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Classification of string links up to self delta-moves and concordance

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For an n–component string link, the Milnor's concordance invariant is defined for each sequence $I = i_1 i_2 \cdots i_m$ $(i_j \in \{1, \ldots, n\})$. Let $r(I)$ denote the maximum number of times that any index appears. We show that two string links are equivalent up to self Δ–moves and concordance if and only if their Milnor invariants coincide for all sequences I with $r(I) \leq 2$.

57M25, 57M27

1 Introduction

For an n–component link L, the *Milnor* $\overline{\mu}$*–invariant* $\overline{\mu}_L(I)$ is defined for each sequence $I = i_1 i_2 \cdots i_m$ $(i_j \in \{1, \ldots, n\})$; see Milnor [13; 14]. Let $r(I)$ denote the maximum number of times that any index appears. For example, $r(1123) = 2$ and $r(1231223) = 3$. It is known that if $r(I) = 1$, then $\overline{\mu}_L(I)$ is a *link-homotopy* invariant [13], where link-homotopy is an equivalence relation on links generated by self crossing changes. Similarly, for a string link L, the Milnor μ*–invariant* $\mu_L(I)$ is defined; see Habegger and Lin [7]. Milnor μ–invariants give a link-homotopy classification for string links.

Theorem 1.1 [7] *Two n–component string links L and L' are link-homotopic if and only if $\mu_L(I) = \mu_{L'}(I)$ for any I with $r(I) = 1$.*

Theorem 1.1 implies the following.

Theorem 1.2 [13; 7] *A link L in S^3 is link-homotopic to the trivial link if and only if $\overline{\mu}_L(I) = 0$ for any I with $r(I) = 1$.*

Although Milnor invariants for sequences I with $r(I) \geq 2$ are not necessarily link-homotopy invariants, they are generalized link-homotopy invariants. In fact, T Fleming and the author [4, Theorem 1.1] showed that $\overline{\mu}$–invariants for sequences I with $r(I) \leq k$ are *self C_k–equivalence* invariants of links in S^3, where the self C_k–equivalence is an equivalence relation on (string) links generated by self C_k–moves, and a C_k–move is

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a local move on links defined by Habiro [9; 10]. This statement holds for μ–invariants of string links as well. The proof is the same as the one of [4, Theorem 1.1] except for using Proposition 3.1 instead of [14, Theorem 7]. The link-homotopy coincides with the self C_1–equivalence. The self C_2–equivalence coincides with the *self Δ–equivalence*, which is an equivalence relation generated by *self Δ–moves*. A *Δ–move* is a local move as illustrated in Figure 1; see Murakami and Nakanishi [15]. The Δ–move is called a self Δ–move if all strands in Figure 1 belong to the same component of a (string) link; see Shibuya [19].

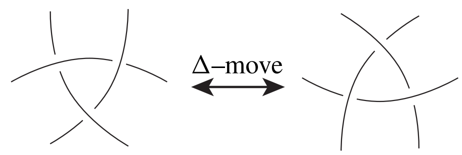

Figure 1

A self Δ–equivalence classification of 2–*component* links was given by Y Nakanishi and Y Ohyama [16] and for 2–component *string* links by Fleming and the author [3]. It is still open for (string) links with at least 3 components.

The following result is a generalization of Theorem 1.2.

Theorem 1.3 [23, Corollary 1.5] *A link L is self Δ–equivalent to the trivial link if and only if $\overline{\mu}_L(I) = 0$ for any I with $r(I) \leq 2$.*

In this paper, we generalize Theorem 1.1 and give a certain geometric characterization for string links whose μ–invariants coincide for all sequences I with $r(I) \leq 2$. It is known that self Δ–equivalence is too fine to give the characterization, ie, there are 2–component string links such that their μ–invariants vanish for all sequences I with $r(I) \leq 2$ and they are not self Δ–equivalent to the trivial string link [3].

Since Milnor invariants are concordance invariants by Casson [1] and concordance does not imply self Δ–equivalence by Nakanishi and Shibuya [17] and Nakanishi, Shibuya and Yasuhara [18], an equivalence relation generated by concordance and self Δ–moves is looser than the self Δ–equivalence and preserves Milnor invariants for all sequences I with $r(I) \leq 2$. We define the equivalence relation as follows. Two (string) links L and L' are *self–Δ concordant* if there is a sequence $L = L_1, \ldots, L_m = L'$ of (string) links such that for each $i\,(\in \{1, \ldots, m-1\})$, L_i and L_{i+1} are either concordant or self Δ–equivalent.

The following is the main result of this paper.

Theorem 1.4 *Two n–component string links L and L' are self–Δ concordant if and only if $\mu_L(I) = \mu_{L'}(I)$ for any I with $r(I) \leq 2$.*

For an n–component string link L, let $L(k)$ be a kn–component string link obtained from L by replacing each component of L with k zero framed parallels of it. By combining Theorem 1.4 and Proposition 3.2, we have the following corollary.

Corollary 1.5 *Two string links L and L' are self–Δ concordant if and only if $L(2)$ and $L'(2)$ are link-homotopic.*

Let $\mathcal{SL}(n)$ be the set of n–component string links, and let $\mathcal{SL}(n)/(\mathrm{s}\Delta + \mathrm{c})$ (resp. $\mathcal{SL}(n)/C_m$) be the set of self–Δ concordance classes (resp. the set of C_m–equivalence classes). Habiro showed that $\mathcal{SL}(n)/C_m$ is a nilpotent group [10, Theorem 5.4]. Since C_{2n}–equivalence for n–component (string) links implies self Δ–equivalence [4, Lemma 1.2], we have that $\mathcal{SL}(n)/(\mathrm{s}\Delta + \mathrm{c})$ is a nilpotent group. Moreover, since the first nonvanishing μ–invariants are additive under the stacking product (for example see Cochran [2] and Habegger and Masbaum [8]), by Theorem 1.4, we have the following proposition.

Proposition 1.6 *The quotient $\mathcal{SL}(n)/(\mathrm{s}\Delta + \mathrm{c})$ forms a torsion–free nilpotent group under the stacking product.*

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2 String links and Milnor invariants

In this section, we summarize the definitions of string links and Milnor invariants of links and string links.

A string link is a generalization of a pure braid defined by Habegger and Lin [7].

2.1 String links

Let D be the unit disk in the plane and let $I=[0,1]$ be the unit interval. Choose n points $p_1,\dots,p_n$ in the interior of D so that $p_1,\dots,p_n$ lie in order on the x–axis, see Figure 2. An *n–component string link* $L=K_1\cup\cdots\cup K_n$ in $D\times I$ is a disjoint union of oriented arcs $K_1,\dots,K_n$ such that each K_i runs from $(p_i,0)$ to $(p_i,1)$ $(i=1,\dots,n)$. A string link $K_1\cup\cdots\cup K_n$ with $K_i=\{p_i\}\times I$ $(i=1,\dots,n)$ is called the *n–component trivial string link* and denoted by $\mathbf{1}_n$. For a string link L in $D\times I$, the *closure* $\mathrm{cl}(L)$ of L is a link in S^3 obtained from L by identifying points of $\partial(D\times I)$ with their images under the projection $D\times I\longrightarrow D$. It is easy to see that every link is the closure of some string link.

Milnor defined in [13; 14] a family of invariants of oriented, ordered links in S^3, known as Milnor $\overline{\mu}$–invariants.

2.2 Milnor invariants of links

Given an n–component link $L=K_1\cup\cdots\cup K_n$ in S^3, denote by G the fundamental group of $S^3\setminus L$, and by G_q the q–th subgroup of the lower central series of G. We have a presentation of G/G_q with n generators, given by a meridian α_i of each component K_i. So, for each $j\in\{1,\dots,n\}$, the longitude l_j of the j–th component of L is expressed modulo G_q as a word in the α_i's (abusing notation, we still denote this word by l_j). The *Magnus expansion* $E(l_j)$ of l_j is the formal power series in noncommuting variables $X_1,\dots,X_n$ obtained by substituting $1+X_i$ for α_i and $1-X_i+X_i^2-X_i^3+\cdots$ for α_i^{-1}, $i=1,\dots,n$. Let $I=i_1i_2\cdots i_{k-1}j$ $(k\le q)$ be a sequence in $\{1,\dots,n\}$. Denote by $\mu_L(I)$ the coefficient of $X_{i_1}\cdots X_{i_{k-1}}$ in the Magnus expansion $E(l_j)$. *Milnor $\overline{\mu}$–invariant* $\overline{\mu}L(I)$ is the residue class of $\mu_L(I)$ modulo the greatest common divisor of all $\mu_L(J)$ such that J is obtained from I by removing at least one index, and permutating the remaining indices cyclicly.

In [7], Habegger and Lin define the Milnor invariants of string links. We also refer the reader to Habegger and Masbaum [8].

2.3 Milnor invariants of string links

In the unit disk D, we chose a point $e\in\partial D$ and loops $\alpha_1,\dots,\alpha_n$ as illustrated in Figure 2. For an n–component string link $L=K_1\cup\cdots\cup K_n$ in $D\times I$ with $\partial K_j=\{(p_j,0),(p_j,1)\}$ $(j=1,\dots,n)$, set $Y=(D\times I)\setminus L$, $Y_0=(D\times\{0\})\setminus L$, and $Y_1=(D\times\{1\})\setminus L$. We may assume that each $\pi_1(Y_t)$ $(t\in\{0,1\})$ with base point (e,t) is the free group $F(n)$ on generators $\alpha_1,\dots,\alpha_n$. We denote the image of α_j in the lower central series quotient $F(n)/F(n)_q$ again by α_j. By Stallings' theorem [22], the inclusions

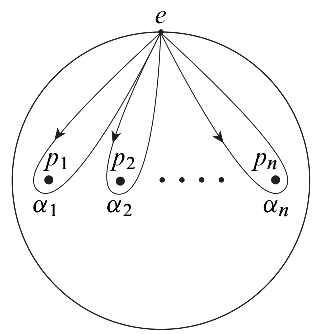

Figure 2

$i_t\colon Y_t \longrightarrow Y$ induce isomorphisms $(i_t)_*\colon \pi_1(Y_t)/\pi_1(Y_t)_q \longrightarrow \pi_1(Y)/\pi_1(Y)_q$ for any positive integer q. Hence the induced map $(i_1)_*^{-1} \circ (i_0)_*$ is an automorphism of $F(n)/F(n)_q$ and sends each α_j to a conjugate $l_j\alpha_j l_j^{-1}$ of α_j, where l_j is the *longitude* of K_j defined as follows. Let γ_j be a zero framed parallel of K_j such that the endpoints $(c_j, t) \in D \times \{t\}$ $(t = 0, 1)$ lie on the x–axis in $\mathbb{R}^2 \times \{t\}$. The longitude $l_j \in F(n)/F(n)_q$ is an element represented by the union of the arc γ_j and the segments $e \times I$, $c_j e \times \{0, 1\}$ under $(i_1)_*^{-1}$. The coefficient $\mu_L(i_1 i_2 \cdots i_{k-1} j)$ $(k \le q)$ of $X_{i_1} \cdots X_{i_{k-1}}$ in the Magnus expansion $E(l_j)$ is well-defined invariant of L, and it is called a *Milnor μ–invariant* of L.

3 Proof of Theorem 1.4

By an argument similar to that in the proof of [14, Theorem 7], we have the following proposition.

Proposition 3.1 (cf [14, Theorem 7]) *Let L'_j $(j = 1, 2)$ be an l–component string link obtained from an n–component string link L_j by replacing each component of L_j with zero framed parallels of it. Suppose that the i–th components of L'_1 and L'_2 correspond to the $h(i)$–th components of L_1 and L_2 respectively. For a sequence $i_1 i_2 \cdots i_m$ of integers in $\{1, 2, \ldots, l\}$, $\mu_{L'_1}(I) = \mu_{L'_2}(I)$ for any subsequence I of $i_1 i_2 \cdots i_m$ if and only if $\mu_{L_1}(J) = \mu_{L_2}(J)$ for any subsequence J of $h(i_1)h(i_2)\cdots h(i_m)$.*

Remark It is shown that for a link L' in S^3 obtained from a link L by taking zero framed parallels of the components of L, if the i–th component of L' corresponds to the $h(i)$–th component of L, then $\overline{\mu}_{L'}(i_1 i_2 \cdots i_m) = \overline{\mu}_L(h(i_1)h(i_2)\cdots h(i_m))$ [14, Theorem 7]. Although this looks stronger than the proposition above, it holds for the residue class $\overline{\mu}$. It does not hold for μ–invariant of string links. In fact, there is the following example: Let L be the 2–component string link illustrated in Figure 3 and L' the 3–component string link illustrated in Figure 3, which is obtained from L by taking two zero framed parallels of the first component. Note that $h(1) = h(2) = 1$

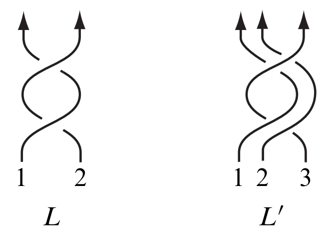

Figure 3

and $h(3) = 2$. Then the Magnus expansion of the 2nd longitude of L is $1 + X_1$ and the expansion of the 3rd longitude of L' is $1 + X_2 + X_1 + X_1X_2$. Hence we have $\mu_L(112) \neq \mu_{L'}(123)$.

Proof It is enough to consider the special case where L'_j $(j = 1, 2)$ is an $(n+1)$–component string link obtained from an n–component string link L_j by replacing the n–th component of L_j with two parallels of it. We may assume that the two parallels are contained in a tubular neighborhood N_j of the n–th component of L_j. Then there is the natural homomorphism from $\pi_1(E_j)(\cong \pi_1(E_j \setminus N_j))$ to $\pi_1(E'_j)$, where E_j and E'_j are the complements of L_j and L'_j respectively. The i–th longitudes l_{ji} $(i = 1, \ldots, n)$ of L_j map to the i–th longitudes l'_{ji} of L'_j, the i–th meridians α_{ji} $(i = 1, \ldots, n-1)$ of L_j map to the i–th meridians α'_{ji} of L'_j, and the n–th meridian α_{jn} maps to $\alpha'_{jn}\alpha'_{jn+1}$. Note that l'_{jn+1} is equal to l'_{jn}. The Magnus expansion $M(l'_{ji})$ of l'_{ji} can be obtained from the expansion

$$M(l_{jh(i)}) = 1 + \sum \mu_{L_j}(h_1 \cdots h_s h(i)) X_{h_1} \cdots X_{h_s}$$

by substituting $M(\alpha'_{jn}\alpha'_{jn+1}) - 1 = X_n + X_{n+1} + X_n X_{n+1}$ for X_n.

Hence, if $\mu_{L_1}(J) = \mu_{L_2}(J)$ for any subsequence J of $h(i_1) \cdots h(i_m)$, then $\mu_{L'_1}(I) = \mu_{L'_2}(I)$ for any subsequence of $i_1 \cdots i_m$. Recall that $h(i) = i$ $(i = 1, \ldots, n-1)$ and $h(n) = h(n+1) = n$.

On the other hand, suppose that there is a subsequence Jh of $h(i_1) \cdots h(i_m)$ such that $\mu_{L_1}(Jh) \neq \mu_{L_2}(Jh)$. We may assume that the length of J is minimal among all such subsequences, ie, for any subsequence $J'(\neq J)$ of J, $\mu_{L_1}(J'h) = \mu_{L_2}(J'h)$. Let $k_1 \cdots k_t k$ be a subsequence of $i_1 \cdots i_m$ with $h(k_1) \cdots h(k_t) = J$ and $h(k) = h$. Note that $\mu_{L'_j}(k_1 \cdots k_t k)$ might not be equal to $\mu_{L_j}(Jh)$ (if $k_1 \cdots k_t$ contains the pattern $n(n+1)$). Since the Magnus expansion $M(l'_{jk})$ can be obtained from $M(l_{jh})$ by substituting $X_n + X_{n+1} + X_n X_{n+1}$ for X_n, there is a set S_J, possibly $S_J = \{J\}$, of subsequences of J such that

$$\mu_{L'_j}(k_1 \cdots k_t k) = \sum_{J' \in S_J} \mu_{L_j}(J'h) \ (j = 1, 2).$$

The minimality of J implies

$$\mu_{L'_1}(k_1 \cdots k_t k) - \mu_{L'_2}(k_1 \cdots k_t k) = \mu_{L_1}(Jh) - \mu_{L_2}(Jh) \neq 0.$$

This completes the proof. □

By combining Theorem 1.1 and Proposition 3.1, we have the following characterization for string links whose μ–invariants coincide for all sequences I with $r(I) \leq k$.

Proposition 3.2 (cf [2, Proposition 9.3]) *Let L_1 and L_2 be n–component string links and k a natural number. Let $L_j(k)$ be a kn–component string link obtained from L_j by replacing each component of L_j with k zero framed parallels of it $(j = 1, 2)$. Then the following are equivalent:*

(1) $\mu_{L_1}(J) = \mu_{L_2}(J)$ *for any J with $r(J) \leq k$.*

(2) $\mu_{L_1(k)}(I) = \mu_{L_2(k)}(I)$ *for any I with $r(I) = 1$.*

(3) *$L_1(k)$ and $L_2(k)$ are link-homotopic.*

Proof Theorem 1.1 implies "(2) ⇔ (3)". We only need to show "(1) ⇔ (2)".

Suppose that the i–th components of $L_1(k)$ and $L_2(k)$ correspond to the $h(i)$–th components of L_1 and L_2 respectively. For a sequence $I = i_1 i_2 \cdots i_m$ of integers in $\{1, 2, \ldots, nk\}$, let $h(I)$ denote $h(i_1)h(i_2)\cdots h(i_m)$.

(1) ⇒ (2) Let I be a sequence of integers in $\{1, 2, \ldots, nk\}$ with $r(I) = 1$. Since $r(h(I)) \leq k$, for any subsequence J of $h(I)$, we have $r(J) \leq k$, and hence $\mu_{L_1}(J) = \mu_{L_2}(J)$. By Proposition 3.1, we have $\mu_{L_1(k)}(I) = \mu_{L_2(k)}(I)$.

(2) ⇒ (1) Let J be a sequence of integers in $\{1, 2, \ldots, n\}$ with $r(J) \leq k$. Then there is a sequence I' of integers in $\{1, 2, \ldots, nk\}$ with $r(I') = 1$ and $h(I') = J$. Since any subsequence I of I' satisfies $r(I) = 1$, $\mu_{L_1(k)}(I) = \mu_{L_2(k)}(I)$. By Proposition 3.1, we have $\mu_{L_1}(J) = \mu_{L_2}(J)$. □

By using Proposition 3.2, we have the following proposition.

Proposition 3.3 *Let L and L' be n–component string links and k a natural number. Then $\mu_L(I) = \mu_{L'}(I)$ for any I with $r(I) \leq k$ if and only if $\mu_{L*\overline{L'}}(I) = 0$ for any I with $r(I) \leq k$, where $*$ is the stacking product and $\overline{L'}$ is the horizontal mirror image of L' with the orientation reversed.*

Note that, for a string link L, $\overline{L}$ is the inverse of L under the concordance, ie, both $\overline{L} * L$ and $L * \overline{L}$ are concordant to a trivial string link.

Proof By Proposition 3.2, $\mu_L(I) = \mu_{L'}(I)$ (resp. $\mu_{L*\overline{L'}}(I) = 0$) for any I with $r(I) \leq k$ if and only if $L(k)$ and $L'(k)$ (resp. $(L * \overline{L'})(k)$ and the kn–component trivial string link $\mathbf{1}_{kn}$) are link-homotopic. Hence it is enough to show that $L(k)$ and $L'(k)$ are link-homotopic if and only if $(L * \overline{L'})(k)$ and $\mathbf{1}_{kn}$ are link-homotopic. Note that $(L * \overline{L'})(k) = L(k) * \overline{L'(k)}$.

If $L(k)$ and $L'(k)$ are link-homotopic, then $L(k) * \overline{L'(k)}$ is link-homotopic to $L'(k) * \overline{L'(k)}$, which is concordant to $\mathbf{1}_{kn}$. Since concordance of string links implies link-homotopy [5; 6][1], $L(k) * \overline{L'(k)}$ is link-homotopic to $\mathbf{1}_{kn}$.

If $L(k) * \overline{L'(k)}$ is link-homotopic to $\mathbf{1}_{kn}$, then $L(k) * \overline{L'(k)} * L'(k)$ is link-homotopic to $L'(k)$. Since $L(k) * \overline{L'(k)} * L'(k)$ is concordant to $L(k)$, $L(k)$ is link-homotopic to $L'(k)$. This completes the proof. □

Two n–component string links L and L' are *weak self Δ–equivalent* if the closure $\mathrm{cl}(L * \overline{L'})$ is self Δ–equivalent to the trivial link.

T Shibuya defined weak self Δ–equivalence for links in S^3 [21], and showed that two links in S^3 are weak self Δ–equivalent if and only if they are self–Δ concordant [20]. (In [21] and [20], the self–Δ concordance is called the Δ–*cobordism*.) Here we give the same result for string links.

Proposition 3.4 (cf [20, Theorem]) *Two string links L and L' are weak self Δ–equivalent if and only if they are self–Δ concordant.*

Before proving the proposition above, we need some preparation.

Let $L = K_1 \cup \cdots \cup K_n$ be an n–component (string) link and b a band attaching a single component K_i with coherent orientation, ie, $b \cap L = b \cap K_i \subset \partial b$ consists

[1] In [5; 6], it was shown that concordance of links in S^3 implies link-homotopy. It still holds for string links. It also follows from Theorem 1.1 since Milnor invariants are concordance invariants.

of two arcs whose orientations from K_i are opposite to those from ∂b. Then $L' = (L \cup \partial b) \setminus \mathrm{int}(b \cap K_i)$, which is a union of an n–component (string) link and a knot, is said to be obtained from L by *fission* (along a band b), and conversely L is said to be obtained from L' by *fusion* [12].

Lemma 3.5 *Let L_1, L_2, L_3 be oriented tangles such that L_2 is obtained from L_1 by a single (self) Δ–move, and that L_3 is obtained from L_2 by a single fusion. Then there is an oriented tangle L_2' such that L_2' is obtained from L_1 by a single fusion, and that L_3 is obtained from L_2' by a single (self) Δ–move.*

Proof Let B be a 3–ball such that $L_1 \setminus B = L_2 \setminus B$, and that the pair of tangles $(B, L_1 \cap B)$ and $(B, L_2 \cap B)$ is a (self) Δ–move. Let b be a fusion band with $L_3 = (L_2 \cup \partial b) \setminus \mathrm{int}(b \cap L_2)$. If b intersects B, then we can move it out of B by an isotopy fixing L_2 since $(B, L_2 \cap B)$ is a trivial tangle. Thus we may assume that b is contained in $L_1 \setminus B$. Let L_2' be a link obtained from L_1 by fusion along b. Then L_3 is obtained from L_2' by a (self) Δ–move, which corresponds to substituting $(B, L_2 \cap B)$ for $(B, L_1 \cap B)$. □

Proof of Proposition 3.4 If L and L' are self–Δ concordant, then $\mu_L(I) = \mu_{L'}(I)$ for any I with $r(I) \leq 2$ [1; 4]. Proposition 3.3 and Theorem 1.3 imply that $\mathrm{cl}(L * \overline{L'})$ is self Δ–equivalent to the trivial link.

Suppose L and L' are weak self Δ–equivalent. Since L is concordant to $L * \overline{L'} * L'$, it is enough to show that $L * \overline{L'} * L'$ and L' are self–Δ concordant. The split sum of L' and $\mathrm{cl}(L * \overline{L'})$ is obtained from $L * \overline{L'} * L'$ by a finite sequence of fission, and $\mathrm{cl}(L * \overline{L'})$ is self Δ–equivalent to the trivial link O. So $L * \overline{L'} * L'$ is obtained from the split sum of L' and O by a sequence of self Δ–moves and fusion. By Lemma 3.5, we can freely choose to perform all fusion first, and then all self Δ–moves. Hence we have that the fusion of L' and O, which is concordant to L', is self Δ–equivalent to $L * \overline{L'} * L'$. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4 By Theorem 1.3, L and L' are weak self Δ–equivalent if and only if $\overline{\mu}_{\mathrm{cl}(L_1 * \overline{L'})}(I) = \mu_{L_1 * \overline{L'}}(I) = 0$ for any I with $r(I) \leq 2$. Proposition 3.3 and Proposition 3.4 complete the proof. □

Remark Although the C_k–move ($k \geq 3$) is not an unknotting operation, it might be reasonable to consider the following question: For two string links L and L' whose components are trivial, if $\mu_L(I) = \mu_{L'}(I)$ for any I with $r(I) \leq k$, then are L and L'

equivalent up to self C_k–move and concordance? The question is still open, but the answer is likely negative. For example, the Hopf link with both components Whitehead doubled, which is a boundary link and thus all its Milnor invariants vanish, is neither self C_3–equivalent [4] nor concordant [11, Section 7.3] to the trivial link.

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