Infinite product decomposition of orbifold mapping spaces

HIROTAKA TAMANOI

Physicists showed that the generating function of orbifold elliptic genera of symmetric orbifolds can be written as an infinite product. We show that there exists a geometric factorization on space level behind this infinite product formula, and we do this in the much more general framework of orbifold mapping spaces, where factors in the infinite product correspond to finite connected coverings of domain spaces whose fundamental groups are not necessarily abelian. From this formula, a concept of geometric Hecke operators for functors emerges. This is a nonabelian geometric generalization of the usual Hecke operators. We show that these generalized Hecke operators indeed satisfy the identity of the usual Hecke operators for the case of 2–dimensional tori.

1 Introduction and summary of results

The elliptic genus of a Spin manifold M refers to the signature of LM; see Landweber [8] and Tamanoi [12]. The elliptic genus of a complex manifold M refers to the S^1–equivariant χ_y–characteristic of its free loop space $LM = \text{Map}(S^1, M)$; see Hirzebruch [6]. These are some of the versions of elliptic genera of M. Since LM is infinite dimensional, the above statements must be interpreted using a localization formula as in Witten [17].

Let G be a finite group. For a G–manifold M, we can consider an orbifold version of the elliptic genus. However, the free loop space $L(M/G)$ on the orbit space is not well behaved. Following Hirzebruch and Höfer [7], we define the orbifold loop space $L_{\text{orb}}(M/G)$ by

$$(1\text{-}1) \qquad L_{\text{orb}}(M/G) \stackrel{\text{def}}{=} \left(\coprod_{g \in G} L_g M \right) / G = \coprod_{(g) \in G_*} \left[L_g M / C_G(g) \right],$$

where G_* is the set of conjugacy classes in G, $C_G(g)$ is the centralizer of g in G, and $L_g M$ is the space of g–twisted loops in M given by

$$(1\text{-}2) \qquad L_g M = \{ \gamma \colon \mathbb{R} \to M \mid \gamma(t+1) = g^{-1}\gamma(t) \text{ for all } t \in \mathbb{R} \}.$$

The centralizer $C(g)$ acts on $L_g M$. Also note that if the order of g is finite and is equal to s, then each twisted loop γ in $L_g M$ is in fact a closed loop of length s. Thus, $L_g M$ also admits an action of a circle $S^1 = \mathbb{R}/s\mathbb{Z}$ of length s.

One could use the more sophisticated language of orbifolds (see for example, Moerdijk [11]), but for our purpose, the above definition suffices.

The orbifold elliptic genus of (M, G), denoted by $\mathrm{ell}_{\mathrm{orb}}(M/G)$, is defined as the S^1–equivariant χ_y–characteristic of $L_{\mathrm{orb}}(M/G)$:

$$(1\text{-}3) \qquad \mathrm{ell}_{\mathrm{orb}}(M/G) = \chi_y^{S^1}\big(L_{\mathrm{orb}}(M/G)\big) = \sum_{(g) \in G_*} \chi_y(L_g M)^{C(g)},$$

where $\chi_y(L_g M)$ is thought of as $R(C(g))$–valued S^1–equivariant χ_y–characteristic computed and made sense through a use of localization formulae. Counting the dimension of coefficient vector spaces, we have

$$(1\text{-}4) \qquad \mathrm{ell}_{\mathrm{orb}}(M/G) \in \mathbb{Z}[y, y^{-1}][\![q]\!],$$

where the powers of q are characters of S^1.

Dijkgraaf, Moore, Verlinde and Verlinde [2] essentially proved a remarkable formula for the generating function of orbifold elliptic genera of symmetric products. This was subsequently extended to symmetric orbifold case by Borisov and Libgober [1]. Here, for an integer $n \geq 0$, the n–th symmetric product of a space X is defined as $\mathrm{SP}^n(X) = X^n/\mathfrak{S}_n$, where the n–th symmetric group $\mathfrak{S}_n$ acts on X^n by permuting n factors. The DMVV and BL formula for the generating function of orbifold elliptic genera of symmetric orbifolds is given by

$$(1\text{-}5) \qquad \sum_{n \geq 0} p^n \, \mathrm{ell}_{\mathrm{orb}}\big(\mathrm{SP}^n(M/G)\big) = \prod_{\substack{n \geq 1 \\ m \geq 0 \\ k \in \mathbb{Z}}} (1 - p^n q^m y^k)^{-c(mn,k)},$$

where
$$\mathrm{ell}_{\mathrm{orb}}(M/G) = \sum_{\substack{m \geq 0 \\ k \in \mathbb{Z}}} c(m,k) q^m y^k \in \mathbb{Z}[y, y^{-1}][\![q]\!].$$

The amazing thing about this formula is that the right hand side of (1-5) is a genus 2 Siegel modular form, up to a simple multiplicative factor. The main motivation of this paper is to understand a geometric origin of this infinite product formula. In fact, we will prove such an infinite product formula on a geometric level, not merely on an algebraic level, as in (1-5).

We can describe this geometric formula in a general context. Let (M, G) be as before, and let Σ be an arbitrary connected manifold with $\Gamma = \pi_1(\Sigma)$. Instead of a loop

space, we consider a mapping space $\mathrm{Map}(\Sigma, M/G)$. As before, this space is not well behaved and the correct space to consider is the orbifold mapping space defined by

$$
\mathrm{Map}_{\mathrm{orb}}(\Sigma, M/G) \overset{\mathrm{def}}{=} \left(\coprod_{\theta \in \mathrm{Hom}(\Gamma, G)} \mathrm{Map}_{\theta}(\widetilde{\Sigma}, M) \right) \Big/ G
$$

(1-6)

$$
= \coprod_{(\theta) \in \mathrm{Hom}(\Gamma, G)/G} \left[\mathrm{Map}_{\theta}(\widetilde{\Sigma}, M)/C(\theta) \right].
$$

Here $\widetilde{\Sigma}$ is the universal cover of Σ, and $\mathrm{Map}_{\theta}(\widetilde{\Sigma}, M)$ is the space of θ–equivariant maps $\alpha\colon \widetilde{\Sigma} \to M$ such that $\alpha(p \cdot \gamma) = \theta(\gamma)^{-1} \cdot \alpha(p)$ for all $p \in \widetilde{\Sigma}$ and $\gamma \in \Gamma$. Note here that we regard the universal cover $\widetilde{\Sigma}$ as a Γ–principal bundle over Σ.

For a variable t and a space X, let $S_t(X) = \coprod_{k \geq 0} t^k \, \mathrm{SP}^k(X)$ be the total symmetric product of X. For convenience, we often write this using the summation symbol as $S_t(X) = \sum_{k \geq 0} t^k \, \mathrm{SP}^k(X)$. In this paper, the summation symbol applied to topological spaces means topological disjoint union.

Theorem A (Infinite product decomposition of orbifold mapping spaces of symmetric products) *Let M be a G–manifold and let Σ be a connected manifold. Then,*

(1-7) $\displaystyle \sum_{n \geq 0} p^n \, \mathrm{Map}_{\mathrm{orb}}\big(\Sigma, \mathrm{SP}^n(M/G)\big)$

$$
\cong \prod_{[\Sigma' \to \Sigma]_{\mathrm{conn.}}} S_{p^{|\Sigma'/\Sigma|}}\big(\mathrm{Map}_{\mathrm{orb}}(\Sigma', M/G)/\mathcal{D}(\Sigma'/\Sigma)\big).
$$

Here the infinite product is taken over all the isomorphism classes of finite sheeted connected covering spaces Σ' of Σ, and $\mathcal{D}(\Sigma'/\Sigma)$ is the group of all deck transformations of the covering space $\Sigma' \to \Sigma$ (which is not necessarily Galois). The number of sheets of this covering is denoted by $|\Sigma'/\Sigma|$.

We will explain the details of the action of $\mathcal{D}(\Sigma'/\Sigma)$ on $\mathrm{Map}_{\mathrm{orb}}(\Sigma', M/G)$ in Section 2.

When $\Sigma = S^1$, the above formula reduces to

(1-8) $\displaystyle \sum_{n \geq 0} p^n \, L_{\mathrm{orb}}\big(\mathrm{SP}^n(M/G)\big) \cong \prod_{r \geq 1} S_{p^r}\big(L_{\mathrm{orb}}^{(r)}(M/G)/\mathbb{Z}_r\big),$

where $L_{\mathrm{orb}}^{(r)}(M/G)$ is the space of orbifold loops of length r. This is the geometric version of the formula (1-5). This formula itself is relatively easy to prove. See Wang and Zhou [16].

The above formula (1-8) for orbifold loop space is an "abelian" case since $\pi_1(S^1) \cong \mathbb{Z}$. The formula in Theorem A is, in a sense, a nonabelian generalization of this orbifold

loop space case. The most interesting case seems to be the one in which Σ is a 2–dimensional surface (regarding it as a world-sheet of a moving string). Here, the genus of the surface can be arbitrary. In physics literature, elliptic genus itself is computed as a path integral over mapping spaces from tori [2].

Restricting the global decomposition formula (1-7) to the subspace of constant orbifold maps and considering their numerical invariants, we recover our previous results in [13; 14]. See Section 3 for a description of these results. We remark that we can apply (generalized) homology and cohomology functors to (1-7) to obtain infinite product decomposition formulas of these homology and cohomology theories.

Another surprising formula discovered by physicists [2] is its connection to Hecke operators. They showed that the right hand side of formula (1-5) can be written in terms of Hecke operators in a very nice way:

$$(1\text{-}9) \qquad \sum_{n \geq 0} p^n \, \mathrm{ell}_{\mathrm{orb}}\big(\mathrm{SP}^n(M/G)\big) = \exp\bigg(\sum_{r \geq 1} p^r \, T(r)\big[\mathrm{ell}_{\mathrm{orb}}(M/G)\big]\bigg),$$

where $T(r)$ is the r–th Hecke operator acting on weight 0 Jacobi forms:

$$(1\text{-}10) \qquad T(r)\bigg[\sum_{\substack{m \geq 0 \\ k \in \mathbb{Z}}} c(m, k) q^m y^k\bigg] = \sum_{ad = r} \frac{1}{a} \sum_{\substack{m \geq 0 \\ k \in \mathbb{Z}}} c(md, k) q^{am} y^{ak}.$$

Is there a corresponding Hecke operator in our geometric context? Such a Hecke operator must assign a certain space to a given space. Our geometric decomposition formula (1-7) suggests what geometric Hecke operators should be. For each positive integer r, we expect the r–th *geometric Hecke operator* $\mathbb{T}(r)$ would act on a space of the form $\mathrm{Map}_{\mathrm{orb}}(\Sigma, M/G)$ and produce a space involving all the connected r–sheeted covering spaces of Σ, as follows.

$$(1\text{-}11) \qquad \mathbb{T}(r)\big[\mathrm{Map}_{\mathrm{orb}}(\Sigma, M/G)\big] \overset{\mathrm{def}}{=} \coprod_{\substack{[\Sigma' \to \Sigma] \\ |\Sigma'/\Sigma| = r}} \mathrm{Map}_{\mathrm{orb}}(\Sigma', M/G)/\mathcal{D}(\Sigma'/\Sigma).$$

The usual Hecke operators use covering spaces of the torus (see Lang [9]), and in [2], they explain the above result (1-9) from this point of view. Our formula (1-11) uses covering spaces of Σ whose fundamental group is not necessarily abelian. Thus, in a sense, our Hecke operator can be thought of as a nonabelian generalization of the usual Hecke operators.

A general discussion of geometric Hecke operators in the framework of functors is more convenient and will be given in Section 4. Let $\mathcal{F}$ be a functor from the category $\mathcal{C}$ of topological spaces and continuous maps to itself. For example, for

a G–manifold M, let $\mathcal{F}_{(M,G)}$ be a contravariant functor from $\mathcal{C}$ to itself given by $\mathcal{F}_{(M,G)}(\Sigma) = \mathrm{Map}_{\mathrm{orb}}(\Sigma, M/G)$. Then, $\mathbb{T}(n)$ acts on the functor $\mathcal{F}$ by the following formula for a connected space Σ:

$$(1\text{-}12) \qquad \big(\mathbb{T}(n)\mathcal{F}\big)(\Sigma) \overset{\mathrm{def}}{=} \coprod_{\substack{[\Sigma'\to\Sigma]_{\mathrm{conn.}} \\ |\Sigma'/\Sigma|=n}} \mathcal{F}(\Sigma')/\mathcal{D}(\Sigma'/\Sigma),$$

where the disjoint union runs over all isomorphism classes of connected n–sheeted covering spaces of Σ. When Σ is not connected, we apply the above construction for each connected component of Σ. In terms of geometric Hecke operators, formula (1-7) can be simply rewritten as

$$(1\text{-}7') \qquad \sum_{n\geq 0} p^n \, \mathrm{Map}_{\mathrm{orb}}\big(\Sigma, \mathrm{SP}^n(M/G)\big) \cong \prod_{r\geq 1} S_{p^r}\Big[\big(\mathbb{T}(r)\mathcal{F}_{(M,G)}\big)(\Sigma)\Big].$$

It is very suggestive to compare this formula with (1-9). If we regard the n–th symmetric product $\mathrm{SP}^n(X)$ as $X^n/n!$, since $\mathfrak{S}_n$ has $n!$ elements, then we can regard $S_p(X)$ as $\exp(pX)$. From this point of view, the analogy between (1-9) and (1-7′) is reasonably precise. However, see also the remark after (4-2).

The name geometric Hecke operator seems appropriate since these operators do satisfy the usual identity when Σ is a genus 1 Riemann surface.

Theorem B (Hecke identity for geometric Hecke operators) *Let T be a torus of dimension 2. Let $\mathcal{F}$ be a functor from the category $\mathcal{C}$ of topological spaces to itself. Then the geometric Hecke operators $\mathbb{T}(n)$, $n \geq 1$, satisfy*

$$(1\text{-}13) \qquad \big((\mathbb{T}(m) \circ \mathbb{T}(n))\mathcal{F}\big)(T) = \sum_{d\,|\,(m,n)} d \cdot \left(\left(\mathbb{T}\left(\frac{mn}{d^2}\right) \circ \mathbb{R}(d)\right)\mathcal{F}\right)(T),$$

where the operator $\mathbb{R}(d)$ on the functor $\mathcal{F}$ is given by

$$(1\text{-}14) \qquad (\mathbb{R}(d)\mathcal{F})(T) = \mathcal{F}\big(R(d)T\big)\big/\mathcal{D}\big(R(d)T/T\big),$$

in which $R(d)T = \tilde{T}/(d \cdot L)$ if $T = \tilde{T}/L$ for some lattice $L \subset \tilde{T} \cong \mathbb{R}^2$.

Thus, $R(d)T$ is a d^2–sheeted covering space of T. The coefficient d in the right hand side of (1-13) means a disjoint topological union of d copies.

Note that (1-13) can be restated in a more familiar form as follows:

$$(1\text{-}13')\qquad
\begin{array}{ll}
\mathbb{T}(m) \circ \mathbb{T}(n) = \mathbb{T}(mn), & \text{if } (m,n) = 1, \\[4pt]
\mathbb{T}(p^r) \circ \mathbb{T}(p) = \mathbb{T}(p^{r+1}) + p \cdot \mathbb{T}(p^{r-1}) \circ \mathbb{R}(p), & \text{if } p \text{ is prime.}
\end{array}$$

As is well known in the theory of modular forms, these identities are equivalent to an Euler product decomposition of the Dirichlet series with the above Hecke operator coefficients. See (4-14).

It would be of interest to investigate relations among the $\mathbb{T}(n)$'s when Σ is a higher genus Riemann surface or a higher dimensional torus whose fundamental group is free abelian.

For a generalization of orbifold elliptic genus to the setting of generalized cohomology theory, see the paper [4] by Ganter.

The organization of this paper is as follows. In Section 2, we prove our main geometric decomposition formula in Theorem A. In Section 3, we specialize our infinite dimensional geometric formula to the finite dimensional subspace of constant orbifold maps, and we deduce various formulae of generating functions of orbifold invariants. In Section 4, after discussing some general properties of geometric Hecke operators on functors, we prove the Hecke identity (1-13).

The main result of this paper, Theorem A, was first announced at a workshop at Banff International Research Station in June 2003.

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2   Infinite product decomposition of orbifold mapping spaces

First, we discuss some general facts of orbifold mapping spaces. For a homomorphism $\theta\colon \Gamma \to G$ and a θ–equivariant map $\alpha\colon \widetilde{\Sigma} \to M$, let $\bar{\alpha}\colon \Sigma \to M/G$ be the induced map on quotient spaces. Thus we have a canonical map $\mathrm{Map}_\theta(\widetilde{\Sigma}, M) \to \mathrm{Map}(\Sigma, M/G)$. Let $C_G(\theta)$ be the centralizer of the image of θ in G. Note that inverse images of this map are $C_G(\theta)$ spaces. The action of $g \in G$ on M has the effect

$$g \cdot \colon \mathrm{Map}_\theta(\widetilde{\Sigma}, M) \longrightarrow \mathrm{Map}_{g \cdot \theta \cdot g^{-1}}(\widetilde{\Sigma}, M),$$

and for every $\alpha \in \mathrm{Map}_\theta(\widetilde{\Sigma}, M)$, we have $\bar{\alpha} = \overline{g \cdot \alpha}$ in $\mathrm{Map}(\Sigma, M/G)$. Thus, we have a canonical map

$$(2\text{-}1)\quad \mathrm{Map}_{\mathrm{orb}}(\Sigma, M/G) \overset{\mathrm{def}}{=} \coprod_{(\theta) \in \mathrm{Hom}(\Gamma, G)/G} \mathrm{Map}_\theta(\widetilde{\Sigma}, M)/C_G(\theta) \longrightarrow \mathrm{Map}(\Sigma, M/G).$$

This map is in general not surjective nor injective.

We consider a necessary condition for a map $f\colon \Sigma \to M/G$ to have a lift to a θ–equivariant map $\tilde{f}\colon \widetilde{\Sigma} \to M$ for some θ. Let η be an arbitrary contractible loop in Σ.

Since $\widetilde{\Sigma} \to \Sigma$ is a covering, η always lifts to a contractible loop $\widetilde{\eta}$ in $\widetilde{\Sigma}$, and hence $\widetilde{f}(\widetilde{\eta})$ is also contractible. Thus, for the existence of a lift $\widetilde{f}$ of a given map f, it is necessary that for every contractible loop η in Σ, $f(\eta) \subset M/G$ lifts to a contractible loop in M.

Next, we discuss a functorial property of orbifold mapping spaces.

Proposition 2.1 (i) *Let M be a G–manifold. Any map $f\colon \Sigma_1 \to \Sigma_2$ between connected manifolds induces a well-defined map*

$$(2\text{-}2) \qquad\qquad f^*\colon \mathrm{Map}_{\mathrm{orb}}(\Sigma_2, M/G) \longrightarrow \mathrm{Map}_{\mathrm{orb}}(\Sigma_1, M/G).$$

(ii) *For two maps $f_1\colon \Sigma_1 \to \Sigma_2$ and $f_2\colon \Sigma_2 \to \Sigma_3$, we have $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$.*

Proof Let Γ_i be the group $\mathcal{D}(\widetilde{\Sigma}_i/\Sigma_i)$ of all deck transformations for the universal cover $\widetilde{\Sigma}_i \to \Sigma_i$ for $i = 1, 2$. Since an isomorphism $\mathcal{D}(\widetilde{\Sigma}_i/\Sigma_i) \cong \pi_1(\Sigma_i)$ depends on the choice of a base point in $\widetilde{\Sigma}_i$, it is better to regard Γ_i as the group of deck transformations rather than as the fundamental group of Σ_i. We choose a lift $\widetilde{f}\colon \widetilde{\Sigma}_1 \to \widetilde{\Sigma}_2$ of f. Then $\widetilde{f}$ induces a homomorphism $\widetilde{f}_*\colon \Gamma_1 \to \Gamma_2$ such that $\widetilde{f}(p \cdot \gamma_1) = \widetilde{f}(p) \cdot \widetilde{f}_*(\gamma_1)$ for all $p \in \widetilde{\Sigma}_1$ and $\gamma_1 \in \Gamma_1$. For a map $\alpha \in \mathrm{Map}_\theta(\widetilde{\Sigma}_2, M)$ with $\theta \in \mathrm{Hom}(\Gamma_2, G)$, we have $\alpha \circ \widetilde{f} \in \mathrm{Map}_{\theta \circ \widetilde{f}_*}(\widetilde{\Sigma}_1, M)$. Hence the composition with $\widetilde{f}$ gives an induced map

$$(2\text{-}3) \qquad \widetilde{f}^*\colon \coprod_{\theta \in \mathrm{Hom}(\Gamma_2, G)} \mathrm{Map}_\theta(\widetilde{\Sigma}_2, M) \to \coprod_{\rho \in \mathrm{Hom}(\Gamma_1, G)} \mathrm{Map}_\rho(\widetilde{\Sigma}_1, M).$$

Obviously, this map commutes with the G–action on M. Hence by quotienting by G, we have a map

$$(2\text{-}4) \qquad\qquad \widetilde{f}^*\colon \mathrm{Map}_{\mathrm{orb}}(\Sigma_2, M/G) \to \mathrm{Map}_{\mathrm{orb}}(\Sigma_1, M/G).$$

We have to verify that this map is independent of the chosen lift $\widetilde{f}$. Let $\widetilde{f}'\colon \widetilde{\Sigma}_1 \to \widetilde{\Sigma}_2$ be another lift of f. By examining the image of one point and using the uniqueness of lifts, we must have that $\widetilde{f}' = \widetilde{f} \cdot \gamma_2$, globally on $\widetilde{\Sigma}_1$, for some uniquely determined $\gamma_2 \in \Gamma_2$. Then, $(\alpha \circ \widetilde{f}')(p_1) = \alpha(\widetilde{f}(p_1) \cdot \gamma_2) = \theta(\gamma_2)^{-1} \cdot (\alpha \circ \widetilde{f})(p_1)$ for all $p_1 \in \widetilde{\Sigma}_1$. Note that $\theta(\gamma_2) \in G$. Thus for all possible choices of lifts $\widetilde{f}$, the collection $\{\alpha \circ \widetilde{f}\}$ is contained in a single G–orbit in $\coprod_{\rho \in \mathrm{Hom}(\Gamma_1, G)} \mathrm{Map}_\rho(\widetilde{\Sigma}_1, M)$. Thus difference of $\widetilde{f}^*$ and $(\widetilde{f}')^*$ in (2-3) disappear after dividing by G, and the map (2-4) is independent of the choice of lifts $\widetilde{f}$. Hence we may simply call it f^* as in (2-2).

The proof of the formula for the induced map of a composition is routine. $\qquad\square$

As an immediate consequence, we have:

Corollary 2.2 *Let $\Sigma' \to \Sigma$ be a connected covering space. Then the group $\mathcal{D}(\Sigma'/\Sigma)$ of all deck transformations acts on $\mathrm{Map}_{\mathrm{orb}}(\Sigma', M/G)$.*

For later use, we give details of this action. As before, let $\mathcal{D}(\widetilde{\Sigma}/\Sigma) = \Gamma$ and $\Sigma' = \widetilde{\Sigma}/H$ for some $H \subset \Gamma$. Then $\mathcal{D}(\Sigma'/\Sigma) \cong N_\Gamma(H)/H$. For $f \in \mathrm{Map}_\rho(\widetilde{\Sigma}', M)$, $u \in N_\Gamma(H)$, and $g \in G$, the action of u, g on f is given by

$$(2\text{-}5) \qquad (u \cdot f)(p) = f(pu), \qquad (g \cdot f)(p) = g \cdot f(p), \quad p \in \widetilde{\Sigma}'.$$

These actions commute, but they do not preserve $\rho \in \mathrm{Hom}(H, G)$. How ρ transforms under these actions can be easily computed and we have the following commutative diagram:

$$(2\text{-}6) \quad
\begin{array}{ccc}
\mathrm{Map}_\rho(\widetilde{\Sigma}', M) & \xrightarrow[\cong]{u \cdot} & \mathrm{Map}_{\rho^{u^{-1}}}(\widetilde{\Sigma}', M) \\
{\scriptstyle g \cdot} \downarrow {\scriptstyle \cong} & & {\scriptstyle g \cdot} \downarrow {\scriptstyle \cong} \\
\mathrm{Map}_{g \cdot \rho \cdot g^{-1}}(\widetilde{\Sigma}', M) & \xrightarrow[\cong]{u \cdot} & \mathrm{Map}_{g \cdot \rho^{u^{-1}} \cdot g^{-1}}(\widetilde{\Sigma}', M),
\end{array}$$

where $\rho^{u^{-1}}(h) = \rho(u^{-1}hu)$ for all $h \in H$. Since $C_G(\rho) = C_G(\rho^{u^{-1}})$, commutativity of this diagram also implies that for $u \in N_\Gamma(H)$,

$$(2\text{-}7) \qquad u \cdot : \mathrm{Map}_\rho(\widetilde{\Sigma}', M) \xrightarrow{\cong} \mathrm{Map}_{\rho^{u^{-1}}}(\widetilde{\Sigma}', M), \quad C_G(\rho)\text{--equivariant.}$$

A global statement is the following for $u \in N_\Gamma(H)$:

$$(2\text{-}8) \qquad u \cdot : \coprod_{\rho \in \mathrm{Hom}(H,G)} \mathrm{Map}_\rho(\widetilde{\Sigma}', M) \xrightarrow{\cong} \coprod_{\rho \in \mathrm{Hom}(H,G)} \mathrm{Map}_\rho(\widetilde{\Sigma}', M), \quad G\text{--equivariant.}$$

In other words, the group $N_\Gamma(H) \times G$ acts on $\coprod_\rho \mathrm{Map}_\rho(\widetilde{\Sigma}', M)$. Also note that the same group $N_\Gamma(H) \times G$ acts on the set $\mathrm{Hom}(H, G)$ by $[(u, g) \cdot \rho](h) = g \cdot \rho^{u^{-1}}(h) \cdot g^{-1}$ for $h \in H$. The effect of changing $u \in N_\Gamma(H)$ by $h \in H$ can be computed as

$$(2\text{-}9) \quad
\begin{aligned}
\rho^{(uh)^{-1}}(\,\cdot\,) &= \rho(h)^{-1} \rho^{u^{-1}}(\,\cdot\,) \rho(h), \\
\rho^{(hu)^{-1}}(\,\cdot\,) &= \rho^{u^{-1}}(h)^{-1} \rho^{u^{-1}}(\,\cdot\,) \rho^{u^{-1}}(h).
\end{aligned}$$

This shows that modification of u by elements in H has the same effect as the conjugation action by elements in G. Hence the map induced from (2-8) on G–orbits is well defined for $\bar{u} \in N_\Gamma(H)/H$, and we have

$$(2\text{-}10) \qquad \bar{u} \cdot : \mathrm{Map}_{\mathrm{orb}}(\Sigma', M/G) \xrightarrow{\cong} \mathrm{Map}_{\mathrm{orb}}(\Sigma', M/G).$$

This is the action in Corollary 2.2.

Since the action of $\mathcal{D}(\Sigma'/\Sigma)$ commutes with the projection map $\pi\colon \Sigma' \to \Sigma$, the action of $\mathcal{D}(\Sigma'/\Sigma)$ on $\mathrm{Map}_{\mathrm{orb}}(\Sigma', M/G)$ commutes with the induced map π^*. In particular, the image of π^* is in the $\mathcal{D}(\Sigma'/\Sigma)$–fixed point subset:

$$(2\text{-}11) \qquad \mathrm{Map}_{\mathrm{orb}}(\Sigma, M/G) \xrightarrow{\pi^*} \mathrm{Map}_{\mathrm{orb}}(\Sigma', M/G)^{\mathcal{D}(\Sigma'/\Sigma)}.$$

We will need an identity on nested equivariant mapping spaces. Let $P \to Z$ be a left Γ–equivariant right G–principal bundle over a left Γ–space Z, where the left Γ–action and the right G–action on P commute. We simply call such a bundle a Γ–G bundle [10]. We studies this concept in detail in Section 3 of [14], where the classification theorem of such bundles is discussed. Note that $\mathrm{Map}_G(P, M)$ is a left Γ–space when P is a Γ–G bundle.

Proposition 2.3 *With notation as above, we have*

$$(2\text{-}12) \qquad \mathrm{Map}_\Gamma\big(\widetilde{\Sigma}, \mathrm{Map}_G(P, M)\big) = \mathrm{Map}_G\big(\widetilde{\Sigma} \underset{\Gamma}{\times} P, M\big).$$

Proof Without equivariance, this identity is obvious. So all we have to check is that the canonical correspondence preserves the correct equivariance property.

Let $f\colon \widetilde{\Sigma} \to \mathrm{Map}_G(P, M)$, and let $u \in \widetilde{\Sigma}$. The Γ–equivariance of f and G–equivariance of $f(u)$ means $f(u\gamma) = \gamma^{-1} \cdot f(u) = f(u) \circ \gamma$ and $f(u)(pg) = g^{-1} f(u)(p)$ for all $\gamma \in \Gamma$, $g \in G$, $p \in P$. Let the canonically corresponding map $\widehat{f}\colon \widetilde{\Sigma} \times P \to M$ be defined by $\widehat{f}(u, p) = f(u)(p)$. The Γ–equivariance of f implies that $\widehat{f}(u\gamma, p) = \widehat{f}(u, \gamma \cdot p)$ for all u, γ, p. Hence $\widehat{f}$ factors through $\widetilde{\Sigma} \times_\Gamma P$ whose elements we denote by $[u, p]$. Using G–equivariance of f, we have $\widehat{f}([u, p]g) = \widehat{f}([u, pg]) = f(u)(pg) = g^{-1} \cdot f(u)(p) = g^{-1}\widehat{f}([u, p])$. Thus, $\widehat{f}$ is G–equivariant.

The obvious inverse correspondence can be similarly checked to behave correctly with respect to equivariance. $\qquad\square$

We examine the left hand side of the formula (1-7). For a positive integer n, let $\mathbf{n} = \{1, 2, \dots, n\}$. Then the wreath product $G_n = G \wr \mathfrak{S}_n$ is defined by

$$(2\text{-}13) \qquad G_n = G \wr \mathfrak{S}_n = \mathrm{Map}(\mathbf{n}, G) \rtimes \mathfrak{S}_n.$$

When M is a G–manifold, the wreath product G_n naturally acts on the Cartesian product M^n, and its quotient space $M^n/G_n = \mathrm{SP}^n(M/G)$ is the n–th symmetric

orbifold of M/G. For detailed information on wreath product, see Section 3 of [14]. To understand (1-7), first we note that

$$(2\text{-}14) \qquad \mathrm{Map}_{\mathrm{orb}}\big(\Sigma, \mathrm{SP}^n(M/G)\big) = \coprod_{(\theta) \in \mathrm{Hom}(\Gamma, G_n)/G_n} \big[\mathrm{Map}_\theta(\widetilde{\Sigma}, M^n)/C_{G_n}(\theta)\big].$$

Let $\mathbf{n} \times G \to \mathbf{n}$ be the trivial G–principal bundle over an n–element set $\mathbf{n}$. Since $\mathrm{Aut}_G(\mathbf{n} \times G) \cong G_n$ [14, Lemma 3-3], the space $\mathrm{Map}_G(\mathbf{n} \times G, M)$ of G–equivariant maps has the structure of left G_n space and we have a G_n–equivariant homeomorphism

$$(2\text{-}15) \qquad\qquad\qquad M^n \cong \mathrm{Map}_G(\mathbf{n} \times G, M).$$

For a given homomorphism $\theta\colon \Gamma \to G_n$, both of the above spaces can be thought of as Γ–spaces. Especially, the trivial G–bundle $\mathbf{n} \times G \to \mathbf{n}$ acquires the structure of a Γ–equivariant G–principal bundle, or simply a Γ–G bundle, via θ. We denote this by $(\mathbf{n} \times G)_\theta$. Now (2-15) and Proposition 2.3 imply that

$$(2\text{-}16) \qquad \begin{aligned} \mathrm{Map}_\theta(\widetilde{\Sigma}, M^n) &\cong \mathrm{Map}_\Gamma\big(\widetilde{\Sigma}, \mathrm{Map}_G((\mathbf{n} \times G)_\theta, M)\big) \\ &= \mathrm{Map}_G\big(\widetilde{\Sigma} \times_\Gamma (\mathbf{n} \times G)_\theta, M\big). \end{aligned}$$

A Γ–G bundle $P \to Z$ is called irreducible if Z is a transitive Γ–set. In this case, $\Gamma \times G$ acts transitively on P. In Section 3 of [14], we classified all the isomorphism classes of irreducible Γ–G bundles. We showed that any irreducible Γ–G bundle must be of the form $P_{H,\rho} = \Gamma \times_\rho G \to \Gamma/H$ for some subgroup $H \subset \Gamma$ and a homomorphism $\rho\colon H \to G$. We also showed that two irreducible Γ–G bundles corresponding to (H_1, ρ_1) and (H_2, ρ_2) are isomorphic as Γ–G bundles if and only if (i) the subgroups H_1 and H_2 are conjugate in Γ, and (ii) when $H_1 = H_2 = H$, we must have $[\rho_1] = [\rho_2] \in \mathrm{Hom}(H, G)/(N_\Gamma(H) \times G)$ [14, Theorem E], where $N_\Gamma(H)$ and G act on $\mathrm{Hom}(H, G)$ by conjugating H and G, respectively.

From now on, an element in $\mathrm{Hom}(H, G)/(N_\Gamma(H) \times G)$ is denoted with a square bracket as in $[\rho]$, and an element in $\mathrm{Hom}(H, G)/G$ is denoted by a round bracket as in (ρ), to distinguish these two kinds of conjugacy classes.

Let $r_\theta(H, \rho)$ be the number of irreducible Γ–G bundles isomorphic to $P_{H,\rho} \to \Gamma/H$ in the irreducible decomposition of $(\mathbf{n} \times G)_\theta \to \mathbf{n}$. Thus,

$$(2\text{-}17) \qquad [(\mathbf{n} \times G)_\theta \to \mathbf{n}] \cong \coprod_{[H]} \coprod_{[\rho]} \coprod^{r_\theta(H,\rho)} [P_{H,\rho} \to \Gamma/H].$$

Here $[H]$ runs over all the conjugacy classes of finite index subgroups of Γ, and for each H, $[\rho]$ runs over the set $\mathrm{Hom}(H, G)/(N_\Gamma(H) \times G)$. By examining the

decomposition of the base space **n** into transitive Γ–sets, we have

(2-18) $$\sum_{[H],[\rho]} r_\theta(H,\rho)|\Gamma/H| = n.$$

Let $\mathbb{P}_{H,\rho} = \widetilde{\Sigma} \times_\Gamma P_{H,\rho}$ and $\Sigma_H = \widetilde{\Sigma} \times_\Gamma (\Gamma/H) = \widetilde{\Sigma}/H$. Then $\mathbb{P}_{H,\rho}$ is a G–bundle over a covering space Σ_H of Σ. Note that in $\mathbb{P}_{H,\rho} \to \Sigma_H \to \Sigma$, for each point in Σ, fibres of these bundles give $P_{H,\rho} \to \Gamma/H$. The above decomposition now implies

(2-19) $$\widetilde{\Sigma} \times_\Gamma [(\mathbf{n} \times G)_\theta \to \mathbf{n}] \cong \coprod_{[H]} \coprod_{[\rho]} \coprod^{r_\theta(H,\rho)} [\mathbb{P}_{H,\rho} \to \Sigma_H].$$

This isomorphism allows us to rewrite (2-16) as

(2-20)
$$\operatorname{Map}_\theta(\widetilde{\Sigma}, M^n) \cong \prod_{[H]} \prod_{[\rho]} \prod^{r_\theta(H,\rho)} \operatorname{Map}_G(\mathbb{P}_{H,\rho}, M)$$
$$\cong \prod_{[H]} \prod_{[\rho]} \prod^{r_\theta(H,\rho)} \operatorname{Map}_\rho(\widetilde{\Sigma}_H, M).$$

The last isomorphism is because $\mathbb{P}_{H,\rho} = \widetilde{\Sigma}_H \times_\rho G$. This gives multiplicative decomposition of each disjoint summand of the right hand side of (2-14). Next, we need to understand the centralizer $C_{G_n}(\theta)$ of the image of the homomorphism $\theta \colon \Gamma \to G_n$ in G_n. One of the main results of [14] is the description of the structure of the centralizer $C_{G_n}(\theta)$. It says that

(2-21) $$C_{G_n}(\theta) \cong \prod_{[H]} \prod_{[\rho]} \left[\operatorname{Aut}_{\Gamma-G}(P_{H,\rho}) \wr \mathfrak{S}_{r_\theta(H,\rho)} \right],$$

where $\operatorname{Aut}_{\Gamma-G}(P_{H,\rho})$ is the group of Γ–equivariant G–principal bundle automorphisms of $P_{H,\rho} \to \Gamma/H$. In terms of the G–bundle $\mathbb{P}_{H,\rho} \to \Sigma_H$ over a covering space, $\operatorname{Aut}_{\Gamma-G}(P_{H,\rho})$ is isomorphic to the group $\operatorname{Aut}_G(\mathbb{P}_{H,\rho})_{\Sigma_H/\Sigma}$ of G–bundle isomorphisms of $\mathbb{P}_{H,\rho}$ whose induced map on Σ_H is a deck transformation of $\Sigma_H \to \Sigma$ [14, Proposition 7-3].

Next we describe the structure of $\operatorname{Aut}_{\Gamma-G}(P_{H,\rho})$. We recall that the group $N_\Gamma(H) \times G$ acts on the set $\operatorname{Hom}(H,G)$ by $(u,g) \cdot \rho = g \cdot \rho^{u^{-1}} \cdot g^{-1}$ for $u \in N_\Gamma(H)$, $g \in G$ and $\rho \in \operatorname{Hom}(H,G)$. Let T_ρ be the isotropy subgroup of this action at ρ:

(2-22) $$T_\rho = \{(u,g) \in N_\Gamma(H) \times G \mid g \cdot \rho^{u^{-1}}(h) \cdot g^{-1} = \rho(h) \text{ for all } h \in H\}.$$

This group T_ρ contains a subgroup $H_\rho = \{(h, \rho(h)) \in T_\rho \mid h \in H\} \cong H$. Then Theorem 4-4 in [14] shows that H_ρ is a normal subgroup of T_ρ and we have the

following exact sequence:

$$(2\text{-}23) \qquad\qquad 1 \to H_\rho \to T_\rho \to \mathrm{Aut}_{\Gamma-G}(P_{H,\rho}) \to 1.$$

Now we are ready to prove Theorem A.

Proof of Theorem A   Using (2-14), (2-18), (2-20), (2-21), we can rewrite the left hand side of (1-8) as

$$\sum_{n\geq 0} p^n \, \mathrm{Map}_{\mathrm{orb}}\big(\Sigma, \mathrm{SP}^n(M/G)\big)$$
$$= \sum_{n\geq 0}\sum_{[\theta]}\prod_{[H]}\prod_{[\rho]} p^{r_\theta(H,\rho)|\Gamma/H|}\left[\left(\prod^{r_\theta(H,\rho)}\mathrm{Map}_\rho(\widetilde{\Sigma}_H, M)\right)\Big/\big(\mathrm{Aut}_{\Gamma-G}(P_{H,\rho})\wr\mathfrak{S}_{r_\theta(H,\rho)}\big)\right]$$

(here $\mathrm{Aut}_{\Gamma-G}(P_{H,\rho}) \cong \mathrm{Aut}_G(\mathbb{P}_{H,\rho})_{\Sigma_H/\Sigma}$ acts on $\mathrm{Map}_\rho(\widetilde{\Sigma}_H, M)\cong\mathrm{Map}_G(\mathbb{P}_{H,\rho}, M)$ by the obvious action)

$$= \sum_{n\geq 0}\sum_{[\theta]}\prod_{[H]}\prod_{[\rho]} p^{r_\theta(H,\rho)|\Gamma/H|}\,\mathrm{SP}^{r_\theta(H,\rho)}\big(\mathrm{Map}_\rho(\widetilde{\Sigma}_H, M)/\mathrm{Aut}_{\Gamma-G}(P_{H,\rho})\big)$$
$$= \prod_{[H]}\prod_{[\rho]}\left[\sum_{r\geq 0} p^{r|\Gamma/H|}\,\mathrm{SP}^r\big(\mathrm{Map}_\rho(\widetilde{\Sigma}_H, M)/\mathrm{Aut}_{\Gamma-G}(P_{H,\rho})\big)\right]$$
$$= \prod_{[H]}\prod_{[\rho]} S_{p^{|\Gamma/H|}}\big(\mathrm{Map}_\rho(\widetilde{\Sigma}_H, M)/\mathrm{Aut}_{\Gamma-G}(P_{H,\rho})\big)$$
$$= \prod_{[H]} S_{p^{|\Gamma/H|}}\left[\coprod_{[\rho]}\mathrm{Map}_\rho(\widetilde{\Sigma}_H, M)/\mathrm{Aut}_{\Gamma-G}(P_{H,\rho})\right].$$

Here in the above formulae, $[\rho] \in \mathrm{Hom}(H, G)/(N_\Gamma(H)\times G)$. On the other hand, since $\mathcal{D}(\Sigma_H/\Sigma) \cong N_\Gamma(H)/H$, we have

$$\mathrm{Map}_{\mathrm{orb}}(\Sigma_H, M/G)/\mathcal{D}(\Sigma_H/\Sigma) = \left[\left(\coprod_{\rho\in\mathrm{Hom}(H,G)}\mathrm{Map}_\rho(\widetilde{\Sigma}_H, M)\right)\Big/G\right]\Big/(N_\Gamma(H)/H)$$
$$= \left(\coprod_{\rho\in\mathrm{Hom}(H,G)}\mathrm{Map}_\rho(\widetilde{\Sigma}_H, M)\right)\Big/(N_\Gamma(H)\times G).$$

Here we recall that the action of G and $N_\Gamma(H)$ commutes, and the action of $H \subset N_\Gamma(H)$ can be absorbed into the action of G. See (2-5), (2-6), (2-8) and (2-9) for details on this. In particular, the action of $(u, g) \in N_\Gamma(H) \times G$ is such that

$$(u, g)\colon \mathrm{Map}_\rho(\widetilde{\Sigma}_H, M) \xrightarrow{\cong} \mathrm{Map}_{g\rho^{u^{-1}}g^{-1}}(\widetilde{\Sigma}_H, M).$$

Since T_ρ in (2-22) is exactly the subgroup which preserves $\rho \in \mathrm{Hom}(H, G)$ under $(N_\Gamma(H) \times G)$–action, in the above identity, we get

$$\mathrm{Map}_{\mathrm{orb}}(\Sigma_H, M/G)/\mathcal{D}(\Sigma_H/\Sigma) = \coprod_{[\rho]} \left(\mathrm{Map}_\rho(\widetilde{\Sigma}_H, M)/T_\rho\right),$$

where $[\rho]$ runs over the orbit set $\mathrm{Hom}(H, G)/(N_\Gamma(H) \times G)$. Next observe that the subgroup H_ρ of T_ρ acts trivially on $\mathrm{Map}_\rho(\widetilde{\Sigma}_H, M)$. To see this, let $(h, \rho(h)) \in H_\rho$ for $h \in H$, and $f \in \mathrm{Map}_\rho(\widetilde{\Sigma}_H, M)$. Then, for any $p \in \widetilde{\Sigma}_H$, we have

$$\left[(h, \rho(h))f\right](p) = \rho(h) \cdot (hf)(p) = \rho(h)f(ph) = \rho(h)\rho(h)^{-1}f(p) = f(p).$$

Thus, H_ρ acts trivially on $\mathrm{Map}_\rho(\widetilde{\Sigma}_H, M)$. Hence quotienting by T_ρ in the above formula can be replaced by quotienting by $T_\rho/H_\rho \cong \mathrm{Aut}_{\Gamma-G}(P_{H,\rho})$. Thus, collecting all the above calculations, we finally have

$$\sum_{n \geq 0} p^n \, \mathrm{Map}_{\mathrm{orb}}\left(\Sigma, \mathrm{SP}^n(M/G)\right) = \prod_{[H]} S_{p^{|\Gamma/H|}}\left(\mathrm{Map}_{\mathrm{orb}}(\Sigma_H, M/G)/\mathcal{D}(\Sigma_H/\Sigma)\right).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

When $G = \{1\}$, we have $\mathrm{Map}_{\mathrm{orb}}(\Sigma, M) = \mathrm{Map}(\Sigma, M)$, and formula (1-8) becomes

$$(2\text{-}24) \quad \sum_{n \geq 0} p^n \, \mathrm{Map}_{\mathrm{orb}}\left(\Sigma, \mathrm{SP}^n(M)\right) \cong \prod_{[\Sigma' \to \Sigma]_{\mathrm{conn.}}} S_{p^{|\Sigma'/\Sigma|}}\left(\mathrm{Map}(\Sigma', M)/\mathcal{D}(\Sigma'/\Sigma)\right).$$

3 Generating functions of finite orbifold invariants

We specialize our main decomposition formula of infinite dimensional orbifold mapping spaces to the finite dimensional subspace of constant orbifold maps. Most of the results in [13; 14] follow from this restricted formula, and we reproduce some of the main results in these papers as corollaries to Theorem A.

Since $\mathrm{Map}_\theta(\widetilde{\Sigma}, M)_{\mathrm{const}} \cong M^{\langle\theta\rangle}$, where $M^{\langle\theta\rangle}$ denotes the fixed point subset of θ, we have

$$(3\text{-}1) \qquad \mathrm{Map}_{\mathrm{orb}}(\Sigma, M/G)_{\mathrm{const}} = \coprod_{(\theta) \in \mathrm{Hom}(\Gamma, G)/G} \left[M^{\langle\theta\rangle}/C(\theta)\right] \overset{\mathrm{def}}{=} C_\Gamma(M/G).$$

As an immediate consequence of Theorem A, we have the following decomposition formula for constant orbifold maps.

Proposition 3.1 *Let M be a G–space and let Γ be an arbitrary group. Then,*

(3-2)
$$\sum_{n \geq 0} p^n C_\Gamma(\mathrm{SP}^n(M/G)) = \prod_{[H]} S_{p^{|\Gamma/H|}}\big(C_H(M/G)/(N_\Gamma(H)/H)\big)$$
$$= \prod_{[H]} S_{p^{|\Gamma/H|}}\bigg(\coprod_{[\rho]}(M^{\langle \rho \rangle}/T_\rho)\bigg),$$

where $[H]$ runs over all the conjugacy classes of finite index subgroups of Γ, and for each $[H]$, $[\rho]$ runs over the set $\mathrm{Hom}(H, G)/\big(N_\Gamma(H) \times G\big)$.

Note that in Theorem A, Γ is the fundamental group of the manifold Σ. But after eliminating Σ by considering constant orbifold maps, Γ can be an arbitrary (discrete) group in Proposition 3.1.

Here we comment on the action of $N_\Gamma(H)/H$ on $C_H(M/G) = (\coprod_\rho M^{\langle \rho \rangle})/G$ in (3-2), where $\rho \in \mathrm{Hom}(H, G)$. In view of (2-5), the action of $N_\Gamma(H)$ commutes with the action of G, and for any $u \in N_\Gamma(H)$ and any $x \in M^{\langle \rho \rangle}$, the action of u on x is such that $u \cdot x = x$, as can be easily verified by (2-5). However, this does not mean that the action of $N_\Gamma(H)$ on $C_H(M/G)$ is trivial. in fact, it is not trivial in general. What happens is that the action of u sends $M^{\langle \rho \rangle}$ to $M^{\langle \rho^{u^{-1}} \rangle}$, where G–conjugacy classes (ρ) and $(\rho^{u^{-1}})$ can be distinct, although these two spaces are identical subspaces of M, since $\langle \rho \rangle = \langle \rho^{u^{-1}} \rangle$ as subgroups of G. For a given $(\rho) \in \mathrm{Hom}(H, G)/G$, let $N_\Gamma^\rho(H)$ be the isotropy subgroup of $N_\Gamma(H)$ at (ρ). Recall that we have an exact sequence of groups [14, Formula (4-6)]:

$$1 \to C_G(\rho) \to T_\rho \to N_\Gamma^\rho(H) \to 1.$$

Thus, $M^{\langle \rho \rangle}/T_\rho = \big(M^{\langle \rho \rangle}/C(\rho)\big)/N_\Gamma^\rho(H)$. We examine the action of $u \in N_\Gamma^\rho(H)$ on $M^{\langle \rho \rangle}/C(\rho)$. By definition, for any $u \in N_\Gamma^\rho(H)$, ρ and $\rho^{u^{-1}}$ are G–conjugate, and thus there exists $g \in G$ such that $\rho^{u^{-1}}(h) = g^{-1}\rho(h)g$ for all $h \in H$. This means that $(u, g) \in T_\rho$. We have

$$M^{\langle \rho \rangle}/C(\rho) \xrightarrow{u \cdot = \mathrm{Id}} M^{\langle \rho^{u^{-1}} \rangle}/C(\rho^{u^{-1}}) = M^{\langle g^{-1}\rho g \rangle}/C(g^{-1}\rho g) \xrightarrow[\cong]{g \cdot} M^{\langle \rho \rangle}/C(\rho),$$

by (2-6). This means that when we apply $u\cdot$, ρ moves within the same G–conjugacy class to $\rho^{u^{-1}}$. To bring it back to ρ, we then apply $g \in G$. Thus, for $u \in N_\Gamma^\rho(H)$ and $x \in M^{\langle \rho \rangle}/C(\rho)$, the action of u on x is given by $u \cdot x = g \cdot x$ where $g \in G$ is an arbitrary element such that $(u, g) \in T_\rho$.

Let $\chi(X)$ be the topological Euler characteristic for a topological space X. In [14], we introduced a notion of an orbifold Euler characteristic associated to a group Γ defined

for a G–manifold M:

$$(3\text{-}3) \qquad \chi_\Gamma(M;G) \overset{\mathrm{def}}{=} \chi\big(C_\Gamma(M/G)\big) = \sum_{(\theta)\in\mathrm{Hom}(\Gamma,G)/G} \chi\big(M^{\langle\theta\rangle}/C(\theta)\big).$$

We observe that when $\Gamma = \mathbb{Z}$,

$$(3\text{-}4) \qquad \chi_\mathbb{Z}(M;G) = \sum_{(g)\in G_*} \chi\big(M^{\langle g\rangle}/C(g)\big) = \frac{1}{|G|} \sum_{gh=hg} \chi\big(M^{\langle g,h\rangle}\big)$$

is the physicist's orbifold Euler characteristic $e_{\mathrm{orb}}(M/G)$ [3]. Here in the last summation, the pair (g,h) runs over the set of commuting pairs of elements. The second identity is due to Lefschetz Fixed Point Formula. Formula (3-3) gives the correct generalization of $e_{\mathrm{orb}}(M/G)$ since it comes from a very natural geometry of orbifold mapping spaces (3-1).

In [14], we introduced a notion of orbifold Euler characteristic of M/G associated to a Γ–set X, denoted by $\chi_{[X]}(M;G)$. When X is a transitive Γ–set of the form $X = \Gamma/H$, it is given by

$$(3\text{-}5) \qquad \chi_{[\Gamma/H]}(M;G) = \chi\big(C_H(M/G)/(N_\Gamma(H)/H)\big)$$

where $\quad C_H(M/G)/(N_\Gamma(H)/H) = \coprod_{[\rho]\in\mathrm{Hom}(H,G)/(N_\Gamma(H)\times G)} M^{\langle\rho\rangle}/\mathrm{Aut}_{\Gamma-G}(P_{H,\rho})$

$$= \coprod_{[\rho]} M^{\langle\rho\rangle}/T_\rho.$$

The second identity above can be proved on topological space level by an argument similar to the last part of the proof of Theorem A.

Now we compute the topological Euler characteristic of both sides of (3-2). We recall that $\chi(S_p(X)) = (1-p)^{-\chi(X)}$.

Corollary 3.2 [14, Theorem C] *Let M be a G–set and let Γ be an arbitrary group. The the generating function of orbifold Euler characteristic associated to Γ of symmetric orbifolds is given by*

$$(3\text{-}6) \qquad \sum_{n\geq 0} p^n \chi_\Gamma(M^n;G_n) = \prod_{[H]} (1-p^{|\Gamma/H|})^{-\chi_{[\Gamma/H]}(M;G)},$$

where $[H]$ runs over all conjugacy classes of finite index subgroups of Γ.

We can rewrite (3-6) in terms of Hecke operators as follows. For a G–manifold, let $\chi_{(M;G)}$ be an integer valued function on the set of discrete groups given by

$$(3\text{-}7) \qquad \chi_{(M;G)}(\Gamma) \overset{\text{def}}{=} \chi\big(C_\Gamma(M/G)\big) = \chi_\Gamma(M;G).$$

For an integer $n \geq 1$, let a Hecke operator $\mathbb{T}(n)$ act on the function $\chi_{(M;G)}$ by

$$(3\text{-}8) \qquad \big[\mathbb{T}(n)\chi_{(M;G)}\big](\Gamma) \overset{\text{def}}{=} \sum_{\substack{[H] \\ |\Gamma/H|=n}} \chi\big(C_H(M/G)\big/(N_\Gamma(H)/H)\big),$$

so that $\mathbb{T}(n)\chi_{(M;G)}$ is another integral function on the set of discrete groups. Then as functions on the set of groups, (3-6) means

$$(3\text{-}9) \qquad \sum_{n \geq 0} p^n \chi_{(M^n;G_n)} = \prod_{n \geq 1}(1 - p^n)^{-\mathbb{T}(n)\chi_{(M;G)}}.$$

Now we consider the case in which Γ is *abelian*. In this case, the action of $N_\Gamma(H) = \Gamma$ on $H \subset \Gamma$ is trivial and so dividing by $N_\Gamma(H)/H$ has no effect. Thus, we have $C_H(M/G)/(N_\Gamma(H)/H) = C_H(M/G)$ and consequently:

Corollary 3.3 *Let Γ be an arbitrary abelian group. For any G–space M, we have*

$$(3\text{-}10) \qquad \sum_{n \geq 0} p^n \chi_\Gamma(M^n; G_n) = \prod_{H}(1 - p^{|\Gamma/H|})^{-\chi_H(M;G)},$$

where the product is over all finite index subgroups H of Γ.

In particular, when $\Gamma = \mathbb{Z}$, the formula (3-7) reduces to

$$(3\text{-}11) \qquad \sum_{n \geq 0} p^n e_{\text{orb}}\big(\text{SP}^n(M/G)\big) = \prod_{r \geq 1}(1 - p^r)^{-e_{\text{orb}}(M/G)}.$$

This is the formula proven in [7] when G is trivial, and for general G in [15].

Instead of Euler characteristic, we can consider other numerical invariants such as signature, spin index, χ_y–characteristic, etc, in suitable categories of manifolds. The formula (3-2) will then provide us with infinite product formula of the corresponding generating functions of orbifold invariants of symmetric orbifolds. What is more interesting in this context is that, since we have a decomposition on the space level, we can apply various (generalized) homology and cohomology functors to obtain infinite product decomposition formulae. This will be discussed in future papers.

4 Geometric Hecke operators for functors

In this section, we prove the Hecke identity (1-13) for 2–dimensional tori. Let $\mathcal{C}$ be the category of topological spaces and continuous maps. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be a covariant (or contravariant) functor. Then it formally follows that whenever $f\colon X \to Y$ is a homeomorphism, the corresponding map $\mathcal{F}(f)\colon \mathcal{F}(X) \to \mathcal{F}(Y)$ (or $\mathcal{F}(Y) \to \mathcal{F}(X)$ in the contravariant case) is also a homeomorphism. In particular, this implies that when X is a G–space, it automatically follows that $\mathcal{F}(X)$ is also a G–space.

The geometric Hecke operator $\mathbb{T}(n)$, $n \geq 1$, acts on a functor $\mathcal{F}$ as follows. For any connected space $X \in \mathcal{C}$,

$$(4\text{-}1) \qquad \left(\mathbb{T}(n)\mathcal{F}\right)(X) \overset{\mathrm{def}}{=} \coprod_{\substack{[X' \to X]_{\mathrm{conn.}} \\ |X'/X|=n}} \mathcal{F}(X')/\mathcal{D}(X'/X),$$

where the disjoint union runs over the isomorphism classes of connected n–sheeted covering spaces X' of X, and $\mathcal{D}(X'/X)$ is the group of all deck transformations of $X' \to X$. When X is not connected, we apply the above construction to each of the connected component.

In general, we do not expect $\mathbb{T}(n)\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ to be a functor. However, see Proposition 4.1 where such a situation does occur.

For the purpose of this paper, the main example of the functor $\mathcal{F}$ is of course the orbifold mapping space functor. Namely, for any G–space M, and any connected space Σ, we let

$$\mathcal{F}_{(M;G)}(\Sigma) = \mathrm{Map}_{\mathrm{orb}}(\Sigma, M/G).$$

Proposition 2.1 shows that this is indeed a contravariant functor in Σ. In terms of this notation, Theorem A can be restated as a formal power series of functors as

$$(4\text{-}2) \qquad \sum_{n \geq 0} p^n \mathcal{F}_{(M^n;G_n)} = \prod_{n \geq 1} S_{p^n}\left(\mathbb{T}(n)\mathcal{F}_{(M;G)}\right).$$

However, in some context, for example in the Grothendieck ring of varieties, it can make sense and can be justified to write $S_p(X) = (1-p)^{-X}$ using powers whose exponents are spaces [5]. For the purpose of our present paper, we can regard $S_p(X)$ as the definition of $(1-p)^{-X}$. This is more appropriate for our purpose since, for example, for Euler characteristic, we have $\chi\left(S_p(X)\right) = (1-p)^{-\chi(X)}$ for any space X. In this point of view, Theorem A has the following form:

$$(4\text{-}3) \qquad \sum_{n \geq 0} p^n \mathcal{F}_{(M^n;G_n)}(\Sigma) = \prod_{n \geq 1} (1-p^n)^{-(\mathbb{T}(n)\mathcal{F}_{(M;G)})(\Sigma)}.$$

By Proposition 4.1 below, this formula can be regarded as a generating function of functors from the category $\mathcal{C}_{\pi_1}$ to $\mathcal{C}$, where $\mathcal{C}_{\pi_1}$ is the category of topological spaces whose morphisms are restricted to those continuous maps inducing isomorphisms on fundamental groups.

By considering constant orbifold maps, we have $\mathcal{F}_{(M;G)}(\Sigma)_{\mathrm{const}} = C_{\pi_1(\Sigma)}(M/G)$. Then, by taking topological Euler characteristic of (4-3) restricted to constant orbifold maps, we recover the formula (3-9). Notice that factors $(1 - p^n)$ in (3-9) are already present in (4-3) on space level.

To define a composition of geometric Hecke operators, we need to have functoriality of geometric Hecke operators in a certain special situation.

Proposition 4.1 *Let $\mathcal{F}: \mathcal{C} \to \mathcal{C}$ be a covariant functor. Let X and Y be connected spaces, and let $f: X \to Y$ be a map such that $f_*: \pi_1(X) \to \pi_1(Y)$ is an isomorphism. Then for every positive integer n, f induces a map*

$$(4\text{-}4) \qquad\qquad f_*: (\mathbb{T}(n)\mathcal{F})(X) \to (\mathbb{T}(n)\mathcal{F})(Y),$$

such that for $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have $(g \circ f)_* = g_* \circ f_*$.

A similar statement holds for contravariant functors.

Proof We fix a base point x_0 of X. Let $p: X' \to X$ be a connected n–sheeted covering space. For each choice of a base point x_0' of X' over x_0, the subgroup $H = p_*\big(\pi_1(X', x_0')\big)$ has index n in $\pi_1(X, x_0)$. Since $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$, where $y_0 = f(x_0)$, is an isomorphism by hypothesis, the subgroup $f_*(H)$ has index n in $\pi_1(Y, y_0)$. Let (Y', y_0') be a connected n–sheeted covering space with base point corresponding to $f_*(H)$. The choice of y_0' is unique up to the action of the group $\mathcal{D}(Y'/Y)$ of deck transformations. Note that since $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism, f_* induces an isomorphism between the corresponding deck transformations

$$\mathcal{D}(X'/X) \xrightarrow[\cong]{f_*} \mathcal{D}(Y'/Y).$$

By the Lifting Theorem in covering space theory, there exists a unique $\mathcal{D}(X'/X)$–equivariant map $\widetilde{f}: X' \to Y'$ such that $\widetilde{f}(x_0') = y_0'$. By the functorial property, we see that $\mathcal{F}(\widetilde{f}): \mathcal{F}(X') \to \mathcal{F}(Y')$ is $\mathcal{D}(X'/X) \cong \mathcal{D}(Y'/Y)$–equivariant. Hence it induces a map on the quotient:

$$\overline{\mathcal{F}(\widetilde{f})}: \mathcal{F}(X')/\mathcal{D}(X'/X) \to \mathcal{F}(Y')/\mathcal{D}(Y'/Y).$$

Different choices of the lift $\widetilde{f}$ are related by the action of deck transformations. Hence the map $\mathcal{F}(\widetilde{f})$ on the orbit space depends only on f. Repeating the above constructions

for each isomorphism class of connected n–sheeted covering spaces of X, we obtain a map

$$(4\text{-}5) \qquad f_*\colon \coprod_{\substack{[X' \to X]_{\text{conn.}} \\ |X'/X|=n}} \mathcal{F}(X')/\mathcal{D}(X'/X) \to \coprod_{\substack{[Y' \to Y]_{\text{conn.}} \\ |Y'/Y|=n}} \mathcal{F}(Y')/\mathcal{D}(Y'/Y).$$

This is the map (4-4). The behavior under the composition of two maps can be easily verified. The argument for contravariant functors is similar. $\qquad\square$

As a special case, let $f\colon X \to X$ be a homeomorphism. There is one point which we have to be careful about in the above construction of f_*. For a connected n–sheeted covering space $p\colon (X', x_0') \to (X, x_0)$, the based covering space $(X'', y_0) \to (X, f(x_0))$ corresponding to the subgroup $f_*\big(p_*(\pi_1(X', x_0'))\big) \subset \pi_1\big(X, f(x_0)\big)$ may not be isomorphic to $X' \to X$ as a covering space over X, although X' and X'' are homeomorphic via a lift $\widetilde{f}\colon X' \xrightarrow{\cong} X''$ of f. Thus, in general, the induced map

$$(4\text{-}6) \qquad f_*\colon \coprod_{\substack{[X' \to X]_{\text{conn.}} \\ |X'/X|=n}} \mathcal{F}(X')/\mathcal{D}(X'/X) \xrightarrow{\cong} \coprod_{\substack{[X' \to X]_{\text{conn.}} \\ |X'/X|=n}} \mathcal{F}(X')/\mathcal{D}(X'/X)$$

shuffles connected components, and it is not easy to control this shuffling. This is an obstacle in studying compositions of Hecke operators given in (4-7) below. However, when $f\colon X \to X$ is a deck transformation of some covering $X \to X_0$, the situation can be completely clarified. In particular, when $\pi_1(X_0)$ is abelian, it turns out that the action of $\mathcal{D}(X/X_0)$ on $(\mathbb{T}(n)\mathcal{F})(X)$ does preserve connected components, and there is a simple relation among various groups of deck transformations involved.

Anyway, as a formal consequence of Proposition 4.1, we have:

Corollary 4.2 *Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be an arbitrary covariant or contravariant functor. If X is a G–space, then for every positive integer n, the space $(\mathbb{T}(n)\mathcal{F})(X)$ is also a G–space.*

Next, we consider compositions of Hecke operators given as follows.

$$\begin{aligned}
\big((\mathbb{T}(m) \circ \mathbb{T}(n))\mathcal{F}\big)(X) &= \mathbb{T}(m)\big(\mathbb{T}(n)\mathcal{F}\big)(X) \\
&= \coprod_{[X']_m} \big[(\mathbb{T}(n)\mathcal{F})(X')\big]\big/ \mathcal{D}(X'/X) \\
&= \coprod_{[X']_m} \bigg[\coprod_{[X'']_n} \mathcal{F}(X'')/\mathcal{D}(X''/X') \bigg]\bigg/ \mathcal{D}(X'/X),
\end{aligned}$$

(4-7)

where $[X']_m$ runs over the set of isomorphism classes of connected m–sheeted covering spaces of X, and for a given X', $[X'']_n$ runs over the set of isomorphism classes of connected n–sheeted covering spaces of X'.

As remarked earlier concerning formula (4-6), the action of the group of deck transformations $\mathcal{D}(X'/X)$ on $(\mathbb{T}(n)\mathcal{F})(X')$ permutes its connected components. We now clarify what happens.

Let $\widetilde{X} \to X$ be the universal cover of X and let $\Gamma = \mathcal{D}(\widetilde{X}/X) \cong \pi_1(X)$ be its group of deck transformations. We regard $\widetilde{X} \to X$ as the right Γ–principal bundle over X. Let $K \subset H \subset \Gamma$ be subgroups such that $|\Gamma/H| = m$ and $|H/K| = n$. We put $X_K = \widetilde{X}/K$ and $X_H = \widetilde{X}/H$. Then $X_H \to X$ is a connected m–sheeted covering of X with $\mathcal{D}(X_H/X) \cong N_\Gamma(H)/H$, and $X_K \to X_H$ is a connected n–sheeted covering of X_H with $\mathcal{D}(X_K/X_H) \cong N_H(K)/K$. Let $g \in N_\Gamma(H) \subset \Gamma$. Then the right multiplication by g induces the following diagram of homeomorphisms and covering spaces:

$$
(4\text{-}8) \qquad
\begin{array}{ccccccc}
\widetilde{X} & \longrightarrow & X_K & \longrightarrow & X_H & \longrightarrow & X \\
{\cdot g}\Big\downarrow{\cong} & & {\cdot g}\Big\downarrow{\cong} & & {\cdot g}\Big\downarrow{\cong} & & \Big\| \\
\widetilde{X} & \longrightarrow & X_{g^{-1}Kg} & \longrightarrow & X_H & \longrightarrow & X.
\end{array}
$$

Since $g \in N_\Gamma(H)$, the map $\cdot g \colon X_H \xrightarrow{\cong} X_H$ is a deck transformation of X_H over X. However, since g may not be in $N_\Gamma(K)$, $\cdot g \colon X_K \xrightarrow{\cong} X_{g^{-1}Kg}$ is only an isomorphism of covering spaces over X. If $g \in N_\Gamma(K)$, then $X_{g^{-1}Kg} = X_K$ and $\cdot g$ induces a deck transformation of X_K over X. For the middle square, when $g \in H \subset N_\Gamma(H)$, $\cdot g$ induces an isomorphism of two coverings X_K and $X_{g^{-1}Kg}$ over X_H. If, furthermore, we have $g \in N_H(K) \subset H$, then $\cdot g$ induces a deck transformation of X_K over X_H. This clarifies the action of $\mathcal{D}(X'/X)$ on $(\mathbb{T}(n)\mathcal{F})(X')$ where $X' = X_H$ and $X'' = X_K$.

The above situation simplifies when the fundamental group of X is abelian. In this case, every element $g \in \Gamma$ induces a deck transformation $\cdot g \colon X_H \xrightarrow{\cong} X_H$ whose lift $\cdot g \colon X_K \xrightarrow{\cong} X_K$ preserves X_K. Also we have $\mathcal{D}(X_K/X_H) \cong H/K$ for any two subgroups $K \subset H \subset \Gamma$. The formula (4-7) now simplifies as follows.

Proposition 4.3 *Let X be a connected space whose fundamental group is abelian. Then, the composition of two geometric Hecke operators is given by*

$$
(4\text{-}9) \qquad \big(\mathbb{T}(m)(\mathbb{T}(n)\mathcal{F})\big)(X) = \coprod_{\substack{H \subset \Gamma \\ |\Gamma/H|=m}} \left[\coprod_{\substack{K \subset H \\ |H/K|=n}} \big(\mathcal{F}(X_K)/\mathcal{D}(X_K/X)\big) \right].
$$

Proof By (4-7), we have

$$\big(\mathbb{T}(m)(\mathbb{T}(n)\mathcal{F})\big)(X)\big) = \coprod_{\substack{H\subset\Gamma \\ |\Gamma/H|=m}} \left[\coprod_{\substack{K\subset H \\ |H/K|=n}} \big(\mathcal{F}(X_K)/\mathcal{D}(X_K/X_H)\big)\right]\Big/\mathcal{D}(X_H/X).$$

Since Γ is abelian, $\mathcal{D}(X_H/X)$ preserves $\mathcal{F}(X_K)/\mathcal{D}(X_K/X_H)$ for each $K\subset H$, this equals

$$\coprod_{\substack{H\subset\Gamma \\ |\Gamma/H|=m}} \coprod_{\substack{K\subset H \\ |H/K|=n}} \left[\big(\mathcal{F}(X_K)/\mathcal{D}(X_K/X_H)\big)\Big/\mathcal{D}(X_H/X)\right].$$

Since $\mathcal{D}(X_K/X_H) = H/K$, $\mathcal{D}(X_H/X) = \Gamma/H$, and $\mathcal{D}(X_K/X) = \Gamma/K$, we have

$$\coprod_{\substack{H\subset\Gamma \\ |\Gamma/H|=m}} \left[\coprod_{\substack{K\subset H \\ |H/K|=n}} \big(\mathcal{F}(X_K)/\mathcal{D}(X_K/X)\big)\right].$$

This completes the proof. □

We continue to assume that the fundamental group of X is abelian. For an integer $d\geq 1$, let $R(d)X$ be the covering space of X corresponding to $d\cdot\pi_1(X)\subset\pi_1(X)$. Let $\mathbb{R}(d)$ act on a functor $\mathcal{F}$ by

$$(4\text{-}10) \qquad (\mathbb{R}(d)\mathcal{F})(X) \overset{\text{def}}{=} \mathcal{F}\big(R(d)X\big)\Big/\mathcal{D}\big((R(d)X)/X\big).$$

As in Proposition 4.1, we can show that any map $f\colon X\to X$ inducing an isomorphism on fundamental groups gives rise to a map

$$(4\text{-}11) \qquad f_*\colon (\mathbb{R}(d)\mathcal{F})(X)\to(\mathbb{R}(d)\mathcal{F})(X).$$

In particular, if X is a G–space, then not only is $\mathcal{F}(X)$ a G–space, but $\big(\mathbb{R}(d)\mathcal{F})(X)\big)$ is also a G–space for all $d\geq 1$.

The main result in this section is the following Hecke identity for geometric Hecke operators for 2–dimensional tori T.

Theorem 4.4 *Let $\mathcal{F}\colon \mathcal{C}\to\mathcal{C}$ be an arbitrary contravariant or covariant functor. Let T be a 2–dimensional torus. Then for every pair of positive integers m and n, the composition of two geometric Hecke operators satisfy*

$$(4\text{-}12) \qquad \big(\mathbb{T}(m)(\mathbb{T}(n)\mathcal{F})\big)(T) = \sum_{d|(m,n)} d\cdot\left(\mathbb{T}\left(\frac{mn}{d^2}\right)(\mathbb{R}(d)\mathcal{F})\right)(T).$$

In particular, $\mathbb{T}(m)$ and $\mathbb{T}(n)$ commute.

In the right hand side of (4-9), the summation symbol means disjoint topological union, and the factor d means a disjoint union of d copies.

For the proof, we first recall the ordinary Hecke identity for lattices. For details, see Lang [9, page 16]. Let $\mathcal{A}$ be the free abelian group generated by rank 2 lattices L of $\mathbb{C}$. The the Hecke operator $T(n)$ for $n \geq 1$ is a map $T(n)\colon \mathcal{A} \to \mathcal{A}$ defined by

$$T(n)(L) = \sum_{[L:L']=n} L' \in \mathcal{A}.$$

Let $R(n)\colon \mathcal{A} \to \mathcal{A}$ be defined by $R(n)L = nL$ consisting of elements $\{n \cdot \ell\}_{\ell \in L} \subset L$. Then Hecke identity says

$$(4\text{-}13) \qquad T(m) \circ T(n)(L) = \sum_{d \mid (m,n)} d \cdot R(d) \circ T\left(\frac{mn}{d^2}\right)(L),$$

for any lattice L. From this formula, it is clear that $T(m)$ and $T(n)$ commute. Also, it is easy to check that $R(d)$ and $T(n)$ commute.

Now we are ready to prove Theorem 4.4.

Proof of Theorem 4.4   Since $\Gamma \cong \pi_1(T) \cong \mathbb{Z}^2$ is free abelian of rank 2, any subgroup of Γ of finite index is also free abelian of rank 2. Applying (4-9) in our context, we obtain

$$\left(\mathbb{T}(m)(\mathbb{T}(n)\mathcal{F})\right)(T) = \coprod_{\substack{H \subset \Gamma \\ |\Gamma/H|=m}} \left[\coprod_{\substack{K \subset H \\ |H/K|=n}} \left(\mathcal{F}(T_K)/(\Gamma/K)\right) \right],$$

where T_K is a covering torus corresponding to an index mn sublattice $K \subset \Gamma$. By the ordinary Hecke identity (4-13), any index mn sublattice K of Γ arising in the above disjoint union is of the form $d \cdot L$ for some integer d dividing (m,n), and for some lattice L of index $(mn)/d^2$ in Γ, and furthermore, there are exactly d such sublattices in the above disjoint union. Hence the right hand side of the above expression can be rewritten as

$$\left(\mathbb{T}(m)(\mathbb{T}(n)\mathcal{F})\right)(T) = \coprod_{d \mid (m,n)} \coprod^{d} \coprod_{\substack{L \subset \Gamma \\ |\Gamma/L|=(mn)/d^2}} \mathcal{F}(T_{d \cdot L}) \big/ (\Gamma/(d \cdot L)).$$

On the other hand,

$$\left(\mathbb{T}\left(\frac{mn}{d^2}\right)(\mathbb{R}(d)\mathcal{F})\right)(T) = \left[\coprod_{\substack{L \subset \Gamma \\ |\Gamma/L| = (mn)/d^2}} (\mathbb{R}(d)\mathcal{F})(T_L)/(\Gamma/L)\right]$$

$$= \coprod_{\substack{L \subset \Gamma \\ |\Gamma/L| = (mn)/d^2}} \left[\mathcal{F}(T_{d \cdot L})/\big(L/(d \cdot L)\big)\right]/(\Gamma/L)$$

$$= \coprod_{\substack{L \subset \Gamma \\ |\Gamma/L| = (mn)/d^2}} \mathcal{F}(T_{d \cdot L})/\big(\Gamma/(d \cdot L)\big).$$

Thus combining the above calculations, we have our formula (4-12). $\qquad\square$

Theorem B is a special case of Theorem 4.4 when $\mathcal{F}(\Sigma) = \text{Map}_{\text{orb}}(\Sigma, M/G)$.

By a general procedure, Theorem 4.4 implies the following "formal" Euler product of operators:

$$(4\text{-}14) \qquad \sum_{n \geq 1} \frac{\mathbb{T}(n)}{n^s} = \prod_{p: \text{ prime}} \left(1 - \mathbb{T}(p)p^{-s} + p \cdot \mathbb{R}(p)p^{-2s}\right)^{-1},$$

on functors $\mathcal{F}$. However, the implications of this Euler product formula in our present context is not clear.

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Department of Mathematics, University of California Santa Cruz
1156 High Street, Santa Cruz, CA 95064, United States

`tamanoi@math.ucsc.edu`