About the macroscopic dimension of certain PSC–Manifolds

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In this note we give a partial answer to Gromov's question about macroscopic dimension filling of a closed spin PSC–Manifold's universal covering.

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1 Introduction

The following definition was given by M Gromov in [3].

Definition 1.1 Let V be a metric space. We say that $\dim_\varepsilon V \leq k$ if there exists a k–dimensional polyhedron P and a proper uniformly co-bounded map $\phi\colon V \to P$ such that $\mathrm{Diam}(\phi^{-1}(p)) \leq \varepsilon$ for all $p \in P$. A metric space V has the macroscopic $\dim_{\mathrm{mc}} V \leq k$ if $\dim_\varepsilon V \leq k$ for some possibly large $\varepsilon < \infty$. If k is as minimal as possible, we say that $\dim_{\mathrm{mc}} V = k$.

Gromov also posed the following conjecture.

Conjecture C1 *Let (M^n, g) be a closed Riemannian n–manifold with torsion free fundamental group and $(\widetilde{M}^n, \widetilde{g})$ be the universal covering of M^n with the pull-back metric. Suppose that $\dim_{mc}(\widetilde{M}^n, \widetilde{g}) < n$. Then $\dim_{mc}(\widetilde{M}^n, \widetilde{g}) < n-1$.*

Remark 1.2 In fact the macroscopic dimension $\dim_{\mathrm{mc}}(\widetilde{M}^n, \widetilde{g})$ of the universal covering $\widetilde{M}^n$ of M does not depend on a particular choice of a Riemannian metric g on M, since the Riemannian manifolds $(\widetilde{M}^n, \widetilde{g})$ and $(\widetilde{M}^n, \widetilde{g}')$ are quasi-isometric for any two metrics g and g' on M.

This conjecture is true for $n = 3$ (see Bolotov [1]). In [2] the author shown that it fails for $n > 3$.

Actually this question arose in M Gromov's works in connection with the study of PSC–manifolds, ie manifolds admitting a Positive Scalar Curvature metric.

The following is Gromov's PSC–conjecture.

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Conjecture C2 *Let (M^n, g) be a closed Riemannian PSC–manifold with torsion free fundamental group, and let $(\widetilde{M}^n, \widetilde{g})$ be the universal covering of M^n with the pull-back metric, then $\dim_{mc}(\widetilde{M}^n, \widetilde{g}) < n-1$.*

Let us also recall the *Gromov–Lawson–Rosenberg conjecture*.

Conjecture 1.3 *Let M^n be a closed spin manifold, $\pi = \pi_1 M^n$, and let $f\colon M^n \to B\pi$ be a classifying map. Then M^n admits a PSC–metric if and only if*

$$A \circ f_*([M^n]_{KO}) = 0$$

in $KO_n(C_r^*(\pi))$, where $[M^n]_{KO} \in KO_n(M^n)$ is the corresponding fundamental class in KO–theory, $C_r^*(\pi)$ is the reduced C^*–algebra of the group π, and

$$A\colon KO_*(B\pi) \to KO_*(C_r^*(\pi))$$

is the assembly homomorphism of homology theories.

Remark 1.4 $f_*[M^n]_{KO}$ depends only on the bordism class $[M^n, f] \in \Omega_n^{\mathrm{Spin}}(B\pi)$ (see Hitchin [5], Gromov–Lawson [4]).

The following important theorem is proved by J Rosenberg.

Theorem 1.5 (Rosenberg [6]) *Let M^n be a spin manifold, $\pi = \pi_1 M^n$, and $f\colon M^n \to B\pi$ be a classifying map. If M^n is a PSC–manifold then $A \circ f_*[M^n]_{KO} = 0$.*

Recall that the *Strong Novikov Conjecture* asserts the following.

Conjecture 1.6 *The assembly map $A\colon KO_*(B\pi) \to KO_*(C_r^*(\pi))$ is a monomorphism.*

In this paper we prove the following theorem.

Main Theorem *Let M^n be a closed spin PSC–manifold and $\pi = \pi_1 M^n$. Suppose that $\mathrm{cd}\,\pi \le n-1$ and that the Strong Novikov Conjecture holds for π. Then Conjecture C2 is true for M^n as well.*

Remark 1.7 Clearly, for the proof of this result it is sufficient to show that the classifying map $f\colon M^n \to B\pi$ can be deformed into the $(n-2)$–skeleton of $B\pi$. In this case the covering map $\widetilde{f}\colon \widetilde{M} \to \widetilde{B}\pi^{(n-2)}$ would yield the result. Also notice that the Main Theorem is nontrivial only in the case $\mathrm{cd}\pi = n-1$.

2 Proof of Main Theorem

Proof Let M be an oriented, closed, n–dimensional, spin manifold with torsion free fundamental group π. Notice that the Main Theorem is trivial for $n=2$. Moreover, since $\mathrm{cd}\pi \neq 2$ for $n=3$, we will assume that $n \geq 4$. Suppose also that $\mathrm{cd}\,\pi = n-1$ and $[M] = 0 \in \Omega_n^{\mathrm{Spin}}(*)$.

Consider the following composition of maps:

$$M \xrightarrow{f} B\pi \xrightarrow{p} B\pi/B\pi^{(n-2)},$$

where p is the factor-map to the factor-space $B\pi/B\pi^{(n-2)}$ of $B\pi$ by its $(n-2)$–skeleton.

Since $\mathrm{cd}\pi = n-1$, we can assume that $\dim B\pi = n-1$ and $B\pi/B\pi^{(n-2)}$ is homeomorphic to a bouquet of $(n-1)$–dimensional spheres.

We can also assume that M is endowed with cellular decomposition having only one cell in each dimensions 0 and n, and that f is a cellular map. Let $f^{(n-2)}$ be the restriction of f to the $(n-2)$–skeleton of M. The first obstruction class $[c_f^{n-1}]$ for the extension of $f^{(n-2)}$ to the $(n-1)$–skeleton belongs to the group $H^{n-1}(M, \pi_{n-2}(B\pi^{(n-2)}))$. Notice that by Hurewicz's theorem

$$\pi_{n-2}(B\pi^{(n-2)}) \cong H_{n-2}(B\pi^{(n-2)}, \mathbb{Z}[\pi])$$

is a free $\mathbb{Z}[\pi]$–module. Hence by Poincaré duality with twisted coefficients

$$H^k(M, \oplus_i \mathbb{Z}[\pi]) \cong H_c^k(\widetilde{M}, \oplus_i \mathbb{Z}) \cong H_{n-k}(\widetilde{M}, \oplus_i \mathbb{Z}). \tag{$*$}$$

Since $H_1(\widetilde{M}, \oplus_i \mathbb{Z}) = 0$, we can extend $f^{(n-2)}$ to a map $f^{(n-1)}\colon M^{(n-1)} \to B\pi^{(n-2)}$ changing (if necessary) $f^{(n-2)}$ on the $(n-2)$–skeleton, but not changing $f^{(n-2)}$ on the $(n-3)$–skeleton.

Since $B\pi$ is a $K(\pi,1)$–space, we can also extend $f^{(n-1)}$ to the map $\hat{f}\colon M \to B\pi$. In the sequel we will denote this map $\hat{f}$ by f.

Notice that the first obstruction $c_f^n \in C^n(M, \pi_{n-1}(B\pi^{(n-2)}))$ for an extension of $f^{(n-1)}$ to all of M can be represented as a composition of $\mathbb{Z}[\pi]$–module homomorphisms:

$$c_f^n\colon C_n(M, \mathbb{Z}[\pi]) \cong \pi_n(M, M^{(n-1)}) \xrightarrow{\partial} \pi_{n-1}(M^{(n-1)}) \xrightarrow{f_*^{(n-1)}} \pi_{n-1}(B\pi^{(n-2)}).$$

Consider the following commutative diagram:

$$\begin{array}{ccc} \pi_n(M, M^{(n-1)}) & \xrightarrow{f_*} & \pi_n(B\pi, B\pi^{(n-2)}) \\ \downarrow \partial & & \downarrow \partial \\ \pi_{n-1}(M^{(n-1)}) & \xrightarrow{f_*^{(n-1)}} & \pi_{n-1}(B\pi^{(n-2)}) \end{array}$$

Notice that $\pi_n(B\pi, B\pi^{(n-2)}) \xrightarrow{\partial} \pi_{n-1}(B\pi^{(n-2)})$ is an isomorphism. This can easily be seen from the exact sequence of the pair $(B\pi, B\pi^{(n-2)})$ since $\pi_i(B\pi) = 0$ for $i \geq 2$.

Recall that $n \geq 4$ and

$$\pi_n(B\pi, B\pi^{(n-2)}) \cong \pi_n(\widetilde{B}\pi, \widetilde{B}\pi^{(n-2)}) \cong \pi_n(\widetilde{B}\pi/\widetilde{B}\pi^{(n-2)})$$

is a free $\mathbb{Z}_2[\pi]$–module.

Using Poincaré duality for the $\mathbb{Z}[\pi]$–module $\Lambda = \pi_n(\widetilde{B}\pi/\widetilde{B}\pi^{(n-2)})$ it is not hard to verify that

$$H^n(M, \Lambda) \cong \Lambda \otimes_{\mathbb{Z}[\pi]} \mathbb{Z} \cong \oplus_i \mathbb{Z}_2.$$

Consider the following commutative diagram:

$$\begin{array}{ccc} \pi_n(M, M^{(n-1)}) & \xrightarrow{f_*} & \pi_n(B\pi, B\pi^{(n-2)}) \\ \downarrow \otimes\mathbb{Z} & & \downarrow \otimes\mathbb{Z} \\ \pi_n(M, M^{(n-1)}) \otimes \mathbb{Z} & \xrightarrow{\bar{f}_*} & \pi_n(B\pi, B\pi^{(n-2)}) \otimes \mathbb{Z} \end{array}$$

Clearly, $\pi_n(M, M^{(n-1)}) \otimes \mathbb{Z} \cong \pi_n(M/M^{(n-1)}) \cong \pi_n(S^n)$ and

$$\pi_n(B\pi, B\pi^{(n-2)}) \otimes \mathbb{Z} \cong \pi_n(B\pi/B\pi^{(n-2)}) \cong \pi_n(\bigvee_i S^{n-1}).$$

We conclude that $[c_f] = (\bar{f}_* \circ \otimes\mathbb{Z})(c)$, where c is a generator of free module $\pi_n(M, M^{(n-1)})$, and $[c_f]$ is represented by the map (as an element of the homotopy group $\pi_n(\bigvee_i S^{n-1})$):

$$M/M^{(n-1)} \cong S^n \xrightarrow{\bar{f}} B\pi/B\pi^{(n-2)} \cong \bigvee_i S^{n-1}.$$

Consider the following commutative diagram:

$$\begin{array}{ccccc} M & \xrightarrow{q} & S^n & \xrightarrow{id} & S^n \\ f\big\downarrow & & \big\downarrow\bar{f} & & \big\downarrow h \\ B\pi & \xrightarrow{p} & \bigvee_i S^{n-1} & \xrightarrow{p_k} & S^{n-1} \end{array} \qquad (**)$$

Suppose that for some k the map $p_k \circ \bar{f}$ is not null-homotopic, where p_k is the projection on the k–factor of the bouquet. The map

$$p_k \circ p \circ f\colon M \to S^{n-1}$$

induces a composition of homomorphisms

$$p_{k*} \circ p_* \circ f_*\colon KO_n(M) \to KO_n(S^{n-1}).$$

Clearly, if

$$p_{k*} \circ p_* \circ f_*[M]_{KO} = (h \circ q)_*[M]_{KO} \neq 0,$$

then $f_*[M]_{KO} \neq 0$ as well.

Lemma 2.1 *Let $\Omega_n^{Spin}(S^{n-1})$ be the n th bordism group of S^{n-1}. Then $[(M, h \circ q)] = [(S^n, h)]$ in $\Omega_n^{Spin}(S^{n-1})$.*

Proof Since $[M] = 0 \in \Omega_n^{\mathrm{Spin}}(*)$, there exists an $(n+1)$–dimensional spin manifold W with $\partial W = M$. Let $B \subset W$ be a small open ball and

$$i\colon D^n \times I \to W \setminus B$$

be a regular normal neighborhood of the transversal segment $i\colon 0 \times I \to W \setminus B$, such that $i(0,0) \in M$ and $i(0,1) \in \partial\bar{B} \cong S^n$. Define the following map:

$$i(D^n \times I) \overset{\text{retraction}}{\longrightarrow} i(D^n \times 0) \overset{\text{quotient}}{\longrightarrow} i(D^n \times 0)/i(\partial D^n \times 0) \cong S^n \overset{h}{\to} S^{n-1}.$$

We can extend it to the map $F\colon W \setminus B \to S^{n-1}$ which is constant outside $i(D^n \times I)$.

Clearly, the restriction $F|_M$ is homotopic to $h \circ q$ in M and the restriction $F|_{\partial\bar{B}}$ is homotopic to h in $\partial\bar{B}$. □

Since $(h \circ q)_*[M]_{KO}$ depends only on the bordism class in $\Omega_n^{\mathrm{Spin}}(S^{n-1})$ (Hitchin [5]), we obtain from Lemma 2.1 that

$$(h \circ q)_*[M]_{KO} = h_*[S^n]_{KO}.$$

We will now show that $[h] \in \pi_1^s$ represents a non-zero element $h_*[S^n]_{KO}$ in $KO_n(S^{n-1})$.

By assumption h is not homotopic to zero. Therefore h must be homotopic to the Hopf $(n-3)$–suspension $H\colon S^n \to S^{n-1}$ which induces a homomorphism $H_*\colon KO_n(S^n) \to KO_n(S^{n-1})$. But $H_*[S^n]_{KO} \neq 0$. Indeed, let $\overline{S}^1$ be a circle with nontrivial spin structure and $pr\colon S^{n-1} \times \overline{S}^1 \to S^n$ be the natural projection. Using framed surgery along generating circle $\overline{S}^1$ it is easy to verify that:

$$[(S^n, H)] = [(S^{n-1} \times \overline{S}^1, pr)] \in \Omega_n^{\mathrm{Spin}}(S^{n-1}).$$

But

$$KO_n(S^{n-1}) = KO_n(\mathbb{R}^{n-1}) \oplus KO_n(*).$$

Moreover, $pr_*[(S^{n-1} \times \overline{S}^1)]_{KO}$ is equal to the generator of

$$KO_n(\mathbb{R}^{n-1}) \cong KO_{n-1}(\mathbb{R}^{n-1}) \otimes KO_1(*) \cong \mathbb{Z} \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$$

and

$$h_*[S^n]_{KO} = H_*[S^n]_{KO} = pr_*[(S^{n-1} \times \overline{S}^1)]_{KO} \neq 0.$$

Thus both $(h \circ q)_*[M]_{KO}$ and $f_*[M]_{KO}$ are non-zero.

Therefore if the Strong Novikov Conjecture is true, then by Theorem 1.5. M does not admit a PSC–metric.

We conclude that if M is a PSC–manifold, then $[\overline{f}] = 0$. Therefore f can be deformed to the $(n-2)$–skeleton of $B\pi$ and by Remark 1.7 $\dim_{\mathrm{mc}} \widetilde{M} \leq n-2$.

In the case when $[M] \neq 0 \in \Omega_n^{\mathrm{Spin}}(*)$ we can consider the manifold $M \times S^1$ representing $0 \in \Omega_{n+1}^{\mathrm{Spin}}(*)$. Clearly, $M \times S^1$ is a PSC–manifold whenever M is a PSC–manifold. Let us consider the following diagram:

$$\begin{array}{ccccccc}
M \times S^1 & \xrightarrow{S} & SM & \xrightarrow{Sq} & S^{n+1} & \xrightarrow{id} & S^{n+1} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Sf} & & \downarrow{\scriptstyle S\overline{f}} & & \downarrow{\scriptstyle Sh} \\
B\pi \times S^1 & \xrightarrow{S} & SB\pi & \xrightarrow{Sp} & \bigvee_i S^n & \xrightarrow{p_k} & S^n
\end{array}$$

where the symbol S means a suspension.

For the natural cell decomposition of $M \times S^1$ the result for M follows from the previous discussion of $M \times S^1$ taking into account that if $h \sim H$, then $Sh \sim SH$. □

Corollary 2.2 *The counterexamples to the Conjecture C1 constructed in [2] do not admit PSC–metrics.*

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