4] shows that θ and η induce an isomorphism on the second homology classes when genus $g \geq 5$, so that $\widetilde{\tau}_g = \eta^* \tau_{g+1} - \theta^* \tau_g$ is a coboundary. Powell [12] proved that the first homology group $H_1(\mathcal{M}_{g,r}; \mathbf{Z})$ is trivial for $g \geq 3$ and $r \geq 0$, so by the universal coefficient theorem, it follows that the cobounding function of $\widetilde{\tau}_g$ is unique.

In this paper we define a $\mathbf{QP}^1$–valued class function m on the mapping class group $\mathcal{M}_{g,2}$ in an explicit way by using information of the first homology group of a mapping torus of $[h] \in \mathcal{M}_{g,2}$. For $[p:q] \in \mathbf{QP}^1$, we define the sign of $[p:q]$ by $\operatorname{sign}([p:q]) := \operatorname{sign}(pq)$. We prove that the sign of the function m cobounds the cocycle $\widetilde{\tau}_g = \eta^* \tau_{g+1} - \theta^* \tau_g$. In particular, it turns out that the cocycle $\widetilde{\tau}_g$ is coboundary for any $g \geq 0$.

This function makes a little bit easy to evaluate the Meyer cocycle on the subgroups consists of mapping classes that fix a curve on the surface. For example, consider the case $g = 1, 2$. We denote by ϕ_1 and ϕ_2 the cobounding functions of τ_1 and τ_2. Since $H_1(\mathcal{M}_{g,2}; \mathbf{Q}) = 0$, the equation $\eta^* \tau_{g+1} = \theta^* \tau_g + \delta m$ means $\eta^* \phi_{g+1} = \theta^* \phi_g + m$ for $g = 1, 2$. In particular, the function ϕ_1 is described explicitly in Meyer [9]. Therefore, our function m helps to describe the cobounding function of the Meyer cocycle for genus 2 and 3 on the subgroup.

In Section 2, we construct a class function m, prove some properties of this function, and calculate the image of the function. In Section 3, we prove that the sign of this function cobounds the difference $\widetilde{\tau}_g = \eta^* \tau_{g+1} - \theta^* \tau_g$. By the definition of the Meyer cocycle τ_g, $\widetilde{\tau}_g(\varphi, \psi)$ is just the difference $\operatorname{Sign} E_{g+1}^{\eta(\varphi), \eta(\psi)} - \operatorname{Sign} E_g^{\theta(\varphi), \theta(\psi)}$, so that we calculate the difference by using the sign of the function m. Moreover we compute the other differences of signature $\operatorname{Sign}(E_{g,2}^{\varphi, \psi}) - \operatorname{Sign}(E_g^{\theta(\varphi), \theta(\psi)})$ and $\operatorname{Sign}(E_{g+1}^{\eta(\varphi), \eta(\psi)}) - \operatorname{Sign}(E_{g,2}^{\varphi, \psi})$ by the function m.

2 Class function $m\colon \mathcal{M}_{g,2} \to \mathbf{QP}^1$

In this section we define the class function on the mapping class group $\mathcal{M}_{g,2}$ stated in the introduction and describe some properties of the function including the nontriviality.

For $[p:q],\ [r:s] \in \mathbf{QP}^1$, we define an addition in $\mathbf{QP}^1$ by

$$[p:q]+[r:s] = \begin{cases} [pr : ps+qr], & \text{if } [p:q] \neq [0:1] \text{ or } [r:s] \neq [0:1] \\ [0:1], & \text{if } [p:q]=[r:s]=[0:1]. \end{cases}$$

The projective line $\mathbf{QP}^1$ forms an additive monoid under this operation with $[1:0]$ the zero element.

In this section, all (co)homology groups are with $\mathbf{Q}$ coefficients.

2.1 Construction of the class function

Before constructing the function, we prepare a fact about homology groups of compact 3–manifolds. Let Y be a compact oriented connected 3–manifold with boundary ∂Y and $i\colon \partial Y \hookrightarrow Y$ the inclusion map. Consider the commutative diagram

$$\begin{array}{ccccc} H^1(Y) & \xrightarrow{i^*} & H^1(\partial Y) & \xrightarrow{\delta^*} & H^2(Y,\partial Y) \\ \downarrow{\scriptstyle \cap[Y]} & & \downarrow{\scriptstyle \cap[\partial Y]} & & \downarrow{\scriptstyle \cap[Y]} \\ H_2(Y,\partial Y) & \xrightarrow{\partial_*} & H_1(\partial Y) & \xrightarrow{i_*} & H_1(Y), \end{array}$$

where the upper and lower rows are the exact sequences of a pair $(Y,\partial Y)$, and the vertical maps are the cap products with the (relative) fundamental classes of Y and ∂Y. By the diagram and Poincaré Duality, it follows that the image of i^* is just its own annihilator with respect to the cup product of $H^1(\partial Y)$

$$\operatorname{Im} i^* = \operatorname{Ann}(\operatorname{Im} i^*).$$

In particular, we have

$$\dim \operatorname{Ker} i_* = \dim \operatorname{Im} i^* = \frac{1}{2} \dim H_1(\partial Y).$$

We define the mapping torus of $\varphi = [h] \in \mathcal{M}_{g,r}$ by

$$X^\varphi := \Sigma_{g,r} \times I/\sim, \quad (x,1) \sim (h(x),0),$$

and $\pi\colon X^\varphi \to I/\partial I = S^1$ by the projection $\pi([x,t]) = [t]$, where $[x,t] \in X^\varphi$ is the equivalent class of $(x,t) \in \Sigma_{g,r} \times I$, and $[t] \in I/\partial I = S^1$ the equivalent class of $t \in I$.

The diffeomorphism type of the mapping torus X^φ does not depend on the choice of the representative h. We fix the orientation on X^φ given by the product orientation on $\Sigma_{g,r} \times I$. Let $i_\varphi\colon \partial X^\varphi \hookrightarrow X^\varphi$ be the inclusion map. In this subsection we denote $\Sigma := \Sigma_{g,2}$, and if we fix $\varphi \in \mathcal{M}_{g,2}$, then we write simply $X := X^\varphi$ and $i := i_\varphi$. Let S_1 and S_2 be the two boundary components of Σ, and $[S_k]$ $(k=1,2)$ the image under the inclusion homomorphism $H_1(S_k) \to H_1(\Sigma)$ of the fundamental homology class.

We consider Σ as a subspace of X by the embedding $\iota\colon \Sigma \hookrightarrow X$ by $x \mapsto [x, 0]$. We choose points $p_1 \in S_1$, $p_2 \in S_2$, and $p \in S^1$, and orientation-preserving homeomorphisms $\iota_1\colon S^1 \to S_1$ and $\iota_2\colon S^1 \to S_2$. We define singular chains $f_k\colon I \to (S_1 \amalg S_2) \times S^1 = \partial X$ $(k = 1, 2, 3, 4)$ by

$f_1(t) = (\iota_1(t), p)$, $f_2(t) = (\iota_2(t), p)$, $f_3(t) = (p_1, t)$ and $f_4(t) = (p_2, t)$ respectively.

Let $e_k \in H_1(\partial X)$ be the homology class of f_k $(k = 1, 2, 3, 4)$. Then the set $\{e_1, e_2, e_3, e_4\}$ forms a basis for $H_1(\partial X)$, and the intersection number

$$e_i \cdot e_j = \begin{cases} 1 & \text{if } j = i + 2, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2$ and $j = 3, 4$. Now we describe the kernel of the homomorphism $i_*\colon H_1(\partial X) \to H_1(X)$. Since e_1 and e_2 lie in the kernel of $(\pi|_{\partial X})_*$ and $\pi_*(e_3) = \pi_*(e_4) = [S^1] \in H_1(S^1)$, we have

$$\text{Ker } i_* \subset \text{Ker } (\pi_* i_*) = \mathbf{Q}e_1 \oplus \mathbf{Q}e_2 \oplus \mathbf{Q}(e_3 - e_4).$$

By the definition of the map f_k, $(i \circ f_k)_*[S^1] = \iota_*[S_k]$, and so $i_*(e_1 + e_2) = \iota_*([S_1] + [S_2]) \in H_1(X)$. Since $S_1 \cup S_2$ is the boundary of Σ, we have $[S_1] + [S_2] = 0 \in H_1(\Sigma)$. Hence

$$\mathbf{Q}(e_1 + e_2) \subset \text{Ker } i_*.$$

As we saw at the beginning of this subsection, $\dim \text{Ker } i_* = \frac{1}{2} \dim H_1(\partial X) = 2$. It follows that $\text{Ker}\, i_* = \mathbf{Q}(e_1 + e_2) \oplus \mathbf{Q}(p(e_3 - e_4) + q e_1)$ for some p, $q \in \mathbf{Q}$. Now we can define a class function.

Definition 2.1 For $\varphi \in \mathcal{M}_{g,2}$, we take $p, q \in \mathbf{Q}$ such that $\text{Ker } i_{\varphi *} = \mathbf{Q}(e_1 + e_2) \oplus \mathbf{Q}(p(e_3 - e_4) + q e_1)$.

We define $m\colon \mathcal{M}_{g,2} \to \mathbf{QP^1}$ by $m(\varphi) = [p : q]$.

Lemma 2.2 *For* $\varphi, \psi \in \mathcal{M}_{g,2}$*,*

$$m(\psi\varphi\psi^{-1}) = m(\varphi).$$

Proof Define $\Psi\colon X^\varphi \to X^{\psi\varphi\psi^{-1}}$ by $\Psi(x, t) = (\psi(x), t)$. Then Ψ maps e_i as defined in $H_1(X^\varphi)$ to the corresponding e_i as defined in $H_1(X^{\psi\varphi\psi^{-1}})$, and the following diagram commutes

$$\begin{array}{ccc} H_1(\partial X^{\varphi}) & \xrightarrow{i_{\varphi *}} & H_1(X^{\varphi}) \\ \downarrow \Psi_* & & \downarrow \Psi_* \\ H_1(\partial X^{\psi\varphi\psi^{-1}}) & \xrightarrow{i_{\psi\varphi\psi^{-1}*}} & H_1(X^{\psi\varphi\psi^{-1}}). \end{array}$$

As we see from the diagram, Ψ_* gives the natural isomorphism between the kernels $\operatorname{Ker}(H_1(\partial X^{\varphi}) \to H_1(X^{\varphi}))$ and $\operatorname{Ker}(H_1(\partial X^{\psi\varphi\psi^{-1}}) \to H_1(X^{\psi\varphi\psi^{-1}}))$. Hence we have $m(\psi\varphi\psi^{-1}) = m(\varphi)$. □

2.2 Some properties and the nontriviality of the class function

By the Serre spectral sequence of the Σ–bundle $\pi\colon X \to S^1$, we have the exact sequence

$$0 \longrightarrow \operatorname{Coker}(\varphi_* - 1) \xrightarrow{\iota_*} H_1(X) \xrightarrow{\pi_*} H_1(S^1) \longrightarrow 0,$$

where $\operatorname{Coker}(\varphi_* - 1)$ is the cokernel of the homomorphism $\varphi_* - 1\colon H_1(\Sigma) \to H_1(\Sigma)$.

Then we have a unique homomorphism $j_{\varphi}\colon \mathbf{Q}e_1 \oplus \mathbf{Q}e_2 \oplus \mathbf{Q}(e_3 - e_4) \to \operatorname{Coker}(\varphi_* - 1)$ such that the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Q}e_1 \oplus \mathbf{Q}e_2 \oplus \mathbf{Q}(e_3 - e_4) & \longrightarrow & H_1(\partial X) & \xrightarrow{\pi_*} & H_1(S^1) & \longrightarrow & 0 \\ & & \downarrow j_{\varphi} & & \downarrow i_* & & \| & & \\ 0 & \longrightarrow & \operatorname{Coker}(\varphi_* - 1) & \xrightarrow{\iota_*} & H_1(X) & \xrightarrow{\pi_*} & H_1(S^1) & \longrightarrow & 0 \end{array}$$

commutes. By the diagram, we have

$$\operatorname{Ker} i_* = \operatorname{Ker} j_{\varphi} \text{ and}$$
$$j_{\varphi}(e_1) = -j_{\varphi}(e_2) = [S_1] \in \operatorname{Coker}(\varphi_* - 1).$$

Now we introduce a cochain $\omega_l \in C^1(\mathcal{M}_{g,2}; H_1(\Sigma))$ defined by Kawazumi [7]. On the fiber $\Sigma = \pi^{-1}(0) \subset X$, pick a path l such that $l(0) \in S_2$ and $l(1) \in S_1$. Define ω_l by

$$\omega_l(\varphi) := [\varphi(l) - l] \in H_1(\Sigma).$$

Then we have the following lemma.

Lemma 2.3

$$j_{\varphi}(e_3 - e_4) = [\omega_l(\varphi)] \in \operatorname{Coker}(\varphi_* - 1).$$

Proof Define a 2–chain $L\colon I\times I\to X$ by $L(s,t)=[l(s),t]$. Its boundary is given by $-i_*(e_3)+\varphi(l)+i_*(e_4)-l\in B_1(X)$. Hence,

$$i_*(e_3-e_4)=\iota_*([\varphi(l)-l])\in H_1(X)$$

Since ι_* is injective, the lemma follows. □

From the lemma, we see the homology class $[\omega_l(\varphi)]\in \mathrm{Coker}\,(\varphi_*-1)$ is independent of the choice of the path l. If $\omega_l(\varphi)=0$, then $j_\varphi(e_3-e_4)=0$.

Remark 2.4 If there exists a path l from a point in S_2 to a point in S_1 which has no common point with the support of a representative of $\varphi\in\mathcal{M}_{g,2}$, then $m(\varphi)=[1:0]$. In particular, $m(id)=[1:0]$, the zero element of the monoid $\mathbf{QP}^1$.

Define the subgroups $\mathcal{I}':=\mathrm{Ker}\,(\mathcal{M}_{g,2}\to\mathrm{Aut}\,(H_1(\Sigma_{g,2};\mathbf{Z}))$ and $\mathcal{I}:=\mathrm{Ker}\,(\mathcal{M}_{g,2}\to\mathrm{Aut}\,(H_1(\Sigma_{g,2},\partial\Sigma_{g,2};\mathbf{Z}))$. For $\varphi\in\mathcal{I}'$, $m(\varphi)=[p:q]$ means $p(\varphi(l)-l)+qe_1=0\in H_1(\Sigma_{g,2};\mathbf{Z})$. This shows that m is homomorphic on $\mathcal{I}'$. For $\varphi\in\mathcal{I}$, $\omega(\varphi)=0\in H_1(\Sigma_{g,2};\mathbf{Z})$. This shows that $m(\varphi)=[1:0]$ for all $\varphi\in\mathcal{I}$.

Remark 2.5 The restriction of m on $\mathcal{I}$ is trivial, and the restriction of m on $\mathcal{I}'$ is a nontrivial monoid homomorphism.

At the beginning of this section, we defined the commutative monoid structure on $\mathbf{QP}^1$. So integral multiples of $m(\varphi)$ are well-defined.

Proposition 2.6 *If $\varphi\in\mathcal{M}_{g,2}$ and $k\in\mathbf{Z}$, then*

$$m(\varphi^k)=km(\varphi).$$

Proof The proposition is trivial for $k=0$ and $k=1$. Assume $k\geq 2$.

Let $m(\varphi)=[p:q]$. By the definition of j_φ, $pj_\varphi(e_3-e_4)=-q[S_1]\in\mathrm{Coker}\,(\varphi_*-1)$. Hence, there exists $v\in H_1(\Sigma)$ such that

$$p[\varphi(l)-l]=-q[S_1]+(\varphi_*-1)v\in H_1(\Sigma).$$

Apply φ^i $(i=0,1,\dots,k-1)$ to the both sides of the equation and sum over i. Then

$$\sum_{i=0}^{k-1}p[\varphi^{i+1}(l)-\varphi^i(l)]=\sum_{i=0}^{k-1}\{-q[S_1]+(\varphi_*^{i+1}(v)-\varphi_*^i(v))\},$$

that is

$$p[\varphi^k(l)-l]=-kq[S_1]+(\varphi_*^k-1)v.$$

Hence, $m(\varphi^k) = [p : kq] = km(\varphi)$ for $k \geq 0$.

By applying φ^{-1} to the equation $p[\varphi(l) - l] = -q[S_1] + (\varphi_* - 1)v$, we have

$$p[\varphi^{-1}(l) - l] = q[S_1] + (\varphi_*^{-1} - 1)v \in H_1(\Sigma).$$

Hence, $m(\varphi^{-1}) = [p : -q] = -m(\varphi)$. Since $m(\varphi^{-k}) = -m(\varphi^k) = -km(\varphi)$ for $k > 0$, the proposition follows for the case $k < 0$. □

Now we compute the image of the function m. In particular, we see that m is nontrivial.

Proposition 2.7 *For $g \geq 1$, m is surjective. For $g = 0$,* $\mathrm{Im}\,(m) = [1 : \mathbf{Z}]$.

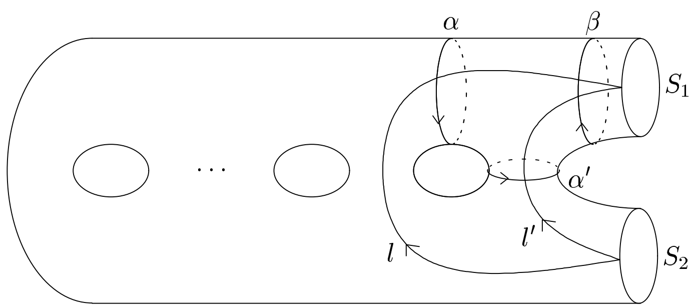

Figure 2

Proof Suppose $g \geq 1$. We choose oriented simple closed curves α, α', and β and paths l and l' as shown in Figure 2. We denote the Dehn twists along a simple closed curve $C \subset \Sigma$ by t_C, and the homology class of C by $[C]$. Then $[\alpha] + [\alpha'] + [\beta] = 0 \in H_1(\Sigma)$ since they bound a 2–chain. For $p \in \mathbf{Z}$, if we denote $\varphi := t_\alpha^p t_{\alpha'} t_\beta^{-1}$, then

$$\begin{aligned} j_\varphi((p+1)(e_3 - e_4)) &= \omega_l(\varphi) + p\omega_{l'}(\varphi) \\ &= [(t_\alpha^p t_{\alpha'} t_\beta^{-1})(l) - l] + p[(t_\alpha^p t_{\alpha'} t_\beta^{-1})(l') - l'] \\ &= p([\alpha] + [\alpha'] + [\beta]) + [\beta] = [\beta] = [S_1]. \end{aligned}$$

Hence, $j_\varphi((p+1)(e_3 - e_4) - e_1) = 0$, so that

$$m(\varphi) = [p + 1 : -1].$$

By Proposition 2.6, we have

$$m(\varphi^{-q}) = -q[p+1 : -1] = \begin{cases} [p+1 : q], & \text{if } p \neq -1 \\ [0 : 1], & \text{if } p = -1. \end{cases} \qquad (q \in \mathbf{Z})$$

Since p and q can run over all integers, we see m is surjective for $g \geq 1$.

For $g = 0$, $\mathcal{M}_{0,2}$ is the infinite cyclic group generated by t_β. Since $m(t_\beta^{-q}) = [1:q]$, we have $\mathrm{Im}\,(m) = [1:\mathbf{Z}]$. □

3 The difference of two Meyer cocycles $\eta^*\tau_{g+1}$ and $\theta^*\tau_g$

In this section (co)homology groups are with $\mathbf{Z}$ coefficient unless specified.

Let $g \geq 0$ be a positive integer. In the introduction, we defined the homomorphisms $\eta\colon \mathcal{M}_{g,2} \to \mathcal{M}_{g+1,0}$ and $\theta\colon \mathcal{M}_{g,2} \to \mathcal{M}_g$ to be the induced maps by sewing a pair of disks and by sewing an annulus onto the surface $\Sigma_{g,2}$ along their boundaries respectively. We denote the Meyer cocycle on the mapping class group of genus g closed orientable surface $\mathcal{M}_g$ by $\tau_g \in Z^2(\mathcal{M}_g)$ and define $\widetilde{\tau}_g \in Z^2(\mathcal{M}_{g,2})$ to be the difference between the Meyer cocycles

$$\widetilde{\tau}_g := \eta^*\tau_{g+1} - \theta^*\tau_g.$$

Let $P := S^2 - \amalg_{i=1}^3 D^2$. In this section, we prove the main theorem and calculate the changes of signature associated with sewing a pair of trivial disk bundles $P \times \amalg_{i=1}^2 D^2$ and sewing a trivial annulus bundles $P \times (S^1 \times I)$ onto $\Sigma_{g,2}$–bundle on the pair of pants P along their boundaries. To state the main theorem, we define the sign of $[p:q] \in \mathbf{QP^1}$ by

$$\mathrm{sign}\,([p:q]) := \mathrm{sign}\,(pq) = \begin{cases} 1 & \text{if } pq > 0, \\ 0 & \text{if } pq = 0, \\ -1 & \text{if } pq < 0. \end{cases}$$

Theorem 3.1 *For $\varphi, \psi \in \mathcal{M}_{g,2}$, we define*

$$\widetilde{\phi}_g(\varphi) := \mathrm{sign}\,(m(\varphi)).$$

Then $\widetilde{\phi}_g$ cobounds the difference $\widetilde{\tau}_g$ between the Meyer cocycles $\eta^*\tau_{g+1}$ and $\theta^*\tau_g$

$$\begin{aligned} \widetilde{\tau}_g(\varphi, \psi) &= \delta\widetilde{\phi}_g(\varphi, \psi) \\ &= \mathrm{sign}\,(m(\varphi)) + \mathrm{sign}\,(m(\psi)) + \mathrm{sign}\,(m((\varphi\psi)^{-1})). \end{aligned}$$

Remark 3.2 Let k be an integer. By Lemma 2.2 and Proposition 2.6, $\widetilde{\phi}_g$ has the properties

$$\begin{aligned} \widetilde{\phi}_g(\psi\varphi\psi^{-1}) &= \widetilde{\phi}_g(\varphi) \text{ and} \\ \widetilde{\phi}_g(\varphi^k) &= \mathrm{sign}\,(k)\widetilde{\phi}_g(\varphi) \end{aligned}$$

for any $g \geq 0$.

3.1 Proof of Main Theorem

In this subsection we prove Theorem 3.1.

In the introduction, we defined compact oriented 4–manifold $E_{g,r}^{\varphi,\psi}$ as a $\Sigma_{g,r}$–bundle on the pair of pants P which has monodromies φ, ψ, and $(\psi\varphi)^{-1} \in \mathcal{M}_{g,r}$ along α, β, and $\gamma \in \pi_1(P)$ respectively, and in Section 2.1, we defined compact oriented 3–manifold $X_{g,r}^{\varphi}$ by the mapping torus of $\Sigma_{g,r} \times I/\sim$ where $(x,1) \sim (h(x),0)$ for $\varphi = [h] \in \mathcal{M}_{g,r}$.

Gluing to $E_{g,2}^{\eta(\varphi),\eta(\psi)}$ the trivial annulus bundle on P along the boundaries of each fiber, we obtain

$$E_{g+1}^{\eta(\varphi),\eta(\psi)} = E_{g,2}^{\varphi,\psi} \cup (-S^1 \times I \times P).$$

Similarly, glue to $X_{g,2}^{\eta(\varphi)}$ the trivial annulus bundle on S^1. Then we have

$$X_{g+1}^{\eta(\varphi)} = X_{g,2}^{\varphi} \cup (-S^1 \times I \times S^1).$$

Define

$$\begin{array}{rccc} G\colon & \partial D^2 \times I & \to & \{1\} \times S^1 \times I. \\ & (x,t) & \mapsto & (1, x, \frac{1+t}{3}). \end{array}$$

By the map G, we can glue $D^2 \times I$ to $I \times S^1 \times I$ as shown in Figure 3.

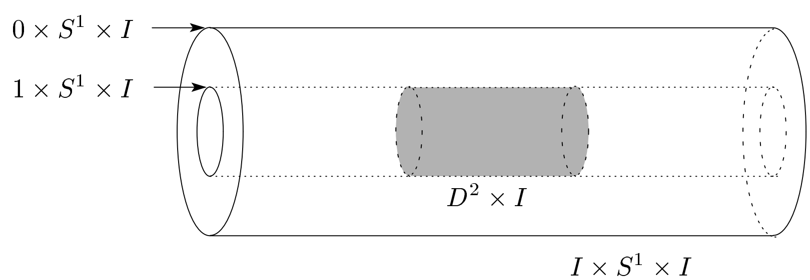

Figure 3: Gluing map G

Glue $D^2 \times I \times P$ to $I \times E_{g+1}^{\eta(\varphi),\eta(\psi)} = (I \times E_{g,2}^{\varphi,\psi}) \cup (-I \times S^1 \times I \times P)$ with the gluing map $G \times id_P\colon \partial D^2 \times I \times P \to \{1\} \times S^1 \times I \times P$. In the same way, glue $D^2 \times I \times S^1$

to $I \times X^{\eta(\varphi)}_{g+1} = (I \times X^{\varphi}_{g,2}) \cup (-I \times S^1 \times I \times S^1)$ with $G \times id_{S^1} \partial D^2 \times I \times S^1 \to \{1\} \times S^1 \times I \times S^1$. Namely, we construct two manifolds

$$\widetilde{E}^{\varphi,\psi} := (I \times E^{\eta(\varphi),\eta(\psi)}_{g+1}) \cup_{G \times id_P} (D^2 \times I \times P)$$

and

$$\widetilde{X}^{\varphi} := (I \times X^{\eta(\varphi)}_{g+1}) \cup_{G \times id_{S^1}} (D^2 \times I \times S^1).$$

Fix the orientations of these manifolds induced from the product orientations of $I \times E^{\eta(\varphi),\eta(\psi)}_{g+1}$ and $I \times X^{\eta(\varphi)}_{g+1}$. To prove main theorem, it suffices to prove Lemma 3.3 and Lemma 3.4 below.

Lemma 3.3

$(\eta^* \tau_{g+1} - \theta^* \tau_g)(\varphi, \psi) = \mathrm{Sign}\, \widetilde{X}^{\varphi} + \mathrm{Sign}\, \widetilde{X}^{\psi} + \mathrm{Sign}\, \widetilde{X}^{(\varphi\psi)^{-1}}$ *for* $\varphi, \psi \in \mathcal{M}_{g,2},\ g \geq 0.$

Lemma 3.4

$$\mathrm{Sign}\, \widetilde{X}^{\varphi} = \mathrm{sign}\,(m(\varphi)) \quad \textit{for}\ \varphi \in \mathcal{M}_{g,2}, \quad g \geq 0.$$

Proof of Lemma 3.3 Note that

$$\widetilde{X}^{\varphi} = \widetilde{E}^{\varphi,\psi}|_{\partial D_1}.$$

Then we can see

$$\begin{aligned} \partial \widetilde{E}^{\varphi,\psi} &= (\widetilde{E}^{\varphi,\psi}|_{\partial D_1} \cup \widetilde{E}^{\varphi,\psi}|_{\partial D_2} \cup \widetilde{E}^{\varphi,\psi}|_{\partial D_3}) \cup E^{\theta(\varphi),\theta(\psi)}_g \cup -E^{\eta(\varphi),\eta(\psi)}_{g+1} \\ &= (\widetilde{X}^{\varphi} \cup \widetilde{X}^{\psi} \cup \widetilde{X}^{(\psi\varphi)^{-1}}) \cup E^{\theta(\varphi),\theta(\psi)}_g \cup -E^{\eta(\varphi),\eta(\psi)}_{g+1}. \end{aligned}$$

Since the Signature is a bordism invariant (for example, see Milnor and Stasheff [10, Lemma 17.3]), we have $\mathrm{Sign}\, \partial \widetilde{E}^{\varphi,\psi} = 0$. By Novikov Additivity, we see that

$$\mathrm{Sign}\,(E^{\eta(\varphi),\eta(\psi)}_{g+1}) - \mathrm{Sign}\,(E^{\theta(\varphi),\theta(\psi)}_g) = \mathrm{Sign}\, \widetilde{X}^{\varphi} + \mathrm{Sign}\, \widetilde{X}^{\psi} + \mathrm{Sign}\, \widetilde{X}^{(\psi\varphi)^{-1}}.$$

Notice that $\widetilde{X}^{(\psi\varphi)^{-1}}$ is diffeomorphic to $\widetilde{X}^{(\varphi\psi)^{-1}}$, so that $\mathrm{Sign}\, \widetilde{X}^{(\psi\varphi)^{-1}} = \mathrm{Sign}\, \widetilde{X}^{(\varphi\psi)^{-1}}$. By the definition of the Meyer cocycle, we have

$$\mathrm{Sign}\,(E^{\eta(\varphi),\eta(\psi)}_{g+1}) = \eta^* \tau_{g+1}(\varphi, \psi), \text{ and } \mathrm{Sign}\,(E^{\theta(\varphi),\theta(\psi)}_g) = \theta^* \tau_g(\varphi, \psi).$$

Define $\widetilde{\phi}(\varphi) = \mathrm{Sign}\,(\widetilde{X}^{\varphi})$; then we have $\delta \widetilde{\phi} = \eta^* \tau_{g+1} - \theta^* \tau_g$. We get the cobounding function $\widetilde{\phi}$. □

Proof of Lemma 3.4 Write simply $X := X^{\eta(\varphi)}_{g+1}$, $X' := X^{\varphi}_{g,2}$, and $Y := \widetilde{X}^{\varphi} = (I \times X) \cup_{G \times id_{S^1}} (D^2 \times I \times S^1)$.

For $i = 0, 1$, define

$$\begin{array}{rccccc} j_i\colon & X & \to & I \times X & \hookrightarrow & Y, \\ & x & \mapsto & (i, x) & & \end{array}$$

where $I \times X \hookrightarrow Y$ is a natural embedding. We will prove there is a exact sequence

$$H_2(X') \xrightarrow{j_{0*}=j_{1*}} H_2(Y) \longrightarrow \mathrm{Ker}\,(H_1(\partial X') \to H_1(X')) \longrightarrow 0.$$

Define the submanifolds $Y_1 := I \times X'$ and $Y_2 := Y - \mathrm{Int}\, Y_1 = (-I \times S^1 \times I \times S^1) \cup_{G \times S^1} (D^2 \times I \times S^1)$. Then we see that

$$Y_1 \simeq X', Y_2 \simeq S^1, Y_1 \cap Y_2 \simeq \partial X' = (S_1 \amalg S_2) \times S^1.$$

By the Meyer–Vietoris exact sequence, we have the exact sequence

$$\begin{array}{ccccccc} H_2(Y_1) \oplus H_2(Y_2) & \longrightarrow & H_2(Y) & \xrightarrow{\partial_*} & H_1(Y_1 \cap Y_2) & \longrightarrow & H_1(Y_1) \oplus H_1(Y_2). \\ \| & & & & \| & & \| \\ H_2(X') \oplus 0 & & & & H_1(\partial X') & & H_1(X') \oplus H_1(S^1). \end{array}$$

Denote the map $H_1(\partial X') \to H_1(X') \oplus H_1(S^1)$ in the above diagram by h. the projection $H_1(\partial X') \to H_1(S^1)$ to the second entry of h is the composite of inclusion homomorphism $H_1(\partial X') \to H_1(X')$ and $\pi_*\colon H_1(X') \to H_1(S^1)$. Therefore,

$$\mathrm{Ker}\,(H_1(\partial X') \to H_1(X') \oplus H_1(S^1)) = \mathrm{Ker}\,(H_1(\partial X') \to H_1(X')).$$

So the sequence is exact.

Next, we will construct the splitting

$$H_2(Y; \mathbf{Q}) = j_{i*} H_2(X'; \mathbf{Q}) \oplus \mathrm{Ker}\,(H_1(\partial X'; \mathbf{Q}) \to H_1(X'; \mathbf{Q})).$$

Note that there exist $p, q \in \mathbf{Q}$ such that

$$\mathrm{Ker}\,(H_1(\partial X'; \mathbf{Q}) \to H_1(X'; \mathbf{Q})) = \mathbf{Q}(e_1 + e_2) \oplus \mathbf{Q}\{p(e_3 - e_4) + q e_1\}$$

as in Section 2. To construct the splitting, we choose elements of inverse images of $e_1 + e_2$, $p(e_3 - e_4) + q e_1$ under $H_2(Y) \to H_1(\partial X')$. Define $\iota_Y\colon \Sigma_{g+1} \to Y$ by

$$\begin{array}{ccccccc} \Sigma_{g+1} & \to & X & \to & I \times X & \hookrightarrow & Y. \\ x & \mapsto & (x, 0) & \mapsto & (0, x, 0). & & \end{array}$$

By the Meyer–Vietoris exact sequence as above, we have

$$\begin{array}{ccccc} H_2(Y) & \to & H_1(Y_1\cap Y_2) & \to & H_1(\partial X'), \\ \iota_{Y*}[\Sigma_{g+1}] & \mapsto & \partial_*\iota_{Y*}[\Sigma_{g+1}] & \mapsto & e_1+e_2 \end{array}$$

so we choose $\iota_{Y*}[\Sigma_{g+1}]$ as an element of the inverse image of e_1+e_2.

Next, we choose an element of the inverse image of $p(e_3-e_4)+qe_1$. Since $p(e_3-e_4)+qe_1\in \mathrm{Ker}\,(H_1(\partial X';\mathbf{Q})\to H_1(X';\mathbf{Q}))$, there exists a singular 2–chain $s\in C_2(X';\mathbf{Q})$ such that

$$\partial s = p(f_3-f_4)+qf_1\in B_1(X';\mathbf{Q}).$$

For $i=0,1$, define $s'_{0i}\colon I\times S^1\to I\times S^1\times I\times S^1\hookrightarrow Y_2$ by $s'_{0i}(t,u)=(i,0,t,u)$. Then we see that

$$[\partial s'_{0i}]=[j_if_3-j_if_4]\in H_1(Y_1\cap Y_2;\mathbf{Q}).$$

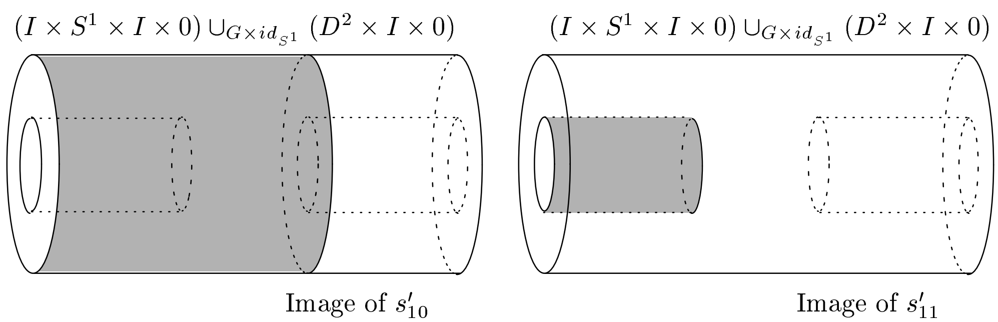

Figure 4: Images of s'_{10} and $s'_{11}\subset Y_2$.

Define $s'_{1i}\colon D^2\to Y_2=(-I\times S^1\times I\times S^1)\cup_{G\times S^1}(D^2\times I\times S^1)\subset Y$ as shown in Figure 4 by

$$s'_{10}(x)=\begin{cases}(6x,1,0) & \in D^2\times I\times S^1 \quad (||x||\le \frac{1}{6}),\\ (2-6||x||,\frac{x}{||x||},\frac{2}{3},0) & \in I\times S^1\times I\times S^1 \quad (\frac{1}{6}\le ||x||\le\frac{1}{3}),\\ (0,\frac{x}{||x||},1-||x||,0) & \in I\times S^1\times I\times S^1 \quad (\frac{1}{3}\le ||x||\le 1),\end{cases}$$

$$s'_{11}(x)=\begin{cases}(\frac{3}{2}x,0,0) & \in D^2\times I\times S^1 \quad (||x||\le\frac{2}{3}),\\ (1,\frac{x}{||x||},1-||x||,0) & \in I\times S^1\times I\times S^1 \quad (\frac{2}{3}\le||x||\le 1).\end{cases}$$

Then, we have $[\partial s'_{1i}]=[j_if_1]\in H_1(Y_1\cap Y_2;\mathbf{Q})$.

The chain $s'_i:=ps'_{0i}+qs'_{1i}$ satisfies

$$[\partial s'_i]=[j_i(p(f_3-f_4)+qf_1)]\in H_1(Y_1\cap Y_2;\mathbf{Q}),$$

so that we have $[\partial(j_i s - s'_i)] = 0 \in H_1(Y_1 \cap Y_2; \mathbf{Q})$.

We see

$$\begin{array}{ccccc} H_2(Y;\mathbf{Q}) & \to & H_1(Y_1 \cap Y_2;\mathbf{Q}) & \to & H_1(\partial X';\mathbf{Q}), \\ [j_i s - s'_i] & \mapsto & \partial_*[j_i s - s'_i] & \mapsto & p(e_3 - e_4) + q e_1 \end{array}$$

so that we can choose $[j_i s - s'_i]$ as an element of the inverse image of $p(e_3 - e_4) + q e_1$.

Now we calculate the intersection form of $H_2(Y;\mathbf{Q})$. Define the subspace $X''_1 = j_1(X) \cup_{G \times id_{S^1}} (D^2 \times 0 \times S^1) \subset Y$. Then we see that X''_1 is a deformation retract of Y. Hence, every element of $H_2(Y;\mathbf{Q})$ is represented by a cycle in X''_1. Therefore, a homology class is included in the annihilator of intersection form in $H_2(Y;\mathbf{Q})$ if it is represented by a cycle which has no common point with X''_1. We see

$$j_0(X') \cap X''_1 = \varnothing \text{ and } \iota_Y(\Sigma_{g+1}) \cap X''_1 = \varnothing,$$

so that the preimage of $\mathbf{Q}(e_1 + e_2)$ and $j_{0*} H_2(X';\mathbf{Q})$ are included in the annihilator of intersection form in $H_2(Y;\mathbf{Q})$.

To describe the signature of Y, it suffices to calculate the self-intersection number of $[j_i s - s'_i] = p(e_3 - e_4) + q e_1$. The cycle $j_i s - s'_i$ satisfies

$$\begin{aligned} \mathrm{Im}\,(j_0 s) \cap (\mathrm{Im}\,(j_1 s) \cup \mathrm{Im}\,(s'_{01}) \cup \mathrm{Im}\,(s'_{11})) &= \varnothing \\ \mathrm{Im}\,(s'_{00}) \cap (\mathrm{Im}\,(j_1 s) \cup \mathrm{Im}\,(s'_{01}) \cup \mathrm{Im}\,(s'_{11})) &= \varnothing \\ \mathrm{Im}\,(s'_{10}) \cap (\mathrm{Im}\,(j_1 s) \cup \mathrm{Im}\,(s'_{11})) &= \varnothing, \end{aligned}$$

so that

$$\begin{aligned} (j_0 s - s'_0) \cdot (j_1 s - s'_1) &= (j_0 s - (p s'_{00} + q s'_{10})) \cdot (j_1 s - (p s'_{01} + q s'_{11})) \\ &= q s'_{10} \cdot p s'_{01}. \end{aligned}$$

If necessary, perturb the chain s'_{01}. Then we see that s'_{01} and s'_{10} intersect only once positively. Hence, we have $\mathrm{Sign}\,(Y) = \mathrm{sign}\,(pq) = \mathrm{sign}\,(m(\varphi))$. □

3.2 Wall's non-additivity formula

In the introduction, we stated the Novikov additivity of Signature. Wall derives a formula from this additivity in a more general case, when two compact oriented smooth $4k$–manifolds are glued along common submanifolds of their boundaries. We will give the specific case of his formula for $k = 1$.

Let Z be a closed oriented smooth 2–manifold, X_-, X_0, X_+ compact oriented smooth 3–manifolds with the boundaries $\partial X_- = \partial X_0 = \partial X_+ = Z$, and Y_-, Y_+ compact oriented smooth 4–manifolds with the boundaries $\partial Y_- = X_- \cup_Z (-X_0)$,

$\partial Y_+ = X_0 \cup_Z (-X_+)$. Here we denote by $M \cup_B (-N)$ the union of two manifolds M and N glued by orientation reversing diffeomorphism of their common boundaries $\partial M = \partial N = B$. Let $Y = Y_- \cup_{X_0} Y_+$ be the union of Y_- and Y_+ glued along submanifolds X_0 of their boundaries. Suppose Y is oriented by the induced orientation of Y_- and Y_+.

Write $V = H_1(Z; \mathbf{R})$; let A, B, and C be the kernels of the maps on first homology induce by the inclusions of Z in X_-, X_0 and X_+ respectively.

We define

$$W := \frac{B \cap (C + A)}{(B \cap C) + (B \cap A)},$$

and a bilinear form Ψ by

$$\begin{array}{rcccl} \Psi\colon & W & \times & W & \to & \mathbf{R}. \\ & (b & , & b') & \mapsto & b \cdot c'. \end{array}$$

Here c' is an element of C such that there exists an element $a' \in A$ such that $a' + b' + c' = 0$, and $b \cdot c'$ denotes the intersection product of b and c'. It is known that Ψ is independent of the choice of c' and well-defined on W. Denote the signature of the bilinear form Ψ by $\mathrm{Sign}\,(V; BCA)$ and the signature of the compact oriented 4–manifold M by $\mathrm{Sign}\, M$. We are now ready to state the formula.

Theorem 3.5 (Wall [13]) $\mathrm{Sign}\, Y = \mathrm{Sign}\, Y_- + \mathrm{Sign}\, Y_+ - \mathrm{Sign}\,(V; BCA)$.

3.3 The differences Sign E_g − Sign $E_{g,2}$ and Sign E_{g+1} − Sign $E_{g,2}$

In this subsection, we calculate the difference of signature associated with sewing the trivial Disk bundles onto the $\Sigma_{g,2}$–bundle.

In the introduction, we defined $E_{g,r}^{\varphi,\psi}$ as a oriented $\Sigma_{g,r}$–bundle on P which has monodromies $\varphi, \psi, (\psi\varphi)^{-1} \in \mathcal{M}_{g,r}$ along $\alpha, \beta, \gamma \in \pi_1(P)$. If we fix $\varphi, \psi \in \mathcal{M}_{g,2}$, we denote simply

$$E_{g,2} := E_{g,2}^{\varphi,\psi},\ E_g := E_g^{\theta(\varphi),\theta(\psi)},\ \text{and}\ E_{g+1} := E_{g+1}^{\eta(\varphi),\eta(\psi)}\ (g \geq 0).$$

Proposition 3.6 $\mathrm{Sign}\,(E_g) - \mathrm{Sign}\,(E_{g,2}) = -\mathrm{sign}\,(m(\varphi) + m(\psi) + m((\varphi\psi)^{-1}))$ *for* $g \geq 0$.

Proof E_g is the union of $E_{g,2}$ and $E_D := (D^2 \amalg D^2) \times P$ glued along their boundaries. Using Non-additivity formula Theorem 3.5, we calculate $\mathrm{Sign}\,(E_g) - \mathrm{Sign}\,(E_{g,2})$.

Define Y_-, Y_+, X_-, X_0, X_+, and Z by

$$Y_- := (\amalg_{j=1}^2 D^2) \times P, \quad Y_+ := E_{g,2},$$
$$X_- := (\amalg_{j=1}^2 D^2) \times \partial P, \quad X_+ := E_{g,2}|_{\partial P}, \quad X_0 := (\amalg_{j=1}^2 \partial D^2) \times P,$$
$$\text{and } Z := (\amalg_{j=1}^2 \partial D^2) \times \partial P, \quad \text{respectively.}$$

Here, by the notation stated in Section 2.1,

$$X_+ = E_{g,2}|_{\partial P} \cong X^{\varphi} \amalg X^{\psi} \amalg X^{(\psi\varphi)^{-1}}, \quad Z \cong \partial X^{\varphi} \amalg \partial X^{\psi} \amalg \partial X^{(\psi\varphi)^{-1}}.$$

Define V, A, B, and C as stated in Section 3.1.

Since $X^{\varphi} = X^{\psi} = X^{(\psi\varphi)^{-1}} = S^1 \times S^1$, we can choose the bases of $H_1(\partial X^{\varphi}; \mathbf{R})$, $H_1(\partial X^{\psi}; \mathbf{R})$, and $H_1(\partial X^{(\psi\varphi)^{-1}}; \mathbf{R})$ as stated in Section 2.1. Denote their bases by $\{e_{11}, e_{12}, e_{13}, e_{14}\}$, $\{e_{21}, e_{22}, e_{23}, e_{24}\}$, and $\{e_{31}, e_{32}, e_{33}, e_{34}\}$ respectively.

Since $Z = \partial X^{\varphi} \amalg \partial X^{\psi} \amalg \partial X^{(\psi\varphi)^{-1}}$, we think of e_{ij} as an element of $H_1(Z; \mathbf{R})$.

Denote $m(\varphi) = [a_1 : b_1]$, $m(\psi) = [a_2 : b_2]$, and $m((\psi\varphi)^{-1}) = [a_3 : b_3]$ respectively. Then we have

$$V = H_1(Z, \mathbf{R}) = \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{4} \mathbf{R}e_{ij},$$
$$A = \mathbf{R}e_{11} \oplus \mathbf{R}e_{21} \oplus \mathbf{R}e_{31} \oplus \mathbf{R}e_{12} \oplus \mathbf{R}e_{22} \oplus \mathbf{R}e_{32},$$
$$B = \mathbf{R}(e_{11} - e_{21}) \oplus \mathbf{R}(e_{11} - e_{31}) \oplus \mathbf{R}(e_{12} - e_{22}) \oplus \mathbf{R}(e_{12} - e_{32}) \oplus \mathbf{R}(e_{13} + e_{23} + e_{33}) \oplus \mathbf{R}(e_{14} + e_{24} + e_{34}),$$
$$C = \bigoplus_{i=1}^{3} \begin{cases} \mathbf{R}(e_{i1} + e_{i2}) \oplus \mathbf{R}(e_{i3} - e_{i4} + m_i e_{i1}) & \text{if } a_i \neq 0 \\ \mathbf{R}e_{i1} \oplus \mathbf{R}e_{i2} & \text{if } a_i = 0. \end{cases}$$

Here we denote $m_i := \frac{b_i}{a_i}$. Hence,

$$B \cap A = \mathbf{R}(e_{11} - e_{21}) \oplus \mathbf{R}(e_{12} - e_{22}) \oplus \mathbf{R}(e_{11} - e_{31}) \oplus \mathbf{R}(e_{12} - e_{32}),$$

$$B \cap C = \begin{cases} \mathbf{R}(e_{11}-e_{21}+e_{12}-e_{22}) \oplus \mathbf{R}(e_{11}-e_{31}+e_{12}-e_{32}) \oplus \\ \mathbf{R}(e_{13}+e_{23}+e_{33}-e_{14}-e_{24}-e_{34}+m_1e_{11}+m_2e_{21}+m_3e_{31}) \\ \text{if } a_i \neq 0 \text{ for } i=1,2,3 \text{ and } m_1+m_2+m_3=0, \\ \mathbf{R}(e_{11}-e_{21}+e_{12}-e_{22}) \oplus \mathbf{R}(e_{11}-e_{31}+e_{12}-e_{32}) \\ \text{if } a_i \neq 0 \text{ for } i=1,2,3 \text{ and } m_1+m_2+m_3 \neq 0, \\ \mathbf{R}(e_{11}-e_{21}+e_{12}-e_{22}) \oplus \mathbf{R}(e_{11}-e_{31}+e_{12}-e_{32}) \\ \text{if } a_1=0, a_2 \neq 0, a_3 \neq 0, \\ \mathbf{R}(e_{11}-e_{21}) \oplus \mathbf{R}(e_{12}-e_{22}) \oplus \mathbf{R}(e_{11}-e_{31}+e_{12}-e_{32}) \\ \text{if } a_1=a_2=0, a_3 \neq 0, \\ \mathbf{R}(e_{11}-e_{21}) \oplus \mathbf{R}(e_{12}-e_{22}) \oplus \mathbf{R}(e_{11}-e_{31}) \oplus \mathbf{R}(e_{12}-e_{32}) \\ \text{if } a_i=0 \text{ for } i=1,2,3, \end{cases}$$

$$B \cap (C+A) = \begin{cases} \mathbf{R}(e_{11}-e_{21}) \oplus \mathbf{R}(e_{12}-e_{22}) \oplus \mathbf{R}(e_{11}-e_{31}) \oplus \mathbf{R}(e_{12}-e_{32}) \\ \oplus \mathbf{R}(e_{13}+e_{23}+e_{33}-e_{14}-e_{24}-e_{34}) \\ \text{if } a_i \neq 0 \text{ for } i=1,2,3, \\ \mathbf{R}(e_{11}-e_{21}) \oplus \mathbf{R}(e_{12}-e_{22}) \oplus \mathbf{R}(e_{11}-e_{31}) \oplus \mathbf{R}(e_{12}-e_{32}) \\ \text{otherwise.} \end{cases}$$

By computing the signature of Ψ, we have

$$\operatorname{Sign}(V; BCA) = \begin{cases} \operatorname{sign}(m_1+m_2+m_3) & \text{if } a_i \neq 0 \text{ for } i=1,2,3, \\ 0 & \text{otherwise.} \end{cases}$$

For example, consider the case when $a_i \neq 0$ for $i=1,2,3$ and $m_1+m_2+m_3 \neq 0$. Then, the space W is generated by the element represented by

$$b := e_{13}+e_{23}+e_{33}-e_{14}-e_{24}-e_{34} \in B \cap (C+A).$$

Choose the elements

$$a := m_1e_{11}+m_2e_{21}+m_3e_{31} \in A \text{ and } c := -\sum_{i=1}^{3}(e_{i3}-e_{i4}+m_ie_{i1}) \in C.$$

Then we see that $a+b+c=0$ and obtain $\Psi(b,b)=b\cdot c=m_1+m_2+m_3$. This shows that $\operatorname{Sign}(V; BCA)=\operatorname{sign}(m_1+m_2+m_3)$. The other cases follow in similar ways.

Hence, we obtain

$$\operatorname{Sign}(V; BCA) = \operatorname{sign}(m(\varphi)+m(\psi)+m((\varphi\psi)^{-1})).$$

By the non-additivity formula, we have

$$\mathrm{Sign}\,(E_g) = \mathrm{Sign}\,(E_D) + \mathrm{Sign}\,(E_{g,2}) - \mathrm{Sign}\,(V; BCA).$$

Since E_D is a trivial bundle $(D^2 \amalg D^2) \times P$, we have $\mathrm{Sign}\,(E_D) = 0$.

This completes the proof of the proposition. □

By Theorem 3.1 and Proposition 3.6, we can calculate the difference of signature $\mathrm{Sign}\,(E_g) - \mathrm{Sign}\,(E_{g,2})$.

Corollary 3.7 *For $g \geq 0$,*

$$\mathrm{Sign}\,(E_{g+1}) - \mathrm{Sign}\,(E_{g,2}) = \mathrm{sign}\,(m(\varphi)) + \mathrm{sign}\,(m(\psi)) + \mathrm{sign}\,(m((\varphi\psi)^{-1})) - \mathrm{sign}\,(m(\varphi) + m(\psi) + m((\varphi\psi)^{-1})).$$

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