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Novikov homology of HNN–extensions and right-angled Artin groups

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We calculate the Novikov homology of right-angled Artin groups and certain HNN–extensions of these groups. This is used to obtain information on the homological Sigma invariants of Bieri–Neumann–Strebel–Renz for these groups. These invariants are subsets of all homomorphisms from a group to the reals containing information on the finiteness properties of kernels of such homomorphisms. We also derive information on the homotopical Sigma invariants and show that one cannot expect any symmetry relations between a homomorphism and its negative regarding these invariants. While it was previously known that these invariants are not symmetric in general, we give the first examples of homomorphisms which are symmetric with respect to the homological invariant, but not with respect to the homotopical invariant.

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1 Introduction

The Sigma invariants of Bieri–Neumann–Strebel–Renz [4; 5] have proven to be an important tool in studying finiteness properties of groups. While they are in general very difficult to compute, there are interesting groups for which they have been completely determined and which give rise to very intriguing examples. These groups include right-angled Artin groups (see Meier, Meinert and VanWyk [19]), and Thompson's group F (see Bieri, Geoghegan and Kochloukova [3]). As an application, Bieri, Geoghegan and Kochloukova [3] use the Sigma invariants of Thompson's group F to show that F contains subgroups of type F_{m-1} which are not of type F_m for all $m \geq 1$; see Section 2 below for the definition of type F_m.

There are two different versions of finiteness properties, one based on homotopical and one based on homological techniques, and it was shown by Bestvina and Brady [1] that they are indeed different. One also has homotopical and homological Sigma invariants which are also different in general [19].

We will give a precise definition in Section 2, but for the moment we can think of the Sigma invariants as certain subsets $\Sigma^k(G)$ and $\Sigma^k(G;\mathbb{Z})$ of $\mathrm{Hom}(G,\mathbb{R})$ for any

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$k \geq 0$ and G a finitely generated group[1]. Here $\Sigma^k(G)$ refers to the homotopical version and $\Sigma^k(G;\mathbb{Z})$ to the homological version.

Given a nonzero homomorphism $\chi\colon G \to \mathbb{R}$, one can always consider the negative homomorphism $-\chi\colon G \to \mathbb{R}$. There are very simple examples of groups and homomorphisms $\chi\colon G \to \mathbb{R}$ which show that the Sigma invariants are not invariant under this antipodal action, possibly the easiest example being the Baumslag–Solitar group $G = \langle a, b \mid a^{-1}ba = b^2\rangle$ with the homomorphism sending a to 1 and b to 0. On the other hand, for right-angled Artin groups the Sigma invariants are invariant under the antipodal action.

Groups for which a computation of the Sigma invariants are quite accessible include HNN–extensions, provided one has information on the Sigma invariants of the groups being extended. Also, for these groups it is easy to break the symmetry of $\Sigma^k(G)$ under the antipodal action. Here one should note that right-angled Artin groups can also be build via HNN–extensions, a fact used by Meier, Meinert and VanWyk in [19] to determine their Sigma invariants, but as the extension is always along only one inclusion one gets the described symmetry in the Sigma invariants.

By forming nonsymmetric HNN–extensions of right-angled Artin groups we show that practically any behaviour under the antipodal action is possible.

Theorem 1.1 *Let p, q be positive integers. Then there exists a group G of type F and a homomorphism $\chi\colon G \to \mathbb{Z}$ with*

$$\chi \in \Sigma^p(G) - \Sigma^{p+1}(G)$$
$$-\chi \in \Sigma^q(G) - \Sigma^{q+1}(G).$$

Recall that a group is of type F if there exists a finite $K(G,1)$. For certain metabelian groups G Kochloukova [15] has given a calculation of $\Sigma^k(G)$ in terms of $\Sigma^1(G)$. Using this result one can obtain other examples satisfying the statement of Theorem 1.1. In these examples we always have $\Sigma^k(G) = \Sigma^k(G;\mathbb{Z})$.

There is also a version of Theorem 1.1 where $\Sigma^k(G)$ is replaced by $\Sigma^k(G;\mathbb{Z})$ and we demand that $\chi, -\chi \notin \Sigma^2(G)$. Recall that we always have $\Sigma^k(G) \subset \Sigma^k(G;\mathbb{Z})$, and for $k \geq 2$ we have $\chi \in \Sigma^k(G)$ if and only if $\chi \in \Sigma^k(G;\mathbb{Z}) \cap \Sigma^2(G)$. Finally, we obtain examples where we only get one of χ and $-\chi$ in $\Sigma^2(G)$.

[1] In fact G should satisfy certain finiteness conditions depending on k.

Theorem 1.2 *There exists a group G of type F and a homomorphism $\chi\colon G \to \mathbb{Z}$ such that for all $p \geq 2$ we have*

$$\chi \in \Sigma^p(G)$$
$$-\chi \in \Sigma^p(G;\mathbb{Z}) - \Sigma^2(G).$$

Theorem 1.2 has interesting consequences for a Theorem of Latour [17] regarding conditions for the existence of a nonsingular closed 1–form within a cohomology class $\chi \in H^1(M;\mathbb{R})$, where M is a high-dimensional closed manifold. One condition demands the contractibility of certain path spaces $\mathcal{M}_\chi$ and $\mathcal{M}_{-\chi}$; see Section 8 for details. In all previously known examples with M a closed manifold, contractibility of $\mathcal{M}_\chi$ was equivalent to contractibility of $\mathcal{M}_{-\chi}$, but using Theorem 1.2 we construct an example where only one of these path spaces is contractible.

We determine the homological Sigma invariants using Novikov homology. It is known that $\chi \in \Sigma^k(G;\mathbb{Z})$ is equivalent to the vanishing of certain Novikov homology groups; see Lemma 2.4 for details. Knowing the exact value of a nonvanishing Novikov homology group gives extra information which is useful for looking at HNN–extensions, as we can use methods from group homology.

It turns out that the Novikov homology of a right-angled Artin group is easily accessible. To make this more precise, recall that for a finite flag complex L the right-angled Artin group G_L is generated by the vertices of L, and two generators commute exactly when the corresponding vertices span a 1–simplex. If $\chi\colon G_L \to \mathbb{R}$ is nonzero on all generators, then

$$H_*(G_L;\widehat{\mathbb{Z}G_L}_\chi) \cong \widehat{\mathbb{Z}G_L}_\chi \otimes_{\mathbb{Z}} \widetilde{H}_{*-1}(L),$$

where the isomorphism is induced by an isomorphism of chain complexes. Here $\widehat{\mathbb{Z}G_L}_\chi$ denotes the Novikov ring; compare Section 2. If χ vanishes on some generators, we get a spectral sequence which carries enough information to determine $\Sigma^k(G_L;\mathbb{Z})$. This gives a purely algebraic and simple calculation of these Sigma invariants. The original calculation of Meier, Meinert and VanWyk [19], which also included the homotopical invariants, used both geometric and algebraic arguments, and a simplification using geometric arguments was done by Bux and Gonzalez [7].

Arguments using Novikov homology will only give information about the homological invariants; in order to understand the homotopical invariants it is necessary to get information about the homotopy type of certain halfspaces. Such a halfspace is defined as $N_L = h^{-1}([0,\infty))$ with $h\colon X_L \to \mathbb{R}$ a map with $h(gx) = \chi(g) + h(x)$, where X_L is the universal cover of a finite $K(G_L,1)$. While in Bux and Gonzalez [7] the first nonvanishing homotopy group of N is determined, Bestvina and Brady showed [1, Theorem 8.6] that N has the homotopy type of a wedge of copies of L, provided all generators of G_L are send to 1. This gives a bit of extra information which is not needed for the Sigma invariants of a right-angled Artin group, but it is useful when considering nonsymmetric HNN–extensions of right-angled Artin groups.

Namely, it turns out that considering subcomplexes K of L leads to naturality in groups $G_K \to G_L$ which is also reflected in the halfspaces, that is, up to homotopy the natural map $N_K \to N_L$ corresponds to inclusion of the wedge of copies of K into the wedge of copies of L. This naturality allows us to understand the halfspaces for nonsymmetric HNN–extensions of right-angled Artin groups, leading to the examples described in Theorems 1.1 and 1.2. Since we need this more refined version of [1, Theorem 8.6], we give a proof in Section 5. Our proof uses in fact different techniques than [1], we exploit the fact that G_L can be build using HNN–extensions of a smaller right-angled Artin group. However, the techniques of [1] can also be used to obtain this result.

2 Sigma invariants and Novikov rings

A group G is said to be of type FP_n, if there is a resolution

$$\cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{1}$$

of free $\mathbb{Z}G$–modules such that F_i is finitely generated for $i \le n$. Here $\mathbb{Z}$ is considered as a trivial $\mathbb{Z}G$–module.

We define

$$S(G) = (\mathrm{Hom}(G,\mathbb{R}) - \{0\})/\mathbb{R}_+,$$

that is, we identify nonzero homomorphisms, if one is a positive multiple of the other. This is a sphere of dimension $\mathrm{rank}(G/[G,G]) - 1$. If $\chi\colon G \to \mathbb{R}$ is a nonzero homomorphism, we still write $\chi \in S(G)$.

Given such χ, we let $G_\chi = \{g \in G \mid \chi(g) \ge 0\}$. If there is a resolution (1) of free $\mathbb{Z}G_\chi$ modules with F_i finitely generated for $i \le k$, we say G_χ *is of type* FP_k. We now set

$$\Sigma^k(G;\mathbb{Z}) = \{\chi \in S(G) \mid G_\chi \text{ is of type } FP_k\}.$$

If G is of type F_n, that is, there exists a $K(G,1)$ with finite n–skeleton, there is a more geometric criterion to check for $\chi \in \Sigma^k(G;\mathbb{Z})$. Let X be the universal cover of the $K(G,1)$ with finite n–skeleton and $\chi\colon G \to \mathbb{R}$ a nonzero homomorphism. Let $h\colon X \to \mathbb{R}$ be a *height function* with respect to χ, that is, we have $h(gx) = \chi(g) + h(x)$ for all $x \in X$ and $g \in G$. To see that such h exist, note that it can easily be defined on the 0–skeleton $X^{(0)}$, and since $\mathbb{R}$ is contractible, the map can always be extended to the higher skeleta.

Now for $r \in \mathbb{R}$, let

$$N^r = \{x \in X \mid h(x) \geq r\}.$$

We call N^r a *halfspace with respect to* χ.

Proposition 2.1 (Bieri–Renz [5]) *Let G be a group of type F_n. For $k \leq n$ we have $\chi \in \Sigma^k(G;\mathbb{Z})$ if and only if there is a real number $r \geq 0$ with the property that the homomorphism $\widetilde{H}_i(N^0) \to \widetilde{H}_i(N^{-r})$, induced by inclusion, is the zero map for all $i < m$, where $\widetilde{H}$ denotes reduced homology.*

This Proposition suggests the definition of another invariant by replacing reduced homology by homotopy. This leads to the homotopical Sigma invariants.

Definition 2.2 Let G be a group of type F_n and $\chi\colon G \to \mathbb{R}$ a nonzero homomorphism. We say $\chi \in \Sigma^k(G)$ if there is a real number $r \geq 0$ with the property that the map $\pi_i(N^0) \to \pi_i(N^{-r})$, induced by inclusion, is the zero map for all $i < m$.

This definition does not depend on the choices involved. Furthermore, using Proposition 2.1 it is easy to see that $\Sigma^1(G;\mathbb{Z}) = \Sigma^1(G)$ and $\Sigma^k(G) \subset \Sigma^k(G;\mathbb{Z})$. However, it follows from the work of Bestvina and Brady [1] that in general $\Sigma^2(G) \neq \Sigma^2(G;\mathbb{Z})$ [7; 19]. Nevertheless, $\Sigma^k(G) = \Sigma^2(G) \cap \Sigma^k(G;\mathbb{Z})$ for $k \geq 2$ [5, Chapter 6].

We now want to describe yet another criterion for the homological Sigma invariant involving Novikov homology. For this we need a completion of the group ring.

Let G be a group and $\chi\colon G \to \mathbb{R}$ a homomorphism. We denote by $\mathbb{Z}^G$ the abelian group of all functions $\lambda\colon G \to \mathbb{Z}$. For $\lambda \in \mathbb{Z}^G$ denote $\operatorname{supp}\lambda = \{g \in G \mid \lambda(g) \neq 0\}$.

Definition 2.3 The *Novikov–Sikorav completion* $\widehat{\mathbb{Z}G}_\chi$ is defined as

$$\widehat{\mathbb{Z}G}_\chi = \{\lambda \in \mathbb{Z}^G \mid \forall\, r \in \mathbb{R} \;\; \operatorname{supp}\lambda \cap \chi^{-1}((-\infty, r]) \text{ is finite}\}.$$

The multiplication is given by the extension of the multiplication of the group ring. The resulting *Novikov homology* is given by

$$H_*(G;\widehat{\mathbb{Z}G}_\chi) = H_*(\widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}G} P_*)$$

where P_* is a free $\mathbb{Z}G$–resolution of the trivial $\mathbb{Z}G$ module $\mathbb{Z}$, that is, ordinary group homology with coefficients in $\widehat{\mathbb{Z}G}_\chi$,viewed as a right $\mathbb{Z}G$–module.

Definition 2.3 is in fact due to Sikorav [24], Novikov's original definition [21] required χ to be injective. We denote this case by

$$\widehat{N}_\chi = \widehat{\mathbb{Z}G/\ker\chi}_\chi.$$

The ring $\widehat{N}_\chi$ has the nice property that it is a Euclidean ring; see Farber [9]. In particular, the homology groups $H_i(G;\widehat{N}_\chi) = H_i(\widehat{N}_\chi \otimes_{\mathbb{Z}G} P_*)$ have a well defined rank, called the Novikov–Betti number $b_i(G;\chi)$, and torsion coefficients for $i \leq n$, where n is such that G is of type FP_n.

The relation to the Sigma invariants is given by the following lemma, a proof of which can be found in Bieri [2, Theorem A.1].

Lemma 2.4 *Let G be a group of type FP_n and $k \leq n$. Then the following are equivalent.*

(1) $\chi \in \Sigma^k(G;\mathbb{Z})$.

(2) $H_i(G;\widehat{\mathbb{Z}G}_\chi) = 0$ *for* $i \leq k$. □

From the Universal Coefficient Spectral Sequence we therefore get:

Corollary 2.5 *Let G be a group of type FP_n and $k \leq n$. If $\chi \in \Sigma^k(G;\mathbb{Z})$, then $H_i(G;M) = 0$ for $i \leq k$ and all right $\widehat{\mathbb{Z}G}_\chi$–modules M.* □

Let G be a group, H a subgroup and $\phi\colon H \to G$ an injective homomorphism. The HNN–extension of G with respect to ϕ is defined as

$$G*_\phi = \langle G, t \mid t^{-1}ht = \phi(h) \text{ for all } h \in H\rangle,$$

that is, the group generated by G and a disjoint element t subject to conjugation relations for elements of H. If ϕ is inclusion, we simply write $G*_H$.

Now if $\chi\colon G \to \mathbb{R}$ is a homomorphism with $\chi|H = \chi\circ\phi$, we can extend χ to $\chi_x\colon G*_\phi \to \mathbb{R}$ for every $x \in \mathbb{R}$ via

$$\chi_x(g) = \chi(g) \text{ and } \chi_x(t) = x.$$

If $x \in \mathbb{R}$ is not of crucial importance, we will simply write $\chi = \chi_x\colon G*_\phi \to \mathbb{R}$.

The inclusion $i\colon G \to G*_\phi$ induces an inclusion of completions $i\colon \widehat{\mathbb{Z}G}_\chi \to \widehat{\mathbb{Z}G*_\phi}_\chi$ and we obtain a long exact sequence (see Brown [6, Chapter VII.9])

$$\cdots \longrightarrow H_n(H;\widehat{\mathbb{Z}G*_\phi}_\chi) \xrightarrow{\alpha} H_n(G;\widehat{\mathbb{Z}G*_\phi}_\chi) \xrightarrow{\beta} H_n(G*_\phi;\widehat{\mathbb{Z}G*_\phi}_\chi) \longrightarrow H_{n-1}(H;\widehat{\mathbb{Z}G*_\phi}_\chi) \longrightarrow \cdots \tag{2}$$

where β is induced by inclusion $i\colon G \to G*_\phi$, and $\alpha = (\phi,t)_* - i_*$, where i_* is induced by the inclusion $i\colon H \to G$ and

$$(\phi,t)\colon (H, \widehat{\mathbb{Z}G*_\phi}_\chi) \to (G, \widehat{\mathbb{Z}G*_\phi}_\chi)$$

is the pair $\phi\colon H \to G$ and $t\colon \widehat{\mathbb{Z}G*_\phi}_\chi \to \widehat{\mathbb{Z}G*_\phi}_\chi$, which is right multiplication by t. Note that Brown [6] uses a different convention in the definition of group homology leading to sign changes compared to [6, Chapter VII.9].

We note the following immediate corollary which is well known; see Meier, Meinert and VanWyk [19] and Meinert [20].

Corollary 2.6 *Let $\chi\colon G*_\phi \to \mathbb{R}$ as above, and G, H of type FP_m for $m \geq 1$.*

(1) *If $\chi|_G \in \Sigma^m(G;\mathbb{Z})$ and $\chi|_H \in \Sigma^{m-1}(H;\mathbb{Z})$, then $\chi \in \Sigma^m(G*_\phi;\mathbb{Z})$.*

(2) *If $\chi \in \Sigma^m(G*_\phi;\mathbb{Z})$ and $\chi|_G \in \Sigma^{m-1}(G;\mathbb{Z})$, then $\chi|_H \in \Sigma^{m-1}(H;\mathbb{Z})$.*

(3) *If $\chi \in \Sigma^m(G*_\phi;\mathbb{Z})$ and $\chi|_H \in \Sigma^m(H;\mathbb{Z})$, then $\chi|_G \in \Sigma^m(G;\mathbb{Z})$.*

(4) *If $\chi|_H = 0$ and $\chi|_G \neq 0$, then $\chi \notin \Sigma^1(G*_\phi;\mathbb{Z})$.* □

3 Right-angled Artin groups

A simplicial complex L is called a *flag complex*, if every finite collection of pairwise adjacent vertices of L spans a simplex in L. We denote the set of vertices by $L^{(0)}$. By a *full subcomplex* of L we mean a subcomplex $\overline{L}$ of L such that $\overline{L}^{(0)} \subset L^{(0)}$ and a finite collection of vertices of $\overline{L}$ spans a simplex in $\overline{L}$ if and only if it spans a simplex in L. Clearly $\overline{L}$ is also a flag complex.

Definition 3.1 Let L be a finite flag complex. The *right-angled Artin group* G_L associated to L is the group with generating set $\{t_1,\dots,t_n\}$ in one-to-one correspondence with the vertex set $L^{(0)} = \{v_1,\dots,v_n\}$, and relations $[t_i,t_j]=1$ precisely if v_i, v_j span a 1–simplex.

If the vertices $v_{i_0},\dots,v_{i_k} \in L^{(0)}$ form a k–simplex in L, we denote this simplex by $[v_{i_0} : \cdots : v_{i_k}]$. We also consider the empty simplex which we denote by [ ] or $\varnothing$.

If L is a finite flag complex, and L^* a full subcomplex, let $L^\dagger$ be the full subcomplex of L spanned by the vertices in $L - L^*$. Given a simplex σ in $L^\dagger$, we write

$$L^*(\sigma) = \mathrm{lk}(\sigma) \cap L^*$$

where $\mathrm{lk}(\sigma)$ is the link of σ in L, that is, the union of all simplices τ in L disjoint from σ, such that $\tau \cup \sigma$ is also a simplex in L. We also allow the empty simplex $\varnothing$ for σ, in which case we get $L^*(\varnothing) = L^*$. Notice that $L^*(\sigma)$ is a full subcomplex of L^* and hence of L. We thus get subgroups $G_{L^*(\sigma)}$ of G_L, which are again right-angled Artin groups.

For a simplex σ in L we write $|\sigma| = k$, if σ is spanned by $k+1$ vertices. We also set $|\varnothing| = -1$.

Remark 3.2 Let the vertex set of L be $\{v_1, \dots, v_n\}$ and let L^* be the full subcomplex of L whose vertex set is $\{v_1, \dots, v_{n-1}\}$. With $K = L^*(v_n)$, we get an HNN–extension

$$G_L = G_{L^*} *_{G_K} .$$

In particular, any right-angled Artin group can be build inductively from the trivial group by HNN–extensions along right-angled Artin subgroups. We can therefore build a $K(G_L, 1)$ complex inductively by using the standard procedure for HNN–extensions, that is, given a $K(G_{L^*}, 1)$ and a $K(G_K, 1)$, we get

$$K(G_L, 1) = K(G_{L^*}, 1) \cup K(G_K, 1) \times [0,1]/\sim, \tag{3}$$

where $(x, j) \sim i(x)$, for $j = 0, 1$, $x \in K(G_K, 1)$ and $i\colon K(G_K, 1) \to K(G_{L^*}, 1)$ a map inducing the inclusion on fundamental group; compare Geoghegan [11, Chapter 7.1].

Let L be a finite flag complex, and $L^{(0)}$ the set of vertices. We write

$$T^n = \prod_{v \in L^{(0)}} S^1,$$

which we think of as a CW–complex, where each circle has the CW–structure with one cell of dimension 0 and 1. That is, for every subset σ of $L^{(0)}$, there is a unique cell $T_\sigma \subset T^n$ with dimension $|\sigma|$ determined by the property that the projection $p_v\colon T_\sigma \to S^1$ is onto if and only if $v \in \sigma$. Let

$$Q_L = \bigcup_{\sigma \in L} T_\sigma \quad \subset \quad T^n$$

be the union of T_σ over all simplices σ in L.

Lemma 3.3 *With the notation above, Q_L is a $K(G_L, 1)$.*

Proof The proof is by induction over the number of vertices in L. The main observation is that Q_L is given via (3), if we use the inclusion $Q_K \subset Q_{L^*}$, so the result follows from Remark 3.2. □

Remark 3.4 Lemma 3.3 can also be proven using results from nonpositively curved geometry; compare for example Bestvina and Brady [1]. These more advanced techniques can give more useful information; compare Remark 8.5.

Let X_L be the universal cover of Q_L. The left $\mathbb{Z}G_L$–chain complex $C_*(X_L)$ can be described as follows. The k–th chain group is freely generated by the $(k-1)$–simplices of L. Here we also consider the empty simplex, which generates C_0. Let us write $\langle v_{i_1} : \cdots : v_{i_k}\rangle$ for the generator corresponding to the simplex $[v_{i_1} : \cdots : v_{i_k}]$. The orientation can be chosen so that

$$\partial(\langle v_{i_1} : \cdots : v_{i_k}\rangle) = \sum_{j=1}^{k} (-1)^j (1 - t_{i_j}) \langle v_{i_1} : \cdots : \widehat{v}_{i_j} : \cdots : v_{i_k}\rangle$$

where $\widehat{v}_{i_j}$ indicates that this vertex is omitted.

Proposition 3.5 *Let L be a flag complex, L^* a full subcomplex, and $L^\dagger$ the full subcomplex spanned by the vertices in $L - L^*$. Let M be a right $\mathbb{Z}G_L$–module. Then there exists a spectral sequence $(E^r_{p\,q})$ with*

$$E^1_{p\,q} = \bigoplus_{\sigma \in (L^\dagger)^{(p-1)}} H_q(G_{L^*(\sigma)}; M),$$

which converges to $H_{p+q}(G_L; M)$. Here $(L^\dagger)^{(p-1)}$ denotes the set of $(p-1)$–simplices in $L^\dagger$; in the case $p = 0$ this set contains the empty simplex.

If $M = r^*N$, where N is a right $\mathbb{Z}G_{L^*}$–module and $r\colon G_L \to G_{L^*}$ is the retraction sending the generators corresponding to vertices of $L^\dagger$ to the trivial element, this spectral sequence collapses at E^1 and we get

$$H_*(G_L; M) \cong \bigoplus_{\sigma \in L^\dagger} H_{*-|\sigma|-1}(G_{L^*(\sigma)}; N),$$

where the direct sum is over all simplices σ in $L^\dagger$, including the empty simplex.

Proof Define a free $\mathbb{Z}G_L$–double complex C_{**} by

$$C_{p\,q} = \bigoplus_{\sigma \in (L^\dagger)^{(p-1)}} \mathbb{Z}G_L \otimes_{\mathbb{Z}G_{L^*(\sigma)}} C_q(L^*(\sigma)).$$

Let us denote the generators of $C_q(L^*(\sigma))$ by $\langle v_{j_1} : \cdots : v_{j_{q+1}}\rangle|_{\langle v_{i_1}:\cdots:v_{i_p}\rangle}$, where $\sigma = [v_{i_1} : \cdots : v_{i_p}] \in (L^\dagger)^{(p-1)}$. Then let $\partial''\colon C_{p\,q} \to C_{p\,q-1}$ be given by

$$\partial''(1 \otimes \langle v_{j_1} : \cdots : v_{j_{q+1}}\rangle|_{\langle v_{i_1}:\cdots:v_{i_p}\rangle}) = (-1)^p \sum_{k=1}^{q+1} (-1)^k \otimes (1 - t_{j_k})\langle v_{j_1} : \cdots : \widehat{v}_{j_k} : \cdots : v_{j_{q+1}}\rangle|_{\langle v_{i_1}:\cdots:v_{i_p}\rangle},$$

and $\partial'\colon C_{p\,q} \to C_{p-1\,q}$ be given by

$$\partial'(1 \otimes \langle v_{j_1} : \cdots : v_{j_{q+1}}\rangle|_{\langle v_{i_1}:\cdots:v_{i_p}\rangle}) = (-1)^{q+1} \sum_{l=1}^{p} (-1)^l (1 - t_{j_l}) \otimes \langle v_{j_1} : \cdots : v_{j_{q+1}}\rangle|_{\langle v_{i_1}:\cdots:\widehat{v}_{i_l}:\cdots:v_{i_p}\rangle}.$$

The total complex $(TC, \partial' + (-1)^p\,\partial'')$ can be identified with $C_*(X_L)$ via

$$1 \otimes \langle v_{j_1} : \cdots : v_{j_{q+1}}\rangle|_{\langle v_{i_1}:\cdots:v_{i_p}\rangle} \leftrightarrow \langle v_{j_1} : \cdots : v_{j_{q+1}} : v_{i_1} : \cdots : v_{i_k}\rangle.$$

The first part of the theorem follows directly.

If M is of the form r^*N for some $\mathbb{Z}G_{L^*}$–module N, we get that

$$1 \otimes \partial'\colon M \otimes_{\mathbb{Z}G_L} C_{p\,q} \to M \otimes_{\mathbb{Z}G_L} C_{p-1\,q}$$

is the zero homomorphism, because t_{j_l} acts trivial on M for $v_{j_l} \in L^\dagger$. Therefore $M \otimes_{\mathbb{Z}G_L} TC_*$ is a direct sum of chain complexes

$$M \otimes_{\mathbb{Z}G_L} \mathbb{Z}G_L \otimes_{\mathbb{Z}G_{L^*(\sigma)}} C_{*-|\sigma|-1}(X_{L^*(\sigma)}) = N \otimes_{\mathbb{Z}G_{L^*}} C_{*-|\sigma|-1}(X_{L^*(\sigma)}).$$

The result follows. □

4 Novikov homology of right-angled Artin groups

In this section, we want to express the Novikov homology of G_L in terms of the flag complex L. Let $\chi\colon G_L \to \mathbb{R}$ be a homomorphism, and let L^* be the full subcomplex of L corresponding to the vertices v_i with $\chi(t_i) \neq 0$. Similarly, let $L^\dagger$ be the full subcomplex of L corresponding to the vertices v_j with $\chi(t_j) = 0$. The retraction $r\colon G_L \to G_{L^*}$, which sends all the generators corresponding to vertices of $L^\dagger$ to 1, induces a ring homomorphism $r\colon \widehat{\mathbb{Z}G_L}_\chi \to \widehat{\mathbb{Z}G_{L^*}}_\chi$.

Theorem 4.1 *Let L be a flag complex, $\chi\colon G_L \to \mathbb{R}$ a homomorphism and L^*, $L^\dagger$ be as above. If M is a right $\widehat{\mathbb{Z}G_L}_\chi$–module which is torsion-free as an abelian group, then there is a spectral sequence $(E^r_{p\,q})$ with*

$$E^1_{p\,q} = \bigoplus_{\sigma \in (L^\dagger)^{(p-1)}} M \otimes_{\mathbb{Z}} \widetilde{H}_{q-1}(L^*(\sigma))$$

converging to $H_{p+q}(G_L; M)$.

If $M = r^* N$ for a right $\widehat{\mathbb{Z}G_{L^*}}_\chi$–module N, the spectral sequence collapses and

$$H_*(G_L; M) \cong \bigoplus_{\sigma \in L^\dagger} N \otimes_{\mathbb{Z}} \widetilde{H}_{*-|\sigma|-2}(L^*(\sigma)).$$

Proof For any simplicial complex L we can look at the reduced chain complex $\widetilde{C}_*(L)$, with $\widetilde{C}_k(L)$ the free abelian group generated by the k–simplices. Note that $\widetilde{C}_{-1}(L) = \mathbb{Z}$ is generated by the empty simplex. Also, let $\widetilde{C}^+_*(L)$ be the suspension of $\widetilde{C}_*(L)$, that is, $\widetilde{C}^+_n = \widetilde{C}_{n-1}(L)$ together with the obvious boundary map.

Now define $\varphi_n\colon \widehat{\mathbb{Z}G_{L^*}}_\chi \otimes_{\mathbb{Z}} \widetilde{C}^+_n(L^*) \to \widehat{\mathbb{Z}G_{L^*}}_\chi \otimes_{\mathbb{Z}G_{L^*}} C_n(X_{L^*})$ by

$$\varphi_n(1 \otimes [v_{i_1} : \cdots : v_{i_n}]) = \prod_{j=1}^{n} (1 - t_{i_j})^{-1} \otimes \langle v_{i_1} : \cdots : v_{i_n} \rangle.$$

Note that, since $[v_{i_1} : \cdots : v_{i_n}]$ is a simplex, all t_{i_j} commute. Also, as $\chi(t_{i_j}) \neq 0$, $1 - t_{i_j}$ is invertible. The inverse is $1 + t_{i_j} + t_{i_j}^2 + \cdots$ or $t_{i_j}^{-1} + t_{i_j}^{-2} + \cdots$, depending on whether $\chi(t_{i_j}) > 0$ or $\chi(t_{i_j}) < 0$.

It follows that φ commutes with the boundary and therefore induces an isomorphism between free $\widehat{\mathbb{Z}G_{L^*}}_\chi$–chain complexes. Now if M is a right $\widehat{\mathbb{Z}G_{L^*}}_\chi$–module, we get

$$\begin{aligned} H_n(G_{L^*}; M) &\cong H_n(M \otimes_{\widehat{\mathbb{Z}G_{L^*}}_\chi} \widehat{\mathbb{Z}G_{L^*}}_\chi \otimes_{\mathbb{Z}} \widetilde{C}^+_*(L^*)) \\ &\cong H_n(M \otimes_{\mathbb{Z}} \widetilde{C}^+_*(L^*)), \end{aligned}$$

and the right-hand side is $M \otimes_{\mathbb{Z}} \widetilde{H}_{n-1}(L^*)$ by the classical Universal Coefficient Theorem, provided that M is torsion-free.

The same argument works for $L^*(\sigma)$ for every simplex σ of $L^\dagger$, so the result follows from Proposition 3.5. □

Let us note two special cases.

Corollary 4.2 *Let $\chi\colon G_L \to \mathbb{R}$ be a homomorphism with $\chi(t_i) \neq 0$ for all generators of G_L. Then*

$$H_n(G_L; \widehat{\mathbb{Z}G_L}_\chi) \cong \widehat{\mathbb{Z}G_L}_\chi \otimes_{\mathbb{Z}} \widetilde{H}_{n-1}(L)$$

for all $n \in \mathbb{Z}$. □

Corollary 4.3 *Let L be a flag complex, $\chi\colon G_L \to \mathbb{R}$ a homomorphism, L^* the full subcomplex generated by the vertices whose image under χ is nonzero, and $L^\dagger$ the full subcomplex generated by the vertices whose image under χ is zero. Then*

$$H_n(G_L; \widehat{N}_\chi) \cong \bigoplus_{\sigma \in L^\dagger} \widehat{N}_\chi \otimes_{\mathbb{Z}} \widetilde{H}_{n-|\sigma|-2}(L^*(\sigma))$$

for all $n \in \mathbb{Z}$.

Proof Simply note that $\widehat{N}_\chi$ viewed as a $\widehat{\mathbb{Z}G_L}_\chi$–module is of the form $r^*\widehat{N}_\chi$ with $\widehat{N}_\chi$ viewed as a $\widehat{\mathbb{Z}G_{L^*}}_\chi$–module. □

Remark 4.4 Since L is a finite simplicial complex, the groups $\widetilde{H}_k(L^*(\sigma))$ are finitely generated abelian groups. If we write $\mathbb{Z}/n = \mathbb{Z}/n\mathbb{Z}$, it is easy to see that

$$\widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \widehat{\mathbb{Z}/nG}_\chi$$

for any group G and homomorphism $\chi\colon G \to \mathbb{R}$. Therefore every nonzero summand in $\widetilde{H}_{n-|\sigma|-2}(L^*(\sigma))$ leads to a nonzero summand in $H_n(G_L; \widehat{N}_\chi)$. In particular, for the Novikov–Betti numbers we obtain

$$b_i(G_L; \chi) = \sum_{\sigma \in L^\dagger} \widetilde{b}_{i-|\sigma|-2}(L^*(\sigma)),$$

where $\widetilde{b}_i$ is the "reduced" Betti number, that is, the rank of $\widetilde{H}_i$, and for the torsion coefficients

$$\max\{q_{i-|\sigma|-2}(L^*(\sigma)) \mid \sigma \in L^\dagger\} \leq q_i(G_L; \chi) \quad \leq \sum_{\sigma \in L^\dagger} q_{i-|\sigma|-2}(L^*(\sigma)).$$

We can therefore recover the homological version of the main theorems of [7; 19].

Corollary 4.5 *Let L be a flag complex, $\chi\colon G_L \to \mathbb{R}$ a homomorphism, L^* the full subcomplex generated by the vertices whose image under χ is nonzero, and $L^\dagger$ the full subcomplex generated by the vertices whose image under χ is zero. Then $\chi \in \Sigma^n(G_L; \mathbb{Z})$ if and only if for every simplex σ of $L^\dagger$ (including the empty simplex) $L^*(\sigma)$ is $(n-|\sigma|-2)$–acyclic.*

Proof If all $L^*(\sigma)$ are $(n-|\sigma|-2)$–acyclic, then by Theorem 4.1, $H_i(G_L;\widehat{\mathbb{Z}G_L}_\chi)=0$ for all $i\le n$ which gives $\chi\in\Sigma^n(G_L;\mathbb{Z})$ by Lemma 2.4.

If $L^*(\sigma)$ is not $(n-|\sigma|-2)$–acyclic for some σ, we see from Corollary 4.3 together with Corollary 2.5 that $\chi\notin\Sigma^n(G_L;\mathbb{Z})$. □

If χ vanishes on certain generators, it is in general very difficult to make precise calculations with Theorem 4.1, but simpler calculations can sometimes be made.

Example 4.6 Let L be a finite flag complex and $\chi\colon G_L\to\mathbb{R}$ a homomorphism which is nonzero on every generator. Extend χ to $\chi\colon G_L\times\mathbb{Z}\to\mathbb{R}$ by sending the extra generator to 0. The augmentation $\varepsilon\colon \widehat{\mathbb{Z}G_L\times\mathbb{Z}}_\chi\to\widehat{\mathbb{Z}G_L}_\chi$ induces a $\widehat{\mathbb{Z}G_L\times\mathbb{Z}}_\chi$–module structure on $\widehat{\mathbb{Z}G_L}_\chi$ and there is a short exact sequence

$$0\longrightarrow\widehat{\mathbb{Z}G_L\times\mathbb{Z}}_\chi\xrightarrow{1-t}\widehat{\mathbb{Z}G_L\times\mathbb{Z}}_\chi\xrightarrow{\varepsilon}\widehat{\mathbb{Z}G_L}_\chi\longrightarrow 0$$

where t corresponds to the generator of $\mathbb{Z}$. The differential d^1 in the spectral sequence of Theorem 4.1 is induced by multiplication with $1-t$, and so $E^2_{0q}=\widehat{\mathbb{Z}G_L}_\chi\otimes_{\mathbb{Z}}\widetilde{H}_{q-1}(L)$. As $E^2_{pq}=0$ for $p\neq 0$, we get

$$H_*(G_L\times\mathbb{Z};\widehat{\mathbb{Z}G_L\times\mathbb{Z}}_\chi)\cong\widehat{\mathbb{Z}G_L}_\chi\otimes_{\mathbb{Z}}\widetilde{H}_{*-1}(L).$$

By Corollary 4.3 we have

$$H_*(G_L\times\mathbb{Z};\widehat{N}_\chi)=(\widehat{N}_\chi\otimes_{\mathbb{Z}}H_{*-1}(L))\oplus(\widehat{N}_\chi\otimes_{\mathbb{Z}}H_{*-2}(L)).$$

Remark 4.7 In [19] the invariant $\Sigma^k(G;\mathbb{Z})$ is also considered for arbitrary commutative rings R. To define the invariant $\Sigma^k(G;R)$ one has to replace the $\mathbb{Z}$ in the definition of $\Sigma^k(G;\mathbb{Z})$ systematically by R, for example, one considers resolutions over RG_χ, and the relevant Novikov homology is $\operatorname{Tor}_p^{RG}(\widehat{RG}_\chi,R)$. The criterion in Corollary 4.5 is then that $\chi\in\Sigma^n(G_L;R)$ if and only if for every simplex σ of $L^\dagger$, $L^*(\sigma)$ is $(n-|\sigma|-2)$–R–acyclic, with a space X being k–R–acyclic if $\widetilde{H}_i(X;R)=0$ for $i\le k$.

The above proof carries over, except that one has to be slightly more careful in two steps. Firstly, in Theorem 4.1 an R–torsion-free $\widehat{RG_L}_\chi$–module M need not be flat over R. However, for Corollary 4.5 we are only interested in the first nonvanishing homology group, and by the universal coefficient spectral sequence this is $M\otimes_R H_i(L^*(\sigma);R)$ for some i and σ.

Secondly, if M is an R–module, it need not be the case that $\widehat{RG_L}_\chi \otimes_R M \cong \widehat{MG_L}_\chi$, as in Remark 4.4. But there is a commutative diagram

$$\begin{array}{ccc} RG_L \otimes_R M & \xrightarrow{\cong} & MG_L \\ \downarrow & & \downarrow \\ \widehat{RG_L}_\chi \otimes_R M & \longrightarrow & \widehat{MG_L}_\chi \end{array}$$

which shows that $\widehat{RG_L}_\chi \otimes_R M$ is nontrivial if and only if M is.

5 The homotopy type of halfspaces

For the homotopical Sigma invariants we want to understand the homotopy type of the halfspaces $N^r = h^{-1}([r,\infty))$ with respect to some χ and a height function h. Let us begin by constructing a specific height function for X_L, the universal cover of Q_L.

We choose a basepoint $* \in X_L$ which is a lift of the unique 0–cell in Q_L. We then get an embedding $G \hookrightarrow X_L$ sending g to $g*$. This can be repeated for every full subcomplex $K \subset L$, resulting in inclusions $i_K^L\colon X_K \hookrightarrow X_L$ sending $*$ to $*$.

Notice that the cells in X_L are cubical in the sense that the characteristic maps for each cell are of the form $\varphi\colon [0,1]^k \to X_L$.

Lemma 5.1 *There exists a collection of height functions $h_K\colon X_K \to \mathbb{R}$ for every full subcomplex $K \subset L$ with the following properties.*

(1) *For every pair $K_1 \subset K_2$ of full subcomplexes of L we have $h_{K_2} \circ i_{K_1}^{K_2} = h_{K_1}$.*

(2) *We have $h_L(*) = 0$.*

(3) *For every cell in X_L there is a characteristic map $\varphi\colon [0,1]^k \to X_L$ such that $h_L \circ \varphi\colon [0,1]^k \to \mathbb{R}$ is linear.*

Proof The proof is by induction on the number of vertices in L. For the empty subcomplex note that $X_\varnothing = \{*\}$, and we let $h_\varnothing(*) = 0$.

Let $v_1, \ldots, v_n$ be the vertices of L, and let L^* be the full subcomplex containing the vertices $v_1, \ldots, v_{n-1}$. We also write $K = L^*(v_n)$. It follows from (3) (see also Geoghegan [11, Chapter 6]) that

(4) $$X_L = G_L \times_{G_{L^*}} X_{L^*} \cup G_L \times_{G_K} X_K \times [0,1]/\sim$$

where $[g,x,0]\sim[g,i_K^{L^*}(x)]$ and $[g,x,1]\sim[gt_n,i_K^{L^*}(x)]$ for all $g\in G_L$, $x\in X_K$. Here $G\times_H X$ is the quotient space of $G\times X$ via the H–action $h\cdot(g,x)=(gh^{-1},hx)$ where H is a subgroup of G and X a space with left H–action.

Assume by induction that h_{L^*} and h_K exist with the required properties. Then define

$$\begin{aligned} h_L([g,x]) &= \chi(g)+h_{L^*}(x) && \text{for } g\in G_L,\ x\in X_{L^*}\\ h_L([g,x,t]) &= \chi(g)+h_K(x)+t\cdot\chi(t_n) && \text{for } g\in G_L,\ x\in X_K. \end{aligned}$$

It is easy to see that this is well defined and has the required properties. □

Let N_L be the maximal subcomplex of X_L contained in $N^0=h_L^{-1}([0,\infty))$. Then the monoid

$$G_L^+=\{g\in G_L\mid \chi(g)\geq 0\}$$

acts on N_L. We can get an inductive description for N_L as in (4). For this let $v_1,\dots,v_n$ be the vertices of L, and let L^* be the full subcomplex containing the vertices $v_1,\dots,v_{n-1}$. We again write $K=L^*(v_n)$. Then

$$N_L=G_L^+\times_{G_{L^*}^+}N_{L^*}\cup G_L^+\times_{G_K^+}N_K\times[0,1]/\sim$$

with $\sim$ as in (4), and where $G_L^+\times_{G_{L^*}^+}N_{L^*}$ is the quotient space of $G_L^+\times N_{L^*}$ via identifying (gh,x) with (g,hx) for $g\in G_L^+$, $h\in G_{L^*}^+$ and $x\in N_{L^*}$.

Remark 5.2 Using the methods of [7] it is easy to see that N_L has the homotopy type of N^0, but we will not need this result. As there is an $r>0$ with $N^r\subset N_L$, we can use N_L and its translates $g\cdot N_L$ for $g\in G_L$ in the definition of $\Sigma^k(G_L)$.

In [1, Th.8.6] it is shown (in the case $\chi(t_i)=1$ for all generators of G_L) that N_L has the homotopy type of a wedge of L's. The statement is not completely precise in the case when L is disconnected; compare the note below [1, Th.8.6].

We now want to give an alternative approach to determining the homotopy type of N_L which will also discuss the functoriality induced by subcomplexes $K\subset L$.

Let us analyze the components of N_L. Since G_L^+ acts on N_L it also acts on $\pi_0(N_L)=\pi_0(N_L,*)$, where $*\in N_L$ is the basepoint. If we denote the component of $x\in N_L$ by $[x]$, it is clear that every component is of the form $g\cdot[*]$ with $g\in G_L^+$.

Lemma 5.3 *Every component of N_L is of the form $g\cdot[*]$ with $g\in\ker\chi$.*

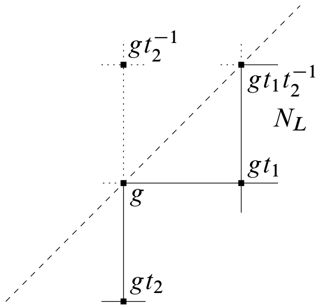

Figure 1

Proof We can assume that $\chi(t_i) \geq 0$ for every generator of G_L. Let $g \cdot [*]$ be a component with $\chi(g) \geq 0$. Clearly $g*$ and gt_i* are in the same component for every generator (compare Figure 1), so we can assume that $g[G,G] = t_1^{k_1} \cdots t_n^{k_n}[G,G]$ with all $n_i \geq 0$, where $[G,G]$ is the commutator subgroup. Then $gt_1^{-k_1} \cdots t_n^{-k_n} \in \ker \chi$, and $g*$ is in the same component as $gt_1^{-k_1} \cdots t_n^{-k_n}*$. □

If we think of the set of components as a discrete space, we get that

$$\pi_L = G_L^+ \times_{G_{L^*}^+} \pi_0(N_{L^*}) \cup G_L^+ \times_{G_K^+} \pi_0(N_K) \times [0,1]/\sim$$

with $[g,[x],0] \sim [g, i_K^{L^*}[x]]$ and $[g,[x],1] \sim [g, i_K^{L^*}[x]]$, is a graph with G_L^+–action, such that $\pi_0(\pi_L) = \pi_0(N_L)$.

Lemma 5.4 *The graph π_L is a forest, that is, a disjoint union of trees.*

Proof Since all the components of π_L are homeomorphic with a homeomorphism induced by some $g \in \ker \chi$, we only have to consider the component Γ containing $[1,[*]] \in G_L^+ \times_{G_{L^*}^+} \pi_0(N_{L^*})$. Let

$$H^+ = \{h \in G_{L^*}^+ \mid h[*] = [*]\},$$

which is a monoid. Note that $[gh^{-1},[*]] = [gh^{-1}, h[*]] = [g,[*]]$ for $h \in H^+$, provided that $\chi(gh^{-1}) \geq 0$.

Two elements $[g_1,[*]], [g_2,[*]] \in \pi_L$ are connected by an edge if and only if there are $h,k \in H^+$ with $g_2 = g_1 h^{\varepsilon} t_n^{\varepsilon'} k^{\varepsilon''}$ and $\varepsilon, \varepsilon', \varepsilon'' \in \{\pm 1\}$. We have to show that edge-loops in Γ are contractible, that is, finite sequences of points $[g_0,[*]], \ldots, [g_k,[*]]$ with $[g_i,[*]]$ and $[g_{i+1},[*]]$ connected by an edge, and such that $[g_0,[*]] = [g_k,[*]] = [1,[*]]$.

Hence there exist $h_i, k_i \in H^+$ with $g_{i+1} = g_i h_i^{\varepsilon_i} t_n^{\varepsilon_i'} k_i^{\varepsilon_i''}$. Therefore

$$1 = h_0^{\varepsilon_0} t_n^{\varepsilon_0'} k_0^{\varepsilon_0''} \cdots h_{n-1}^{\varepsilon_{n-1}} t_n^{\varepsilon_{n-1}'} k_{n-1}^{\varepsilon_{n-1}''}.$$

But by the Normal Form Theorem for HNN–extensions [18, Chapter IV] we get that there is an i and a subword $t_n^{\varepsilon_{i-1}'} k_{i-1}^{\varepsilon_{i-1}''} h_i^{\varepsilon_i} t_n^{\varepsilon_i'}$ with $k_{i-1}^{\varepsilon_{i-1}''} h_i^{\varepsilon_i} \in K$ and $\varepsilon_{i-1}' = -\varepsilon_i'$. Therefore

$$\begin{aligned} g_{i+1} &= g_{i-1} h_{i-1}^{\varepsilon_{i-1}} t_n^{\varepsilon_{i-1}'} k_{i-1}^{\varepsilon_{i-1}''} h_i^{\varepsilon_i} t_n^{\varepsilon_i'} k_i^{\varepsilon_i''} \\ &= g_{i-1} h_{i-1}^{\varepsilon_{i-1}} k_{i-1}^{\varepsilon_{i-1}''} h_i^{\varepsilon_i} k_i^{\varepsilon_i''} \end{aligned}$$

and $[g_{i+1}, [*]] = [g_{i-1} h_{i-1}^{\varepsilon_{i-1}} k_{i-1}^{\varepsilon_{i-1}''} h_i^{\varepsilon_i} k_i^{\varepsilon_i''}, [*]] = [g_{i-1}, [*]]$. Therefore the loop represented by $[g_{i-1}, [*]]$, $[g_i, [*]]$ and $[g_{i+1}, [*]]$ is null-homotopic, and the result follows by induction. □

Recall that $v_1, \dots, v_n$ are the vertices of the flag complex L. Define an equivalence relation on the set of vertices by $v_i \sim v_j$ if they are in the same component. Denote an equivalence class by $*_i$ and embed $*_i$ into L by choosing a representative. This defines a basepoint in every component of L.

Lemma 5.5 *Let $\chi\colon G_L \to \mathbb{R}$ be a homomorphism with $\chi(t_i) > 0$ for all generators of G_L, and let $g \in G_L$. If t_i, t_j are generators of G_L such that $v_i \sim v_j$, then $gt_i^r[*] = gt_j^s[*]$ for all $r, s \geq 0$ with $gt_i^r, gt_j^s \in G_L^*$.*

Proof We have that $*$ and $t_k *$ are connected by a 1–cell in N_L for all generators t_k. So if $h \in G_L^+$, we get $h*$ and ht_k* are connected by a 1–cell in N_L. So for $k \geq 0$ we get

$$gt_i^r[*] = [gt_i^r *] = [gt_i^{r+k} *] = gt_i^{r+k}[*].$$

Also if t_i and t_j commute, we get

$$gt_i^r[*] = gt_i^r t_j^s[*] = gt_j^s t_i^r[*] = gt_j^s[*].$$

If v_i and v_j are in the same component of L, there is a finite sequence of generators $t_i = t_{i_0}, \dots, t_{i_k} = t_j$ with t_{i_m} and $t_{i_{m+1}}$ commuting, and we get

$$gt_i^r[*] = gt_{i_1}^{r_1}[*] = \cdots = gt_{i_{k-1}}^{r_{k-1}}[*] = gt_j^s[*]$$

by the argument above. □

Proposition 5.6 *Let $\chi\colon G_L \to \mathbb{R}$ a homomorphism, L^* the full subcomplex containing the vertices with $\chi(t_i) \neq 0$ and let $L^\dagger$ be the full subcomplex containing the vertices with $\chi(t_j) = 0$. Then the following are equivalent.*

(1) $\chi \in \Sigma^1(G_L)$.

(2) N_L *is connected.*

(3) L^* *is connected, and for every vertex* $v_i \in L^\dagger$ *we have* $L^*(v_i)$ *is nonempty.*

Proof (1) $\Rightarrow$ (3) follows from Corollary 4.5, because $\Sigma^1(G) = \Sigma^1(G;\mathbb{Z})$.

(2) $\Rightarrow$ (1) follows from the Definition.

(3) $\Rightarrow$ (2) is proven by induction on the number of vertices in $L^\dagger$. If $L^\dagger = \varnothing$, we can assume that $\chi(t_i) > 0$ for all generators, as replacing t_i with t_i^{-1} induces an automorphism of G_L.

Let $g \in G_L^+$, so that $g[*]$ is a component of N_L. If g is a word which uses only positive powers of the generators t_i, it is clear that $g[*] = [*]$. Otherwise let $g = ht_j^{-1}w$, with w a word which uses only positive powers of the generators. But by Lemma 5.5 there is a $r \geq 1$ with $ht_j^{-1}w[*] = ht_j^{r-1}[*]$. Therefore we can reduce the number of negative powers in g without changing the component. By induction, we get $g[*] = [*]$.

If $\chi(t_n) = 0$, let $\bar{L}$ be the full subcomplex of L containing all vertices except v_n. Then $N_{\bar{L}}$ is connected by (3) and the induction assumption. Also, there is a t_i in G_{L^*} which commutes with t_n. If $g \in G_{\bar{L}}$, we get $g[*] = [*]$ by the connectivity of $N_{\bar{L}}$. If $g = ht_n^s w$ with $w \in G_{\bar{L}}$, let $r \geq 0$ such that $w^{-1}t_i^r \in G_{\bar{L}}^+$. Then $g[*] = ht_n^s w w^{-1} t_i^r[*] = ht_i^r t_n^s[*] = ht_i^r[*]$, as $t_n^s[*] = [*]$. Therefore we can reduce the occurrences of t_n, which shows by induction that $g[*] = [*]$. $\square$

We will for now assume that $\chi(t_i) > 0$ for all generators t_i of G_L.

Define
$$M_L = G_L \times L \cup \pi_0(N_L)/\sim$$
where $(g, *_i) \sim gt_i^r[*]$ for $g \in G_L$ and $r \geq 0$ such that $\chi(gt_i^r) \geq 0$, and $(g,x) \sim g[*]$ for $g \in G_L^+$ and all $x \in L$. In words, M_L has a copy of L for every $g \in G_L$ with $\chi(g) < 0$, and basepoints are identified with certain components of N_L. Clearly, M_L has a G_L^+–action.

Example 5.7 If L is connected, we get $\pi_0(N_L)$ is a point by Proposition 5.6, and M_L is a wedge of copies of L, one for each $g \in G_L$ with $\chi(g) < 0$.

If $K \subset L$ is a full subcomplex, the inclusion need not preserve basepoints. In fact, K can have more components than L, but we can choose basepoints for the components of K as above for L. Then choose a map $j_K^L\colon K \to L$ homotopic to the inclusion which sends basepoints to basepoints. If $K_1 \subset K_2 \subset L$ then we get maps with $j_{K_2}^L \circ j_{K_1}^{K_2} \simeq j_{K_1}^L$. This induces equivariant maps $\varphi_K^L\colon M_K \to M_L$ with $\varphi_{K_2}^L \circ \varphi_{K_1}^{K_2} \simeq \varphi_{K_1}^L$ equivariantly.

Proposition 5.8 *For every full subcomplex $K \subset L$ there is an equivariant map $\psi_K\colon N_K \to M_K$ which is an unequivariant homotopy equivalence, and such that for $K_1 \subset K_2 \subset L$ the diagram*

$$\begin{array}{ccc} N_{K_1} & \xrightarrow{i_{K_1}^{K_2}} & N_{K_2} \\ \downarrow \psi_{K_1} & & \downarrow \psi_{K_2} \\ M_{K_1} & \xrightarrow{\varphi_{K_1}^{K_2}} & M_{K_2} \end{array}$$

commutes up to equivariant homotopy.

Proof The proof is by induction on the number of vertices in L. For 0 or 1 vertex the statement is clear.

Using induction, we get

$$\begin{aligned} N_L &= G_L^+ \times_{G_{L^*}^+} N_{L^*} \cup G_L^+ \times_{G_K^+} N_K \times [0,1]/\sim \\ &\simeq G_L^+ \times_{G_{L^*}^+} M_{L^*} \cup G_L^+ \times_{G_K^+} M_K \times [0,1]/\approx \end{aligned}$$

with $[g,y,0] \approx [g,\varphi_K^{L^*}(y)]$ and $[g,y,1] \approx [gt_n,\varphi_K^{L^*}(y)]$ for $g \in G_L^+$ and $y \in N_K$, via an equivariant map with domain N_L. The right-hand-side written out is

$$\Big(\big(G_L \times L^* \cup G_L^+ \times_{G_{L^*}^+} \pi_0(N_{L^*})\big)/\sim\Big) \cup \Big(\Big(\big(G_L \times K \cup G_L^+ \times_{G_K^+} \pi_0(N_K)\big)/\sim\Big) \times [0,1]\Big)\Big/\approx \tag{5}$$

with identifications as before. If we do the $\approx$–identification in two steps, we get

$$N_L \simeq (G_L \times L^* \cup \pi_L/\sim) \cup (G_L \times K \times [0,1])\big/\approx \tag{6}$$

with $(g,x,0) \approx (g, j_K^{L^*}(x))$ and $(g,x,1) \approx (gt_n, j_K^{L^*}(x))$ for $x \in K$ and $g \in G_L$, and for $s \in [0,1]$ we also identify $(g,x,s) \approx [g,[*],s]$ for $g \in G_L^+$, and $(g,*_i,s) \approx [gt_i^r,[*],s]$, provided $\chi(gt_i^r) \geq 0$.

This space is the forest π_L, together with copies of L^* wedged to it, one for each $g \in G_L$ with $\chi(g) < 0$, and such that the copies of L^* corresponding to g and gt_n are connected via $K \times [0,1]$.

Now if $\chi(g) < 0$, but $\chi(gt_n) \geq 0$, we get that L^* corresponding to g is being coned off along K. Denote this as $C_K L^*$, which is homotopy equivalent to L. Also, if form $L^* \cup K \times [0,1] \cup C_K L^*$ by identifying $K \times \{0,1\}$ with copies in L^* and $C_K L^*$, it is easy to see that the result is homotopy equivalent to the wedge of L.

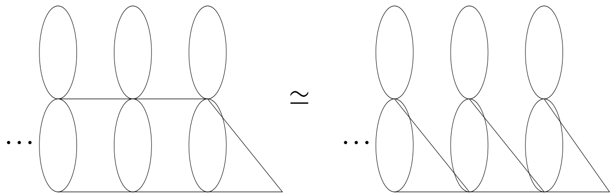

Figure 2

In the right-hand-side of (6) we have infinite sequences of such objects; compare Figure 2. It follows that the right hand side is homotopy equivalent to M_L by this and collapsing the forest π_L to its components.

To see that the construction is natural with respect to full subcomplexes, note that we can do the above construction for every subcomplex containing v_n and so that they are natural up to equivariant homotopy. If we consider a subcomplex not containing v_n, naturality follows by induction. □

For $r \in \operatorname{im}\chi$ let $N_L^r = hN_L$, where $h \in G_L$ satisfies $\chi(h) = r$. Also let

$$M_L^r = G_L \times L \cup \pi_0(N_L^r)/\sim$$

where $(g, *_i) \sim [gt_i^s *]$ for $g \in G_L$ and $s \geq 0$ such that $\chi(gt_i^s) \geq r$, and $(g, x) \sim [g*]$ for $\chi(g) \geq r$. For $r < s$ we have an obvious projection $p^{sr}\colon M_L^s \to M_L^r$.

If $g \in G_L$ satisfies $\chi(g) = r$, we get a homeomorphism $u_g\colon M_L \to M_L^r$ by $u_g([h, x]) = [gh, x]$ and $u_g([x]) = [gx]$ for $(h, x) \in G_L \times L$ and $[x] \in \pi_0(N_L)$. If $v_{g^{-1}}\colon N_L^r \to N_L$ denotes left-multiplication by g^{-1}, we get that

$$\psi_L^r = u_g \circ \psi_L \circ v_{g^{-1}}\colon N_L^r \to M_L^r$$

is a homotopy equivalence which is G_L^+ equivariant, and for $r < s$ the diagram

(7)
$$\begin{array}{ccc} N_L^s & \longrightarrow & N_L^r \\ \Big\downarrow \psi_L^s & & \Big\downarrow \psi_L^r \\ M_L^s & \xrightarrow{p^{sr}} & M_L^r \end{array}$$

commutes, as is easily seen from the cases with r or s equal 0.

For ψ_L to be a homotopy equivalence, we need χ to be nonzero on all generators, and we will not attempt to describe the homotopy type of N_L in general. But the next Lemma contains partial information which is useful for determining $\Sigma^2(G_L)$ in general.

Lemma 5.9 *Let $\chi\colon G_L \to \mathbb{R}$ a homomorphism, L^* the full subcomplex containing the vertices with $\chi(t_i) \neq 0$ and let $L^\dagger$ be the full subcomplex containing the vertices with $\chi(t_j) = 0$. For every $s \in \operatorname{im}\chi$ there is a retraction $\rho^s\colon N^s_L \to N^s_{L^*}$ which makes the diagram*

$$\begin{array}{ccc} N^s_L & \xrightarrow{\subset} & N^r_L \\ \downarrow{\rho^s} & & \downarrow{\rho^r} \\ N^s_{L^*} & \xrightarrow{\subset} & N^r_{L^*} \end{array}$$

commute for every $s > r \in \operatorname{im}\chi$.

Proof The proof is by induction on the number of vertices in $L^\dagger$ with the induction start $L^\dagger$ empty being trivial. Assume the statement holds for $\overline{L}$ a full subcomplex L containing L^*. If $v_n \notin \overline{L}$, let K be the full subcomplex of $\overline{L}$ containing all vertices adjacent to v_n. Then if L' is the full subcomplex of L containing $\overline{L}$ and v_n, we get

$$N^s_{L'} = G^+_{L'} \times_{G^+_{\overline{L}}} N^s_{\overline{L}} \cup G^+_{L'} \times_{G^+_K} N^s_K \times [0,1]/\sim \tag{8}$$

with identifications as in (4). If $\rho\colon G_{L'} \to G_{\overline{L}}$ is the retraction obtained by sending t_n to 1, we get a retraction $\rho^s\colon N^s_{L'} \to N^s_{\overline{L}}$ defined by $\rho^s([g,x]) = \rho(g)\cdot x$ for $g \in G^+_{L'}$ and $x \in N^s_{\overline{L}}$, and $\rho^s([g,x,t]) = r(g)\cdot i^{\overline{L}}_K$ for $g \in G^+_{L'}$, $x \in N^s_{\overline{L}}$ and $t \in [0,1]$. It is clear that the resulting diagram for $s > r$ commutes. □

Theorem 5.10 *Let $\chi\colon G_L \to \mathbb{R}$ a homomorphism, L^* the full subcomplex containing the vertices with $\chi(t_i) \neq 0$ and let $L^\dagger$ be the full subcomplex containing the vertices with $\chi(t_j) = 0$. Then the following are equivalent.*

(1) $\chi \in \Sigma^2(G_L)$.

(2) *N_L is simply connected.*

(3) *L^* is simply connected, for every vertex $v_i \in (L^\dagger)^{(0)}$ we have $L^*(v_i)$ is connected, and for every 1–simplex $\sigma \in (L^\dagger)^{(1)}$ we have $L^*(\sigma)$ is nonempty.*

Proof (1) ⇒ (3) holds by the following argument: Since $\chi \in \Sigma^2(G_L)$ implies $\chi \in \Sigma^2(G_L;\mathbb{Z})$, Corollary 4.5 implies that we only need to show that L^* is simply connected. If L^* is not simply connected, then for all $r < 0$ the homomorphism $\pi_1(N_{L^*}) \to \pi_1(N^r_{L^*})$ induced by inclusion is nontrivial, as follows from Proposition 5.8 together with diagram (7). This implies $\pi_1(N_L) \to \pi_1(N^r_L)$ is nontrivial for all $r < 0$ by Lemma 5.9, which by definition implies $\chi \notin \Sigma^2(G)$.

(3) ⇒ (2) is again shown by induction on the number vertices in $L^\dagger$. If $L^\dagger$ is empty, the result follows from Proposition 5.8. Let L', $\overline{L}$ and K be as in the proof of Lemma 5.9,

and assume inductively that $N_{\overline{L}}$ is simply connected. Note that $K^* = L^*(v_n)$ is connected, and $K^*(v_i) = L^*([v_i : v_n])$ for every vertex $v_i \in K - L^*$. Therefore N_K is connected by Proposition 5.6. It follows from (8) and a Seifert–Van Kampen argument that $N_{L'}$ is simply connected. After finitely many steps we get N_L simply connected.

(2) $\Rightarrow$ (1) follows from the definition. □

Remark 5.11 Since $\Sigma^k(G) = \Sigma^k(G;\mathbb{Z}) \cap \Sigma^2(G)$ for $k \geq 2$, Theorem 5.10 and Corollary 4.5 recover the main theorems of [7; 19].

6 Nontrivial HNN–extensions of right-angled Artin groups

Given a finite flag complex L and a full subcomplex K, we can form a new flag complex $\overline{L}$ by adding a vertex and requiring that it is adjacent to every vertex of K. The resulting right-angled Artin group $G_{\overline{L}}$ is a trivial HNN–extension of G_L along G_K; compare Remark 3.2.

We now want to look at the situation where we have two full subcomplexes K_1, K_2 of L which are isomorphic as simplicial complexes. Such an isomorphism induces an isomorphism of groups $\phi\colon G_{K_1} \to G_{K_2}$, and we can form the HNN–extension

$$G = G_L *_\phi,$$

The extra generator of G is denoted by t.

Any homomorphism $\chi\colon G_L \to \mathbb{R}$ with $\chi \circ \phi = \chi|_{G_{K_1}}$ extends to homomorphisms $\chi_x\colon G \to \mathbb{R}$ by setting $\chi_x(t) = x$ for any $x \in \mathbb{R}$. We will usually drop the subscript x in $\chi\colon G \to \mathbb{R}$.

If we assume $\chi(t_i) \neq 0$ for all generators of G_L, the exact sequence (2) becomes:

$$\widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}} \widetilde{H}_{n-1}(K_1) \xrightarrow{\alpha} \widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}} \widetilde{H}_{n-1}(L) \longrightarrow H_n(G;\widehat{\mathbb{Z}G}_\chi) \longrightarrow$$
$$\widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}} \widetilde{H}_{n-2}(K_1) \xrightarrow{\alpha} \widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}} \widetilde{H}_{n-2}(L) \longrightarrow \cdots$$

Note for example, that L connected implies $\chi \in \Sigma^1(G;\mathbb{Z})$. Also, if the Betti numbers of K_1 and L are different, we get nonvanishing $H_*(G;\widehat{\mathbb{Z}G}_\chi)$ independent of $\chi(t)$. By looking at the chain complex description in the proof of Theorem 4.1, we see that $\alpha\colon \widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}} \widetilde{H}_*(K_1) \to \widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}} \widetilde{H}_*(L)$ is given by

$$\alpha(1 \otimes z) = t \otimes j_*(z) - 1 \otimes i_*(z) \tag{9}$$

for $z \in \widetilde{H}_*(K_1)$, with $i_*\colon \widetilde{H}_*(K_1) \to \widetilde{H}_*(L)$ is induced by inclusion and j_* is induced by the isomorphism $K_1 \to K_2$ followed by inclusion. Now i_* and j_* can induce quite different maps on homology, and we want to construct examples where $K_1 \hookrightarrow L$ is a homotopy equivalence, while $K_2 \hookrightarrow L$ is not.

Definition 6.1 Let $f\colon K \to L$ be a simplicial map between finite simplicial complexes. We call f a *full simplicial embedding*, if it is injective as a continuous map, and if vertices $v_0, \dots, v_k$ span a k–simplex in K if and only if $f(v_0), \dots, f(v_k)$ span a k–simplex in L for all $k \geq 0$.

Example 6.2 Let $K = \{0, 1\}$ and $L = [0, 1]$ with two vertices 0 and 1. Then the inclusion $K \to L$ is not a full embedding. But if we subdivide L by adding a vertex $\frac{1}{2}$, the inclusion becomes a full embedding. Note that $G_K = F_2$, $G_L = \mathbb{Z}^2$, and the map induced by inclusion $i_*\colon F_2 \to \mathbb{Z}^2$ is clearly not injective. However, with L' the barycentric subdivision of L, we get $G_{L'} = F_2 \times \mathbb{Z}$, and we have an injection $i_*\colon G_K \to G_{L'}$.

Lemma 6.3 *Let $f\colon K \to L$ be an injective simplicial map between finite flag complexes. Then f is a full embedding if and only if the following holds: two vertices v_0, v_1 in K span a 1–simplex if and only if $f(v_0)$ and $f(v_1)$ span a 1–simplex in L.*

In this case, G_K is a retract of G_L.

Proof The "if and only if" statement is clear since flag complexes are determined by their 1–skeleton. The retraction for the induced homomorphism $i_*\colon G_K \to G_L$ is defined as follows: if t_i is a generator corresponding to the vertex $u_i \in L^{(0)}$, we set $r(t_i) = t_j \in G_K$, if there is a vertex $v_j \in K^{(0)}$ with $f(v_j) = u_i$, and we set $r(t_i) = 1 \in G_K$, if u_i is not in the image. From the full embedding condition it follows that r respects all relations in G_L, and we have $ri_* = \mathrm{id}_{G_K}$. □

Lemma 6.4 *Let K, L be finite flag complexes and $f\colon K \to L$ a simplicial map. Then there exists a finite flag complex M containing L as a deformation retract, and a full simplicial embedding $g\colon K \to M$ homotopic to f.*

Proof We first want to replace f by a simplicial map $g'\colon K \to M'$ which is injective on vertices. Let v_0, v_1 be vertices with $f(v_0) = f(v_1) = u \in L$. Form a new simplicial complex L_1 by forming the cone over the star of u, that is, we add one vertex u', and whenever there is a k–simplex σ involving u, we add the $(k+1)$–simplex $\sigma \cup \{u'\}$ to L_1. Then L_1 is still a flag complex and it contains L as a deformation retract. If we define $f_1\colon K \to L_1$ by $f_1(v) = f(v)$ for $v \neq v_1$, $f_1(v_1) = u'$, we get a simplicial map homotopic to $i \circ f\colon K \to L_1$, which is slightly less noninjective than f. If f_1 is not injective, we repeat this process finitely many times.

So assume that $g'\colon K \to M'$ is injective on vertices, M' contains L as a deformation retract and $i \circ f$ is homotopic to g'. Let v_0, v_1 be two vertices of K which do not form a 1–simplex, but such that $f(v_0)$ and $f(v_1)$ form a 1–simplex in M'. Let $u_0 = g'(v_0)$ and $u_1 = g'(v_1)$, and let M^* be the full subcomplex of M' containing u_1 and all vertices u adjacent to u_1, except u_0. Now form M_1 by coning off M^*, that is, we add a vertex u_1', and for every k–simplex σ in M^* we add the $(k+1)$–simplex $\sigma \cup \{u_1'\}$. Again M_1 deformation retracts to M', and defining $g_1\colon K \to M_1$ by $g_1(v) = g'(v)$ for $v \neq v_1$ and $g_1(v_1) = u_1'$ gives an injective simplicial map homotopic to $i \circ g'$.

Also, if v_2, v_3 are vertices in K not forming a 1–simplex, then $g_1(v_2), g_1(v_3)$ form a 1–simplex in M_1 if and only if $\{v_2, v_3\} \neq \{v_0, v_1\}$ and $g'(v_2), g'(v_3)$ form a 1–simplex. As there are only finitely many such pairs, we can repeat the argument finitely many times to end up with the desired full embedding. □

Let K be a finite flag complex and $f\colon K \to K$ a continuous map. By the simplicial approximation theorem, there is $r \geq 0$ and a simplicial map $f'\colon K^{[r]} \to K$ homotopic to f, where $K^{[r]}$ is the r–th barycentric subdivision of K, which is also a flag complex. By Lemma 6.4 we can find a full embedding $g\colon K^{[r]} \to M$ with M a flag complex containing K as a deformation retract.

Lemma 6.5 *There exists a finite flag complex L with the following properties:*

(1) *$K^{[r]}$ and M are full subcomplexes of L with inclusions $i\colon K^{[r]} \to L$ and $j\colon M \to L$.*

(2) *The set of vertices is the disjoint union of the vertices of $K^{[r]}$ and M.*

(3) *The full embedding $i\colon K^{[r]} \to L$ is a homotopy equivalence.*

(4) *The full embedding $j \circ g\colon K^{[r]} \to L$ is homotopic to $i \circ f\colon K \to L$.*

Proof If $r = 0$, let $L = K \times [0,1] \cup_{K \times \{1\}} M$ with the standard subdivision of $K \times [0,1]$.

If $r > 0$, let $s\colon K^{[r-1]} \to K$ be a simplicial approximation of the identity, and $M(s)$ the simplicial analogue of a mapping cylinder as constructed in [12, page 183]. By checking the construction there, one sees that $M(s)$ is a flag complex, as $K^{[r-1]}$ and K are. Furthermore, the set of vertices is the disjoint union of the vertices of $K^{[r]}$ and K, so $K^{[r]}$ and K are fully embedded in $M(s)$. Also, there is a deformation retraction $r_t\colon M(s) \to K$ with $r_1 \circ i = s\colon K^{[r]} \to K$. In particular, the inclusion $i\colon K^{[r]} \to M(s)$ is a homotopy equivalence. Now let $L = M(s) \cup_K M$. It is clear that L has the desired properties. □

Definition 6.6 Let $\widehat{\mathbb{N}}$ be the set of all nonnegative integers together with an element ∞. We order this set by the usual order of integers together with $p < \infty$ for all nonnegative integers p.

For the next theorem, we set

$$\Sigma^{\infty}(G;\mathbb{Z}) = \bigcap_{p\geq 1} \Sigma^{p}(G;\mathbb{Z})$$
$$\Sigma^{\infty+1}(G;\mathbb{Z}) = \varnothing.$$

Theorem 6.7 *Let $p,q \in \widehat{\mathbb{N}}$. Then there exists a group G of type F and a homomorphism $\chi\colon G \to \mathbb{R}$ with*

$$\chi \in \Sigma^{p}(G;\mathbb{Z}) - \Sigma^{p+1}(G;\mathbb{Z})$$
$$-\chi \in \Sigma^{q}(G;\mathbb{Z}) - \Sigma^{q+1}(G;\mathbb{Z}).$$

The homomorphism can be chosen to have image in $\mathbb{Z}$.

Proof We can assume $p \neq q$ for otherwise we can find a right-angled Artin group and χ with that property. Without loss of generality let $p > q$.

If $q \geq 1$, let K be a flag complex realizing S^q, and $f\colon S^q \to S^q$ a map of degree 2. Let L be the flag complex arising from Lemma 6.5 and M the one-point union of L and a flag complex realizing S^p (in case $p = \infty$ we set $M = L$). Then let G be the HNN–extension of G_M along $G_{K^{[r]}}$ using the two full embeddings from Lemma 6.5. To define $\chi\colon G \to \mathbb{R}$, let $\chi(t_i) = 1$ for every generator $t_i \in G_M$ and let $\chi(t) = 1$. Then $\alpha\colon \widehat{\mathbb{Z}G}_{\chi} \otimes_{\mathbb{Z}} \widetilde{H}_*(S^q) \to \widehat{\mathbb{Z}G}_{\chi} \otimes_{\mathbb{Z}} \widetilde{H}_*(S^q \vee S^p)$ is by Lemma 6.5 and (9) the map

$$\alpha(1 \otimes z) = t \otimes f_*(z) - 1 \otimes i_*(z).$$

In degree q this is the map $\alpha(x) = x(2t-1)$ for $x \in \widehat{\mathbb{Z}G}_{\chi}$, which is an isomorphism since $\chi(t) > 0$. If we look at $-\chi$, we still get the formula for α, but this time it is a map $\widehat{\mathbb{Z}G}_{-\chi} \to \widehat{\mathbb{Z}G}_{-\chi}$ and α is injective but not surjective. It follows that $-\chi \in \Sigma^q(G;\mathbb{Z}) - \Sigma^{q+1}(G;\mathbb{Z})$ from Lemma 2.4 and (2), while $\chi \in \Sigma^{q+1}(G;\mathbb{Z})$. If $p < \infty$, note that $H_{p+1}(G;\widehat{\mathbb{Z}G}_{\chi}) \neq 0$, since $\widetilde{H}_p(M) \neq 0$ while $\widetilde{H}_p(K) = 0$. Therefore $\chi \in \Sigma^p(G;\mathbb{Z}) - \Sigma^{p+1}(G;\mathbb{Z})$.

If $q = 0$ we have to use a slightly different technique. We assume $p < \infty$, for otherwise the Baumslag–Solitar group $G = \langle s,t \mid t^{-1}st = s^2\rangle$ will do.

Let L be a finite flag complex subdividing S^p, and M the union of L with two vertices v, w with v being adjacent to exactly one vertex of L; see Figure 3. Furthermore, let K be the two vertices v, w.

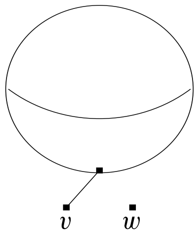

Figure 3

We denote the generators of $G_K = F_2$ by r, s where r corresponds to v. Let $\theta\colon G_K \to G_M$ be given by $\theta(r) = r$ and $\theta(s) = s^2$. Then let $G = G_M *_\theta$ and $\chi\colon G \to \mathbb{R}$ is given by sending every generator other than s to 1, and $\chi(s) = 0$.

In Theorem 4.1 we get $E^1_{1\,0} = E^1_{0\,p+1} = \widehat{\mathbb{Z}G}_\chi$ as the only nonzero terms, therefore $H_i(G_M; \widehat{\mathbb{Z}G}_\chi) \cong \widehat{\mathbb{Z}G}_\chi$ for $i = 1, p+1$ and $H_i(G_M; \widehat{\mathbb{Z}G}_\chi) = 0$ otherwise. Similarly $H_*(G_K; \widehat{\mathbb{Z}G}_\chi) \cong \widehat{\mathbb{Z}G}_\chi \otimes_\mathbb{Z} \tilde{H}_{*-2}(\varnothing)$ as $E^1_{u\,v} = 0$ for $u \neq 1$. Note that $\varnothing = K^*(w)$ and $\tilde{H}_{-1}(\varnothing) = \mathbb{Z}$. The long exact sequence (2) contains

$$\cdots \longrightarrow H_2(G; \widehat{\mathbb{Z}G}_\chi) \longrightarrow \widehat{\mathbb{Z}G}_\chi \xrightarrow{\alpha} \widehat{\mathbb{Z}G}_\chi \longrightarrow H_1(G; \widehat{\mathbb{Z}G}_\chi) \longrightarrow 0$$

where $\alpha = (t,\theta)_* - i_*$. First note that i_* is an isomorphism. To see this, look at the short exact sequence of chain complexes

$$0 \longrightarrow \widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}G_K} C_*(G_K) \longrightarrow \widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}G_M} C_*(G_M) \longrightarrow Q_* \longrightarrow 0$$

where Q_* is the free $\widehat{\mathbb{Z}G}_\chi$–chain complex with $Q_i = \widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}G_L} C_i(G_L)$ for $i \neq 0, 2$, $Q_0 = 0$ and $Q_2 = \widehat{\mathbb{Z}G}_\chi \oplus \widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}G_L} C_2(G_L)$, where the extra summand in Q_2 comes from the edge between v and L. Without the extra summand in Q_2 we would get $H_1(Q_*) = \widehat{\mathbb{Z}G}_\chi$ and $H_i(Q_*) = H_i(G_L; \widehat{\mathbb{Z}G}_\chi)$ for $i \neq 1$. The boundary of the extra summand is $\partial(x) = x(1-r)$ and as $(1-r)$ is invertible, we get $H_*(Q_*) = H_*(G_L; \widehat{\mathbb{Z}G}_\chi) = \widehat{\mathbb{Z}G}_\chi \otimes_\mathbb{Z} \tilde{H}_{*-1}(S^p)$.

From the long exact sequence it follows that $i_*\colon H_1(G_K; \widehat{\mathbb{Z}G}_\chi) \to H_1(G_M; \widehat{\mathbb{Z}G}_\chi)$ is surjective. As we know that these are free $\widehat{\mathbb{Z}G}_\chi$–modules of rank one, it is injective as well by [2; 16] (note that injectivity is clear for $p > 1$). Now $(t,\theta)_*$ increases the value of χ, so α is an isomorphism in degree 1. Therefore $H_i(G; \widehat{\mathbb{Z}G}_\chi) = 0$ for $i \leq p$. Also $H_{p+1}(G; \widehat{\mathbb{Z}G}_\chi) \cong H_{p+1}(G_M; \widehat{\mathbb{Z}G}_\chi) \neq 0$ as follows again from the long exact sequence for HNN–extensions. Therefore $\chi \in \Sigma^p(G;\mathbb{Z}) - \Sigma^{p+1}(G;\mathbb{Z})$ by Lemma 2.4.

We need to show that $-\chi \notin \Sigma^1(G;\mathbb{Z})$. For this we need to analyze α closer in the case when t is send to a negative value. Define a chain map $\tau_*\colon C_*(G_K) \to C_*(G_M)$

with $\tau_*(gx) = \theta(g)\tau_*(x)$ for $g \in G_K$. To do this, note that θ extends naturally to $\theta\colon \mathbb{Z}G_K \to \mathbb{Z}G_M$ with this property. So let $\tau_0\colon C_0(G_K) = \mathbb{Z}G_K \to \mathbb{Z}G_M = C_0(G_M)$ be given by $\tau_0 = \theta$.

Also $C_1(G_K) = \mathbb{Z}G_K \oplus \mathbb{Z}G_K$ and $C_1(G_M) = \mathbb{Z}G_M \oplus \mathbb{Z}G_M \oplus (\mathbb{Z}G_M)^k$ and we define

$$\tau_1(x, y) = (\theta(x), \theta(y)(s+1), 0). \tag{10}$$

As $\tau_0(\partial(x, y)) = \theta(x)(r-1) + \theta(y)(s^2-1)$, we see that this induces the required chain map τ_*. We want to show that $H_1(G; \widehat{N}_{-\chi}) \neq 0$. Observe that $\widehat{N}_{-\chi}$ can be identified with the Laurent series ring $\mathbb{Z}((t^{-1}))$ whose elements are of the form $\sum_{n=-\infty}^{u} m_n t^n$, and the $\widehat{\mathbb{Z}G}_{-\chi}$–module structure is given by the ring homomorphism sending all generators except s to t, and s being send to 1. The map $(t, \theta)_*\colon H_1(G_K; \widehat{N}_{-\chi}) \to H_1(G_L; \widehat{N}_{-\chi})$ is easily seen by using (10) to be

$$(t, \theta)_* \left(\sum_{n=-\infty}^{u} m_n t^n \right) = 2 \sum_{n=-\infty}^{u} m_n t^{n+1}.$$

Therefore $\alpha\colon \widehat{N}_{-\chi} \to \widehat{N}_{-\chi}$ is given by $\alpha(x) = x(2t-1)$ which is not surjective. Therefore $H_1(G; \widehat{N}_{-\chi}) \neq 0$ which implies $-\chi \notin \Sigma^1(G; \mathbb{Z})$ by Corollary 2.5. □

We can define $\Sigma^\infty(G)$ and $\Sigma^{\infty+1}(G)$ analogously. We will see in the next section that the examples constructed above in fact satisfy $\chi \in \Sigma^p(G) - \Sigma^{p+1}(G; \mathbb{Z})$ and $-\chi \in \Sigma^q(G) - \Sigma^{q+1}(G; \mathbb{Z})$.

7 The homotopy type of halfspaces II

To study the homotopical invariant $\Sigma^k(G)$, we consider the following situation. We have a finite flag complex L and a finite flag complex K together with two full simplicial embeddings $i_0\colon K \to L$ and $i_1\colon K \to L$. As before this gives rise to the HNN–extension G along the injections $G_K \to G_L$ induced by i_0 and i_1.

Note that G admits a finite $K(G, 1)$ given by

$$Q = Q_L \cup Q_K \times [0, 1]/\sim$$

where $(x, 0) \sim i_0(x)$ and $(x, 1) \sim i_1(x)$.

If X denotes the universal cover of Q, we get

$$X = G \times_{G_L} X_L \cup G \times_{G_K} X_K \times [0, 1]/\sim$$

with the usual identifications. Furthermore, for $\chi\colon G\to\mathbb{R}$ as in Section 6, we get a height function $h\colon X\to\mathbb{R}$ by $h([g,x])=\chi(g)+h_L(x)$ for $g\in G$, $x\in X_L$ and $h([g,y,s])=\chi(g)+h_K(y)+\chi(t)\cdot s$ for $g\in G$, $y\in X_K$ and $s\in[0,1]$. Here h_L and h_K are the height functions from Lemma 5.1.

Let us assume that χ is nonzero on all the generators of G. Let us also assume that $\chi(t)=1$ for the extra generator $t\in G$. The case $\chi(t)=-1$ is then handled by interchanging the role of i_0 and i_1. Let N be the maximal subcomplex of X contained in $h^{-1}([0,\infty))$. With $G^+=\{g\in G\mid \chi(g)\geq 0\}$, we get

$$N=G^+\times_{G_L^+}N_L\cup G^+\times_{G_K^+}N_K\times[0,1]/\sim$$

with the usual identifications.

Lemma 7.1 *Let K, L be finite connected flag complexes and $i_0,i_1\colon K\to L$ full simplicial embeddings, and $\chi\colon G\to\mathbb{R}$ a homomorphism with $\chi(t_i)\neq 0$ for all generators $t_i\in G_L$ corresponding to vertices of L, and $\chi(t)=1$ for the extra generator of G.*

(1) *If $i_{0\#}\colon \pi_1(K)\to\pi_1(L)$ is an isomorphism, then $\chi\in\Sigma^2(G)$.*

(2) *If $i_{1\#}\colon \pi_1(K)\to\pi_1(L)$ is injective and the normal closure of $\pi_1(K)$ in $\pi_1(L)$ is not the whole group, then $-\chi\notin\Sigma^2(G)$.*

Proof To see (1), we want to show that N is simply connected. By Proposition 5.8 we get that N is homotopy equivalent to

$$\Big(G^+\times_{G_L^+}\bigvee_{g\in G_L^-}L\Big)\cup\Big(G^+\times_{G_K^+}\bigvee_{g\in G_K^-}K\times[0,1]\Big)\Big/\sim$$

where the identifications of $[g,x,0]$ and $[g,x,1]$ are induced by the inclusions $i_0\colon K\to L$ and $i_1\colon K\to L$ respectively, and $G_L^-=\{g\in G_L\mid \chi(g)<0\}$. Let π be the subcomplex given by

$$\pi=G^+\times_{G_L^+}\{*\}\cup G^+\times_{G_K^+}\{*\}\times[0,1]/\sim.$$

Then π is a tree by an argument similar to the proof of Lemma 5.4. By collapsing this tree, we get

$$N\simeq\bigvee_{g\in G^-}L\cup\bigvee_{g\in G^-}K\wedge[0,1]_+/\sim$$

with the following identifications. Let ${}_gx$ be an element of the copy of K corresponding to $g\in G^-=\{g\in G\mid \chi(g)<0\}$. Then $({}_gx,0)\sim{}_gj_0(x)$, where $j_0\colon K\to L$ sends the basepoint of K to the basepoint of L and is homotopic to $i_0\colon K\to L$, and

${}_g j_0(x)$ means we consider $j_0(x)$ as an element of the copy of L corresponding to $g \in G^-$. Similarly $({}_g x, 1) \sim {}_{gt} j_1(x)$, provided that $\chi(gt) < 0$. If $\chi(gt) \geq 0$, we identify $({}_g x, 1)$ with the basepoint $*$. Recall that K and L are considered based spaces as in Section 5; also $[0,1]_+$ is the interval with a disjoint base point, and $K \wedge [0,1]_+ = K \times [0,1]_+ / K \vee [0,1]_+$.

Now let $\mathcal{F}$ be a finite subset of G^- with the following property: if $g \in \mathcal{F}$, then $\chi(gt) \geq 0$ or $gt \in \mathcal{F}$. Such sets can be ordered by inclusion, and we define

$$N_{\mathcal{F}} = \bigvee_{g \in \mathcal{F}} L \cup \bigvee_{g \in \mathcal{F}} K \wedge [0,1]_+ / \sim$$

as a subcomplex of N. Then $\pi_1(N) \cong \varinjlim \pi_1(N_{\mathcal{F}})$ where the direct limit is taken over all such finite sets $\mathcal{F}$.

For $g \in G^- - \mathcal{F}$ with $gt \in \mathcal{F}$ or $\chi(gt) \geq 0$, we can write

$$N_{\mathcal{F} \cup \{g\}} = (N_{\mathcal{F}} \cup K \wedge [0,1]_+) \cup (N_{\mathcal{F}} \vee L)$$

with

$$N_{\mathcal{F}} \vee K = (N_{\mathcal{F}} \cup K \wedge [0,1]_+) \cap (N_{\mathcal{F}} \vee L).$$

Note that $N_{\mathcal{F}} \cup K \wedge [0,1]_+ \simeq N_{\mathcal{F}}$. By the Seifert-van Kampen theorem we have a push-out diagram:

(11)
$$\begin{array}{ccc} \pi_1(N_{\mathcal{F}}) * \pi_1(K) & \xrightarrow{\mathrm{id}_\# * i_{0\#}} & \pi_1(N_{\mathcal{F}}) * \pi_1(L) \\ \downarrow & & \downarrow \\ \pi_1(N_{\mathcal{F}}) & \longrightarrow & \pi_1(N_{\mathcal{F} \cup \{g\}}) \end{array}$$

So if $i_{0\#}$ is an isomorphism, we get that $\pi_1(N_{\mathcal{F}}) \to \pi_1(N_{\mathcal{F} \cup \{g\}})$ is an isomorphism. Since $\pi_1(N_\varnothing) = 1$, this shows that N is simply connected, which proves (1).

Instead of studying $-\chi$, we keep the discussion above, but interchange the role of i_0 and i_1. So let us assume that $i_{0\#}\colon \pi_1(K) \to \pi_1(L)$ is injective, and if H is the quotient of $\pi_1(L)$ by the normal closure of $i_{0\#}(\pi_1(K))$ in $\pi_1(L)$, we get that H is nontrivial. We want to show that in this situation $\chi \notin \Sigma^2(G)$.

From (11) we get that $\pi_1(N_{\mathcal{F}}) \to \pi_1(N_{\mathcal{F} \cup \{g\}})$ is injective. Also, we get that $\pi_1(N_{\mathcal{F}}) \to \pi_1(N_{\mathcal{F} \cup \{g\}})$ surjects onto H, by letting $\pi_1(L) \to H$ be the quotient map and sending $\pi_1(N_{\mathcal{F}})$ to 1, and using the push-out property of (11). This shows that $\pi_1(N)$ is nontrivial.

To get $\chi \notin \Sigma^2(G)$, we have to show that the image of $i_\#\colon \pi_1(N) \to \pi_1(gN)$ is nontrivial for all $g \in G^-$, where $i\colon N \to gN$ is inclusion. For all $s \in \operatorname{im} \chi$ we can define $N_{\mathcal{F}}^s$ by using wedges for g with $\chi(g) < s$. For $s < 0$, it is easy to see that there is an obvious projection $N_{\mathcal{F}} \to N^s_{\mathcal{F}}$ which induces a surjection on π_1 by using (11) and the analogous diagram for $N^s_{\mathcal{F}}$. Therefore $i_\#\colon \pi_1(N) \to \pi_1(gN)$ is surjective for all $g \in G^-$, which proves (2). □

Remark 7.2 If both i_0 and i_1 induce isomorphisms on fundamental group, we get of course $\pm\chi \in \Sigma^2(G)$. Note that Lemma 7.1 applies to the examples used for Theorem 6.7 with $q \geq 1$. Therefore the homological Sigma invariant can be replaced by the homotopical Sigma invariant in these examples.

To get examples with

$$\chi \in \Sigma^p(G;\mathbb{Z}) - (\Sigma^{p+1}(G;\mathbb{Z}) \cup \Sigma^2(G))$$
$$-\chi \in \Sigma^q(G;\mathbb{Z}) - (\Sigma^{q+1}(G;\mathbb{Z}) \cup \Sigma^2(G))$$

for $p, q \geq 1$, one can choose $L' = L \vee A$ with L as in the proof of Theorem 6.7 and A a finite flag complex with vanishing reduced homology and nontrivial fundamental group. Then condition (2) of Lemma 7.1 applies to both inclusions of K into L', ensuring that $\pm\chi \notin \Sigma^2(G)$.

Theorem 7.3 *There exists a group G of type F and a homomorphism $\chi\colon G \to \mathbb{R}$ such that*

$$\chi \in \Sigma^\infty(G)$$
$$-\chi \in \Sigma^\infty(G;\mathbb{Z}) - \Sigma^2(G).$$

The homomorphism can be chosen to have image in $\mathbb{Z}$.

Proof The binary icosahedral group has a presentation

$$I = \langle x, y \mid x^2 = y^3 = (xy)^5 \rangle$$

and is nontrivial and perfect (see Kervaire [14]), so we can find a CW–complex with one 0–cell, two 1–cells and two 2–cells. From the Euler characteristic, we see that this CW–complex has vanishing reduced homology, so let A be a subdivision which is a flag complex. Pick a vertex $* \in A$ as a basepoint and let $K = A \vee A$. We claim there is a map $f\colon K \to K$ inducing an injection on π_1 and such that the normal closure of the image is not the whole group. Note that $\pi_1(K) \cong I * I$, and a presentation is given by

$$\pi_1(K) \cong \langle x, y, \bar{x}, \bar{y} \mid x^2 = y^3 = (xy)^5, \bar{x}^2 = \bar{y}^3 = (\bar{x}\bar{y})^5 \rangle.$$

Now define $\varphi\colon \pi_1(K) \to \pi_1(K)$ by $\varphi(x) = x$, $\varphi(y) = y$, $\varphi(\bar{x}) = \bar{x}^{-1}x\bar{x}$ and $\varphi(\bar{y}) = \bar{x}^{-1}y\bar{x}$. It is clear that φ is injective, and if $p\colon \pi_1(K) \to I$ denotes the projection to the second factor I of $I * I$, we get that the image of φ is contained in the kernel of p, which is clearly not the whole group.

Since K is a 2–dimensional complex, we can realize φ by a continuous function $f\colon K \to K$. Now let L be the finite flag complex from Lemma 6.5, and G the HNN–extension of G_L along $G_{K^{[r]}}$ and the two full simplicial embeddings $K^{[r]} \to L$. We define $\chi\colon G \to \mathbb{R}$ by sending every generator to 1, so $\chi \in \Sigma^2(G)$ and $-\chi \notin \Sigma^2(G)$ by Lemma 7.1. Since $\widetilde{H}_*(L) = \widetilde{H}_*(K) = 0$ we get $H_*(G;\widehat{\mathbb{Z}G}_{\pm\chi}) = 0$ from (2) and Theorem 4.1. Therefore $\pm\chi \in \Sigma^\infty(G;\mathbb{Z})$ and the result follows from Lemma 2.4 and the fact that $\Sigma^k(G) = \Sigma^k(G;\mathbb{Z}) \cap \Sigma^2(G)$ for $k \geq 2$. □

We can combine the examples for Theorem 6.7 and Theorem 7.3 to get

Theorem 7.4 *For every pair $p, q \in \widehat{\mathbb{N}}$ there exists a group of type F and a homomorphism $\chi\colon G \to \mathbb{R}$ such that*

$$\begin{aligned} \chi &\in \Sigma^p(G) - \Sigma^{p+1}(G) \\ -\chi &\in \Sigma^q(G;\mathbb{Z}) - (\Sigma^{q+1}(G;\mathbb{Z}) - \Sigma^2(G)). \end{aligned}$$

The homomorphism can be chosen to have image in $\mathbb{Z}$.

Proof We will sketch the proof as the techniques are very similar to previous arguments.

We consider various cases. If $p, q \geq 2$ let K, L and f be as in the proof of Theorem 7.3. If $p \geq q$ let $K_1 = K \vee S^q$. Also let $f_1\colon K_1 \to K_1$ be the wedge of f with a map of degree 2 on S^q. Let L_1 be the result from Lemma 6.5 and let $L_2 = L_1 \vee S^p$, provided $p < \infty$. It is easy to see that the standard construction gives G and χ with the desired properties.

If $q > p$, let $\bar{K} = S^p$, $g\colon S^p \to S^p$ a map of degree 2 and $\bar{L}$ the flag complex arising from Lemma 6.5. Note that $\bar{L}$ has the homotopy type of S^p. We set $L_1 = L \vee \bar{L} \vee S^q$. Now let $\bar{G}$ be the HNN–extension of G_{L_1} along the two full embeddings of $K^{[r]}$ in L_1 and χ as usual. Then $\chi \in \Sigma^2(\bar{G})$, $-\chi \in \Sigma^2(\bar{G};\mathbb{Z}) - \Sigma^2(\bar{G})$ and the only nonzero Novikov homology groups are

$$H_q(\bar{G};\widehat{\mathbb{Z}\bar{G}}_{\pm\chi}) = \widehat{\mathbb{Z}\bar{G}}_{\pm\chi} = H_p(\bar{G};\widehat{\mathbb{Z}\bar{G}}_{\pm\chi}).$$

We still have two injections $G_{\bar{K}^{[s]}} \to \bar{G}$ arising from the full embeddings $\bar{K}^{[s]} \to \bar{L}$ so we can form another HNN–extension G along these injections and the usual $\chi\colon G \to \mathbb{R}$ by sending the extra generator to 1. As before it follows from (2) that $\chi \in \Sigma^p(G;\mathbb{Z}) - \Sigma^{p+1}(G;\mathbb{Z})$ and $-\chi \in \Sigma^q(G;\mathbb{Z}) - \Sigma^{q+1}(G;\mathbb{Z})$. Furthermore, it follows from [19, Theorem 5.2] that $\chi \in \Sigma^2(G)$, while it follows from [10, Proposition 10] that $-\chi \notin \Sigma^2(G)$, since the example from Theorem 7.3 is a retract of G and χ.

If $q \leq 1$, we can use the examples from Theorem 6.7, except that for $q = 0$ and $p \geq 2$ we did not actually show that $\chi \in \Sigma^2(G)$. To see this note that N_L is homotopy equivalent to a disjoint union of wedges of S^p, in particular its components are simply connected. Also N_K is a forest. One easily sees that the halfspace N is connected (recall $\chi \in \Sigma^1(G)$), and with an argument similar to the proof of Lemma 5.4 we see that it is simply connected. We omit the details.

If $p \leq 1$, use the examples from Theorem 6.7, but take the one point union of L with A, where A is a non–simply connected flag complex with $\widetilde{H}_*(A) = 0$. The resulting $\chi\colon G \to \mathbb{R}$ will satisfy $-\chi \notin \Sigma^2(G)$ by [10, Proposition 10], since G_A with $\chi\colon G_A \to \mathbb{R}$ sending every generator to 1 is a retract of this. □

8 Closed 1–forms without singularities

Even though we cannot expect a lot of symmetry in the Sigma invariants with respect to the antipodal map, we obtain the following rather peculiar symmetry condition for $\Sigma^k(G;\mathbb{Z})$.

Proposition 8.1 *Let G be a group of type F_k with $k \geq 2$, and let $\chi\colon G \to \mathbb{R}$ be a nonzero homomorphism. Assume there exists a smooth closed connected manifold M with $G = \pi_1(M)$ whose universal cover $\widetilde{M}$ is $(k-1)$–connected, and such that*

$$C_*(M;\widehat{\mathbb{Z}G}_\chi) = \widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}G} C_*(\widetilde{M})$$

is chain-contractible, where $C_*(\widetilde{M})$ is the simplicial chain complex over $\mathbb{Z}G$ obtained from a smooth triangulation of M. Then $\pm\chi \in \Sigma^k(G;\mathbb{Z})$.

Notice that $\widetilde{M}$ is certainly 1–connected, so we get $\pm\chi \in \Sigma^2(G;\mathbb{Z})$ provided that $C_*(M;\widehat{\mathbb{Z}G}_\chi)$ is chain-contractible.

Proof We have the universal coefficient spectral sequence with

$$E^2_{pq} = \operatorname{Tor}_p^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, H_q(\widetilde{M}))$$

converging to $H_{p+q}(M;\widehat{\mathbb{Z}G}_\chi)$. Since $\widetilde{M}$ is $(k-1)$–connected, we have $E^2_{pq} = 0$ for $q = 1,\ldots,k-1$, so

$$E^\infty_{p0} = \operatorname{Tor}_p^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, H_0(\widetilde{M})) = H_p(G;\widehat{\mathbb{Z}G}_\chi)$$

for $p \leq k$. As our assumption is $H_*(M;\widehat{\mathbb{Z}G}_\chi) = 0$, we get $H_p(G;\widehat{\mathbb{Z}G}_\chi) = 0$ for $p \leq k$, which means $\chi \in \Sigma^k(G;\mathbb{Z})$ by Lemma 2.4.

Let $C^*(\widetilde{M}) = \mathrm{Hom}_{\mathbb{Z}G}(C_*(\widetilde{M}), \mathbb{Z}G)$. As $C_*(\widetilde{M})$ is viewed as a left $\mathbb{Z}G$–chain complex, this is a right $\mathbb{Z}G$–chain complex, but we can view it as a left complex by using the orientation-involution on $\mathbb{Z}G$. Then Poincaré duality gives a chain homotopy equivalence $C_*(\widetilde{M}) \simeq C^{n-*}(\widetilde{M})$ of free left $\mathbb{Z}G$–chain complexes, where n denotes the dimension of M. Therefore $\widehat{\mathbb{Z}G}_{-\chi} \otimes_{\mathbb{Z}G} C_*(\widetilde{M}) \simeq \widehat{\mathbb{Z}G}_{-\chi} \otimes_{\mathbb{Z}G} C^{n-*}(\widetilde{M})$, and the latter is isomorphic to $\mathrm{Hom}_{\widehat{\mathbb{Z}G}_\chi}(\widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}G} C_{n-*}(\widetilde{M}), \widehat{\mathbb{Z}G}_\chi)$ via

$$\Phi\colon \widehat{\mathbb{Z}G}_{-\chi} \otimes_{\mathbb{Z}G} C^{n-*}(\widetilde{M}) \longrightarrow \mathrm{Hom}_{\widehat{\mathbb{Z}G}_\chi}(\widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}G} C_{n-*}(\widetilde{M}), \widehat{\mathbb{Z}G}_\chi)$$
$$\lambda \otimes \varphi \;\mapsto\; \Phi(\lambda \otimes \varphi)\colon 1 \otimes x \mapsto \varphi(x)\bar{\lambda}.$$

Note that the involution on $\mathbb{Z}G$ extends to an anti-ring-homomorphism $\bar{\cdot}\colon \widehat{\mathbb{Z}G}_{-\chi} \to \widehat{\mathbb{Z}G}_\chi$, so that both complexes are indeed free left $\widehat{\mathbb{Z}G}_{-\chi}$–chain complexes. A chain contraction for $\widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}G} C_*(\widetilde{M})$ therefore induces a chain contraction for $\widehat{\mathbb{Z}G}_{-\chi} \otimes_{\mathbb{Z}G} C_*(\widetilde{M})$. Thus the spectral sequence argument above also applies to $\widehat{\mathbb{Z}G}_{-\chi}$ and we get $-\chi \in \Sigma^k(G;\mathbb{Z})$. □

The condition that $C_*(M;\widehat{\mathbb{Z}G}_\chi)$ is chain-contractible is a necessary condition for the existence of a nonsingular closed 1–form ω representing $\chi \in H^1(M;\mathbb{R}) \cong \mathrm{Hom}(\pi_1(M),\mathbb{R})$. Here nonsingular means that $\omega_x \neq 0$ for all $x \in M$.

In [17], Latour gives various conditions which are necessary and sufficient for the existence of a nonsingular closed 1–form ω on a closed smooth manifold M within a given cohomology class $\chi \in H^1(M;\mathbb{R})$, provided that $\dim M \geq 6$. Let us quickly recall these conditions.

If ω is any closed 1–form representing χ, the pullback of ω to $\widetilde{M}$ is exact and gives a height function $h\colon \widetilde{M} \to \mathbb{R}$ with respect to χ. A map $\gamma\colon [0,\infty) \to \widetilde{M}$ is called a *path to infinity with respect to* χ, if $\lim_{t\to\infty} h \circ \gamma(t) = \infty$. Pick a basepoint $x_0 \in \widetilde{M}$. We then let

$$\mathcal{M}_\chi = \{\gamma\colon [0,\infty) \to \widetilde{M} \mid \gamma(0) = x_0, \gamma \text{ is a path to infinity w.r.t. } \chi\}.$$

This set is topologized with the compact-open topology together with a "control at infinity", that is, a subbasis for the topology is given by the following open sets: For $a, b \in [0,\infty)$ and U open in $\widetilde{M}$ let

$$W(a,b;U) = \{\gamma \in \mathcal{M}_\chi \mid \gamma([a,b]) \subset U\}$$

and for $a, A \in [0,\infty)$ let

$$W(a,A) = \{\gamma \in \mathcal{M}_\chi \mid \forall t \geq a \;\; h(\gamma(t)) - h(\gamma(0)) > A\}.$$

If there exists a nonsingular closed 1–form ω representing χ, it is easy to see that $\widetilde{M}$ is diffeomorphic to $N\times\mathbb{R}$ with N a smooth manifold, and a height function is given by projection to $\mathbb{R}$. It is then easy to see that both $\mathcal{M}_\chi$ and $\mathcal{M}_{-\chi}$ are contractible.

On the other hand, if $\mathcal{M}_\chi$ is contractible, it can be shown that

$$C_*(M;\widehat{\mathbb{Z}G}_\chi)=\widehat{\mathbb{Z}G}_\chi\otimes_{\mathbb{Z}G}C_*(\widetilde{M})$$

is chain-contractible [17], where $G=\pi_1(M)$. As this is a finitely generated free chain complex over $\widehat{\mathbb{Z}G}_\chi$, one can look at its Whitehead torsion $\tau(M;\chi)$ in an appropriate quotient of $K_1(\widehat{\mathbb{Z}G}_\chi)$. We will not define this quotient, but remark that it is a quotient of the ordinary Whitehead group $\mathrm{Wh}(\pi)$ and in fact vanishes if and only if $\mathrm{Wh}(\pi)$ vanishes [22; 23].

The main result of Latour is then:

Theorem 8.2 [17] *Let M be a smooth closed connected manifold of dimension at least 6, and $\chi\in H^1(M;\mathbb{R})$. Then χ can be realized by a nonsingular closed 1–form if and only if $\mathcal{M}_\chi$ and $\mathcal{M}_{-\chi}$ are contractible and $\tau(M;\chi)$ vanishes.*

The condition that $\mathcal{M}_\chi$ is contractible is known to be equivalent to the following two conditions [17; 10].

(1) $\chi\in\Sigma^2(G)$.

(2) $C_*(M;\widehat{\mathbb{Z}G}_\chi)$ is chain-contractible.

Since $C_*(M;\widehat{\mathbb{Z}G}_\chi)$ is chain contractible if and only if $C_*(M;\widehat{\mathbb{Z}G}_{-\chi})$ is chain contractible one can ask whether $\mathcal{M}_\chi$ is contractible if and only if $\mathcal{M}_{-\chi}$ is contractible. In other words, one can ask whether the analogue of Proposition 8.1 also holds for the homotopical Sigma invariant.

Based on the work of Bestvina and Brady [1], Damian [8] has constructed an example of a manifold where $C_*(M;\widehat{\mathbb{Z}G}_\chi)$ is chain-contractible, but neither $\mathcal{M}_\chi$ nor $\mathcal{M}_{-\chi}$ are contractible. We now give an example of a manifold M where only one of $\mathcal{M}_\chi$ and $\mathcal{M}_{-\chi}$ is contractible. The construction is in fact completely analogous to the construction in [8], replacing [1] with Theorem 7.3. For the convenience of the reader, we will repeat the construction.

Theorem 8.3 *There exists a closed connected smooth manifold M of dimension at least 6 and a nonzero $\chi\in H^1(M;\mathbb{R})$ such that $\mathcal{M}_\chi$ is contractible, but $\mathcal{M}_{-\chi}$ is not contractible.*

Proof Let G be the group from Theorem 7.3, which has a finite $K(G,1)$ denoted Q, say of dimension n. Embed this $K(G,1)$ into $\mathbb{R}^{2n+3}$ and let W be a regular neighborhood of Q, which we can think of as a smooth compact manifold with boundary. Let $M = \partial W$, which is of dimension $2n+2$ and homotopy equivalent to $W - Q$ by the properties of regular neighborhoods. By transversality we get that every pair of maps $(D^{i+1}, S^i) \to (W, M)$ factors through $(W - Q, M)$ up to homotopy for $i \leq n+1$, since D^{n+2} can avoid the n–dimensional Q in $(2n+3)$–space. Therefore $\pi_i(M) \cong \pi_i(W) \cong \pi_i(Q)$ for $i \leq n+1$. In particular, the universal cover $\widetilde{M}$ is $(n{+}1)$–connected, and $\pi_1(M) = G$. The universal coefficient spectral sequence with $E^2_{pq} = \mathrm{Tor}_p^{\mathbb{Z}G}(\widehat{\mathbb{Z}G}_\chi, H_q(\widetilde{M}))$ converging to $H_{p+q}(M; \widehat{\mathbb{Z}G}_\chi)$ satisfies $E^2_{pq} = 0$ for $p{+}q \leq n{+}1$ by Theorem 7.3 and the fact that the space $\widetilde{M}$ is $(n{+}1)$–connected. Therefore $H_i(M; \widehat{\mathbb{Z}G}_\chi) = 0$ for $i \leq n+1$. The same argument gives $H_i(M; \widehat{\mathbb{Z}G}_{-\chi}) = 0$ for $i \leq n+1$. Using Poincaré duality we get $H_*(M; \widehat{\mathbb{Z}G}_{\pm\chi}) = 0$, and $C_*(M; \widehat{\mathbb{Z}G}_{\pm\chi})$ is chain-contractible as this is a free complex. Now $\mathcal{M}_\chi$ is contractible as we have $\chi \in \Sigma^2(G)$, but $\mathcal{M}_{-\chi}$ is not contractible, as $-\chi \notin \Sigma^2(G)$. □

Remark 8.4 The dimension of M is in fact much bigger than 6. The dimension of K used in Lemma 6.5 is 2 and $\dim L = \max\{\dim K + 1, \dim M\}$. The simplicial approximation $g\colon K^{[r]} \to K$ used in Theorem 7.3 is far from injective which increases $\dim M$. In any case $\dim L \geq 3$. Since $n = \dim L + 2$, we get that M is at least 12–dimensional.

Remark 8.5 Since $C_*(M; \widehat{\mathbb{Z}G}_\chi)$ is chain contractible in the previous theorem, one can ask about the Whitehead torsion arising this way. Now G is an HNN–extension of a right-angled Artin group via two isomorphic right-angled Artin subgroups. If we look at the universal cover X of Q, we see that X is a nonpositively curved space by the same argument that each X_L is a nonpositively curved space; compare Bestvina and Brady [1] (the link of each vertex is a flag complex). Therefore the Whitehead group $\mathrm{Wh}(G)$ vanishes by [13] and the torsion is trivial.

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