Minimal entropy and geometric decompositions in dimension four

PABLO SUÁREZ-SERRATO

We show vanishing results about the infimum of the topological entropy of the geodesic flow of homogeneous smooth four-manifolds. We prove that any closed oriented *geometric* four-manifold has zero minimal entropy if and only if it has zero simplicial volume. We also show that if a four-manifold M admits a geometric decomposition in the sense of Thurston and does not have geometric pieces modelled on hyperbolic four-space $\mathbb{H}^4$, the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ or the product of two hyperbolic planes $\mathbb{H}^2 \times \mathbb{H}^2$ then M admits an $\mathcal{F}$–structure. It follows that M has zero minimal entropy and collapses with curvature bounded from below. We then analyse whether or not M admits a metric whose topological entropy coincides with the minimal entropy of M and provide new examples of manifolds for which the minimal entropy problem cannot be solved.

37B40, 57M50; 22F30, 53D25

1 Introduction

A *model geometry*, in the sense of W P Thurston, is a complete simply connected Riemannian manifold X such that the group of isometries acts transitively on X and contains a discrete subgroup with a finite volume quotient. The maximal four dimensional geometries were classified by R Filipkiewicz [12]. In this note we will focus on the minimal entropy problem for smooth 4–manifolds M which are geometrisable in the sense of Thurston; M is diffeomorphic to a connected sum of manifolds which admit a decomposition into pieces which are modelled on a Thurston geometry.

The minimal entropy $\mathrm{h}(M)$ of a closed smooth manifold M is the infimum of the topological entropy $\mathrm{h}_{\mathrm{top}}(g)$ of the geodesic flow of g over the family of C^∞ Riemannian metrics on M with unit volume. A metric g_0 is *entropy minimising* if it achieves this infimum $\mathrm{h}_{\mathrm{top}}(g_0) = \mathrm{h}(M)$. When such a metric exists we say the minimal entropy problem can be solved for M.

The minimal entropy $\mathrm{h}(M)$ of an n–manifold M is related to its simplicial volume $||M||$, volume entropy $\lambda(M)$ and minimal volume $\mathrm{MinVol}(M)$ according to the

Published: 25 February 2009 DOI: 10.2140/agt.2009.9.365

inequalities noticed by M Gromov [14], A Manning [28] and by G Besson, G Courtois and S Gallot [4] and G P Paternain [33]:

$$\frac{n^{n/2}}{n!}||M|| \leq \lambda(M)^n \leq \mathrm{h}(M)^n \leq (n-1)^n \,\mathrm{MinVol}(M).$$

The simplicial and minimal volumes were defined by Gromov in the seminal paper [14]. Both the simplicial volume and volume entropy are known to be homotopy invariant; see I K Babenko [2] and M Brunnbauer [5]. However, L Bessières [3] has shown that the minimal volume $\mathrm{MinVol}(M)$ depends on the differentiable structure of M. In fact D Kotschick has proven that even the vanishing of $\mathrm{MinVol}(M)$ depends on the differentiable structure [23]. As the question of whether the minimal entropy is a homotopy invariant is still unresolved, it is interesting to calculate it and compare it with the invariants mentioned above.

The instrument we will use to show that these invariants vanish is a generalisation of a local torus action, called an $\mathcal{F}$–structure. J Cheeger and Gromov showed in [6] that if a manifold M admits a *polarised* $\mathcal{F}$–structure then $\mathrm{MinVol}(M) = 0$. The simplest example of a polarised $\mathcal{F}$–structure is a free S^1–action on M. Paternain and J Petean proved that if M admits any $\mathcal{F}$–structure then $\mathrm{h}(M) = 0$, M collapses with curvature bounded from below and the Yamabe invariant of M is nonnegative [34].

In dimension four there exist smooth manifolds that admit $\mathcal{F}$–structures and which are homeomorphic to manifolds that do not admit them; see Paternain and Petean [34] and C LeBrun [27]. The results in this paper provide a basis of examples with which to compare manifolds in the same homeomorphism class.

The relevant definitions will be reviewed in the following sections.

Let **H** and **V** be the following sets of four dimensional geometries:

$$\mathbf{H} = \{\mathbb{H}^4, \mathbb{H}^2\times\mathbb{H}^2, \mathbb{H}^2_{\mathbb{C}}\}$$

$$\mathbf{V} = \left\{\begin{matrix} \mathbb{S}^4 & \mathbb{C}\mathrm{P}^2 & \mathbb{S}^3\times\mathbb{E} & \mathbb{H}^3\times\mathbb{E} \\ \widetilde{\mathrm{SL}}_2\times\mathbb{E} & \mathrm{Nil}^3\times\mathbb{E} & \mathrm{Nil}^4 & \mathrm{Sol}^4_1 \\ \mathbb{S}^2\times\mathbb{E}^2 & \mathbb{H}^2\times\mathbb{E}^2 & \mathrm{Sol}^4_{m,n} & \mathrm{Sol}^4_0 \\ \mathbb{S}^2\times\mathbb{S}^2 & \mathbb{S}^2\times\mathbb{H}^2 & \mathbb{E}^4 & \mathbb{F}^4 \end{matrix}\right\}$$

Together **H** and **V** constitute all the four-dimensional geometries that admit finite volume quotients. Comprehensive descriptions of all these geometries have been provided by J Hillman [16, page 133] and C T C Wall [43], where the relationship between geometric and complex structures is explored.

Our first result is about compact manifolds which are modelled on a single geometry, these are called *geometric* manifolds.

Theorem A *Let M be a smooth oriented and closed geometric four-manifold. The following notions are equivalent:*

(i) *M is modelled on a geometry in* $\mathbf{V}$.

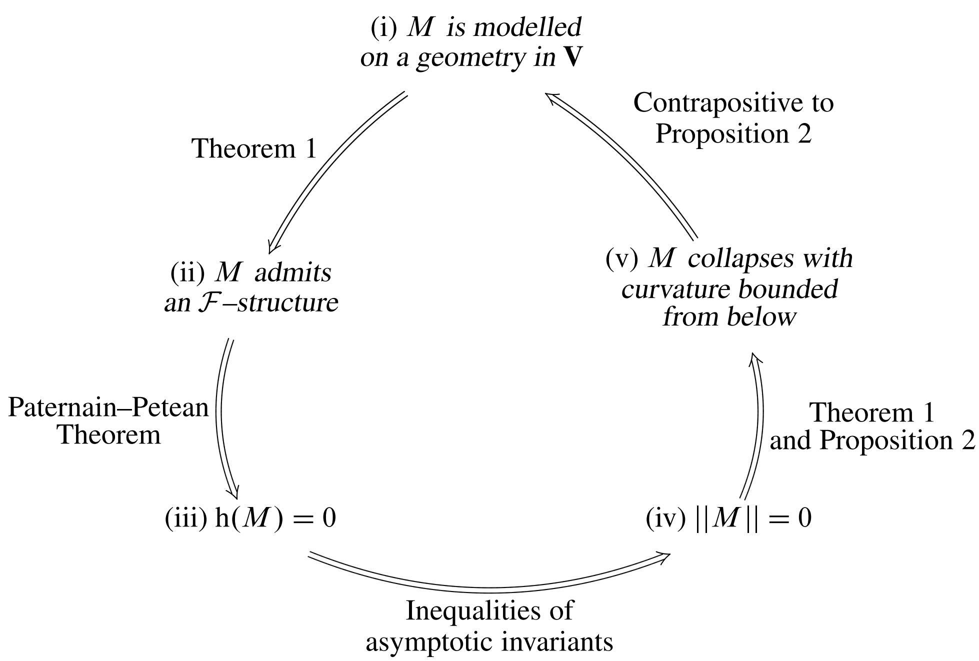
(ii) *M admits an $\mathcal{F}$–structure.*

(iii) *M has zero minimal entropy,* $\mathrm{h}(M) = 0$.

(iv) *The simplicial volume of M vanishes,* $||M|| = 0$.

(v) *M collapses with curvature bounded from below,* $\mathrm{Vol}_K(M) = 0$.

The main novel ingredient here is the proof that (i) implies (ii) and is shown in Theorem 1. The other equivalences follow from the inequalities above and an application of results of Gromov and Thurston shown in Proposition 2. This can be seen in the next diagram which summarises the proof of Theorem A.



The definitions and properties of these terms will appear in the following sections. Relying on results of Thurston and Gromov [14] we can then see that the contents of Theorem A can be rephrased in the following way: *Let M be an oriented closed smooth geometric four-manifold, then* $\mathrm{h}(M) > 0$ *if and only if* $||M|| \neq 0$ *if and only if M is modelled on a geometry in* $\mathbf{H}$.

J W Anderson and Paternain showed in [1, Theorem 2.9] that for a geometric 3–manifold M it is equivalent for its simplicial volume, minimal entropy or minimal volume to vanish and for M to be a *graph manifold*. If a geometric 3–manifold M admits a geometric structure modelled on a geometry which is not $\mathbb{H}^3$ then M is a graph manifold. By the results of Besson, Courtois and Gallot [4] if M is modelled on $\mathbb{H}^3$ then the minimal entropy of M is strictly positive and it is achieved by the hyperbolic metric. In the same vein Theorem A shows that vanishing of the minimal entropy is an obstruction to the manifold being of *hyperbolic type* in the extended sense of it being modelled on a geometry in $\mathbf{H}$.

A manifold M is said to admit a *geometric decomposition* if it admits a finite collection of 2–sided hypersurfaces S such that each component of $M - S$ is geometric. A manifold is *geometrisable in the sense of Thurston* if it is diffeomorphic to a connected sum of manifolds with a geometric decomposition. After inspecting every possible geometric decomposition and every type of geometrisable smooth four-manifold, we can extend Theorem A to the geometrisable case. The main result of this paper is the following theorem.

Theorem B *Let M be a closed orientable smooth four-manifold which is geometrisable in the sense of Thurston. If all of the geometric pieces of M are modelled on geometries in $\mathbf{V}$ then M admits an $\mathcal{F}$–structure. Consequently* $\mathrm{h}(M) = \mathrm{Vol}_K(M) = 0$.

Therefore all closed geometrisable smooth four-manifolds M which are known to have $\|M\| = 0$ also have $\mathrm{h}(M) = 0$. It should be noted that there are no known examples of manifolds with zero simplicial volume and positive minimal entropy. So Theorem B also shows that such an example can not be constructed in dimension four by means of geometric decompositions.

The minimal entropy problem for geometric four-manifolds has been treated by Paternain and Petean in [35]. They have shown that if M admits a geometric structure modelled on a geometry in $\mathbf{Z} := \{\mathbb{S}^4, \mathbb{C}\mathrm{P}^2, \mathbb{S}^3 \times \mathbb{E}, \mathrm{Nil}^3 \times \mathbb{E}, \mathrm{Nil}^4, \mathbb{S}^2 \times \mathbb{E}^2, \mathbb{S}^2 \times \mathbb{S}^2, \mathbb{E}^4\}$ then M admits a metric with zero topological entropy. Whereas if M is modelled on $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^3 \times \mathbb{E}$, $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$, $\mathbb{H}^2 \times \mathbb{E}^2$, Sol^4_1, Sol^4_0 or $\mathrm{Sol}^4_{m,n}$ then the fundamental group of M has exponential growth. This implies that any smooth metric on M has positive topological entropy by a result of Manning [28]. It follows from Theorem A that the minimal entropy problem cannot be solved for a manifold M modelled on any of these geometries, since we can endow them with $\mathcal{T}$–structures.

On the other hand, for manifolds modelled on $\mathbb{H}^4$ and $\mathbb{H}^2_{\mathbb{C}}$ the work of Besson, Courtois and Gallot implies that the minimal entropy problem is solved by their respective *hyperbolic* and *locally symmetric* metrics [4]. Moreover, finite volume manifolds modelled on these two geometries have positive simplicial volume. The minimal entropy problem for manifolds M modelled on the geometry $\mathbb{H}^2 \times \mathbb{H}^2$ remains open.

A possible candidate for an entropy-minimising metric on M could be the product metric on $\mathbb{H}^2 \times \mathbb{H}^2$ inherited by M as a quotient. Modulo the case of $\mathbb{H}^2 \times \mathbb{H}^2$ manifolds, Theorem A provides a complete solution to the minimal entropy problem for geometric four-manifolds:

Corollary C *Let M be a closed orientable smooth geometric four-manifold, which is not modelled on $\mathbb{H}^2 \times \mathbb{H}^2$. Then the minimal entropy problem can be solved for M if and only if M is modelled on a geometry from $\mathbf{H}$ or $\mathbf{Z}$.*

We show in Lemma 21 that if an orientable 4–manifold M admits a proper geometric decomposition then its fundamental group $\pi_1(M)$ is not trivial and has exponential growth. So the only manifolds considered in Theorem B with nontrivial fundamental group for which the minimal entropy can be solved are the geometric ones modelled on a geometry from $\mathbf{Z}$. Furthermore if M is simply connected and is a connected sum, then by another result of Paternain and Petean in [34] there exist only two such closed orientable 4–manifolds which admit a metric of zero topological entropy, $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$. In the context of these results Theorem B implies:

Corollary D *Let M be a closed orientable geometrisable four-manifold with a proper geometric decomposition into pieces modelled on a geometry in $\mathbf{V}$ or nontrivial connected sums of manifolds modelled on a geometry in $\mathbf{V}$. Then the minimal entropy problem can be solved for M if and only if M is diffeomorphic to $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ or $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$.*

On the organisation of this paper Section 2 contains definitions and the statement of Theorem 1. Proposition 2 is shown in Section 3. The proof of Theorem 1 is a case by case analysis of the geometries in **V** and it occupies Sections 4 to 12. All the results are collected in Section 13, where the proofs of Theorems 1, A and B can be found.

Acknowledgements The results found here are part of my doctoral thesis. It is my pleasure to thank my supervisor Gabriel Paternain for his constant support, energetic help and kind encouragement. I also owe thanks to Dieter Kotschick for his interest in the ideas presented here as well as for noticing a couple of mistakes and commenting on a previous version of this work. Likewise I would like to thank Ivan Smith and Jim Anderson for all the corrections and suggestions they made to my thesis, which have certainly improved the presentation here. I am also grateful to Alberto Verjovsky and Peter Scott. Their experience and insight was a very positive influence. Finally I also wish to say thanks to Jimmy Petean and Dan Jane.

Financial support from the Consejo Nacional de Ciencia y Tecnología, CONACyT México and the Secretaría de Educación Pública, SEP México during my time as a PhD student is gratefully acknowledged.

2 Preliminaries

The simplicial volume $||M||$ of a closed orientable manifold is defined as the infimum of $\Sigma_i |r_i|$ where r_i are the coefficients of a *real* cycle representing the fundamental class of M.

For a closed connected smooth Riemannian manifold (M, g), let $\mathrm{Vol}(M, g)$ be the volume of g and let K_g be its sectional curvature. We define the following minimal volumes as in [14]:

$$\mathrm{MinVol}(M) := \inf_g \{\mathrm{Vol}(M, g) \;:\; |K_g| \leq 1\}$$
$$\mathrm{Vol}_K(M) := \inf_g \{\mathrm{Vol}(M, g) \;:\; K_g \geq -1\}$$

If $\mathrm{Vol}_K(M)$ is zero then the simplicial volume of M is also zero. This follows from Bishop's comparison theorem; see Paternain and Petean [34].

A $\mathcal{T}$–structure on a smooth closed manifold M is a finite open cover $\{U_i\}_{i=1}^k$ of M with a nontrivial torus action on each U_i such that the intersections of the open sets are invariant (under all corresponding torus actions) and the actions commute. A $\mathcal{T}$–structure is called *polarised* if the torus actions on each U_i are locally free and in the intersections the dimensions of the orbits (of the corresponding torus action) are constant. The structure is called *pure* if the dimension of the orbits is constant.

By definition an $\mathcal{F}$–structure on a closed manifold M is given by:

(1) A finite open cover $\{U_1, \dots, U_N\}$;

(2) $\pi_i \colon \widetilde{U_i} \to U_i$ a finite Galois covering with group of deck transformations Γ_i, $1 \leq i \leq N$;

(3) A smooth torus action with finite kernel of the k_i–dimensional torus: $\phi_i \colon T^{k_i} \to \mathrm{Diff}(\widetilde{U_i})$, $1 \leq i \leq N$;

(4) A homomorphism $\Psi_i \colon \Gamma_i \to \mathrm{Aut}(T^{k_i})$ such that

$$\gamma(\phi_i(t)(x)) = \phi_i(\Psi_i(\gamma)(t))(\gamma x)$$

for all $\gamma \in \Gamma_i$, $t \in T^{k_i}$ and $x \in \widetilde{U_i}$;

(5) For any finite subcollection $\{U_{i_1}, \dots, U_{i_l}\}$ such that $U_{i_1\cdots i_l} := U_{i_1} \cap \cdots \cap U_{i_l} \neq \varnothing$ the following compatibility condition holds: let $\widetilde{U}_{i_1\cdots i_l}$ be the set of points $(x_{i_1}, \dots, x_{i_l}) \in \widetilde{U}_{i_1} \times \cdots \times \widetilde{U}_{i_l}$ such that $\pi_{i_1}(x_{i_1}) = \cdots = \pi_{i_l}(x_{i_l})$. The set $\widetilde{U}_{i_1\cdots i_l}$ covers $\pi_{i_j}^{-1}(U_{i_1\cdots i_l}) \subset \widetilde{U}_{i_j}$ for all $1 \leq j \leq l$. Then we require that ϕ_{i_j} leaves $\pi_{i_j}^{-1}(U_{i_1} \cdots i_l)$ invariant and it lifts to an action on $\widetilde{U}_{i_1\cdots i_l}$ such that all lifted actions commute.

An $\mathcal{F}$–structure is said to be *pure* if all the orbits of all actions at a point, for every point have the same dimension. We will say an $\mathcal{F}$–structure is *polarised* if the smooth torus action ϕ_i above is fixed point free for every U_i. A $\mathcal{T}$–structure is simply an $\mathcal{F}$–structure where all the coverings π_i in (2) are trivial. In this case (4) is satisfied and (5) just means the respective actions commute on overlaps. The existence of a $\mathcal{T}$–structure implies the existence of an $\mathcal{F}$–structure.

The existence of a polarised $\mathcal{F}$–structure on a manifold M implies the minimal volume $\mathrm{MinVol}(M)$ is zero [6]. The interested reader is invited to consult further examples there as well. One of the main contributions of this paper is the proof of the following result, which is found in the last section.

Theorem 1 *Let M be a closed orientable geometric four-manifold modelled on a geometry in* $\mathbf{V}$*. Then M admits a $\mathcal{F}$–structure.*

3 Geometric manifolds of positive simplicial volume

Proposition 2 *If M is a closed oriented geometric four-manifold modelled on a geometry in* $\mathbf{H}$ *then $\|M\| > 0$.*

Proof First assume M is closed, oriented and modelled on either $\mathbb{H}^4$ or $\mathbb{H}^2_{\mathbb{C}}$. Then M admits a metric of negative sectional curvature and by the Thurston–Inoue–Yano theorem in [14; 18] we have that $\|M\| > 0$. In fact this shows that any M with *finite volume* modelled on $\mathbb{H}^4$ and $\mathbb{H}^2_{\mathbb{C}}$ has positive simplicial volume.

Let M be a closed manifold modelled on $\mathbb{H}^2 \times \mathbb{H}^2$. We can use Gromov's Proportionality Principle from [14] to see that $\|M\| > 0$. If the universal coverings of two closed Riemannian manifolds M and N are isometric then $\|M\|/\mathrm{Vol}(M) = \|N\|/\mathrm{Vol}(N)$. Consider the product of two closed hyperbolic surfaces $N = S_1 \times S_2$. The smooth manifold N is modelled on $\mathbb{H}^2 \times \mathbb{H}^2$. Because the simplicial volume of a product of closed manifolds is bounded from below by the product of their respective simplicial volumes we have $\|N\| \geq C\|S_1\|\|S_2\| > 0$ for some constant C. Therefore $\|M\| > 0$ for any closed manifold M modelled on $\mathbb{H}^2 \times \mathbb{H}^2$. □

4 Circle foliations

Proposition 3 *Let M be a closed manifold foliated by circles. Suppose M admits a metric such that the circles are geodesics. Then M admits a polarised $\mathcal{T}$–structure.*

Proof By a theorem of A W Wadsley [42], the foliation by circles gives rise to an orbifold bundle or Seifert fibration. This means that locally we have the following model for the foliation near a fixed leaf L [11, Theorem 4.3]. There exists a finite group $G \subset \mathrm{O}(n)$ (where $\dim M = n+1$) and a homomorphism $\psi\colon \pi_1(L) = \mathbb{Z} \to \mathrm{O}(n)$. Let $\widetilde{L}$ be the covering of L corresponding to the kernel of ψ. That $\widetilde{L}$ is compact follows from [11, Theorem 4.3]. Then G acts on $\widetilde{L}$ by deck transformations and we can consider the quotient $(\widetilde{L} \times D^n)/G$, where D^n is the unit ball in $\mathbb{R}^n$. Theorem 4.3 in [11] asserts the existence of G and ψ and a diffeomorphism between $(\widetilde{L} \times D^n)/G$ and a neighbourhood of L preserving the leaves. When L is S^1, $\widetilde{L}$ is also a circle and G can only be a cyclic group $\mathbb{Z}_m$. The obvious circle action on $S^1 \times D^n$ clearly descends to a circle action on $(S^1 \times D^n)/G$ and thus locally we always have a locally free circle action, whose orbits are precisely the leaves of the foliation.

If one can coherently orient all the leaves we would have a circle action on M, but if not, we still have a $\mathcal{T}$–structure, since "opposite" actions still commute. Let us make this a bit more precise: The leaf space B is an orbifold and M is an orbifold bundle over B. So we may cover B with compatible open sets such that the transition maps of the bundle have values in $\mathrm{O}(2)$ (the fibres are circles and $\mathrm{Diff}(S^1)$ deformation retracts onto $\mathrm{O}(2)$). But given $h \in \mathrm{O}(2)$ we obviously have $hlh^{-1} \in \mathrm{SO}(2)$ for any $l \in \mathrm{SO}(2)$. Thus if we conjugate the obvious circle action of S^1 on itself by an element of $\mathrm{O}(2)$ we obtain a new circle action commuting with the original one. Thus M has a $\mathcal{T}$–structure. □

Some of the four dimensional geometries are foliated by $\mathbb{R}$. These foliations descend to circle foliations on their geometric manifolds, and define a $\mathcal{T}$–structure.

Theorem 4 *Every closed geometric four-manifold M modelled on any of the geometries $\mathbb{X}^4$ in $\{\mathbb{S}^3 \times \mathbb{E},\ \mathbb{H}^3 \times \mathbb{E},\ \widetilde{\mathrm{SL}}_2 \times \mathbb{E},\ \mathrm{Nil}^3 \times \mathbb{E},\ \mathrm{Sol}^3 \times \mathbb{E},\ \mathrm{Nil}^4,\ \mathrm{Sol}_1^4\}$ admits a polarised $\mathcal{T}$–structure.*

Proof In each of the geometries $\mathbb{S}^3 \times \mathbb{E}$, $\mathbb{H}^3 \times \mathbb{E}$, $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$, $\mathrm{Nil}^3 \times \mathbb{E}$ or $\mathrm{Sol}^3 \times \mathbb{E}$ we have a trivial foliation given by the product with the Euclidean factor. In the case of $\mathrm{Nil}^4 = \mathbb{R}^3 \ltimes_\theta \mathbb{R}$, with $\theta(t) = (t, t, t^2/2)$, it is given by the $\mathbb{R}$ factor on the right hand side of the semidirect product. For the remaining geometry of solvable Lie type

$$\mathrm{Sol}_1^4 = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & \alpha & b \\ 0 & 0 & 1 \end{pmatrix} : \alpha, a, b, c \in \mathbb{R}, \alpha > 0 \right\},$$

the $\mathbb{R}$ we are interested in is given by elements of the form $\alpha = 1$, $a = b = 0$ and $c \in \mathbb{R}$. This foliation on $\mathbb{X}^4$ descends to a foliation $\mathcal{F}$ on any quotient $M = \mathbb{X}^4/\Gamma$ under the action of a discrete group of isometries Γ and the leaves of $\mathcal{F}$ are all circles. By Proposition 3 M admits a polarised $\mathcal{T}$–structure. □

5 Seifert fibred geometries

5.1 Seifert fibrations

Let S be a closed geometric manifold modelled on $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}^1$, Nil^4, $\mathrm{Sol}^3 \times \mathbb{E}^1$ or $\mathbb{S}^3 \times \mathbb{E}^1$. It was shown by M Ue in [39; 40] that S is a Seifert fibred space. We will review the description of these structures locally—in a neighbourhood of a point in S—this will allow us to furnish these manifolds with $\mathcal{T}$–structures. We refer to [39; 40] throughout this section for details. The reader familiar with this data may want to skip to Lemma 6 and then straight to Theorem 7 which are the main results of this section.

Definition 5 A smooth oriented 4–manifold S is Seifert fibred if it is the total space of an orbifold bundle $\pi\colon S \to B$ with general fibre a torus over a 2–orbifold B.

Notice that the class of Seifert 4–manifolds contains all the compact complex surfaces diffeomorphic to elliptic surfaces X with $c_2(X) = 0$ and also contains examples which do not admit any complex structure [43].

Let $\pi\colon S \to B$ be a Seifert fibration, with S a geometric manifold modelled on one of the geometries mentioned above and B the orbifold base of the fibration. Denote by T^2 be the standard torus and $G \subset \mathrm{O}(2)$ a discrete subgroup, viewed as a group of Euclidean isometries. For any point $p \in B$ there exists a neighbourhood U of p such that $\pi^{-1}(U)$ is diffeomorphic to $(T^2 \times D^2)/G$ for some $G \subset \mathrm{O}(2)$. Here T^2 is parametrised by two unit complex circles $S^1 \times S^1 \subset \mathbb{C}^2$, D^2 is the open unit complex disk $|z| \leq 1$ in $\mathbb{C}$ and G is the stabiliser at p, which acts freely on $T^2 \times D^2$.

5.2 Local description

For G nontrivial, there are three cases to consider, cyclic groups of rotations $\mathbb{Z}_m$, reflection groups $\mathbb{Z}_2$ and dihedral groups D_{2m}.

(1) $G \cong \mathbb{Z}_m \cong \langle \rho \rangle$, where ρ is a rotation of $2\pi/m$. This isotropy subgroup corresponds to cone points of cone angle $2\pi/m$. The action $\rho\colon T^2 \times D^2 \to T^2 \times D^2$ is given by $\rho(x, y, z) = (x - a/m, y - b/m, e^{2\pi i/m} z)$, where $x, y \in S^1$ and $z \in D^2$ with $\gcd(m, a, b) = 1$. The fibre over $p = 0$ is called a multiple torus of type (m, a, b).

(2) $G \cong \mathbb{Z}_2 \cong \langle \ell \rangle$, where ℓ is a reflection on the second factor of T^2 and on D^2. Now the action $\ell\colon T^2 \times D^2 \to T^2 \times D^2$ is given by $\ell(x, y, z) = (x + \frac{1}{2}, -y, \overline{z})$. This is the isotropy subgroup corresponding to points on a reflector line or circle. In this case the fibre over p is a Klein bottle K and $\pi^{-1}(U)$ is a nontrivial D^2–bundle over K.

(3) $G \cong D_{2m} = \langle \ell, \rho : \ell^2 = \rho^m = 1, \ell\rho\ell^{-1} = \rho^{-1} \rangle$, for $m \in \mathbb{Z}$. This is a dihedral group, the isotropy subgroup of corner reflector points of angle π/m, with the actions $\rho, \ell\colon T^2 \times D^2 \to T^2 \times D^2$ given by,

$$\rho(x, y, z) = (x, y - b/m, e^{2\pi i/m} z), \quad \ell(x, y, z) = (x + \tfrac{1}{2}, -y, \overline{z}).$$

Informally, the fibre over p is a Klein bottle whose fundamental domain is $(1/m)$–times that of the fibre of the reflector point near p. We call this fibre a multiple Klein bottle of type $(m, 0, b)$.

5.3 Local S^1–actions

Lemma 6 *Let $\pi\colon S \to B$ be a Seifert fibration for the 4–manifold S. For every point $p \in B$ there exists a neighbourhood U of p diffeomorphic to D^2/G with $G \subset \mathrm{O}(2)$ such that $\pi^{-1}(U) \cong (T^2 \times D^2)/G$ admits an S^1 action which commutes with the action of G on $T^2 \times D^2$.*

Proof Take $U \subset B$ such that $\pi^{-1}(U)$ is diffeomorphic to $(T^2 \times D^2)/G$. Here we can define an S^1–action. We will do this by first lifting the quotient by G to $(T^2 \times D^2)$ and then showing that the S^1 action commutes in $(T^2 \times D^2)$ with the actions of all the different possible isotropy groups G. Hence this S^1–action will be well defined in the quotient $(T^2 \times D^2)/G \cong \pi^{-1}(U) \subset S$. This defines a local S^1–action on S. Define $\varphi\colon S^1 \times (T^2 \times D^2) \to (T^2 \times D^2)$ by $\varphi(\theta, x, y, z) = (x + \theta, y, z)$. For each of the three cases for G short computations show that φ commutes with the generators of G as described in Section 5.2. □

So we now know that given an orbifold chart U of B, we can construct an S^1–action on its preimage $\pi^{-1}(U)$ which is equivariant with respect to the action of G on $T^2 \times D^2$.

5.4 Description along the singular set

The following picture along the reflector circles is taken from [39; 40]. Let l and h be the curves in T^2 represented by $\mathbb{R}/\mathbb{Z} \times \{0\}$ and $\{0\} \times \mathbb{R}/\mathbb{Z}$ respectively. A choice of such a pair (l, h) is called a framing for T^2. The boundary of B consists of a disjoint union of circles C_i, each of which we call a reflector circle. Let N_i be an annulus bounded by C_i and a curve γ_i parallel to C_i. In order to clarify the structure of S near C_i we now describe $\pi^{-1}(N_i)$. Say the corner reflectors $p_1, \dots, p_s$ on C_i are of type $(m_1, 0, b_1), \dots, (m_s, 0, b_s)$, with respect to the framing (l_i, h_i) of the general fibre over some base point of N_i.

Understanding the fibres over these corner reflectors is simplified if we consider the double cover DB of B, with the projection $p\colon DB \to B$ obtained by identifying 2 copies of B along the reflector circles. Let DN_i be the suborbifold of DB covering N_i and $D\pi\colon DS \to DB$ be the fibration induced from $\pi\colon S \to B$. Then S is the quotient of DS by a free involution ι which is the lift of the reflection l which switches both copies of DB. The action of ι on the reflection point near the base point is identical to that of l in case (3) above. In the presentation of $\pi_1(S)$ in [39] the map ι satisfies

$$(\star) \qquad \iota^2 = l \quad \text{and} \quad \iota h \iota^{-1} = h^{-1}.$$

The corner reflector point p_j is covered by a cone point $q_j \in DB$ and the fibre over q_j is a multiple torus of type $(m_j, 0, b_j)$.

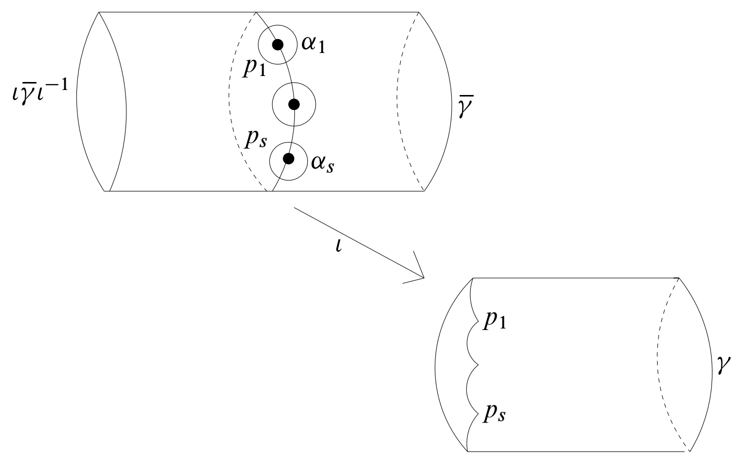


Figure 1: Local picture along a reflector circle

Take the oriented meridional circle α_j centered at p_j as in figure 1. Then the lifts $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_s$ of the curves $\alpha_1, \dots, \alpha_s$ can be taken to satisfy in $\pi_1(S)$ the following relations:

$$\widetilde{\alpha}_i^{m_i} h^{b_i} = 1 \quad (i = 1, \dots, s),$$
$$\iota\widetilde{\alpha}_s\iota^{-1} = \widetilde{\alpha}_s^{-1}, \;\; \iota\widetilde{\alpha}_{s-1}\iota^{-1} = \widetilde{\alpha}_s^{-1}\widetilde{\alpha}_{s-1}^{-1}\widetilde{\alpha}_s, \dots, \iota\widetilde{\alpha}_1\iota^{-1} = \widetilde{\alpha}_s^{-1}\widetilde{\alpha}_{s-1}^{-1} \cdots \widetilde{\alpha}_1^{-1}\widetilde{\alpha}_2 \cdots \widetilde{\alpha}_s.$$

We can describe the monodromy of the fibration along a reflector circle. Let V denote the union of small disk neighbourhoods around each corner reflector point p_j and take $\overline{\gamma}$ and $\iota\overline{\gamma}\iota^{-1}$ as in figure 1. Then the curve represented by $\overline{\gamma}^{-1}\alpha_1\alpha_2\cdots\alpha_s\iota\overline{\gamma}\iota^{-1}$ is null-homologous in $DN - V$. Hence the monodromy matrix A along $\overline{\gamma}$ with respect to the framing (l,h) must satisfy $JAJ^{-1}=A$ where

$$J=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

is a monodromy matrix for ι. Then we must have that $A=\pm I$, where I is the identity matrix.

Another piece of information that we need in order to describe the Seifert fibration is the obstruction to extending the fibration over a neighbourhood of each reflector circle C_i. This is called the *Euler class* of C_i. Consider a lift $\widetilde{\gamma}$ of γ, which also determines a lift $\iota\widetilde{\gamma}\iota^{-1}$. Then we have the relation $\widetilde{\gamma}^{-1}\widetilde{\alpha}_1\cdots\widetilde{\alpha}_s\iota\widetilde{\gamma}\iota^{-1}=l^ah^b$ in $\pi_1(S)$. The Euler class (a,b) of C_i is the obstruction to extending $\gamma\cup\iota\gamma\iota^{-1}\cup\alpha_1\cup\cdots\cup\alpha_s$ to the cross section on $\pi^{-1}(DN-V)$. We have that $a=-1$ if the monodromy A around γ is $-I$ and $a=0$ if $A=I$. The value of b depends on the choices of the lifts $\widetilde{\gamma}$, $\widetilde{\alpha_i}$ of γ, α.

5.5 Global description of a Seifert fibration

We can now give a global description of our Seifert fibred manifold S.

Let N_i be a tubular neighbourhood of each reflector circle C_i, with boundaries C_i and γ_i as in the figures above. Fix a base point near C_i and a framing (l,h) of the general fibre satisfying ($\star$). Denote by $|B|$ the topological space underlying the orbifold B. Let p_i be a cone point and D_i a disk neighbourhood of p_i. If we fix the lift $\widetilde{\gamma}_i$ of γ_i, then the fibration over the complement $B_0=B-\cup_i N_i$ is described by the following information:

(1) The monodromy matrices $A_i, B_i\in \mathrm{SL}(2,\mathbb{Z})$ along the set of standard generators s_i, t_i (for $i=1,\dots,g$) of $\pi_1|B_0|$ if $|B_0|$ is orientable.

(1') The monodromy matrices $A'_i\in \mathrm{GL}(2,\mathbb{Z})$ with $\det A'_i=-1$ along the set of standard generators v_i (for $i=1,\dots,g$) of $\pi_1|B_0|$ if $|B_0|$ is nonorientable.

(2) The type (m_i,a_i,b_i) of the multiple torus over the cone point p_i, (for $i=1,\dots,t$).

(3) The obstruction (a',b') to extending a section over B_0 to a section over all of B (see Ue [39; 40]).

The fibration over N_i is described as before with respect to the framing (l_i,h_i) of the general fibre on N_i and the lift ι_i of the reflection along C_i (where $(l_1,h_1)=(l,h)$) satisfying $\iota_i^2=l_i$ and $\iota_i h_i\iota_i^{-1}=h_i^{-1}$.

Then $\pi_1(N_i)$ is attached to $\pi^{-1}(B_0)$ so that $(l_i, h_i) = (l, h)P_i$ for some $P_i \in \mathrm{SL}(2, \mathbb{Z})$ with $P_1 = \mathrm{Id}$. This implies that if we take the lift $\widetilde{\delta}_i$ of the curve δ_i, then the monodromy along $\widetilde{\delta}_i$ is $B_i = P_i J P_i^{-1} J$ with respect to (l, h). It is possible to take $\widetilde{\delta}_i$ so that $\iota_i = \widetilde{\delta}_i \iota$, because $\iota_i \widetilde{\delta}_i \iota^{-1} = \widetilde{\delta}_i^{-1}$ and so $\widetilde{\delta}_i \iota_i \widetilde{\delta}_i \iota^{-1} = l^{s+1} h^t$ for some $s, t \in \mathbb{Z}$ (recall that $\iota^2 = l$).

As a final step, we describe the relations between the monodromies. Let $I_i = \pm\,\mathrm{Id}$ be the monodromy along C_i, and A_i, B_i (or A_i' in case B_0 is not orientable) be the monodromies along the standard curves on B_0 as before. Then $\prod [A_i, B_i] \prod I_i = \mathrm{Id}$ (or $\prod A_i^2 \prod I_i = \mathrm{Id}$). The Seifert fibration of S is determined by the information above. Using this description we can now show the existence of $\mathcal{T}$–structures on Seifert fibred four-manifolds.

5.6 Seifert Fibrations and polarised $\mathcal{T}$–structures

Theorem 7 *Every smooth closed and oriented Seifert fibred four-manifold S admits a polarised $\mathcal{T}$–structure.*

Proof Let $\pi\colon S \to B$ be a Seifert fibred smooth 4–manifold, over the orbifold B. So S is the total space of an orbifold bundle with general fibre a torus, over the 2–orbifold B. Let N_i be open annular neighbourhoods of the circle reflectors C_i of B. Take $B_0 = B - \bigcup_{i=1}^r N_i$. Let $\mathcal{U}$ be an open covering for B_0 such that for U in $\mathcal{U}$ we have that $\pi^{-1}(U) \cong (T^2 \times D^2)/G$ as in Lemma 6 above. So G is either trivial or isomorphic to $\mathbb{Z}_p$. As B_0 is compact we may choose a finite subcovering $\{U_i\}$ of $\mathcal{U}$.

Notice that for G trivial we have that $(T^2 \times D^2)/G = T^2 \times D^2$ and for $G \cong \mathbb{Z}_p$, we have that $(T^2 \times D^2)/G$ is diffeomorphic to $T^2 \times D^2$. Denote $\pi^{-1}(B_0)$ by S', the restriction $\pi|_{S'}\colon S' \to B_0$ is an orbifold bundle with fibre T^2. The singular points of B_0 are all cone points. For each $p \in U_i \cap U_j$ call the local trivialisations $\Phi_i\colon \pi^{-1}(U_i) \to T^2 \times D^2$ and $\Phi_j\colon \pi^{-1}(U_j) \to T^2 \times D^2$. These give rise to the transition functions $\Phi_j \circ \Phi_i^{-1}(x, y, z) = (\Psi_{ij}(z), z)$.

Let (l, h) be a framing for the general fibre $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$, as explained above. The diffeomorphism group $\mathrm{Diff}(T^2)$ of T^2 retracts to $T^2 \ltimes \mathrm{GL}(2, \mathbb{Z})$. So the transition functions $\Psi_{ij}(z)$ can be regarded as elements of $\mathrm{GL}(2, \mathbb{Z})$ and the structural group of the orbifold bundle reduces to this linear one. Showing that the actions φ_i and φ_j commute in the intersections $\pi^{-1}(U_i) \cap \pi^{-1}(U_j)$ is an easy exercise in linear algebra. Therefore they define a $\mathcal{T}$–structure on $\pi^{-1}(B_0)$.

Now we exhibit a $\mathcal{T}$–structure on the neighbourhoods N_i of the reflector circles C_i. Consider $\overline{N}_i$, the closure of N_i. Each N_i is covered by open subsets in which we defined the circle actions. We extend these to cover $\bar{N}_i$ and take a finite subcovering $\{V_k\}$. Let γ_i denote the boundary of N_i which is not C_i. We claim that the corresponding actions in the sets U_j in B_0 and V_k in N_i commute in the intersection of these sets. This follows from the fact that the fibration on N_i is described with respect to the framing (l_i, h_i) of the general fibre on N_i, as we will now see. Recall that $\pi^{-1}(N_i)$ is attached to $\pi^{-1}(B_0)$ so that $(l_i, h_i) = P_i(l, h)$ for some $P_i \in \mathrm{SL}(2, \mathbb{Z})$, following the global description of a Seifert bundle above. Once more the matrices involved here behave well with respect to the actions we defined on $\pi^{-1}(N_i)$ and $\pi^{-1}(B_0)$, meaning that the actions commute in the intersection of these sets. Therefore we have a $\mathcal{T}$–structure on the Seifert fibred 4–manifold S, which is polarised since T^2 acts freely on itself. □

Corollary 8 *Every closed geometric four-manifold M modelled on one of the geometries $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}^1$, Nil^4, $\mathrm{Sol}^3 \times \mathbb{E}^1$ or $\mathbb{S}^3 \times \mathbb{E}^1$ admits a polarised $\mathcal{T}$–structure.*

6 Solvable Lie geometries

We have already dealt with the geometries Sol^4_1 and $\mathrm{Sol}^3 \times \mathbb{E} = \mathrm{Sol}^4_{n,n}$ in Theorem 4 where we gave manifolds modelled on them a locally free S^1–action. We will now focus on the remaining cases. Recall that if we glue two manifolds along components of their boundary using isotopic diffeomorphisms the resulting manifolds are diffeomorphic (see for example Hirsch [17]).

Theorem 9 *If M is an orientable geometric four-manifold modelled on Sol^4_0 or $\mathrm{Sol}^4_{m,n}$ when $(m \neq n)$, then M admits a polarised $\mathcal{T}$–structure.*

Proof By results of Hillman [16] and Cobb [7, page 176] M is diffeomorphic to $M_f = (T^3 \times I)/\mathbb{s}$, where $(x, 0)\mathbb{s}(f(x), 1)$ for some diffeomorphism f of T^3. Its fundamental group is $\mathbb{Z}^3 \rtimes_A \mathbb{Z}$ for some $A \in \mathrm{SL}(3, \mathbb{Z})$ since M is orientable. Corollary 20 shows that any diffeomorphism of T^3 is isotopic to an affine transformation; see also Ivanov [19]. In this case f is isotopic to the transformation induced by A on T^3. Denote by M_A the mapping torus of T^3 under the transformation induced by A. That M_f is diffeomorphic to M_A follows from the fact that mapping tori of isotopic diffeomorphisms are diffeomorphic. Let φ_t denote the action of T^3 on $T^3 \times \{t\}$ by translations, and $\overline{\varphi_t}$ the lift to $\mathbb{R}^3$. In order to define a $\mathcal{T}$–structure on M_A using the actions φ_t, we must now verify that φ_0 commutes with φ_1 when conjugated by the transformation induced by A on T^3. A simple calculation (which is a particular case of Lemma 18 below) shows

$$A^{-1} \circ \overline{\varphi_1} \circ A \circ \overline{\varphi_0} = \overline{\varphi_0} \circ A^{-1} \circ \overline{\varphi_1} \circ A.$$

Therefore φ_0 commutes φ_1 on M_A. Notice that the dimension of every orbit is 3 and that the action φ_t is locally free. Hence we have endowed M_A, and therefore M, with a polarised $\mathcal{T}$–structure. □

7 Sphere foliations

We will now see that when a compact closed manifold M is modelled on a geometry of type $\mathbb{S}^2 \times \mathbb{X}^2$, where $\mathbb{X}^2$ is a 2–dimensional geometry, M admits a $\mathcal{T}$–structure.

Theorem 10 *A smooth orientable closed 4–manifold which is foliated by S^2 or $\mathbb{R}\mathrm{P}^2$ admits a $\mathcal{T}$–structure.*

Proof Suppose M is a smooth orientable closed 4–manifold which admits a codimension 2 foliation with leaves S^2 or $\mathbb{R}\mathrm{P}^2$. If all the leaves are homeomorphic then the projection to the leaf space is a submersion and M is the total space of an S^2 or an $\mathbb{R}\mathrm{P}^2$ bundle over a surface. Then M is known to admit effective S^1–actions (see Melvin [30] and Melvin and Parker [31]). Assume that the leaves are not all homeomorphic. Having such a foliation is equivalent to having an S^2 orbifold bundle over a 2–orbifold (see Ehresman [10], Epstein [11], Eells and Verjovsky [9] and Molino [32]), as such a foliation is Riemannian.

Denote by F the orbit space of the foliation and $\pi \colon M \to F$ the orbifold bundle. Ehresman's structure theorem [10; 11] implies its singularities may only be isolated points and provides the following description; for any point $p \in F$ and a small neighbourhood U of p, $\pi^{-1}(U)$ is diffeomorphic to $(S^2 \times D^2)/G$. Here G is a discrete subgroup of O(2) which acts freely on $S^2 \times D^2$. Because $\mathbb{Z}_2$ is the only such group that acts freely on S^2 the only possible singularities for the orbifold bundle correspond to projective planes $\mathbb{R}\mathrm{P}^2$ over the set of singular points $p_i \in F$.

Consider an open neighbourhood V_i of p_i, let $V = \cup V_i$. Then the restriction of π to $E = F - V$ is a fibre bundle with total space $N \subset M$ and fibres S^2. Melvin and Parker have shown that N admits an S^1 action given by rotations in the fibres [30; 31]. Moreover, they show that the structure group of $N \to E$ is contained in $\mathrm{O}(2)$. Since $\mathrm{Diff}(S^2)$ retracts to O(3) and preserves fibres, the transition maps are either isotopic to the identity or the antipodal map.

We will now show that this is also the case for $M \to F$. This will allow us to extend this action to a $\mathcal{T}$–structure on M. Let r_α denote the rotation of S^2 with respect to the axis α and a the antipodal map. An easy exercise in linear algebra shows these two transformations commute, that is $r_\alpha \circ a = a \circ r_\alpha$. Therefore the following diagram commutes,

$$\begin{array}{ccc} S^2 & \xrightarrow{r_\alpha} & S^2 \\ \downarrow & & \downarrow \\ \mathbb{R}\mathrm{P}^2 & \xrightarrow{r_{[\alpha]}} & \mathbb{R}\mathrm{P}^2 \end{array}$$

where $r_{[\alpha]}$ denotes the rotation of $\mathbb{R}\mathrm{P}^2$ with fixed point the class of α.

For a neighbourhood V_i of a singular point p_i, we can lift the preimage $\pi^{-1}(V_i)$ to $S^2 \times D^2$. The action of r_α on $S^2 \times D^2$ commutes with the quotient of $\mathbb{Z}_2$, thus defining an S^1 action on $(S^2 \times D^2)/\mathbb{Z}_2$ which is diffeomorphic to $\pi^{-1}(V_i)$.

The holonomy around ∂V_i is $\mathbb{Z}_2$, so that the maps that attach $\pi^{-1}(V_i)$ to N in order to obtain M are either isotopic to the identity or to the antipodal map. In the case of the identity there is nothing to prove. If the attaching map is isotopic to the antipodal map it suffices to note that the rotations on S^2 which are defined on N and $\pi^{-1}(V_i)$ both commute with the antipodal map a. Therefore they define a $\mathcal{T}$–structure on M. □

Conveniently enough, Hillman has shown that if a manifold M admits a geometric decomposition into pieces modelled on geometries of the type $\mathbb{S}^2 \times \mathbb{X}^2$ then M is foliated by S^2 or $\mathbb{R}\mathrm{P}^2$ [16]. We use this description to see that we have also proved the following two results.

Corollary 11 *Any smooth orientable 4–manifold M with a geometric decomposition into pieces of the type $\mathbb{S}^2 \times \mathbb{X}^2$ admits a $\mathcal{T}$–structure.*

Corollary 12 *A closed manifold M modelled on a geometry of type $\mathbb{S}^2 \times \mathbb{X}^2$, where $\mathbb{X}^2$ is a 2–dimensional geometry, admits a $\mathcal{T}$–structure.*

In these cases it is the best we can hope for. In general such a manifold M might not admit a *polarised* $\mathcal{T}$–structure because M could have positive Euler characteristic $\chi(M) > 0$ and therefore its minimal volume could not vanish.

8 Geometric decompositions

Definition 13 We say that an n–manifold M admits a geometric decomposition if it has a finite collection of disjoint 2–sided hypersurfaces S such that each component of $M - \bigcup S$ is geometric of finite volume.

In other words, each component of $M - \bigcup S$ is homeomorphic to X/Γ, for some geometry $\mathbb{X}$ and a lattice Γ. We shall call the hypersurfaces S *cusps* and the components of $M - \bigcup S$ *pieces* of M. The decomposition is *proper* if the set of cusps is nonempty.

8.1 Dimension four

Hillman [16, page 138] brought together various results and organised them to show that if a closed 4–manifold M admits a geometric decomposition then either

(1) M is geometric,

(2) M has a codimension 2 foliation with leaves S^2 or $\mathbb{R}\mathrm{P}^2$,

(3) the pieces of M have geometry $\mathbb{H}^4$, $\mathbb{H}^3 \times \mathbb{E}$, $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$,

(4) the pieces of M have geometry $\mathbb{H}^2_{\mathbb{C}}$ or $\mathbb{F}^4$ or

(5) the pieces of M all have geometry $\mathbb{H}^2 \times \mathbb{H}^2$.

This follows from inspecting the various possible types of cusps that appear in a geometric decomposition.

Geometry	Cusps	Reference
$\mathbb{H}^n$	flat	Eberlein [8]
$\mathbb{H}^3 \times \mathbb{E}$, $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$	flat	Hillman [16]
$\mathbb{S}^2 \times \mathbb{H}^2$	$\mathbb{S}^2 \times \mathbb{E}$–manifolds	Hillman [16]
$\mathbb{F}^4$	Nil^3–manifolds	Hillman [16]
$\mathbb{H}^2_{\mathbb{C}}$	Nil^3–manifolds	Goldman [13]
irreducible $\mathbb{H}^2 \times \mathbb{H}^2$	Sol^3–manifolds	Shimizu [38]
reducible $\mathbb{H}^2 \times \mathbb{H}^2$	graph manifolds	Shimizu [38]

These are the only geometries we need to consider, because if a geometry is of solvable or compact type every lattice has compact quotient [36].

8.2 Geometrisable 4–manifolds and positive simplicial volume

A manifold is called *geometrisable* if it is diffeomorphic to a connected sum of manifolds which admit geometric decompositions. Given a manifold N with a geometric decomposition, if the fundamental group of the hypersurfaces of the decomposition injects into $\pi_1(N)$ we say that the geometric decomposition is π_1–injective.

In dimension 4, Hillman observed that, except for reducible pieces modelled on the geometry $\mathbb{H}^2 \times \mathbb{H}^2$, the inclusion of a cusp into the closure of a piece induces a monomorphism on the fundamental group. So modulo reducible $\mathbb{H}^2 \times \mathbb{H}^2$–pieces, every geometric decomposition in dimension four is π_1–injective [16, page 139].

Proposition 14 *Let M be a geometrisable smooth four-manifold. If a piece of the decomposition of M is modelled on $\mathbb{H}^4$ or on $\mathbb{H}^2_{\mathbb{C}}$ then $\|M\| > 0$.*

Proof Let N denote a piece of M modelled on $\mathbb{H}^4$ or $\mathbb{H}^2_{\mathbb{C}}$. The manifold N has finite volume and negative curvature, which implies $\|N\| > 0$. The cusps of manifolds modelled in the geometries $\mathbb{H}^4$ and $\mathbb{H}^2_{\mathbb{C}}$ are either flat or Nil^3–manifolds, respectively. This implies that we can cut the cusps S of N off from M (by Gromov's Cutting-Off theorem [14, page 58]), because the fundamental group of any cusp of N is amenable. This means the simplicial volume of M is not affected when we take the cusps of N off. In other words, if $N^0 := M - (N \bigcup S)$ then the cutting off theorem implies $\|M\| = \|N^0\| + \|N\|$. Therefore $\|M\| \geq \|N\| > 0$. □

Conjecture 15 *If a smooth orientable four-manifold M admits a proper geometric decomposition into pieces modelled on $\mathbb{H}^2 \times \mathbb{H}^2$ then $\mathrm{h}(M) \neq 0$.*

Remark The issues of uniqueness of a decomposition, or even uniqueness of the pieces involved in a decomposition are subtle open questions (but they are not directly relevant to the results of this paper). In the work of M Kreck, W Lück and P Teichner topological and smooth counterexamples to the Kneser conjecture can be found [24]. However, they have also shown that given a splitting of the fundamental group of a smooth four-manifold M there does exist a unique *stable* decomposition of M [25]. Here stable means up to adding copies of $S^2 \times S^2$.

9 Mixed Euclidean cases $\mathbb{H}^3 \times \mathbb{E}$, $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$

In this section we will deal with the manifolds in case (3) of Hillman's Theorem which do not have pieces modelled on $\mathbb{H}^4$.

9.1 Generalities on the isometry group of $\mathbb{X}$

Definition 16 A Riemannian manifold M is reducible if M is isometric to the Riemannian product $M_1 \times M_2$ of two manifolds, M_1 and M_2 of positive dimension. If M is not reducible, then it is said to be *irreducible*.

In general if we have a simply connected Riemannian product $N \times M$ where M is Euclidean and N is irreducible (a de Rham decomposition in the notation of [8]) then $\mathrm{Iso}(N \times M) = \mathrm{Iso}(N) \times \mathrm{Iso}(M)$ (see Kobayashi and Nomizu [22, page 240]). Thus $\mathrm{Iso}(\mathbb{H}^3 \times \mathbb{E}) = \mathrm{Iso}(\mathbb{H}^3) \times \mathrm{Iso}(\mathbb{E})$, $\mathrm{Iso}(\mathbb{H}^2 \times \mathbb{E}^2) = \mathrm{Iso}(\mathbb{H}^2) \times \mathrm{Iso}(\mathbb{E}^2)$ and $\mathrm{Iso}(\widetilde{\mathrm{SL}}_2 \times \mathbb{E}) = \mathrm{Iso}(\widetilde{\mathrm{SL}}_2) \times \mathrm{Iso}(\mathbb{E})$.

The identity components of these groups are:

$$\begin{aligned}
\mathrm{Iso}_0(\mathbb{H}^3 \times \mathbb{E}) &= \mathrm{PSL}(2,\mathbb{C}) \times \mathbb{R}, \\
\mathrm{Iso}_0(\mathbb{H}^2 \times \mathbb{E}^2) &= \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{Iso}^+(\mathbb{E}^2), \\
\mathrm{Iso}_0(\widetilde{\mathrm{SL}}_2 \times \mathbb{E}) &= \mathrm{Iso}_0(\widetilde{\mathrm{SL}}_2) \times \mathbb{R}.
\end{aligned}$$

9.2 Lattices

Let $\Gamma \subset \mathrm{Iso}(\mathbb{X})$ be a discrete subgroup which acts freely on $\mathbb{X}$ such that $M := \mathbb{X}/\Gamma$ is a complete orientable manifold with finite volume.

By a theorem of Wang (cf [36, 8.27]) the lattice Γ meets the radical R of the connected Lie group $\mathrm{Iso}_0(\mathbb{X})$ in a lattice. The radicals are Euclidean and may be described as follows. For $\mathbb{H}^3 \times \mathbb{E}$, the radical is the copy of $\mathbb{R}$ given by the translations on the $\mathbb{E}$ factor. For $\mathbb{H}^2 \times \mathbb{E}^2$, the radical is a copy of $\mathbb{R}^2$ given by the translations on the $\mathbb{E}^2$ factor. For $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$ it is also $\mathbb{R}^2$, with one copy of $\mathbb{R}$ coming from translations on the $\mathbb{E}$ factor and the other coming from the center of $\mathrm{Iso}_0(\widetilde{\mathrm{SL}}_2)$. Thus $\Gamma \cap R$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$.

9.3 $\mathcal{F}$–Structures on flat manifolds

The isometry group of $\mathbb{E}^n$ is the semidirect product of $\mathbb{R}^n$ and $\mathrm{O}(n)$. Let $\rho\colon \mathrm{O}(n) \to \mathrm{Aut}(\mathbb{R}^n)$ be the map $\rho(B)(x) = Bx$. Let $\Gamma \subset \mathrm{Iso}(\mathbb{E}^{\mathrm{n}})$ be a cocompact lattice and $M := \mathbb{E}^n/\Gamma$ a compact flat manifold. Let $p\colon \Gamma \to \mathrm{O}(n)$ be the homomorphism $p(t,\alpha) = \alpha$, where $(t,\alpha) \in \mathbb{R}^n \times \mathrm{O}(n)$. The Bieberbach theorem guarantees that Γ meets the translations in a lattice (ie the kernel of p is isomorphic to $\mathbb{Z}^n$) and $p(\Gamma)$ is a finite group G. Then M is finitely covered by the torus $\mathbb{R}^n/\ker(p)$ and the deck transformation group of this finite cover is G.

Note that for any $\alpha \in G$, $\rho(\alpha)$ maps $\ker(p)$ to itself because

$$(u,\alpha) \circ (s,I) \circ (u,\alpha)^{-1} = (\rho(\alpha)s, I)$$

and thus if $(s,I) \in \Gamma$, then $(\rho(\alpha)s, I) \in \Gamma$.

Hence the map $\rho\colon \mathrm{O}(n) \to \mathrm{Aut}(\mathbb{R}^n)$ induces a map $\psi =\colon G \to \mathrm{Aut}(\mathbb{T}^n = \mathbb{R}^n/\ker(p))$. As an action ϕ of $\mathbb{T}^n$ on $\mathbb{R}^n/\ker(p)$ we take $x \mapsto x + t$. To see that this defines an $\mathcal{F}$–structure we check the condition $\alpha(\phi(t)(x)) = \phi(\psi(\alpha)(t))(\alpha(x))$ for $\alpha \in G$ which just says $\alpha(x+t) = \alpha(x) + \alpha(t)$.

9.4 Ends of hyperbolic manifolds

The following description may be found in Eberlein [8]. Let $\Gamma \subset \mathrm{Iso}(\mathbb{H}^n)$ be a lattice and let $M := \mathbb{H}^n/\Gamma$. If M is not compact, then it has finitely many ends (or cusps) and the ends are in one-to-one correspondence with conjugacy classes of subgroups of Γ that contain parabolic elements. For each end there is a point $x \in \mathbb{H}(\infty)$ (the sphere at infinity) such that if we let Γ_x be the stabiliser of x, then Γ_x consists only of parabolic elements which leave every horosphere L at x invariant. The horosphere L is flat with the induced metric (this can be easily seen in the upper-half space model with horospheres given by x_n constant) and thus $N := L/\Gamma_x$ is a compact flat manifold. A horocyclic neighbourhood U of the end is given by the projection of open horoballs in $\mathbb{H}^n$. The set U is a warped Riemannian product of the flat metric on N and $(0,\infty)$ whose metric is given by $e^{-2t}\, ds_N^2 + dt^2$.

9.5 $\mathcal{F}$–Structures on quotients of $\mathbb{H}^3 \times \mathbb{E}$ and ends

Let Γ be a lattice in $\mathrm{Iso}(\mathbb{H}^3 \times \mathbb{E})$. By the discussion in Section 9.2, there exists s_0 such that $\Gamma \cap R$ contains the translations generated by $(x,t) \mapsto (x, t+s_0)$ (and only them).

Consider the projection homomorphism $\mathrm{Iso}(\mathbb{H}^3 \times \mathbb{E}) \mapsto \mathrm{Iso}(\mathbb{E}) \mapsto \mathbb{Z}_2$ (recall that $\mathrm{Iso}(\mathbb{E})$ is the semidirect product of $\mathbb{R}$ with $\mathrm{O}(1) = \mathbb{Z}_2$). Then we have a homomorphism $\Gamma \mapsto \mathbb{Z}_2$. Its kernel is an index 2 subgroup $\Gamma_0 \subset \mathrm{Iso}(\mathbb{H}^3) \times \mathbb{R}$. The manifold $M_0 = \mathbb{X}/\Gamma_0$ is a 2-1 cover of M. But M_0 admits a circle action since the action of $\mathbb{R}$, $(x,t) \mapsto (x, t+s)$ descends to a circle action on M_0. The action may not descend to M, but M is still foliated by circles. In any case we obtain in this way an $\mathcal{F}$–structure on M. Where $\Psi\colon \mathbb{Z}_2 \to \mathrm{Aut}(S^1)$ on the non trivial element of $\mathbb{Z}_2$ is just $t \mapsto -t$.

Let us now take a look at the ends of M. Let $p_1\colon \mathrm{Iso}(\mathbb{H}^3 \times \mathbb{E}) \to \mathrm{Iso}(\mathbb{H}^3)$ be the projection on the first factor. The group $p_1(\Gamma)$ is a lattice in $\mathrm{Iso}(\mathbb{H}^3)$ isomorphic to $\Gamma/\mathbb{Z}$. The ends of M arise from the ends of the hyperbolic 3–orbifold $\mathbb{H}^3/p_1(\Gamma)$. Note that the action of $p_1(\Gamma)$ on $\mathbb{H}^3$ is not necessarily free and the fixed points create the orbifold nature of the quotient. By Selberg's lemma [37], $p_1(\Gamma)$ does contain a finite index subgroup which acts freely on $\mathbb{H}^3$.

For each end of M, there is a point $x \in \mathbb{H}^3(\infty)$ and a horosphere L through x. The set $P = L \times \mathbb{E}$ is a copy of Euclidean 3–space which inherits the flat metric from

$\mathbb{H}^3 \times \mathbb{E}$. If we let Γ_P be the elements of Γ which project under p_1 to $\mathrm{Stab}(x)$, then the horocyclic neighbourhood $V = P/\Gamma_P \times (0, \infty)$ is the end of M. Then, on V we have a canonical $\mathcal{F}$–structure given by Section 9.3.

Note that the $\mathcal{F}$–structure we defined on M before using the $\mathbb{R}$–action on the $\mathbb{E}$–factor is compatible with the one we just described at the ends. In fact on V the $\mathbb{R}$–action does descend to a circle action leaving P/Γ_P invariant.

9.6 Gluing $\mathbb{H}^3 \times \mathbb{E}$ pieces

In this subsection we suppose that M is a closed orientable geometrisable 4–manifold with pieces modeled on $\mathbb{H}^3 \times \mathbb{E}$ and we show how to put a polarised $\mathcal{F}$–structure on M.

In order to prove this, the situation we need to consider is the following. Let $M_i = \mathbb{H}^3 \times \mathbb{E}/\Gamma_i$ for $i = 1, 2$ and suppose M_i has one end of the form $P_i \times (0, \infty)$ for $i = 1, 2$ and $f\colon P_1 \to P_2$ is a diffeomorphism. The manifolds P_i are flat. We wish to show that $M = M_1 \cup_f M_2$ has an $\mathcal{F}$–structure. The diffeomorphism type of M only depends on the isotopy class of f. We will use the fact that on a flat 3–manifold any diffeomorphism is isotopic to an affine map, so from now on we will suppose that f is affine (this follows from either [29] or [26]).

Now according to the previous subsection we have $\mathcal{F}$–structures on each of the ends. These structures will be compatible when the gluing map is affine. Indeed we only need to observe that in $\mathbb{R}^n$, an affine map has the form $f(x) = Ax + b$, where A is an invertible matrix and $b \in \mathbb{R}^n$ a fixed vector. Hence if we conjugate by f the $\mathbb{R}^n$–action by translations $x \mapsto x + u$ we obtain $x \mapsto x + Au$ and these two actions commute. So we have a polarised $\mathcal{F}$–structure on M.

9.7 $\mathcal{F}$–Structures on quotients of $\mathbb{H}^2 \times \mathbb{E}^2$ and ends

Let Γ be a lattice in $\mathrm{Iso}(\mathbb{H}^2 \times \mathbb{E}^2)$. By the discussion in Section 9.2 above, there exist linearly independent vectors $w_1, w_2 \in \mathbb{R}^2$ such that Γ contains the translations generated by $(x, y) \mapsto (x, y + w_i)$, for $i = 1, 2$ (and only them).

Consider the projection homomorphism $\mathrm{Iso}(\mathbb{H}^2 \times \mathbb{E}^2) \to \mathrm{Iso}(\mathbb{E}^2) \to \mathrm{O}(2)$ (recall that $\mathrm{Iso}(\mathbb{E}^2)$ is the semidirect product of $\mathbb{R}^2$ with $\mathrm{O}(2)$). Then we have a homomorphism $\Gamma \mapsto \mathrm{O}(2)$ with image a finite group G. Its kernel is a finite index subgroup $\Gamma_0 \subset \mathrm{Iso}(\mathbb{H}^2) \times \mathbb{R}^2$. The manifold $M_0 = \mathbb{X}/\Gamma_0$ is a finite cover of M with G as deck transformation group. But M_0 admits a 2–torus action since the action of $\mathbb{R}^2$, $(x, y) \mapsto (x, y + u)$ descends to a 2–torus action on M_0. The action may not descend to M, but M is still foliated by tori. In any case we obtain in this way an $\mathcal{F}$–structure on M, where $\Psi\colon G \to \mathrm{Aut}(T^2)$ is given exactly by the Ψ of Section 9.3.

Let us now take a look at the ends of M. Let $p_1\colon \mathrm{Iso}(\mathbb{H}^2 \times \mathbb{E}^2) \to \mathrm{Iso}(\mathbb{H}^2)$ be the projection on the first factor. The group $p_1(\Gamma)$ is a lattice in $\mathrm{Iso}(\mathbb{H}^2)$ isomorphic to $\Gamma/\mathbb{Z}^2$. The ends of M arise from the ends of the hyperbolic 2–orbifold $\mathbb{H}^2/p_1(\Gamma)$.

For each end of M, there is a point $x \in \mathbb{H}^2(\infty)$ and a horosphere L through x. The set $P = L \times \mathbb{E}^2$ is a copy of Euclidean 3–space which inherits the flat metric from $\mathbb{H}^2 \times \mathbb{E}^2$. If we let Γ_P be the elements of Γ which project under p_1 to $\mathrm{Stab}(x)$, then the horocyclic neighbourhood $V = P/\Gamma_P \times (0,\infty)$ is the end of M. Then, on V we have a canonical $\mathcal{F}$–structure given by Section 9.3.

Note that the $\mathcal{F}$–structure we defined on M before using the $\mathbb{R}^2$–action on the $\mathbb{E}^2$–factor is compatible with the one we just described at the ends.

9.8 $\mathcal{F}$–Structures on quotients of $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$ and ends

Let Γ be a lattice in $\mathrm{Iso}(\widetilde{\mathrm{SL}}_2 \times \mathbb{E})$. Since $\widetilde{\mathrm{SL}}_2$ does not admit orientation reversing isometries and M is orientable we see that $\Gamma \subset \mathrm{Iso}(\widetilde{\mathrm{SL}}_2) \times \mathbb{R}$. Recall that we have the sequence

$$0 \to \mathbb{R} \to \mathrm{Iso}(\widetilde{\mathrm{SL}}_2) \to \mathrm{Iso}(\mathbb{H}^2) \to 1$$

and $\mathbb{R}$ is central in $\mathrm{Iso}_0(\widetilde{\mathrm{SL}}_2)$. Hence $\mathrm{Iso}(\widetilde{\mathrm{SL}}_2) \times \mathbb{R}$ contains a copy of $\mathbb{R}^2$. By the discussion in Section 9.2, there exist linearly independent vectors $w_1, w_2 \in \mathbb{R}^2$ such that Γ intersects $\mathbb{R}^2$ in the lattice $\mathbb{Z}w_1 + \mathbb{Z}w_2$.

We have a homomorphism

$$\Gamma \to \mathrm{Iso}(\widetilde{\mathrm{SL}}_2) \to \mathrm{Iso}(\mathbb{H}^2) \to \mathbb{Z}_2 = \mathrm{Iso}(\mathbb{H}^2)/\mathrm{PSL}(2,\mathbb{R}).$$

The kernel of this homomorphism gives an index 2 subgroup $\Gamma_0 \subset \mathrm{Iso}_0(\widetilde{\mathrm{SL}}_2) \times \mathbb{R}$ and the manifold $M_0 = \mathbb{X}/\Gamma_0$ is a 2-1 cover of M. But M_0 admits a 2–torus action since the action of $\mathbb{R}^2$ on $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$ descends to a 2–torus action on M_0. The action may not descend to M, but M is still foliated by tori. In any case we obtain in this way an $\mathcal{F}$–structure on M, where $\Psi\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{T}^2)$ is given by $(t_1,t_2) \mapsto (-t_1,t_2)$.

Let us now take a look at the ends of M. Let $p_1\colon \mathrm{Iso}(\widetilde{\mathrm{SL}}_2 \times \mathbb{E}) \to \mathrm{Iso}(\mathbb{H}^2)$ be the composition of the projection on the first factor with $\mathrm{Iso}(\widetilde{\mathrm{SL}}_2) \to \mathrm{Iso}(\mathbb{H}^2)$. The group $p_1(\Gamma)$ is a lattice in $\mathrm{Iso}(\mathbb{H}^2)$ isomorphic to $\Gamma/\mathbb{Z}^2$. The ends of M arise from the ends of the hyperbolic 2–orbifold $\mathbb{H}^2/p_1(\Gamma)$. For each end of M, there is a point $x \in \mathbb{H}^2(\infty)$ and a horosphere L through x. The Lie group $\widetilde{\mathrm{SL}}_2$ is an $\mathbb{R}$–bundle over $\mathbb{H}^2$. So inside $\widetilde{\mathrm{SL}}_2$ we now get a copy F of Euclidean 2–space given by those $\mathbb{R}$–lines over L.

The set $P = F \times \mathbb{E}$ is a copy of Euclidean 3–space which inherits the flat metric from $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$. If we let Γ_P be the elements of Γ which project under p_1 to $\mathrm{Stab}(x)$, then the horocyclic neighbourhood $V = P/\Gamma_P \times (0,\infty)$ is the end of M. Then, on V we have a canonical $\mathcal{F}$–structure given by Section 9.3. As in the other cases, the $\mathcal{F}$–structure we defined on M before using the $\mathbb{R}^2$–action on $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$ is compatible with the one we just described at the ends. In fact, on V we do have a 2–torus action leaving P/Γ_P invariant.

Theorem 17 *A closed orientable and geometrisable* 4*–manifold with pieces modelled only on* $\mathbb{H}^3 \times \mathbb{E}$, $\mathbb{H}^2 \times \mathbb{E}^2$ *or* $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$ *admits a polarised* $\mathcal{F}$*–structure.*

Proof Sections 9.5, 9.7 and 9.8 exhibit $\mathcal{F}$–structures on each piece, such that at the flat ends we have the canonical $\mathcal{F}$–structure defined in Section 9.3. Gluing by affine diffeomorphisms ensures compatibility on the overlaps as explained in Section 9.6. By inspection we see that the structure is polarised. □

10 Manifolds which decompose into $\mathbb{F}^4$–pieces

10.1 The geometry $\mathbb{F}^4$

Suppose we have a finite volume manifold M modelled on $\mathbb{F}^4$. The fundamental group Γ of M is a lattice in $\mathbb{R}^2 \ltimes \mathrm{SL}(2,\mathbb{R})$. It must meet $\mathbb{R}^2$ in $\mathbb{Z}^2$, otherwise the volume of M would not be finite. Denote by $\overline{\Gamma}$ the image of Γ in $\mathrm{SL}(2,\mathbb{R})$ and notice that $\overline{\Gamma} = \Gamma/\mathbb{Z}^2$. We can now see that $M = \mathbb{F}^4/\Gamma$ is an elliptic surface over $B = \mathbb{H}^2/\overline{\Gamma}$, where B is a noncompact orbifold [43, page 150].

The identity component of $\mathrm{Iso}(\mathbb{F}^4)$ coincides with $\mathrm{Iso}^+(\mathbb{F}^4)$ and is given by the semidirect product $\mathbb{R}^2 \ltimes_\alpha \mathrm{SL}(2,\mathbb{R})$, with α the natural action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^2$. Let $\Gamma \subset \mathrm{Iso}^+(\mathbb{F}^4)$ be a lattice, so that $M = \mathbb{F}^4/\Gamma$ is a finite volume manifold modelled on $\mathbb{F}^4$. Let $p\colon \mathbb{R}^2 \ltimes_\alpha \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ be the projection homomorphism. By the same theorem of Wang [36, 8.27] which we used in Section 9.2, Γ meets $\mathbb{R}^2$ in a lattice isomorphic to $\mathbb{Z}^2$. The quotient $\Gamma/\mathbb{Z}^2$ is isomorphic to $p(\Gamma)$. As in the case of flat manifolds, the structure of semidirect product implies that if $A \in p(\Gamma)$, then A maps $\Gamma \cap \mathbb{R}^2$ to itself. Thus we have an induced homomorphism $\psi\colon p(\Gamma) \to \mathrm{Aut}(\mathbb{T}^2 = \mathbb{R}^2/(\Gamma \cap \mathbb{R}^2))$. The manifold M is $\mathbb{T}^2 \times \mathbb{H}^2$ modulo the action of $p(\Gamma)$, where it acts on $\mathbb{T}^2$ via ψ and on $\mathbb{H}^2$ in the usual way. The quotient $B := \mathbb{H}^2/p(\Gamma)$ is a hyperbolic orbifold of finite volume and hence M is an orbifold bundle over B. If B is smooth, ie $p(\Gamma)$ acts without fixed points, then M is a torus bundle over B with structure group $\mathrm{SL}(2,\mathbb{Z})$ and ψ is precisely its holonomy.

The ends of M arise from parabolic elements in $p(\Gamma)$. If $L \subset \mathbb{H}^2$ is an appropriate horosphere left invariant by a parabolic element $A \in p(\Gamma)$, then the cusp will have the form $P \times (0,\infty)$, where $P = (\mathbb{T}^2 \times L)/\mathbb{Z}$, and this $\mathbb{Z}$ is generated by A. This exhibits the boundary of the ends as torus bundles over the circle.

10.2 Affine transformations of Lie groups

Let G be a Lie group and $\mathrm{Aut}(G)$ be the group of continuous automorphisms of G. Then the group $\mathrm{Aff}(G)$ of affine transformations of G is isomorphic to the semidirect product $A(G) := G \ltimes \mathrm{Aut}(G)$ with the operation,

$$(g_1,\alpha_1)(g_2,\alpha_2) = (g_1\alpha_1(g_2), \alpha_1\alpha_2), \quad g_1, g_2 \in G, \quad \alpha_i \in \mathrm{Aut}(G).$$

It has a Lie group structure and acts on G by $(g,\alpha)x = g\alpha(x)$ for $(g,\alpha) \in A(G)$, $x \in G$.

The left inverse of (g,α) is $(g,\alpha)^{-1} = ((\alpha^{-1}(g))^{-1}, \alpha^{-1})$.

$$\begin{aligned}(g,\alpha)^{-1}(g,\alpha) &= ((\alpha^{-1}(g))^{-1}, \alpha^{-1})(g,\alpha)\\ &= ((\alpha^{-1}(g))^{-1}\alpha^{-1}(g), \alpha^{-1}\alpha) = (e, \mathrm{Id})\end{aligned}$$

It was first noticed by Kamber and Tondeur in [20] that the action of $A(G)$ on G defines an isomorphism $i\colon A(G) \to \mathrm{Aff}(G)$. The following lemma is useful for computations.

Lemma 18 *Let G be a Lie group, ρ and σ elements of the centre of L and $A \in \mathrm{Aff}(G)$. Then $A^{-1}\rho A\sigma = \sigma A^{-1}\rho A$.*

Proof Let $\rho_A = A^{-1}\rho A$, where $A = (g,\alpha)$ in $A(G) \cong \mathrm{Aff}(G)$, $\rho = (\rho, \mathrm{Id})$ and $\sigma = (\sigma, \mathrm{Id})$. If ρ is in the centre of G then for an α in $\mathrm{Aut}(G)$ and a g in G we have that $\rho g = g\rho \Rightarrow \alpha(\rho g) = \alpha(g\rho) \Rightarrow \alpha(\rho)\alpha(g) = \alpha(g)\alpha(\rho)$.

We now compose the above elements to see that $\rho_A = (\alpha^{-1}(\rho), \mathrm{Id})$:

$$\begin{aligned}\rho_A &= A^{-1}\rho A = A^{-1}[(\rho,\mathrm{Id}) \circ (g,\alpha)]\\ &= A^{-1} \circ (\rho g, \alpha) = ((\alpha^{-1}(g))^{-1}, \alpha^{-1}) \circ (\rho g, \alpha)\\ &= ((\alpha^{-1}(g))^{-1}\alpha^{-1}(\rho g), \alpha^{-1}\alpha) = ((\alpha^{-1}(g))^{-1}\alpha^{-1}(g\rho), \mathrm{Id}))\\ &= ((\alpha^{-1}(g))^{-1}\alpha^{-1}(g)\alpha^{-1}(\rho), \mathrm{Id})) = (\alpha^{-1}(\rho), \mathrm{Id})\end{aligned}$$

The above calculation implies:

$$\begin{aligned}\rho_A\sigma &= (\alpha^{-1}(\rho), \mathrm{Id}) \circ (\sigma, \mathrm{Id}) = (\alpha^{-1}(\rho)\sigma, \mathrm{Id})\\ &= (\sigma\alpha^{-1}(\rho), \mathrm{Id}) = (\sigma, \mathrm{Id}) \circ (\alpha^{-1}(\rho), \mathrm{Id}) = \sigma\rho_A\end{aligned}$$

Therefore $A^{-1}\rho A\sigma = \sigma A^{-1}\rho A$. □

10.3 $\mathcal{T}$–Structures on manifolds with decomposition into $\mathbb{F}^4$–pieces

Theorem 19 *Let M be a closed orientable complete four-manifold with a geometric decomposition into orientable pieces modelled only on $\mathbb{F}^4$. Then M admits a $\mathcal{T}$–structure.*

Proof First we will see how the $\mathbb{F}^4$–pieces of the geometric decomposition on M admit a $\mathcal{T}$–structure. Let N denote one such $\mathbb{F}^4$–piece. Then N is an open elliptic surface over the base B. Let m be the number of cusps of B and p_i one such cusp. Denote by E_i the end of N corresponding to the cusp p_i of B. We know that E_i is a Nil^3–manifold and $\pi_1(E)$ is isomorphic to Γ_{k_i} as above, for some $k_i \in \mathbb{Z}$.

Consider a small horocyclic neighbourhood U of p_i. Let $B^0 := B - U$ and $N^0 \to B^0$ be the corresponding elliptic surface obtained by restriction. Identify the boundary ∂N^0 of N^0 with itself using the identity to form the double DN^0 of N^0. Now DN^0 is a compact elliptic surface over the double of B^0, so DN^0 admits a $\mathcal{T}$–structure whose orbits are the elliptic fibres [34, Theorem 5.10]. When we restrict the $\mathcal{T}$–structure on DN^0 to N^0 we obtain a $\mathcal{T}$–structure on N^0. Recall that E_i is a T^2–bundle over S^1 with geometric monodromy

$$\mu_i = \begin{pmatrix} 1 & k_i \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

The monodromy around the boundary of U is also μ_i. This allows us to extend the $\mathcal{T}$–structure on N^0 to N, because the action of T^2 on the elliptic fibres behaves well with respect to any element of $\mathrm{SL}(2, \mathbb{Z})$. This holds in particular for μ_i and the corresponding actions will commute after conjugation by μ_i.

A collar neighbourhood of E_i in N is diffeomorphic to $E_i \times (0, \infty)$, which is $\mathrm{Nil}^3/\Gamma_{k_i} \times (0, \infty) := V$. Both actions leave V invariant, as they leave every slice $E_i \times \{t\}$ invariant for every $t \in (0, \infty)$. As translations along the z axis in Nil^3, given by $[x, y, z] \overset{\sigma}{\mapsto} [x, y, z + \sigma]$, $\sigma \in \mathbb{R}$ are central in Nil^3. They descend to an S^1–action σ on V given by $([x, y, z], t) \overset{\sigma}{\mapsto} ([x, y, z + \sigma], t)$. On $N \cap V$ the structures τ and σ commute:

$$\begin{aligned} \tau\sigma([x, y, z], t) &= \tau([x, y, z + \sigma], t) = ([x + \tau_1, y + \tau_2, z + \sigma], t) \\ &= \sigma([x + \tau_1, y + \tau_2, z], t) = \sigma\tau([x, y, z], t) \end{aligned}$$

Assume N_1 and N_2 are two $\mathbb{F}^4$–manifolds which are glued along E_1 and E_2, components of their respective boundaries. Let $h\colon E_1 \to E_2$ be the gluing diffeomorphism; we will see in the next section that h is isotopic to an affine transformation $\alpha\colon E_1 \to E_2$. When we use isotopic diffeomorphisms to identify boundary components we obtain diffeomorphic manifolds [17]. So it is enough to work with α, as we are interested in the existence of a $\mathcal{T}$–structure up to diffeomorphism.

Define $\rho_\alpha := \alpha^{-1}\rho\alpha$, we need to show that $\rho_\alpha\sigma = \sigma\rho_\alpha$ for the S^1–actions ρ, σ on E_1 and E_2 which are induced by translations along the z–axis of Nil^3. The affine transformation α lifts to an affine transformation A of Nil^3 which is $\pi_1(E_1)$–invariant. That is A sends $\pi_1(E_1)$–orbits to $\pi_1(E_1)$–orbits in Nil^3. Both ρ and σ lift to translations along the z–axis of Nil^3, which we will call $\widetilde{\rho}$ and $\widetilde{\sigma}$. Therefore $\rho_\alpha\sigma = \sigma\rho_\alpha$ follows from $\widetilde{\rho_A}\widetilde{\sigma} = \widetilde{\sigma}\widetilde{\rho_A}$ on Nil^3, where again $\widetilde{\rho_A} := A^{-1}\widetilde{\rho}A$. This was shown in Lemma 18, as both $\widetilde{\rho}$ and $\widetilde{\sigma}$ are central in Nil^3. By repeating the same procedure on each geometric piece of M and on each pair of identified cusps, we give M a $\mathcal{T}$–structure. □

11 Diffeomorphisms of flat 3–manifolds and Nil^3–manifolds

In his review on problems in low dimensional topology [21, page 137], R Kirby points out that the following is a consequence of the results of P Scott and W Meeks [29]. Let M be a 3–manifold modelled on $\mathbb{R}^3$ or Nil^3, $\mathrm{Aff}(M)$ denote the group of affine transformations of M and $\mathrm{Diff}(M)$ the group of diffeomorphisms of M. The inclusion $\mathrm{Aff}(M) \hookrightarrow \mathrm{Diff}(M)$ induces an isomorphism on components

$$\pi_0(\mathrm{Aff}(M)) \overset{\cong}{\hookrightarrow} \pi_0(\mathrm{Diff}(M)).$$

Remark It follows that every diffeomorphism of a compact flat 3–manifold or Nil^3–manifold is isotopic to an affine transformation.

A proof of this result can be reproduced by induction from the results on periodic diffeomorphisms shown in [29]—this was communicated to the author by Scott. An alternative approach has been suggested by A Verjovsky [41]. It uses the following deep fact about closed one forms on 3–manifolds. Laudenbach and Blank showed in [26] that two closed nonsingular 1–forms on a 3–manifold are isotopic if and only if they are cohomologous.

Corollary 20 *Any diffeomorphism g of T^3 is isotopic to an affine transformation.*

Proof (Verjovsky) The map g induces a linear map on $H^1(T^3)$. Composing g with the inverse of this linear map, we can assume that $g^*\colon H^1(T^3) \to H^1(T^3)$ is the identity. Let $p\colon T^3 = S^1 \times S^1 \times S^1 \to S^1$ be the projection onto the first factor and let $\omega = \pi^*(d\theta)$, where $d\theta$ denotes the metric on S^1. The form ω is a closed nonsingular 1–form on T^3. Since $g^*\omega$ is cohomologous to ω, by the Laudenbach–Blank theorem, $g^*\omega$ and ω are isotopic through an isotopy $(h_t)_{\{0\leq t\leq 1\}}$ such that $h_0 = \mathrm{Id}$ and $h_1^*(g^*\omega) = \omega$.

The map $f = g \circ h_1$, fixes ω in cohomology and therefore fixes each torus $T_\theta = \{e^{2\pi i\theta}\} \times T^2$. Let f_θ be the restriction of f to T_θ. The map $\theta \mapsto f_\theta$ defines a loop $S^1 \to \mathrm{Diff}(T^2)$; in fact the image of this loop lies in $\mathrm{Diff}_0(T^2)$, the subgroup of diffeomorphisms isotopic to the identity, because f_θ induces the identity in the cohomology of T^2. Because $\mathrm{Diff}_0(T^2)$ retracts to the group of translations and therefore we can retract our loop to a map $S^1 \to T^2$. This map is homotopic to a constant, since it is the identity in (co)-homology. □

12 Fundamental groups of geometrisable manifolds

Let M be an orientable smooth four-manifold which admits a proper geometric decomposition. A standard argument using the Seifert–van Kampen theorem shows $\pi_1(M)$ is isomorphic to an amalgamated product $A *_C B$ or to an HNN–extension $A*_C^\phi$. Here A is the fundamental group of one of the geometric pieces.

A free product with amalgamation $A *_C B$ where C is a subgroup of both A and B is *nondihedral* if the two inclusions $C \subset A$ and $C \subset B$ are strict and if, moreover, the index of C is not 2 in both A and B. An HNN–extension $A*_C^\phi$, where ϕ is an isomorphism from some subgroup C of A onto some subgroup C' of A is *nonsemidirect* if at least one of the inclusions $C \subset A$ or $C' \subset A$ is strict.

It was shown by P de la Harpe that if a group Γ is isomorphic to either a nondihedral amalgamated product $A *_C B$ or to a nonsemidirect HNN–extension $A*_C^\phi$, then Γ is of exponential growth [15]. A straightforward consequence is:

Lemma 21 *The fundamental group of a smooth four-manifold M with a proper geometric decomposition has exponential growth. So for any smooth Riemannian metric g on M we have that* $\mathrm{h_{top}}(g) > 0$.

Recall that the fundamental group of a connected sum is the free product of the fundamental groups of the summands. If A and B are two finitely generated groups, then the free product $A * B$ contains a free product of rank 2 unless A and B are trivial, A is trivial and B is of order 2, or A and B both have order 2. Therefore if M and N are differentiable manifolds with $\pi_1(M) = A$ and $\pi_1(N) = B$, then $\pi_1(M \# N)$ will grow exponentially and again $\mathrm{h_{top}}(g) > 0$ for any smooth metric g on $M \# N$, unless $\pi_1(M)$ and $\pi_1(N)$ are trivial, $\pi_1(M)$ is trivial and $\pi_1(N)$ has order 2, or $\pi_1(M)$ and $\pi_1(N)$ both have order 2. These arguments complete a proof of Corollary D.

13 Proofs of the main results

13.1 Proof of Theorem 1

Proof In each case we can construct an $\mathcal{F}$–structure.

For $\mathbb{S}^4$ and $\mathbb{C}\mathrm{P}^2$, the only manifolds modelled on these geometries are S^4 and $\mathbb{C}\mathrm{P}^2$. These two manifolds have S^1–actions so they admit a $\mathcal{T}$–structure.

If M is modelled on $\mathbb{S}^3 \times \mathbb{E}$, $\mathbb{H}^3 \times \mathbb{E}$, $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$, $\mathrm{Nil}^3 \times \mathbb{E}$, Nil^4 or Sol^4_1, then M is foliated by geodesic circles. By Theorem 4, this foliation allows us to define a polarised $\mathcal{T}$–structure.

In the case of M being modelled on $\mathbb{S}^2 \times \mathbb{E}^2$ or $\mathbb{H}^2 \times \mathbb{E}^2$, M is Seifert fibred and hence M admits a polarised $\mathcal{T}$–structure. We saw in Theorem 7 how to define circle actions in the fibres which behave well with respect to the Seifert fibration, in that they commute with the structure group and so define a $\mathcal{T}$–structure.

When M is modelled on $\mathrm{Sol}^4_{m,n}$ or Sol^4_0, M is actually diffeomorphic to a mapping torus of T^3. With this description M can be given a polarised $\mathcal{T}$–structure, as was explained in Theorem 9.

We have constructed in Corollary 12 a $\mathcal{T}$–structure on foliated manifolds whose leaves are S^2 or $\mathbb{R}P^2$, which includes all the cases of type $\mathbb{X}^2 \times \mathbb{S}^2$ with $\mathbb{X}^2$ a two-dimensional geometry. The idea here is that the S^2 leaves can be rotated consistently, endowing M with a $\mathcal{T}$–structure.

As shown in Section 9.3, all flat manifolds admit an $\mathcal{F}$–structure.

Therefore if M is modelled on a geometry in $\mathbf{V}$ then M admits a $\mathcal{T}$–structure. □

13.2 Proof of Theorem A

Proof (i) ⇒ (ii) This is content of Theorem 1.

(ii) ⇒ (iii) This is Theorem A of Paternain and Petean in [34].

(iii) ⇒ (iv) Follows directly from the string of inequalities between the asymptotic invariants mentioned in the introduction.

(iv) ⇒ (v) If M is modelled on $\mathbf{H}$ then $||M|| > 0$ by Proposition 2. Therefore $||M|| = 0$ implies M is modelled on a geometry in $\mathbf{V}$.

It follows from Theorem 1 that M admits a $\mathcal{T}$–structure and by Paternain and Petean's Theorem A in [34], M collapses with curvature bounded from below.

(v) $\Rightarrow$ (i) Again we will show the contrary. Let M be a manifold modelled on a geometry in $\mathbf{H}$. Then Proposition 2 implies $||M|| > 0$. As a consequence of the results in [34] and [6], $||M||$ bounds $\mathrm{Vol}_K(M)$ (up to some constants depending only on the dimension n) from below. Therefore for some constant c_n we have

$$0 < ||M|| \leq c_n \mathrm{Vol}_K(M)$$

so M can not collapse with curvature bounded from below. □

13.3 Proof of Theorem B

Proof If the connected sum components of M admit a $\mathcal{T}$–structure then this extends to M under the connected sum. The same is true for $\mathcal{F}$–structures if one of the open sets of the $\mathcal{F}$–structure has a trivial covering [34, page 437]. In all the cases in $\mathbf{V}$ we may achieve this as can be seen from the proofs of Theorems 1, 12, 17 and 19. □

References

[1] **J W Anderson**, **G P Paternain**, *The minimal entropy problem for* 3*–manifolds with zero simplicial volume*, from: "Geometric methods in dynamics, I", Astérisque 286 (2003) xvii–xviii, 63–79 MR2052297

[2] **I K Babenko**, *Asymptotic invariants of smooth manifolds*, Izv. Ross. Akad. Nauk Ser. Mat. 56 (1992) 707–751 MR1208148

[3] **L Bessières**, *Un théorème de rigidité différentielle*, Comment. Math. Helv. 73 (1998) 443–479 MR1633371

[4] **G Besson**, **G Courtois**, **S Gallot**, *Volume et entropie minimale des espaces localement symétriques*, Invent. Math. 103 (1991) 417–445 MR1085114

[5] **M Brunnbauer**, *Homological invariance for asymptotic invariants and systolic inequalities*, GAFA 18 (2008) 1087–1117

[6] **J Cheeger**, **M Gromov**, *Collapsing Riemannian manifolds while keeping their curvature bounded. I*, J. Differential Geom. 23 (1986) 309–346 MR852159

[7] **R J Cobb**, *Infra-solvmanifolds of dimension four*, Bull. Austral. Math. Soc. 62 (2000) 347–349 MR1786217

[8] **P Eberlein**, *Lattices in spaces of nonpositive curvature*, Ann. of Math. (2) 111 (1980) 435–476 MR577132

[9] **J Eells**, **A Verjovsky**, *Harmonic and Riemannian foliations*, Bol. Soc. Mat. Mexicana (3) 4 (1998) 1–12 MR1625589

[10] **C Ehresmann**, *Les connexions infinitésimales dans un espace fibré différentiable*, from: "Colloque de topologie (espaces fibrés), Bruxelles, 1950", Georges Thone, Liège (1951) 29–55 MR0042768

[11] **D B A Epstein**, *Foliations with all leaves compact*, Ann. Inst. Fourier (Grenoble) 26 (1976) viii, 265–282 MR0420652

[12] **R O Filipkiewicz**, *Four-dimensional geometries*, PhD thesis, University of Warwick (1984)

[13] **W M Goldman**, *Complex hyperbolic geometry*, Oxford Math. Monogr., Oxford Science Publ, The Clarendon Press, Oxford Univ. Press, New York (1999) MR1695450

[14] **M Gromov**, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. (1982) 5–99 (1983) MR686042

[15] **P de la Harpe**, *Topics in geometric group theory*, Chicago Lectures in Math., Univ. of Chicago Press (2000) MR1786869

[16] **J A Hillman**, *Four-manifolds, geometries and knots*, Geom. Topol. Monogr. 5, Geom. Topol. Publ., Coventry (2002) MR1943724

[17] **M W Hirsch**, *Differential topology*, Graduate Texts in Math. 33, Springer, New York (1994) MR1336822 Corrected reprint of the 1976 original

[18] **H Inoue**, **K Yano**, *The Gromov invariant of negatively curved manifolds*, Topology 21 (1982) 83–89 MR630882

[19] **N V Ivanov**, *Diffeomorphism groups of Waldhausen manifolds*, J. Sov. Math. 12 (1979) 115–118

[20] **F W Kamber**, **P Tondeur**, *Flat manifolds with parallel torsion*, J. Differential Geometry 2 (1968) 385–389 MR0254868

[21] **R Kirby**, **editor**, *Problems in low-dimensional topology*, from: "Geometric topology (Athens, GA, 1993)", AMS/IP Stud. Adv. Math. 2, Amer. Math. Soc. (1997) 35–473 MR1470751

[22] **S Kobayashi**, **K Nomizu**, *Foundations of differential geometry, Vol. I*, Wiley Classics Library, Wiley, New York (1996) MR1393940 Reprint of the 1963 original

[23] **D Kotschick**, *Entropies, volumes, and Einstein metrics* `arXiv:math.DG/0410215`

[24] **M Kreck**, **W Lück**, **P Teichner**, *Counterexamples to the Kneser conjecture in dimension four*, Comment. Math. Helv. 70 (1995) 423–433 MR1340102

[25] **M Kreck**, **W Lück**, **P Teichner**, *Stable prime decompositions of four-manifolds*, from: "Prospects in topology (Princeton, NJ, 1994)", (F Quinn, editor), Ann. of Math. Stud. 138, Princeton Univ. Press (1995) 251–269 MR1368662

[26] **F Laudenbach**, **S Blank**, *Isotopie de formes fermées en dimension trois*, Invent. Math. 54 (1979) 103–177 MR550181

[27] **C LeBrun**, *Kodaira dimension and the Yamabe problem*, Comm. Anal. Geom. 7 (1999) 133–156 MR1674105

[28] **A Manning**, *Topological entropy for geodesic flows*, Ann. of Math. (2) 110 (1979) 567–573 MR554385

[29] **W H Meeks, III**, **P Scott**, *Finite group actions on* 3*–manifolds*, Invent. Math. 86 (1986) 287–346 MR856847

[30] **P Melvin**, 2*–sphere bundles over compact surfaces*, Proc. Amer. Math. Soc. 92 (1984) 567–572 MR760947

[31] **P Melvin**, **J Parker**, 4*–manifolds with large symmetry groups*, Topology 25 (1986) 71–83 MR836725

[32] **P Molino**, *Riemannian foliations*, Progress in Math. 73, Birkhäuser, Boston (1988) MR932463 Translated from the French by G Cairns, With appendices by G Cairns, Y Carrière, É Ghys, E Salem and V Sergiescu

[33] **G P Paternain**, *Geodesic flows*, Progress in Math. 180, Birkhäuser, Boston (1999) MR1712465

[34] **G P Paternain**, **J Petean**, *Minimal entropy and collapsing with curvature bounded from below*, Invent. Math. 151 (2003) 415–450 MR1953264

[35] **G P Paternain**, **J Petean**, *Entropy and collapsing of compact complex surfaces*, Proc. London Math. Soc. (3) 89 (2004) 763–786 MR2107014

[36] **M S Raghunathan**, *Discrete subgroups of Lie groups*, Ergebnisse der Math. und ihrer Grenzgebiete 68, Springer, New York (1972) MR0507234

[37] **A Selberg**, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. (N.S.) 20 (1956) 47–87 MR0088511

[38] **H Shimizu**, *On discontinuous groups operating on the product of the upper half planes*, Ann. of Math. (2) 77 (1963) 33–71 MR0145106

[39] **M Ue**, *Geometric* 4*–manifolds in the sense of Thurston and Seifert* 4*–manifolds, I*, J. Math. Soc. Japan 42 (1990) 511–540 MR1056834

[40] **M Ue**, *Geometric* 4*–manifolds in the sense of Thurston and Seifert* 4*–manifolds, II*, J. Math. Soc. Japan 43 (1991) 149–183 MR1082428

[41] **A Verjovsky**, Personal communication

[42] **A W Wadsley**, *Geodesic foliations by circles*, J. Differential Geometry 10 (1975) 541–549 MR0400257

[43] **C T C Wall**, *Geometric structures on compact complex analytic surfaces*, Topology 25 (1986) 119–153 MR837617

CIMAT, Jalisco S/N, Col. Valenciana
CP: 36240 Guanajuato, Gto, México.

p.suarez-serrato@cantab.net

Received: 21 April 2008 Revised: 5 February 2009