The Lawson homology for Fulton–MacPherson configuration spaces

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In this paper, we compute the Lawson homology groups and Deligne–Beilinson cohomology groups for the Fulton–MacPherson configuration spaces.

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1 Introduction

The purpose of this paper is to give formulas for the Lawson homology groups and Deligne–Beilinson cohomology groups of the Fulton–MacPherson configuration spaces $X[n]$ for $n \in \mathbb{N}$. The Lawson homology groups of $X[n]$ can be decomposed as a direct sum of Lawson homology groups of certain Cartesian products of X with shifts of bidegrees. A similar decomposition holds for Deligne–Beilinson cohomology groups.

All varieties in the paper are defined over $\mathbb{C}$. Let X be a d–dimensional projective variety and let $\mathcal{Z}_p(X)$ be the space of algebraic p–cycles on X. The *Chow group* $\mathrm{Ch}_p(X)$ of p–cycles is defined to be $\mathcal{Z}_p(X)$ modulo the rational equivalence (see Fulton [8, Section 1.3]). The *Lawson homology* $L_pH_k(X)$ of p–cycles is defined as

$$L_pH_k(X) := \pi_{k-2p}(\mathcal{Z}_p(X)) \quad \text{for} \quad k \geq 2p \geq 0,$$

where $\mathcal{Z}_p(X)$ is provided with a natural compactly generated topology (see Friedlander [6] and Lawson [13]). Set $L_pH_k(X) = L_0H_k(X)$ for $p < 0$. (See Lawson [14] for the general background on Lawson homology.)

Let $F(X,n) \subset X^n$ be the complement of the diagonals, ie,

$$F(X,n) := \{(x_1, \dots, x_n) \in X^n : x_i \neq x_j, \forall i \neq j\}.$$

For each subset $I \subseteq [n] := \{1, \dots, n\}$ with at least two elements, denote by $\mathrm{Bl}_\Delta(X^I)$ the blow-up of the Cartesian product X^I along its small diagonal (for example when $I = \{1,4,5\}$, $X^I = \{(x_1,x_4,x_5)\} \cong X^3$ and its small diagonal Δ consists of points satisfying $x_1 = x_4 = x_5$). The Fulton–MacPherson configuration space is defined as follows.

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Definition–Theorem 1.1 (Fulton–MacPherson [9, Theorems 1 and 3]) Let X be a smooth projective variety over $\mathbb{C}$. The closure of the natural locally closed embedding

$$F(X,n) \hookrightarrow X^n \times \prod_{|I|\geq 2} \mathrm{Bl}_\Delta(X^I)$$

is nonsingular, and the boundary is a simple normal crossing divisor. The closure is called the *Fulton–MacPherson configuration space* and is denoted by $X[n]$.

Let us introduce some combinatorial definitions first.

Definition 1.1 Fix a positive integer n. Denote $[n] := \{1,\dots,n\}$.

Two subsets $I, J \subseteq [n] := \{1,2,\dots,n\}$ are called *overlapped* if $I \cap J$ is a nonempty proper subset of I and of J.

A *nest* $\mathcal{S}$ of $[n]$ is a set of subsets of $[n]$ such that each subset $I \in \mathcal{S}$ contains at least two elements and any two subsets $I, J \in \mathcal{S}$ are not overlapped.

Next we explain some definitions in graph theory. All trees in the paper will be rooted trees.

Definition 1.2 A *(rooted) tree* is a finite graph in which any two nodes are connected by exactly one path, together with a root node.

The *tree-order* is the partial ordering on the nodes of a tree with $u < v$ if and only if $u \neq v$ and the unique path from the root to v passes through u. (So the root is the minimal node in a tree.)

A *forest* is a disjoint union of trees; it has a natural partial order induced from the tree-order. (A tree is a forest by definition.)

A *leaf* is a maximal node in a forest.

In a forest, a node v is called a *child* of a node u if $u < v$ and there is no node w satisfying $u < w < v$.

We will explain in Section 2.2 that there is a 1–1 correspondence between the set of nests of $[n]$ and the set of forests with n leaves, such that each element of the nest labels a node of the forest.

Definition 1.3 Let $\mathcal{S}$ be a nest of $[n]$.

Define $c(\mathcal{S})$ to be the number of connected components of the forest corresponding to $\mathcal{S}$.

Define $c_I(\mathcal{S})$ (or c_I if no ambiguity arises) to be the number of children of the node with label I.

For a nonempty nest $\mathcal{S}$, define the set $M_{\mathcal{S}}$ of lattice points in the integer lattice $\mathbb{Z}^{\mathcal{S}}$ as

(1) $$M_{\mathcal{S}} := \{\underline{\mu} = (\mu_I)_{I\in\mathcal{S}} : 1 \le \mu_I \le (c_I - 1)\dim X - 1,\ \mu_I \in \mathbb{Z}\}.$$

Define the norm $\|\underline{\mu}\| := \sum_{I\in\mathcal{S}} \mu_I$ for $\underline{\mu} \in M_{\mathcal{S}}$. For $\mathcal{S} = \varnothing$, we assume that there is only one lattice point $\underline{0}$ in $M_{\mathcal{S}}$ whose norm is 0.

The first main theorem asserts that the Lawson homology group of $X[n]$ can be decomposed as a direct sum of the Lawson homology groups of the Cartesian products of X with shifts of bidegrees.

Theorem 1.2 *Let X be a smooth projective variety defined over $\mathbb{C}$. Then for each pair of integers p, k such that $k \ge 2p \ge 0$, there is an isomorphism of Lawson homology groups*

$$L_pH_k(X[n]) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu}\in M_{\mathcal{S}}} L_{p-\|\underline{\mu}\|}H_{k-2\|\underline{\mu}\|}(X^{c(\mathcal{S})}),$$

where the direct sum $\bigoplus_{\mathcal{S}}$ run over all the nests $\mathcal{S}$ of $[n]$.

As a consequence of the above theorem, we obtain the following more explicit formula for $L_pH_k(X[n])$. We explain some notation first: generalizing R Stanley's notation $[x^i]F(x)$ in [20] (which gives the coefficient of x^i in the power series $F(x)$) to two variables x and t, we define $[x^i t^n/n!]F(x,t) = a_{in}$ for a power series

$$F(x,t) := \sum_{j,k} a_{jk}\frac{x^j t^k}{k!}.$$

Theorem 1.3 *Let X be a smooth d–dimensional projective variety defined over $\mathbb{C}$. Then for each pair of integers p, k such that $k \ge 2p \ge 0$, there is an isomorphism of Lawson homology groups*

$$L_pH_k(X[n]) \cong \bigoplus_{\substack{1\le m\le n\\ 0\le i\le p}} L_{p-i}H_{k-2i}(X^m)^{\oplus[x^i t^n/n!]N^m/m!}$$

where $N := N(x,t) = \sum_{i\ge 1} h_i(x)(t^i/i!)$ is the exponential generating function of polynomials $h_i(x)$ determined by the identity

$$(1-x)x^d t + (1-x^{d+1}) = \exp(x^d N) - x^{d+1}\exp(N).$$

We also obtain a formula similar to Theorem 1.2 for Deligne–Beilinson cohomology.

Theorem 1.4 *Let X be a smooth projective variety defined over $\mathbb{C}$. Then for each pair of integers p, k, there is an isomorphism of Deligne–Beilinson cohomology groups*

$$H^k_{\mathcal{D}}(X[n],\mathbb{Z}(p)) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu}\in M_{\mathcal{S}}} H^{k-2\|\underline{\mu}\|}_{\mathcal{D}}\big(X^{c(\mathcal{S})},\mathbb{Z}(p-\|\underline{\mu}\|)\big).$$

Similarly, as a consequence of the above theorem, we obtain the following more explicit formula for $H^k_{\mathcal{D}}(X[n],\mathbb{Z}(p))$.

Theorem 1.5 *Let X be a smooth d–dimensional projective variety defined over $\mathbb{C}$. Then for each pair of integers p, k, there is an isomorphism of Deligne–Beilinson cohomology groups*

$$H^k_{\mathcal{D}}(X[n],\mathbb{Z}(p)) \cong \bigoplus_{\substack{1\le m\le n\\ 0\le i\le p}} H^{k-2i}_{\mathcal{D}}\big(X^m,\mathbb{Z}(p-i)\big)^{\oplus[x^i t^n/n!]N^m/m!}$$

where N is the same as in Theorem 1.3.

The key tools used to prove the main results are the blow-up formula for Lawson homology proved by the first author [10] and the method for computing the Chow groups of the Fulton–MacPherson configuration space of the second author [15]. The structure of the paper is as follows: In Section 2, we briefly review background material about Lawson homology, as well as the construction on Fulton–MacPherson spaces and we compute the Lawson homology groups of the Fulton–MacPherson configuration space thus constructed. In Section 3, we briefly review background material about Deligne–Beilinson cohomology and we compute the Deligne–Beilinson cohomology for the Fulton–MacPherson configuration space. In Section 4, we compare Lawson homology with integral singular homology using simple examples. In Section 5, we write down the formula in Theorem 1.3 explicitly for $n=2$ and $n=3$.

2 Lawson homology groups of Fulton–MacPherson spaces

In this section, we prove Theorem 1.2 and Theorem 1.3. According to the construction, the Fulton–MacPherson configuration space $X[n]$ is obtained by successively blowing up X^n along (the strict transforms of) its diagonals in a suitable order, where each blow-up is along a nonsingular subvariety. Therefore, we can calculate the Lawson homology groups of $X[n]$ by successively applying the blow-up formula for Lawson homology (see Theorem 2.1).

2.1 Lawson homology

Recall that for a morphism $f\colon W \to V$ between projective varieties, there exist induced homomorphisms $f_*\colon L_pH_k(W) \to L_pH_k(V)$ for all $k \geq 2p \geq 0$. Furthermore, it has been shown that if W and V are smooth and projective, there are Gysin "wrong way" homomorphisms $f^*\colon L_pH_k(V) \to L_{p-c}H_{k-2c}(W)$, where $c = \dim(V) - \dim(W)$ (see Peters [19]).

Let X be a smooth projective variety and Y be a smooth subvariety of X of codimension r with the natural embedding $i_0\colon Y \hookrightarrow X$. Let $\sigma\colon \widetilde{X}_Y \to X$ be the blow-up of X along Y, let D be the exceptional divisor and $\pi\colon D \to Y$ and $i\colon D \hookrightarrow \widetilde{X}_Y$ be the natural morphisms. Set $U := X - Y \cong \widetilde{X}_Y - D$. Denote by j_0 the inclusion $U \hookrightarrow X$ and j the inclusion $U \hookrightarrow \widetilde{X}_Y$. That is, we have a diagram:

$$\begin{tikzcd}
D \arrow[r, hook, "i"] \arrow[d, "\pi"] & \widetilde{X}_Y \arrow[d, "\sigma"] & U = \widetilde{X}_Y - D \arrow[l, hook', "j"'] \arrow[d, "="] \\
Y \arrow[r, hook, "i_0"] & X & U = X - Y \arrow[l, hook', "j_0"']
\end{tikzcd}$$

It is well known that $\pi\colon D \to Y$ gives a projective bundle of rank $r-1$ by identifying D with $\mathbb{P}(N_{Y/X})$. (Note: We adopted the "old fashioned" geometric notation for the projective bundle $\mathbb{P}(E)$ associated with a vector bundle E used in Fulton's book [8, B.5.5], instead of Grothendieck's notation in EGA.)

Moreover, we have (see Voison [23, page 271])

$$\mathcal{O}_{\widetilde{X}_Y}(D)|_D = \mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1).$$

Denote by h the class of $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1)$ in $\mathrm{Pic}(D)$. We have $h = [-D]|_D$ and $-h = i^*i_*\colon L_qH_m(D) \to L_{q-1}H_{m-2}(D)$ for $0 \leq 2q \leq m$ [7, Theorem 2.4; 19, Lemma 11]. The last equality can be equivalently regarded as a Lefschetz operator

$$-h = i^*i_*\colon L_qH_m(D) \to L_{q-1}H_{m-2}(D), \quad 0 \leq 2q \leq m.$$

The following result is essential to the proof of Theorem 1.2.

Theorem 2.1 (Lawson homology for a blow-up [10, Theorem 1.2]) *Let X be a smooth projective variety and $Y \subset X$ be a smooth subvariety of codimension r. Let $\sigma\colon \widetilde{X}_Y \to X$ be the blow-up of X along Y, $\pi\colon D = \sigma^{-1}(Y) \to Y$ the natural map, and $i\colon D = \sigma^{-1}(Y) \to \widetilde{X}_Y$ the exceptional divisor of the blow-up. Then for each p, k*

with $k \geq 2p \geq 0$*, we have the isomorphism*

$$I_{p,k}\colon \left\{ \bigoplus_{1\leq j\leq r-1} L_{p-j}H_{k-2j}(Y) \right\} \oplus L_pH_k(X) \xrightarrow{\cong} L_pH_k(\widetilde{X}_Y)$$

given by
$$I_{p,k}(u_1,\dots,u_{r-1},u) = \sum_{j=1}^{r-1} i_*(h^j\cdot \pi^* u_j) + \sigma^* u.$$

Remark 2.2 The above theorem proved in [10] can be generalized without difficulty to l–adic Lawson homology (see Friedlander [6]) for nonsingular projective varieties X over any algebraic closed field of characteristic p where $(p,l)=1$.

2.2 The Fulton–MacPherson configuration spaces

The Fulton–MacPherson configuration spaces $X[n]$ were discovered around 1989 [9]. In their original paper, Fulton and MacPherson used it to construct a differential graded algebra which is a model for $F(X,n)$ in the sense of Sullivan. Axelrod and Singer constructed a similar compactification in the setting of smooth manifolds [1, Section 5.1]. The space $\mathbb{P}^1[n]$ is closely related the Deligne–Mumford compactification $\bar{M}_{0,n}$ of the moduli space of nonsingular rational curves with n marked points (see Keel [12]).

The following notation is needed. Fix a positive integer n. There is a 1–1 correspondence between the set of nests of $[n]$ and the set of forests with n leaves which sends a nest $\mathcal{S}$ to a forest F as follows: the n leaves in F are labeled by $1,2,\dots,n$. Each element I of $\mathcal{S}$ gives a node of F, labeled by I. Two nodes with labels I and J are connected by an edge if $I \subsetneq J$ and there does not exist $K\in\mathcal{S}$ such that $I\subsetneq K\subsetneq J$. Denote by $c(\mathcal{S})$ the number of connected components of the forest corresponding to $\mathcal{S}$ and denote by $c_I(\mathcal{S})$, or c_I, the number of children of the node with label I. Below is an example.

Example 2.3 Let $n=7$ and $\mathcal{S}=\{1234,12,56\}$. See Figure 1.

In this example, $c(\mathcal{S})=3$, $c_{12}=c_{56}=2$ and $c_{1234}=3$.

For a nonempty nest $\mathcal{S}$, define the set $M_{\mathcal{S}}$ as in equation (1).

Example 2.4 Let $n=7$ and $\mathcal{S}=\{1234,12,56\}$ be as in Example 2.3 and let $d=\dim X$. Then

$$M_{\mathcal{S}}=\{(\mu_{1234},\mu_{12},\mu_{56})\in\mathbb{Z}^3 : 1\leq\mu_{1234}\leq 2d-1,\ 1\leq\mu_{12}\leq d-1,\ 1\leq\mu_{56}\leq d-1\}.$$

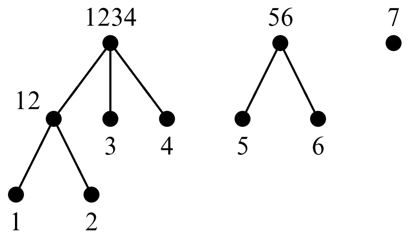

Figure 1: The forest corresponding to the nest $\mathcal{S} = \{1234, 12, 56\}$

Consider the following ordered set of diagonals

$$\underbrace{\Delta_{12\cdots n}}_{\binom{n}{n}}, \underbrace{\Delta_{12\cdots(n-1)}, \Delta_{12\cdots(n-2),n}, \ldots, \Delta_{23\cdots n}}_{\binom{n}{n-1}}, \ldots, \underbrace{\Delta_{12}, \ldots, \Delta_{n-2,n}, \Delta_{n-1,n}}_{\binom{n}{2}} \tag{2}$$

which induces an order on $\{I \subseteq [n] : |I| \geq 2\}$:

$$\{1,2,\ldots,n\} < \{1,2,\ldots,n-1\} < \{1,2,\ldots,n-2,n\} < \cdots < \{n-2,n\} < \{n-1,n\}.$$

The following lemma is needed for the proof of Theorem 1.2. This lemma is implicit in De Concini and Procesi [3], MacPherson and Procesi [18, Section 5.1], and has been pointed out, but not as explicitly as below, in Thurston [21, Proposition 3.5 and 3.6] in the situation of real manifolds.

Lemma 2.5 *$X[n]$ can be constructed by successively blowing up (the strict transforms of) the $2^n - n - 1$ diagonals of X^n in the order (2).*

Proof A proof is given in [16, Proposition 2.13] by the second author and we will not reproduce it here. The idea is to prove inductively for a more general situation – the wonderful compactification of an arrangement – using the notion due to De Concini and Procesi of *building sets of arrangements*. The interested reader is referred to [16, Theorem 1.3] for a conclusion on orders of blow-ups in the construction of the wonderful compactification spaces, which in particular applies to Fulton–MacPherson spaces. □

Proof of Theorem 1.2 Since the Fulton–MacPherson configuration space $X[n]$ can be constructed by a sequence of blow-ups, we can obtain its Lawson homology group by successively applying the formula of one blow-up (Theorem 2.1).

The notion of nest appears naturally when we decompose the Lawson homology groups of $X[n]$ in terms of Lawson homology groups of Cartesian products of X. Take X^n as the ambient variety. For $I \subseteq [n]$ with at least two elements, it can be shown (as a special case of [15, Proposition 2.7]) that the strict transform $\widetilde{\Delta}_I$ (which will be blown up along) is obtained by successively blowing up Δ_I along the centers $\Delta_J \cap \Delta_I$ for ($J \supsetneq I$) or ($J < I$ and $J \cap I = \varnothing$), so $\{I, J\}$ is a nest; take Δ_I as the ambient variety instead of X^n. The strict transform of $\Delta_I \cap \Delta_J$ (which will be blown up along which we blow up is obtained) is obtained by successively blowing up $\Delta_I \cap \Delta_J$ along those $\Delta_I \cap \Delta_J \cap \Delta_K$ such that K satisfies

- $I \cap J \cap K = \varnothing$ and $K < J$,
- $K \supsetneq I$ and $K \cap J = \varnothing$ or
- $K \supsetneq J$ and $K \cap I = \varnothing$,

so the set $\{I, J, K\}$ is a nest. In general, consider a nest $\mathcal{S} = \{I_1, I_2, \dots, I_\ell\}$ which is arranged in the order compatible with (2), that is, the elements in $\mathcal{S}$ is ordered as the order of Δ_{I_j}, $1 \le j \le \ell$ in (2). Then $\mathcal{S}$ determines a chain of polydiagonals (ie intersections of diagonals) of X^n:

$$\bigcap_{j=1}^{\ell} \Delta_{I_j} \subset \bigcap_{j=2}^{\ell} \Delta_{I_j} \subset \bigcap_{j=3}^{\ell} \Delta_{I_j} \subset \cdots \subset \bigcap_{j=\ell-1}^{\ell} \Delta_{I_j} \subset \Delta_{I_\ell} \subset X^n.$$

The codimension of the i–th polydiagonal in the $(i+1)$–th polydiagonal equals $(c_{I_i}-1)\dim X$. Blowing up along the strict transform of Δ_{I_i} makes $(c_{I_i}-1)\dim X - 1$ copies of the Lawson homology groups (with shifted bidegrees) of the strict transform of the i–th polydiagonal $\bigcap_{j=i}^{\ell} \Delta_{I_j}$ contributing to the Lawson homology group of the strict transform of the $(i+1)$–th polydiagonal $\bigcap_{j=i+1}^{\ell} \Delta_{I_j}$. By successively applying the formula of a blow-up (Theorem 2.1), one sees that for each lattice point $(\mu_{I_1}, \dots, \mu_{I_\ell}) \in \mathbb{Z}^\ell$ such that

$$1 \le \mu_{I_i} \le (c_{I_i} - 1)\dim X - 1,$$

there is a copy of the Lawson homology group of $X^{c(\mathcal{S})}$ (with shifted bidegree) in the decomposition of the Lawson homology group of $X[n]$. More precisely, we have the following summand in the decomposition of $L_pH_k(X[n])$:

$$L_{p-\|\underline{\mu}\|}H_{k-2\|\underline{\mu}\|}(X^{c(\mathcal{S})}),$$

where $\|\underline{\mu}\| := \sum_{i=1}^{\ell} \mu_{I_i}$ (notice that $\bigcap_{j=1}^{\ell} \Delta_{I_j} \cong X^{c(\mathcal{S})}$). Let $\mathcal{S}$ run through all the nests of $[n]$, we obtain all the direct summands of the decomposition of $L_pH_k(X[n])$, and the theorem follows. □

Corollary 2.6 *When $p = 0$, Theorem 1.2 is reduced to a formula of the singular homology groups with integer coefficients for $X[n]$.*

Now we proceed to prove Theorem 1.3. A similar result for the Chow group and Chow motive of $X[n]$ was proved in [15]. For the readers' convenience we briefly recall the proof with the necessary adaptations to the present context. We need to recall the compositional formula of exponential generating functions. Let f be a map from $\mathbb{Z}_{\geq 0}$ to a field K of characteristic 0. We denote by $E_f(t)$ the exponential generating function of f, ie,

$$E_f(t) = \sum_{n\geq 0} f(n)\frac{t^n}{n!}.$$

Theorem 2.7 (Compositional formula [20, Theorem 5.1.4]) *Let K be a field of characteristic 0. For any finite set S, denote by $\Pi(S)$ the set of ordered partitions of S. Given functions $f\colon \mathbb{Z}_{\geq 0} \to K$ with $f(0) = 0$ and $g\colon \mathbb{Z}_{\geq 0} \to K$ with $g(0) = 1$, define a new function $h\colon \mathbb{Z}_{\geq 0} \to K$ by $h(0) = 1$ and*

$$h(|S|) = \sum_{\{B_1,\dots,B_k\}\in\Pi(S)} f(|B_1|)f(|B_2|)\cdots f(|B_k|)g(k), \text{ for } |S| > 0.$$

Then
$$E_h(t) = E_g(E_f(t)).$$

Proof of Theorem 1.3 By Theorem 1.2, it suffices to show that for any positive integers m and i,

$$\sum_{c(\mathcal{S})=m} \sum_{\substack{\underline{\mu}\in M_{\mathcal{S}} \\ \|\underline{\mu}\|=i}} 1 = \left[\frac{x^i t^n}{n!}\right]\frac{N^m}{m!}, \quad \text{(where } \mathcal{S} \text{ is a nest of } [n])$$

which is equivalent to showing that for any positive integer m,

$$\sum_{c(\mathcal{S})=m} \sum_{\underline{\mu}\in M_{\mathcal{S}}} x^{\|\underline{\mu}\|} = \left[\frac{t^n}{n!}\right]\frac{N^m}{m!}, \quad \text{(where } \mathcal{S} \text{ is a nest of } [n]). \tag{3}$$

First we consider the special situation $m = 1$. Let K be the polynomial ring $\mathbb{C}[x]$. Define $f\colon \mathbb{Z}_{\geq 0} \to K$ by

$$f(n) := \sum_{c(\mathcal{S})=1} \sum_{\underline{\mu}\in M_{\mathcal{S}}} x^{\|\underline{\mu}\|}, \quad \text{(where } n > 0 \text{ and } \mathcal{S} \text{ is a nest of } [n]) \tag{4}$$

and $f(0) = 0$. We need to show that

$$f(n) = \left[\frac{t^n}{n!}\right]N, \quad \text{for } n > 0.$$

Fix an ordered partition $\{B_1, \dots, B_k\} \in \Pi([n])$. Consider those nests $\mathcal{S}$ such that $c(\mathcal{S}) = 1$ and $B_1, \dots, B_k$ are the maximal elements in

$$(\mathcal{S} \setminus [n]) \cup \{\{1\}, \dots, \{n\}\},$$

in other words, consider those trees such that $B_1, \dots, B_k$ are the labels of the children of the root. Each child of the root is the root of a subtree where we can use induction. Then we get a recursive relation

$$f(n) = \sum_{\{B_1,\dots,B_k\} \in \Pi([n])} f(|B_1|) f(|B_2|) \cdots f(|B_k|) g(k), \quad \text{for } n > 1,$$

where $f(1) = 1$ and $g(k)$ is the function

$$g(k) = \begin{cases} \sum_{i=1}^{dk-d-1} x^i = (x^{dk-d} - x)/(x-1) & \text{if } \ k > 1; \\ 0 & \text{if } \ k = 1; \\ 1 & \text{if } \ k = 0. \end{cases}$$

Then
$$\begin{aligned} E_g(t) &= 1 + \sum_{k=2}^{\infty} \left(\frac{x^{dk-d} - x}{x-1} \right) \frac{t^k}{k!} \\ &= 1 + \frac{1}{x^d(x-1)} \sum_{k=2}^{\infty} \frac{(x^d t)^k}{k!} - \frac{x}{x-1} \sum_{k=2}^{\infty} \frac{t^k}{k!} \\ &= 1 + \frac{1}{x^d(x-1)} (\exp(x^d t) - 1 - x^d t) - \frac{x}{x-1} (e^t - 1 - t) \\ &= 1 + t + \frac{\exp(x^d t) - 1}{x^d(x-1)} - \frac{x(e^t - 1)}{x-1}. \end{aligned}$$

Define $h(n)$ to be the right hand side of 2.2 for $n > 0$ and define $h(0) = 1$. Then $h(n)$ and $f(n)$ only differ at $n = 0, 1$. Therefore

$$E_h(t) = E_f(t) - t + 1.$$

By the compositional formula of exponential generating functions (Theorem 2.7),

$$E_f(t) - t + 1 = E_g(E_f(t)).$$

Denote $N := E_f(t)$, we need to show that N satisfies the identity in Theorem 1.3. This is indeed the case, since

$$N - t + 1 = 1 + N + \frac{\exp(x^d N) - 1}{x^d(x-1)} - \frac{x(e^N - 1)}{x-1},$$

or equivalently,

$$(1-x)x^d t + (1-x^{d+1}) = \exp(x^d N) - x^{d+1}\exp(N).$$

Thus we have proved (3) in the special case $m = 1$.

Now we prove (3) for $m > 1$, ie, the case when forests have m disjoint trees. First, notice that

$$\sum_{c(\mathcal{S})=m} \sum_{\underline{\mu}\in M_{\mathcal{S}}} x^{\|\underline{\mu}\|} = [y^m]F(n), \tag{5}$$

where $\displaystyle F(n) = \sum_{\{B_1,\dots,B_k\}\in\Pi([n])} f(|B_1|)f(|B_2|)\cdots f(|B_k|)y^k, \quad \text{for } n \geq 1$

and f is defined in (4). Indeed, fixing a partition $\{B_1,\dots,B_m\}\in\Pi([n])$, we define $\mathfrak{S}$ to be the set of nests $\mathcal{S}$ such that $c(\mathcal{S}) = m$ and the maximal elements in $\mathcal{S}\cup\{\{1\},\dots,\{n\}\}$ are $B_1,\dots,B_m$. Define a function n: (nests) $\to\mathbb{Z}$ by

$$n(\mathcal{S}) := \text{ number of leaves in the forest corresponding to } \mathcal{S}.$$

For each connected component (which is a tree) in the forest corresponding to $\mathcal{S}$ we can use identity (3) in the case $m = 1$. Then we have

$$\sum_{\mathcal{S}\in\mathfrak{S}} \sum_{\underline{\mu}\in M_{\mathcal{S}}} x^{\|\underline{\mu}\|} = \prod_{i=1}^{m}\Bigg(\sum_{\substack{c(\mathcal{S}_i)=1\\ n(\mathcal{S}_i)=|B_i|}} \sum_{\underline{\mu}\in M_{\mathcal{S}_i}} x^{\|\underline{\mu}\|}\Bigg) = f(|B_1|)f(|B_2|)\cdots f(|B_m|).$$

Equation (5) immediately follows by adding up different partitions in $\Pi([n])$.

Applying again the compositional formula for $g(k) = y^k$, we obtain

$$F(n) = \left[\frac{t^n}{n!}\right]\exp(yN).$$

Therefore

$$[y^m]F(n) = [y^m]\left[\frac{t^n}{n!}\right]\exp(yN) = \left[\frac{t^n}{n!}\right][y^m]\exp(yN) = \left[\frac{t^n}{n!}\right]\frac{N^m}{m!}.$$

This proves (3), and the theorem follows. □

3 Deligne–Beilinson cohomology of Fulton–MacPherson space

3.1 Deligne–Beilinson cohomology

Let X be a complex manifold of complex dimension d. Let Ω_X^k the sheaf of holomorphic k–form on X. The *Deligne complex of level* p is the complex of sheaves

$$\underline{\mathbb{Z}}_{\mathcal{D}}(p)\colon 0 \to \mathbb{Z} \stackrel{(2i\pi)^p}{\to} \Omega_X^0 \to \Omega_X^1 \to \Omega_X^2 \to \cdots \to \Omega_X^{p-1} \to 0$$

The hypercohomology of the complex is called the *Deligne–Beilinson cohomology* of X of level p:

$$H^*_{\mathcal{D}}(X, \mathbb{Z}(p)) := \mathbb{H}^*(X, \underline{\mathbb{Z}}_{\mathcal{D}}(p))$$

There is a multiplication of complexes

$$\nu\colon \mathbb{Z}(p)_{\mathcal{D}} \otimes \mathbb{Z}(q)_{\mathcal{D}} \to \mathbb{Z}(p+q)_{\mathcal{D}}$$

defined as

$$\nu(x \bullet y) = \begin{cases} x \cdot y, & \text{if} \quad \deg x = 0, \\ x \wedge dy, & \text{if} \quad \deg x > 0 \text{ and } \deg y = q > 0, \\ 0, & \text{otherwise.} \end{cases}$$

This gives a product structure on the Deligne–Beilinson cohomology as follows:

$$\cup\colon H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \otimes_{\mathbb{Z}} H^{k'}_{\mathcal{D}}(X, \mathbb{Z}(q)) \to H^{k+k'}_{\mathcal{D}}(X, \mathbb{Z}(p+q)). \tag{6}$$

For details, the reader is referred to Esnault and Viehweg [5].

Let X be an d–dimensional compact Kähler manifold. The Hodge filtration

$$\cdots \subseteq F^p H^k(X, \mathbb{C}) \subseteq F^{p-1} H^k(X, \mathbb{C}) \subseteq \cdots \subseteq F^0 H^k(X, \mathbb{C}) = H^k(X, \mathbb{C})$$

is defined by

$$F^p H^k(X, \mathbb{C}) = \bigoplus_{i \geq p} H^{i,k-i}(X).$$

We denote by p_X^k the natural quotient map

$$p_X^k\colon H^k(X, \mathbb{C}) \to H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C}).$$

It was proved (see Esnault and Viehweg [5, Corollary 2.4] and Voisin [22, Proposition 12.26]) that we have the following long exact sequence:

$$\begin{aligned} \cdots \to H^{k-1}(X, \mathbb{C})/F^p H^{k-1}(X, \mathbb{C}) &\to H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \\ &\to H^k(X, \mathbb{Z}) \to H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C}) \to \cdots \end{aligned}$$

Now assume that X is projective, and Y, D are the same as defined in Section 2.1. Denote by h the class of $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1)$ under the first Chern class $c_1\colon H^1(D, \mathcal{O}_D^*) \to H^2_{\mathcal{D}}(D, \mathbb{Z}(1))$, ie, $h = c_1(\mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1)) \in H^2_{\mathcal{D}}(D, \mathbb{Z}(1))$ (see [5, page 88]).

The following proposition was proved in [5, Proposition 8.5].

Proposition 3.1 [5] *The Deligne–Beilinson cohomology $H^k_{\mathcal{D}}(D, \mathbb{Z}(p))$ of the projective bundle $\pi\colon D \to Y$ is given by the isomorphism*

$$\bigoplus_{0\le j\le r-1} H_{\mathcal{D}}^{k-2j}(Y, \mathbb{Z}(p-j)) \cdot h^j \xrightarrow{\cong} H^k_{\mathcal{D}}(D, \mathbb{Z}(p)),$$

where $H_{\mathcal{D}}^{k-2j}(Y, \mathbb{Z}(p-j)) \cdot h^j = \{\alpha \cup h^j \mid \alpha \in H_{\mathcal{D}}^{k-2j}(Y, \mathbb{Z}(p-j))\}$.

Moreover, Barbieri Viale proved the following blow-up formula for Deligne–Beilinson cohomology:

Theorem 3.2 [2, Section A.3] *Let X, Y, D, $\widetilde{X}_Y$, Y be as above. Then for each p, k with $p \ge r \ge 0$, we have the isomorphism*

$$(7)\quad I_{p,k}\colon \left\{ \bigoplus_{1\le j\le r-1} H_{\mathcal{D}}^{k-2j}(Y, \mathbb{Z}(p-j)) \right\} \oplus H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \xrightarrow{\cong} H^k_{\mathcal{D}}(\widetilde{X}_Y, \mathbb{Z}(p)).$$

Remark 3.3 Barbieri Viale proved a general result, including the blow-up formula for étale cohomology, to Theorem 3.2.

Similarly, we compute the Deligne–Beilinson cohomology for Fulton–MacPherson configuration spaces.

Proof of Theorem 1.4 The proof is similar to the proof of Theorem 1.2: We use the fact that the Fulton–MacPherson configuration space $X[n]$ can be constructed by a sequence of blow-ups, and then successively apply the blow-up formula for Deligne–Beilinson cohomology (Theorem 3.2). We only point out the difference here:

For each lattice point $(\mu_{I_1}, \dots, \mu_{I_\ell}) \in \mathbb{N}^\ell$ such that $1 \le \mu_{I_i} \le (c_{I_i} - 1)\dim X - 1$, there is a copy of the Deligne–Beilinson cohomology of $X^{c(\mathcal{S})}$ in the decomposition of the Deligne–Beilinson cohomology of $X[n]$. More precisely, the summand for $\underline{\mu} \in M_{\mathcal{S}}$ in the decomposition of $H^k_{\mathcal{D}}(X[n], \mathbb{Z}(p))$ is

$$H_{\mathcal{D}}^{k-2\|\underline{\mu}\|}\big(X^{c(\mathcal{S})}, \mathbb{Z}(p-\|\underline{\mu}\|)\big).$$

Let $\mathcal{S}$ run through all the nests of $[n]$, we obtain all the direct summands of the decomposition of $H^k_{\mathcal{D}}(X[n], \mathbb{Z}(p))$, and then the theorem follows. □

Remark 3.4 By using the same method, we can compute the étale cohomology for Fulton–MacPherson configuration spaces.

Remark 3.5 The decomposition of Lawson homology (Theorem 1.2) and Deligne–Beilinson cohomology (Theorem 1.4) of the Fulton–MacPherson configuration spaces can be generalized with no difficulty to the wonderful compactifications of arrangements of subvarieties, since the latter compactifications can also be constructed by a sequence of blow-ups along smooth centers (for the definition and the blow-up construction of these compactifications, see the second author's paper [16]).

4 Comparing to homology

(1) Let X be a smooth d–dimensional complex projective variety. Then for each integer $k \geq 0$, there is an isomorphism of singular homology groups

$$H_k(X[n]) \cong \bigoplus_{1\leq m\leq n} H_k(X^m)^{\oplus [t^n/n!]N^m/m!}$$

where $N := N(x,t)$ is defined as in Theorem 1.3. This is a direct result of Theorem 1.3 and the Dold–Thom Theorem [4] which implies that $L_0H_k(V) \cong H_k(V)$ for any complex projective variety V. In particular, the integral singular homology of $X[n]$ depends only the integral singular homology of X. If there is no torsion in $H_k(X)$ for all $k \geq 0$, then there is no torsion in $H_j(X[n])$ for all $j \geq 0$. In particular, $H_k(C[n])$ has no torsion for any smooth complex projective curve C since by the Künneth formula $H_k(C^m)$ has no torsion for all integers k and m.

(2) If X is a smooth complex cellular variety, ie, X admits a filtration by closed subvarieties $\varnothing = X_{-1} \subset X_0 \subset \cdots \subset X_n = X$ such that $X_i - X_{i-1} \cong \mathbb{C}^{m_i}$ for some positive integer m_i, then

$$L_pH_k(X[n]) \cong H_k(X[n])$$

for all integers $k \geq 2p \geq 0$. This follows from Theorem 1.3 and a result of Lima-Filho [17], since the product of cellular varieties is cellular.

(3) In contrast to the singular homology where $H_k(X[n])$ is always finitely generated for any k, n and smooth projective variety X, the Lawson homology group $L_pH_k(X[n])$ can be infinitely generated even if X is a smooth projective curve. This statement follows from Theorem 1.3 and a result of the authors [11] where we have found smooth algebraic curves C such that $L_pH_{2p}(C^n)$ is not finitely generated for $1 \leq p \leq n-2$ if $n \geq 3$.

5 Examples

Example 5.1 (The Lawson homology group of $X[2]$) The morphism $\pi\colon X[2]\to X^2$ is a blow-up along the diagonal Δ_{12}. Theorem 1.3 asserts

$$L_pH_k(X[2])\cong L_pH_k(X^2)\oplus\left\{\bigoplus_{j=1}^{d-1}L_{p-j}H_{k-2j}(X)\right\}.$$

Example 5.2 (The Lawson homology group of $X[3]$) Note that $X[3]$ is the blow-up of X^3 first along small diagonal Δ_{123}, then along three disjoint strict transforms of diagonals Δ_{12}, Δ_{13} and Δ_{23}. Apply again Theorem 1.3, we have

$$\begin{aligned}L_pH_k(X[3])\cong L_pH_k(X^3)&\oplus\left\{\bigoplus_{j=1}^{d-1}\big(L_{p-j}H_{k-2j}(X^2)\big)^{\oplus 3}\right\}\\&\oplus\left\{\bigoplus_{j=1}^{2d-1}\big(L_{p-j}H_{k-2j}(X)\big)^{\oplus\min\{3j-2,6d-3j-2\}}\right\}.\end{aligned}$$

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