Surgery presentations of coloured knots and of their covering links

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We consider knots equipped with a representation of their knot groups onto a dihedral group D_{2n} (where n is odd). To each such knot there corresponds a closed 3–manifold, the (irregular) dihedral branched covering space, with the branching set over the knot forming a link in it. We report a variety of results relating to the problem of passing from the initial data of a D_{2n}–coloured knot to a surgery presentation of the corresponding branched covering space and covering link. In particular, we describe effective algorithms for constructing such presentations. A by-product of these investigations is a proof of the conjecture that two D_{2n}–coloured knots are related by a sequence of surgeries along ± 1–framed unknots in the kernel of the representation if and only if they have the same coloured untying invariant (a $\mathbb{Z}_n$–valued algebraic invariant of D_{2n}–coloured knots).

57M12; 57M25

1 Introduction

The starting point for this work was the authors' desire to explore the quantum topology of covering spaces as a means of acquiring a deeper understanding of how quantum invariants actually encode topological information. Recent results in the case of cyclic covering spaces (see eg Garoufalidis and Kricker [9; 10]) suggest the existence of such a theory.

Having understood the cyclic case, the natural next step is to consider the branched dihedral covering spaces. These spaces have long played an important role in knot theory, dating back to Reidemeister's use of the linking matrix of a knot's dihedral covering link to distinguish knots with the same Alexander polynomial [21] (see also eg Perko [19]). More recently they have also been used in investigations of knot concordance (eg Gilmer [12]). In addition, branched dihedral covers are useful in 3–manifold topology: for example, it turns out that every 3–manifold is a 3–fold branched dihedral covering space over some knot (see eg Burde and Zieschang [2, Theorem 11.11]).

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Quantum invariants for 3–manifolds are typically constructed using surgery presentations. To investigate the quantum topology of covering spaces, then, it seems we need a combinatorial theory of surgery presentations of covering spaces.

The cyclic case is well-known. Recall that there is a famous trick for obtaining surgery presentations of n–fold cyclic covers for any natural number n (see eg Rolfsen [22, Chapter 6D]). We wish to generalize this trick to dihedral covers, so we'll begin by reviewing how it goes.

One first performs crossing changes to untie the knot by introducing ± 1–framed unknots along which surgery is carried out. The unknots are chosen to have linking number zero with the knot. After this step, we have a surgery presentation of the given knot as a ± 1–framed link L lying in the complement of an unknot U, where each component of L has linking number zero modulo n with U. For the purpose of generalization, this last condition can be restated: every component of L lies in the kernel of the mod n linking homomorphism $\mathrm{Link}_n\colon H_1(\overline{S^3 - N(U)}) \twoheadrightarrow \mathbb{Z}_n$. If this condition is satisfied, the construction of a surgery presentation of the cyclic cover can now be completed by lifting L to the n–fold cyclic cover of S^3 branched over U, which is of course again S^3.

We would like analogous procedures for classes of covering spaces corresponding to other groups, in particular to the dihedral groups. The key feature which permitted construction in the cyclic case was the existence of a knot (the unknot) which every other knot could be transformed into via surgeries in the kernel of the mod n linking homomorphism, and whose branched cyclic cover could be constructed explicitly.

To discuss how this generalizes it's worth introducing a few definitions.

Definition 1 (G–coloured knots) For a finite group G and a closed orientable 3–manifold M, define a *G–coloured knot in M* to be a pair (K, ρ) of an oriented knot $K \subset M$ and a surjective representation $\rho\colon \pi_1(\overline{M - N(K)}) \twoheadrightarrow G$. Unless otherwise specified it will be assumed that $M = S^3$.

Definition 2 (Surgery in $\ker \rho$) Let (K, ρ) be a G–coloured knot in a 3–manifold M. If $L \subset M - K$ is a framed link[1] each of whose components is specified by a curve lying in $\ker \rho$ then we can perform surgery along L to obtain a new G–coloured knot (K', ρ') in a 3–manifold M', as follows:

- Remove tubular neighbourhoods $N(L_i)$ of the components L_i of L and reattach them to $\overline{M - \bigcup N(L_i)}$ so as to match the meridional discs to the framing curves. This gives the knot K' in the 3–manifold M'.

[1] A framed link in M is a link in which each component L_i comes equipped with a simple closed curve on the boundary of a regular neighbourhood of L_i which is parallel to L_i in the regular neighbourhood.

- To specify the induced representation ρ', we must state the value it takes for an arbitrary curve γ in $M' - K'$. Homotope γ into $M - \bigcup N(L_i)$, then evaluate it in the restriction of ρ. This value is well-determined because the components of L lie in the kernel of ρ.

In this situation we say that (K', ρ') has been obtained from (K, ρ) by *surgery in* $\ker\rho$.

Definition 3 (Complete set of base-knots) A *complete set of base-knots*[2] for a group G is a set Ψ of G–coloured knots (K_i, ρ_i) in 3–manifolds M_i, such that any G–coloured knot (K, ρ) in S^3 can be obtained from some $(K_i, \rho_i) \in \Psi$ by surgery in $\ker\rho_i$.

To generalize the procedure from the cyclic case to some other group G, we must find a complete set of base-knots whose desired covering spaces (and covering links) we know how to construct explicitly, and into whose covering spaces we know how to lift surgery presentations for any G–coloured knot.

This paper deals with the case when G is the dihedral group D_{2n} with n any odd integer—the group of permutations of the vertices of a regular polygon with n sides. Its presentation is

$$D_{2n} := \left\{ t, s \,\middle|\, t^2 = s^n = 1,\ tst = s^{-1} \right\}.$$

As permutations on the set of vertices of the regular polygon, these generators correspond to

$$t = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & n & n-1 & \cdots & 3 & 2 \end{pmatrix}$$

and

$$s = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \end{pmatrix}.$$

Elements in D_{2n} of the form s^a are called *rotations*, and elements of the form ts^a are called *reflections*. The cyclic group of rotations $\mathcal{C}_n := \langle s \rangle$ is a normal subgroup in D_{2n}.

Convention 1 We'll present a D_{2n}–colouring ρ of a knot $K \subset S^3$ by labeling every arc of a knot diagram for K by the image under ρ of the corresponding Wirtinger generator[3]. More generally, we can present a D_{2n}–colouring of a knot K in a closed 3–manifold M by a diagram of a link $L \cup K_1$ in S^3 where:

[2] The term base-knot imitates base-point.

[3] Note that this is a different convention from the one used for a Fox n–colouring of a knot, in which an arc which we would label by ts^a is labeled simply by a (see Fox [7]).

- L is integer-framed, and surgery along L turns (S^3, K_1) into (M, K).
- Every arc of the diagram is labeled by an element of D_{2n}.
- Wirtinger relations are satisfied.
- When the framing curve of any component of L is expressed as a product of Wirtinger generators of $\pi_1(S^3 - N(L \cup K_1))$, the product of the corresponding labels is $1 \in D_{2n}$.

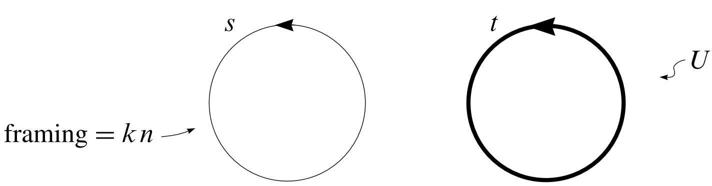
Our goal in this paper is to give combinatorial procedures for constructing surgery presentations of the irregular dihedral branched covering space corresponding to some D_{2n}–coloured knot, together with the covering link it contains. In Section 2 we'll recall exactly what these phrases refer to.

Roughly speaking, we'll describe two approaches to this problem, corresponding to two different complete sets of base-knots for D_{2n}. The sets of base-knots will be introduced shortly. The construction of their corresponding dihedral covering spaces, covering links, and how to lift surgery presentations in the complement of the base-knot will be discussed in detail in Section 3 and Section 4.

The untying approach

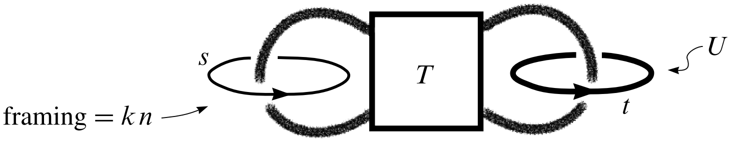
This first approach begins with exactly the same procedure for untying knots as is used when constructing surgery presentations of cyclic covers. It may be viewed as an adaptation of that approach to the case of D_{2n}.

Theorem 1 *Consider the following diagram, which (according to Convention 1) depicts a D_{2n}–coloured unknot U in the $(kn, 1)$–lens-space that results from surgery on the separated kn–framed unknot.*



This set of D_{2n}–coloured knots for $k = 0, 1, \ldots, n-1$ is a complete set of base-knots for D_{2n}.

The previous theorem is saying that every D_{2n}–coloured knot in S^3 has a presentation (in the sense of Convention 1) of the following form, where $0 \leq k < n$:



Here, T is some framed tangle, and each of the thick lines depicts a number of blackboard parallel strands. Each arc of the framed link L' which results from T together with the thick lines is labeled $1 \in D_{2n}$. After surgery on $L := L' \cup C$, where C is the s–labeled kn–framed unknot, U becomes the desired D_{2n}–coloured knot (K, ρ) in S^3.

We remark that it will follow from the proof that L can be chosen so that each of its components has linking zero with U.

As an example, see the surgery presentation for a D_{14}–coloured 5_2 knot given in Figure 1.

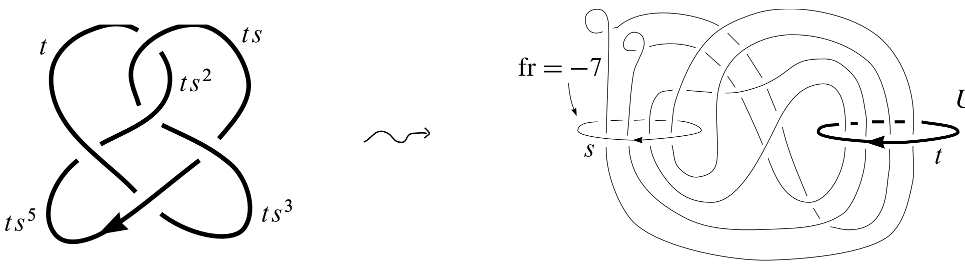


Figure 1: A surgery presentation for a D_{14}–coloured 5_2 knot

Theorem 2 *A surgery presentation for the irregular dihedral branched covering space M determined by the D_{2n}–coloured knot (K, ρ), and for the covering link $\widetilde{K}$ of K sitting inside M is as shown in Figure 2. In that figure, a small zero near an introduced surgery component means it has zero framing and $\widetilde{U}_1 \cup \cdots \cup \widetilde{U}_{(n+1)/2}$ is the covering link of U, becoming $\widetilde{K}$ after the surgeries are performed.*

Band projection approach

This approach is based on a choice of band projection for a Seifert surface F of the D_{2n}–coloured knot. Such a choice determines a basis for $H_1(F)$. To a basis for $H_1(F)$

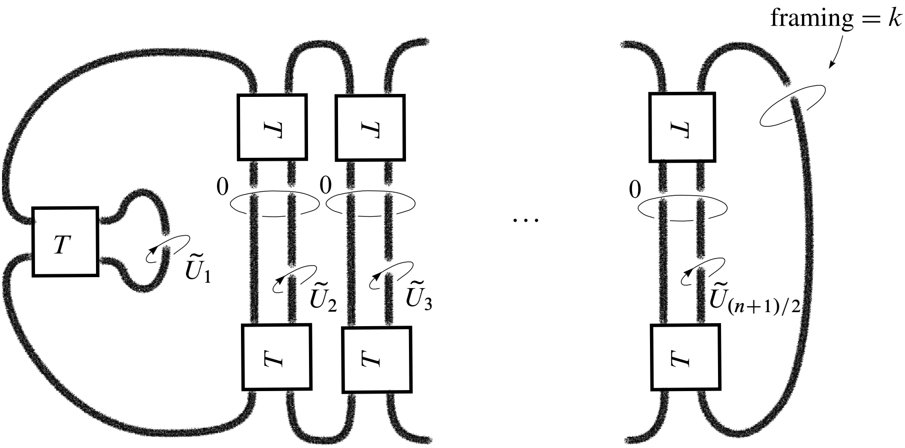


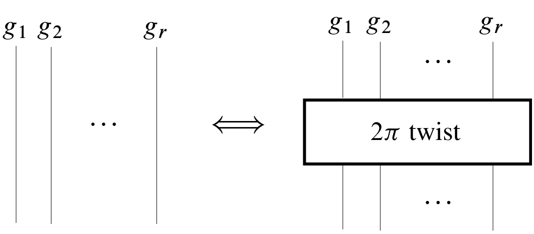
Figure 2: The surgery presentation for the covering space in the untying approach

there corresponds a Seifert matrix and a *colouring vector* (to be defined in Section 4.1.1). The colouring vector determines the D_{2n}–colouring of the knot (Lemma 14). The heart of this approach will be realizing algebraic operations on the Seifert matrix and colouring vector by sliding bands and performing ± 1–framed surgeries on unknots lying in the complement of the Seifert surface and in $\ker \rho$. While this method seems to be less efficient in practice, it is a stronger theoretical result because it arises from an equivalence relation on D_{2n}–coloured knots in S^3 whose corresponding equivalence classes can be detected with a certain algebraic invariant: the coloured untying invariant (see Moskovich [16]).

Definition 4 We say that two D_{2n}–coloured knots (K_1, ρ_1) and (K_2, ρ_2) in S^3 are ρ–equivalent if one can be obtained from the other by a sequence of surgeries on ± 1–framed unknots in $\ker \rho$.

We alert the reader that we are restricting to surgeries along ± 1–framed unknots, so that this is an equivalence relation on D_{2n}–coloured knots in S^3.

This equivalence can be defined as an equivalence relation on coloured knot diagrams without reference to surgery in the following way:



with $\prod_{i=1}^{r} g_i = 1 \in D_{2n}$.

Theorem 3 *Any D_{2n}–coloured knot (K,ρ) is ρ–equivalent to one of the D_{2n}–coloured knots of Figure 3 for $k = 0, 1, \ldots, n-1$. This implies that this set of knots (the pretzel knots $p((2k+1)n, 1, -n)$ for $k = 0, 1, \ldots, n-1$ with the specified colouring) is a complete set of base-knots for D_{2n}.*

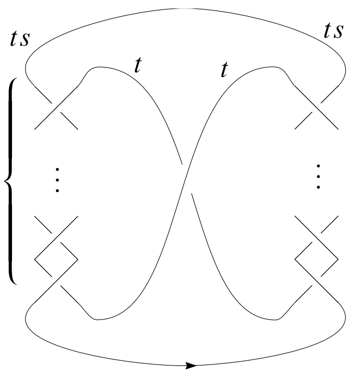


Figure 3: The base-knots in the band projection approach

Because too much extra notation would need to be introduced at this point in order to explain it, an explicit construction of the irregular branched dihedral covering spaces corresponding to the set of base-knots of Theorem 3 is pushed off to Section 4.4.

We remark that our proof shows that the same theorem holds if the ρ–equivalence relation is strengthened to require that the ±1–framed unknots not only lie in $\ker\rho$, but moreover have linking zero with K.

Having just defined a new equivalence relation, several questions immediately arise. How many equivalence classes are there? Can they be detected with algebraic information?

What we would really like is a theorem characterizing these classes in terms of a readily computable algebraic invariant. There are many prototypes for this in the recent literature. One example is the result of Murakami–Nakanishi [17], which is closely related to results of Matveev [15], which characterizes Δ–equivalence classes of links in terms of their linking matrices. Another is the result of Habiro [13] classifying knots, all of whose finite-type invariants up to a certain degree are equal, via surgery along tree claspers. Yet another is the work of Naik–Stanford [18] which links S–equivalence classes of knots to double-delta moves. The influence of this point of view on recent research should be clear.

Moskovich [16, Section 6] defined a nontrivial function from D_{2n}–coloured knots to $\mathbb{Z}_n$. Its value for a D_{2n}–coloured knot in S^3, in terms of a Seifert matrix S and a vector $\vec{w}$ which determines the D_{2n}–colouring ρ, is given by the formula

$$\mathrm{cu}(K,\rho) = \frac{2(\vec{w}^T \cdot S \cdot \vec{w})}{n} \bmod n.$$

The value $\mathrm{cu}(K,\rho) \in \mathbb{Z}_n$ was called the *coloured untying invariant*[4] of (K,ρ). It was proven there that cu is invariant under surgery in $\ker\rho$, and so, in particular, is a constant function on ρ–equivalence classes (see also Litherland and Wallace [14]). It was also shown there that every possible value is realized by some D_{2n}–coloured knot. These facts imply that the number of ρ–equivalence classes is at least n. On the other hand, because the complete sets of base-knots in Theorem 1 and in Theorem 3 both have cardinality n, it follows that n is also an upper bound for the number of ρ–equivalence classes. Thus a by-product of our constructions is:

Corollary 4 *Two D_{2n}–coloured knots have the same coloured untying invariant if and only if they are ρ–equivalent. In particular, the number of ρ–equivalence classes of D_{2n}–coloured knots is n.*

For n prime this was [16, Conjecture 1], where it was proved for $n = 3$ and for $n = 5$. This conjecture was also the subject of [14], where the machinery of Cochran–Gerges–Orr [5] was used to put an upper bound of $2n$ on the number of ρ–equivalence classes.

The view from here

As stated at the beginning of the introduction, our motivation is to develop a theory of quantum topology for dihedral covering spaces and covering links. How to proceed? Many tantalizing hints can be found in the literature.

One possible route would be to generalize recent results in the cyclic case [8; 9] due to Garoufalidis and Kricker. The results culminate in a universal formula for the LMO invariant of a cyclic branched cover in terms of the rational lift of the loop expansion of the Kontsevich invariant [10]. This rational lift may be viewed as a version of the Kontsevich invariant coloured by the canonical representation $\pi_1(\overline{S^3 - N(K)}) \to \mathbb{Z}$.

Using the surgery presentations in this paper, one should be able to obtain a version of these constructions where the colouring group is D_{2n} instead of $\mathbb{Z}$. Taking the "1–loop part" will give an analogue to the Alexander polynomial. The 2–loop part should determine the Casson(–Walker–Lescop) invariant for an irregular dihedral covering space M (which can be any 3–manifold).

[4] The authors thank Pat Gilmer for informing them that he had earlier considered this quantity in relation to Casson–Gordon invariants [11].

Another clue for the shape of such a theory is a mysterious formula for the Rohlin invariant of a dihedral branched covering space that was discovered in the seventies by Cappell and Shaneson [3; 4]. Recall that the Rohlin invariant is the mod 2 reduction of the Casson–Walker invariant, which is the unique finite type invariant of degree 1.

The theory of knot concordance has long been a blind spot for "traditional" quantum topology. The classical invariants which access this type of information are typically constructed from systems of covering spaces. One of our longer term goals is to develop sufficient technology to make contact with these constructions.

Odds and ends

The paper concludes in Section 5 with a variety of odds and ends which are immediate corollaries of the constructions in the previous sections. First, the choice of a complete set of base-knots in Theorem 3, which was made after trial and error, is of course not the only one possible. Some other choices are also worth mentioning. By choosing the twist knots in Figure 4 as a complete set of base-knots, we can prove that the surgery link in Theorem 1 can be chosen to have linking number zero with the component labeled by s. In that figure, $m = 1-(n+1)^2/2$ if $(n+1)/2$ is even, while if $(n+1)/2$ is odd then $m = 2-(n^2+1)/2$. Choosing the torus knots of Figure 5 gives a picture that is easy to lift (see Moskovich [16] for the $n=3$ and $n=5$ cases) but is not a natural end-point for our algorithms. Using it we can prove that a 3–manifold with D_{2n}–symmetry has a surgery presentation with D_{2n}–symmetry, extending a "visualization" result of Przytycki and Sokolov [20] and of Sakuma [25]. Finally, we may choose a complete set of base-knots which differ only by the choice of their D_{2n}–colouring, as shown in Figure 6.

Although the methods in this paper are elementary, the results appear to be new. Swenton [27], and independently Yamada [28] for $n=3$, give quite different algorithms for translating from dihedral covering presentations to surgery presentations, "forgetting" the knot.

Some further problems

- Explore the relationship between the untying approach and the band projection approach. In particular, how can one calculate the coloured untying invariant of a D_{2n}–coloured knot in a 3–manifold other than S^3?
- Explore the possibility of using Goeritz surfaces instead of Seifert surfaces in the band projection approach, giving the torus knots as a complete set of base knots directly.

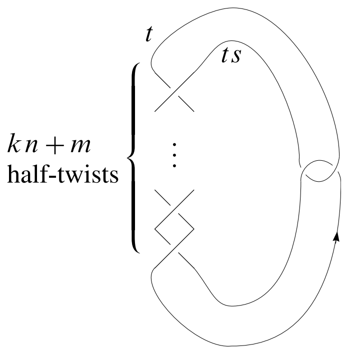


Figure 4

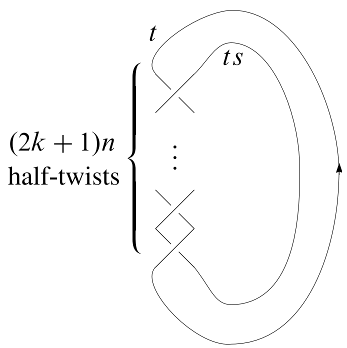


Figure 5

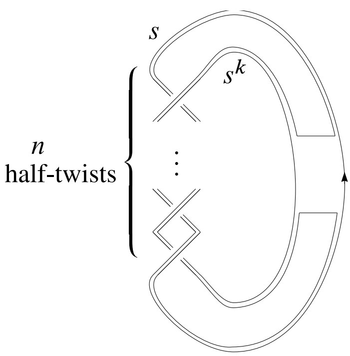


Figure 6

- Find minimal complete sets of base-knots for groups other than dihedral groups. Use these to find presentations for other classes of covering spaces.
- Extend the results of this paper to D_{2n}–coloured algebraically split links (the extension to boundary links is straightforward).

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2 Dihedral branched covering spaces

In this section we recall the way in which (K,ρ) presents a closed 3–manifold M.

We first recall how a *monodromy representation* characterizes an (unbranched) covering space. Let pr: $\widetilde{X} \twoheadrightarrow X$ be an n–fold (unbranched) covering space of a closed 3–manifold X with basepoint $*$. An oriented loop $\ell \subset X$ based at $*$ lifts to a collection of distinct paths $\ell_1, \ldots, \ell_n$ each starting and ending at one of the n preimages $*_1, \ldots, *_n$ of $*$ in $\widetilde{X}$. Sending the initial point of each of these paths to its endpoint gives a permutation of $*_1, \ldots, *_n$, inducing a representation

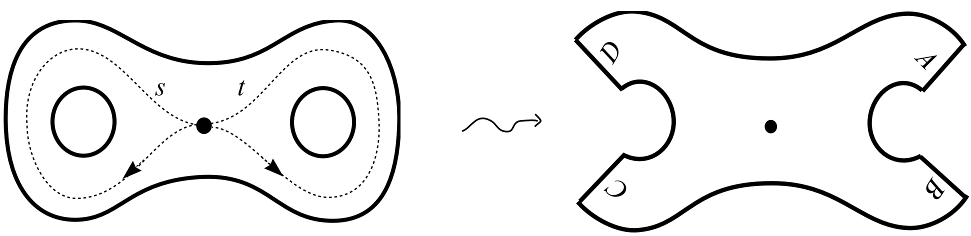
$$\pi_1(X, *) \to \mathrm{Sym}(\mathrm{pr}^{-1}(*))$$

which is unique up to relabeling lifts of the basepoint. Choosing a different basepoint in X modifies the representation via some bijection $\mathrm{pr}^{-1}(*) \simeq \mathrm{pr}^{-1}(*')$.

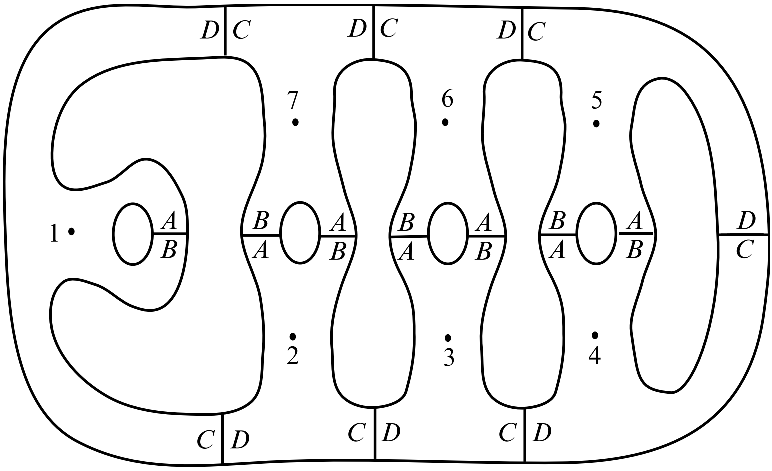
The theory of covering spaces tells us that two covering spaces are equivalent (that is, homeomorphic by a homeomorphism respecting the covering map) if and only if their monodromy representations are the same (after some relabeling). Thus we can specify a covering space by means of a representation $\pi_1(X, *) \to \mathrm{Sym}(\mathrm{pr}^{-1}(*))$.

From an intuitive cut-and-paste point of view it is natural to present a covering space by means of its monodromy. This allows one to construct it by cutting the base space into cells, taking the appropriate number of copies of each cell, and gluing them together according to the representation. The following example, which plays a part in the proof of Theorem 2, is a good illustration of this.

Example 1 Consider a genus two handlebody equipped with base point and a representation ρ from its fundamental group onto D_{14}. To construct the covering space whose monodromy group is given by this representation, we begin by cutting the handlebody into a cell:



Now we take seven copies of this cell, and glue them together according to ρ.



We now construct M. Begin from a D_{2n}–coloured knot (K,ρ), and consider the n–sheeted (unbranched) covering space $\widetilde{X}$ of the knot complement X of K (the closure in S^3 of the complement of a tubular neighbourhood $N(K)$ of K) with monodromy given by ρ, where D_{2n} is thought of as a subgroup of $\mathrm{Sym}(*_1,\ldots,*_n)$.

Consider the boundary of this covering space. What is it? Well, the D_{2n}–colouring ρ sends a meridian to a reflection, and the longitude may be chosen so that it is sent to 1. It follows that the boundary of $\widetilde{X}$ is a collection of $(n+1)/2$ tori—$(n-1)/2$ two-sheeted coverings and 1 one-sheeted covering of the boundary torus $\partial N(K)$ of X. Glue $(n+1)/2$ solid tori into these boundary components, longitude to longitude, such that a meridional disc is glued into some lift of a power of the meridian downstairs.

This is the desired space M: the branched dihedral covering space of S^3 associated to the D_{2n}–coloured knot (K,ρ). The cores of the glued-in tori, with orientations induced by the orientation of K, form the covering link $\widetilde{K}$.

Remark It is more usual to specify a covering space by a conjugacy class of subgroups of $\pi_1(X)$ (corresponding to the image of $\pi_1(\widetilde{X})$ under the projection). If a covering space is determined by a monodromy representation then the corresponding class of subgroups is given by taking the stabilizer of a chosen element in $\mathrm{Sym}(*_1,\ldots,*_n)$.

Remark The 3–manifold M is usually referred to as the *irregular* branched dihedral covering space associated to (K,ρ) (as we referred to it in the introduction), because it corresponds to the preimage under ρ of $\langle t\rangle$ which is not a normal subgroup of D_{2n}.

3 Untying approach

This approach consists of two steps. The first is to obtain a surgery presentation of a D_{2n}–coloured knot (K,ρ) in the complement of an unknot in a lens-space (one of the base-knots of Theorem 1). Such a presentation is called a *separated dihedral surgery presentation* of (K,ρ). The second step is to lift the separated dihedral surgery presentation to a surgery presentation of the dihedral branched covering space and of the covering link.

3.1 Obtaining a separated dihedral surgery presentation

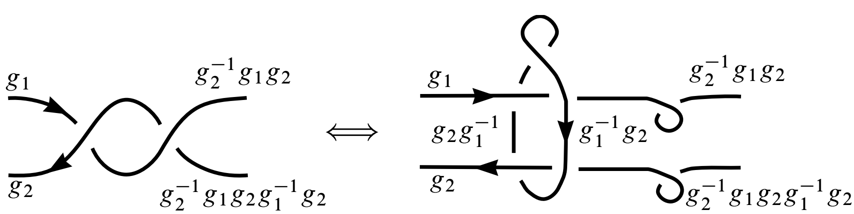
The construction consists of three steps: use surgery to untie the knot, perform handleslides to concentrate the nontrivial labels onto a single surgery component, and finish with another round of surgery to untie that surgery component. We'll also describe some moves which put the labels and surgery curves in the resulting diagram in a standard form.

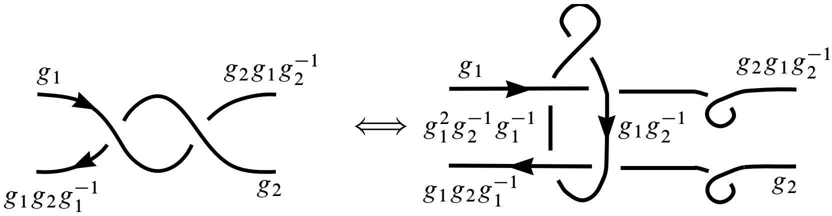
3.1.1 Untying the knot We can untie any knot K by crossing changes, realized by surgery on ± 1–framed unknots which have linking number zero with K. This allows us to present K as a ± 1–framed link L in the complement of a standard unknot $U \subset S^3$, such that surgery on L recovers $K \subset S^3$. In the following section we generalize this procedure to D_{2n}–coloured knots.

Note that the arcs of a knot in S^3 are all coloured by reflections (elements of the form $ts^a \in D_{2n}$) for the following reason. Near a crossing where the over-crossing arc is labeled g_1, the under-crossing arcs must be labeled g_2 and either $g_1^{-1}g_2g_1$ or $g_1g_2g_1^{-1}$ for some $g_2 \in D_{2n}$. So if any arc is labeled by a rotation then all arcs in the knot diagram would be labeled by rotations (because $\mathcal{C}_n \triangleleft D_{2n}$) which would contradict surjectivity of the D_{2n}–colouring ρ.

When performing surgery, the colours of the arcs of the introduced surgery component are induced as follows:

Lemma 5 *Let g_1 and g_2 be elements in D_{2n}. The local moves depicted below induce colours on the added surgery components as shown. (The two strands "being twisted" can be from the knot or from surgery components.)*

(3-1) 

(3-2) 

Proof The precise claim is that there is a PL–homeomorphism h between the two spaces, taking the knot in one space onto the knot in the other space, such that the pulled-back representation of the knot group is as shown. The homeomorphism h is to cut $\overline{S^3 - T}$ along a disc spanning T, a tubular neighbourhood of the introduced surgery component, do a 2π twist in the appropriate direction, then reglue the disc and the solid torus.

The label on an arc of the right-hand diagram is determined by the image under h of a path representing the appropriate element of the fundamental group. For example, we obtained the label $g_1^{-1} g_2$ in the first local move by finding the image under h of

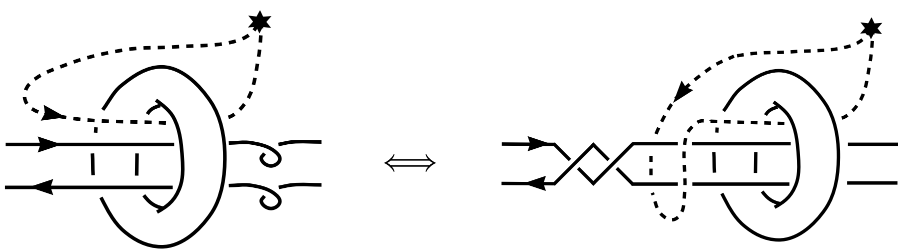
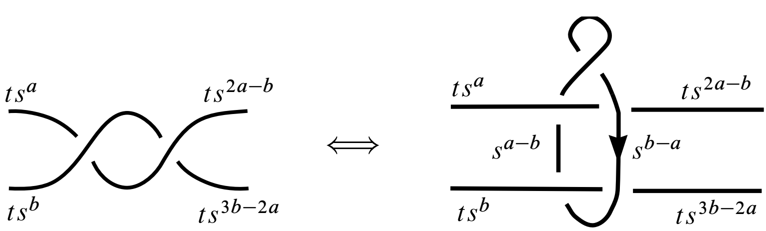


Figure 7

the Wirtinger generator corresponding to the appropriate meridian of the introduced surgery curve in Figure 7. □

Shortly, we'll use this lemma to untie D_{2n}–coloured knots. In that case we'll have $g_1 = ts^a$ and $g_2 = ts^b$ for some $a, b \in \mathbb{Z}_n$, so that $g_1^{-1} g_2 = s^{b-a}$, and the typical move will look like:



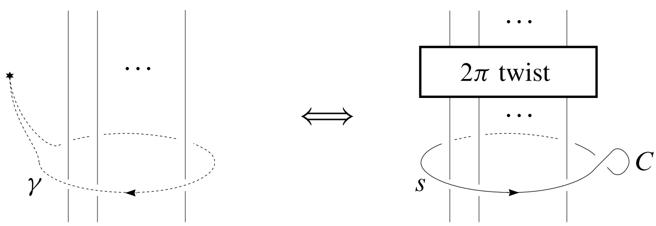
Here is what we are going to do:

(1) First, we'll do a special surgery to ensure that the first of our surgery components has an arc labeled s.

(2) Then, we'll untie the knot using Lemma 5.

At each step in this untying process, arcs of the knot are labeled by reflections, and arcs of the introduced surgery components are labeled by rotations. Because conjugation of a rotation by a reflection inverts it, while conjugation by another rotation leaves it unchanged, for each introduced surgery component there exists $j \in \mathbb{Z}_n$ such that all of its arcs are labeled either s^j or s^{-j}.

We begin with the special surgery: Because ρ is surjective, there exists an element $\gamma \in \pi_1(\overline{S^3 - N(K)})$ such that $\rho(\gamma) = s \in D_{2n}$. It is possible to arrange the knot diagram so that a representative for γ appears as on the left hand side of the following diagram. It is also possible to choose γ to have linking number zero with the knot (this is because it must have even linking number to map to s, and because each arc of the knot is labeled by an element of order 2, so that we can change the sense in which γ crosses an arc, which changes the linking number by 2, but does not change the image of γ under ρ). We can now introduce a surgery component C as shown on the right hand side, and the induced label will be s, as shown.

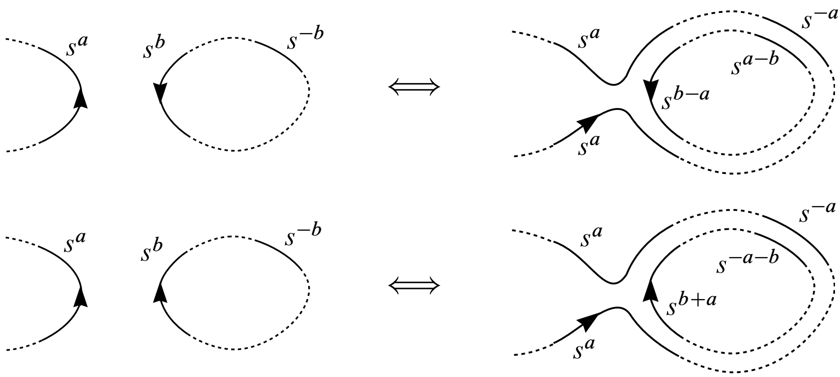


After introducing this special surgery component C, which we'll call the *distinguished surgery component*, use Lemma 5 to completely untie the resulting knot. After these manipulations we have a link $U \cup C \cup L_1 \cup \cdots \cup L_\mu$ in S^3, where:

- U is the unknot.
- C is the distinguished surgery component (each arc of which is labeled either s or s^{-1}).
- $L_1, \ldots, L_\mu$ are further surgery components whose arcs are labeled by various rotations.

3.1.2 Handleslides The second step of the construction is to perform handleslides so as to arrange that every surgery component except for the distinguished surgery component C has all its arcs labeled $1 \in D_{2n}$. The following lemma tells us how labels transform under handleslides.

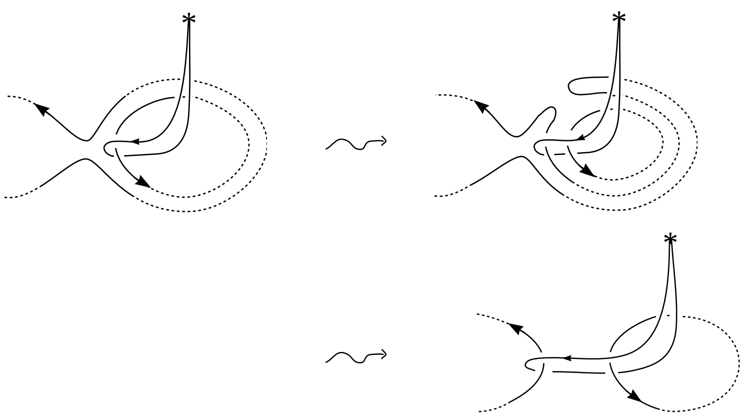
Lemma 6 *Two diagrams that differ by one of the moves shown below present equivalent D_{2n}–coloured knots. (The displayed components are surgery components.)*



Proof When two diagrams are related by a handleslide, the corresponding spaces are related by a PL–homeomorphism which is the identity outside a genus two handlebody containing the two involved components and the "path" of the slide.

To observe how labels transform: pick a curve representing some arc in the right-hand diagram, isotope the curve so that it lies outside the genus two handlebody corresponding to a handle slide which will take us to the left-hand diagram, then read off what that curve maps to in the left-hand diagram.

For example, the label s^{b-a}, above can be obtained as shown below:



Note further that once we know what the label on one of the arcs of a component is, the labels on all of its arcs are determined by the fact that they must induce a well-defined representation onto D_{2n}. □

By Lemma 6 we may repeatedly perform handleslides until all of the surgery components except the distinguished surgery component C has each of its arcs labeled 1, for the following reason. Recall that C has an arc labeled s. For each $1 < i \leq \mu$ let $a_i \in \mathbb{Z}_n$ be an element such that some arc of L_i is labeled s^{a_i}. The effect of sliding C over L_i is to replace s^{a_i} either by s^{a_i-1} or by s^{a_i+1} depending on which version of the handleslide is used. Thus sliding C over L_i repeatedly either a_i times or $n-a_i$ times kills the labels of the arcs of L_i. Do this for each $L_1, \ldots, L_\mu$.

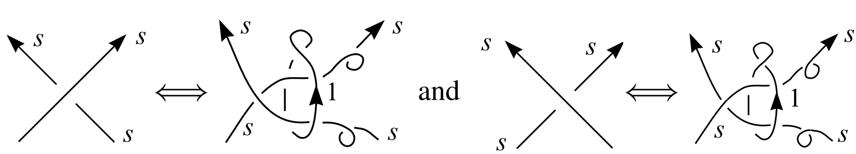
Remark Readers who try some examples will find that this second step can add significant complexity to the construction. However things are not so bad when $n = 3$. The reason is that there will only ever be a single handleslide required to kill the label on a surgery component, because $1 + 2 = 0 \bmod 3$ or $1 - 1 = 0 \bmod 3$. Note further that in this situation the surgery components will remain framed unknots after the handleslides.

3.1.3 Putting the presentation into a standard form After the first two steps we have a diagram where:

- The knot K has been untied and is in its standard position U.
- There are a number of surgery components, each of which has linking number zero with the knot.
- Every surgery component except C has all of its arcs labeled $1 \in D_{2n}$.
- The distinguished surgery component C has each of its arcs labeled either s or s^{-1}.

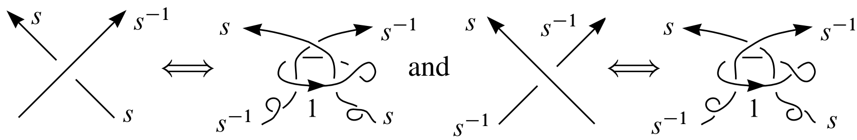
The final step is to add extra surgery components so that the two component sublink $U \cup C$ becomes a standard two component unlink. We will require that the surgery components introduced to make this happen have all of their arcs labeled $1 \in D_{2n}$.

In the neighbourhood of a crossing in C, either all arcs will be labeled s, in which case we can reverse the crossing by:

(3-3)


or the crossing will have one incident arc labeled s and another incident arc labeled s^{-1}, which can be dealt with by:

(3-4)

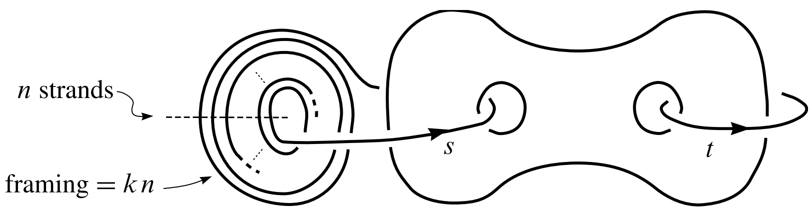


Thus, we can reverse any crossing on C by introducing such components, untying C and unlinking it from U.

Notice that the framing of the distinguished component must end up a multiple of n. This is because labels induce a well-defined representation of $\pi_1(\overline{S^3 - N(K)})$ onto D_{2n}, so contractible curves map to 1. The framing curve of every surgery component (in particular, of the distinguished component) bounds a disc in the corresponding torus being glued in and is thus contractible. Since each arc of C is labeled s, and C is separated from U, its framing curve f maps to $s^{\mathrm{Link}(f,C)}$. This implies that $\mathrm{Link}(f,C) = 0 \bmod n$.

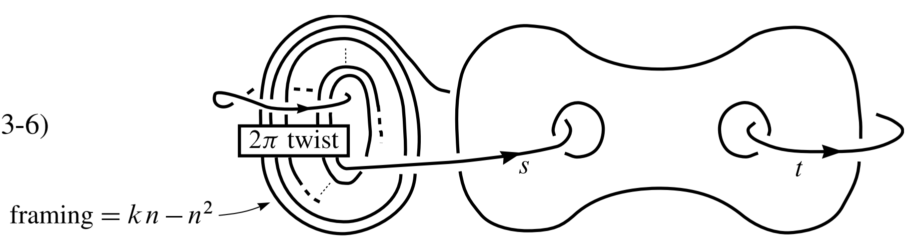
It is possible to introduce extra surgery components into the presentation which will change that framing by n^2. It follows that k may be chosen so that $0 \le k < n$. To do this, coil the distinguished surgery component into n parallel strands:

(3-5)



Then add a ± 1–framed surgery component. (Choose $+1$ to increase framing by n^2, and -1 to decrease framing by n^2.)

(3-6)



The distinguished surgery component may now be tied in a knot[5], but we can untie it using surgery along components whose arcs are all labeled $1 \in D_{2n}$, as shown in Equation (3-3). Observe that such moves do not change the framing of the distinguished

[5]For $p = 3$ for example, (3-5) and (3-6) will tie the distinguished surgery component into a trefoil.

surgery component because the linking number of the introduced surgery components with C is zero.

All arcs of U are labeled by some reflection $ts^a \in D_{2n}$. Because n is odd, there exists an integer b such that $a+2b = 0 \bmod n$. By repeating the ambient isotopy of Figure 8 b times, we may conjugate the label ts^a by s^b, so that the label on the arcs of U becomes $ts^{a+2b} = t$. We thus obtain a presentation for (K,ρ) in which all arcs of U are labeled t.

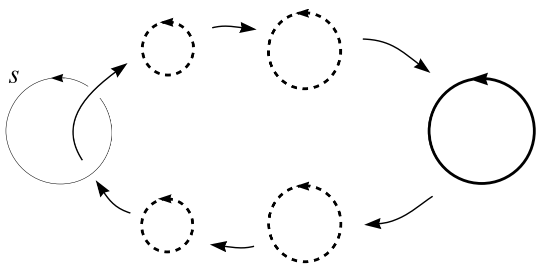


Figure 8: Ambient isotopy of the surgery picture to conjugate the label on U by s

To summarize:

Proposition 7 *Any D_{2n}–coloured knot (K,ρ) has a separated dihedral surgery presentation, ie it has an surgery presentation $L = C \cup L_1 \cup \cdots \cup L_m$ such that:*

- *The distinguished surgery component C has all its arcs labeled s and has framing kn with $0 \le k < n$.*
- *All arcs of the other components $L_1,\ldots,L_m$ are labeled 1.*
- *All arcs of U are labeled t.*
- *$C \cup U$ is the standard 2–component unlink.*

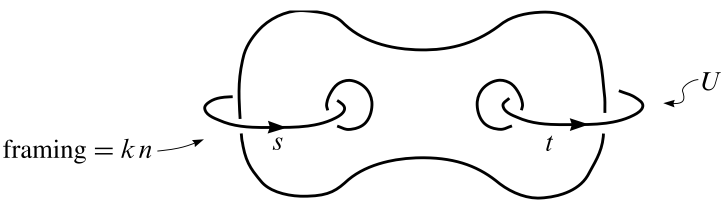
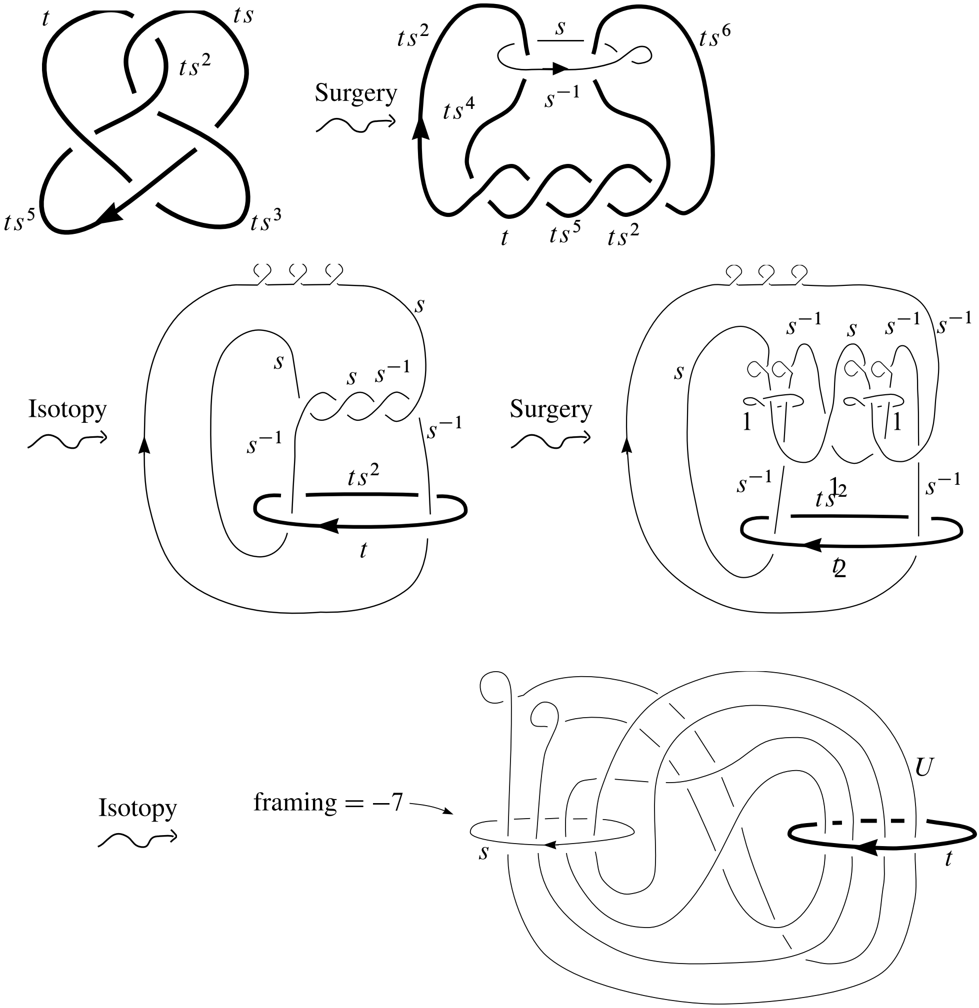


Figure 9: A separated dihedral surgery presentation

Observe that this Proposition is equivalent to our Theorem 1.

In Section 5.1.1 we will additionally show that a separated dihedral surgery presentation may be chosen such that the components $L_1, \ldots, L_m$ all have linking number zero with the distinguished component C.

3.1.4 Example: The D_{14}–coloured 5_2 knot As an example, let's see how we obtain a separated dihedral surgery presentation of a D_{14}–coloured 5_2 knot.



3.2 Constructing the cover

Take a separated dihedral surgery presentation of some D_{2n}–coloured knot (K, ρ). It consists of a framed link L in a genus two handlebody H, embedded into a link in the way shown in Figure 9. Our goal in this section is to lift this picture to a surgery presentation of M, the n–fold dihedral covering space of S^3 branched over the knot K whose monodromy is given by ρ.

Our starting point is Figure 10, which tells us how to use the separated dihedral surgery presentation to construct the knot complement $X := \overline{S^3 - N(K)}$. This is achieved by doing surgery on L, attaching 2–handles to the curves A and B, and finishing by attaching a ball to the resulting S^2 boundary component.

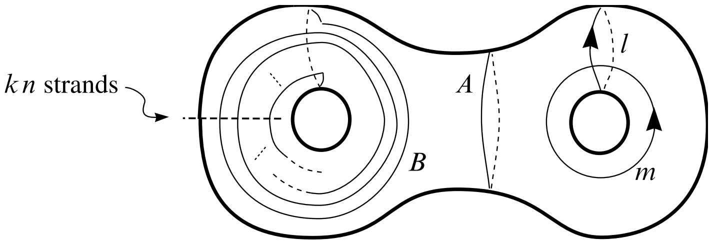
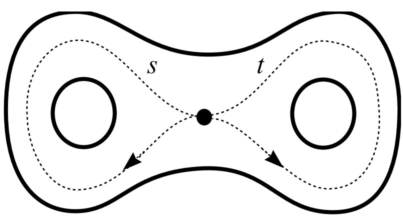


Figure 10: How to construct $X := \overline{S^3 - N(K)}$

The knot complement X comes equipped with the representation $\rho\colon \pi_1(X) \twoheadrightarrow D_{2n}$. The value that ρ takes on the generators of the fundamental group of H are as follows:



Observe that A and B, the attaching circles of the 2–handles, lie in the kernel of ρ as required (because they bounds discs in $\overline{S^3 - N(K)}$).

On the boundary of H we have also marked the meridian m and a choice of longitude l of K. This data will be referred to below as the *peripheral markings*. We recover S^3 with the knot K embedded in it by gluing a solid torus $N(K)$, displayed in Figure 11, into the boundary of X (a torus), so as to match up the curves m and l.

With these preliminaries in hand, we can now describe the construction of M. The first step is to construct $\widetilde{X}_\rho$, which is defined to be the (unbranched) covering space of X whose monodromy is specified by ρ. The following steps construct $\widetilde{X}_\rho$.

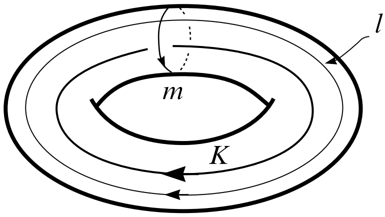


Figure 11: The solid torus $N(K)$ with embedded knot K

(1) Take $\widetilde{H}_\rho$, the n–fold covering space of H whose monodromy is specified by ρ. Lift the surgery link in H to $\widetilde{H}_\rho$ and do surgery on that link.
(2) Lift A and B, the 2–handle attaching circles, to systems of curves $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ on $\widetilde{H}_\rho$.
(3) Attach 2–handles to these systems of curves.
(4) Attach a ball to each of the n resulting S^2 boundary components.

Figure 13 shows how the attaching circles and peripheral markings lift to $\widetilde{H}_\rho$, in the special case that $n=7$. The general case is clear from this picture.

Consider now the boundary of $\widetilde{X}_\rho$, the space we have just constructed. Inspecting Figure 13 we observe that it consists of $(n+1)/2$ tori:

$$\partial(\widetilde{X}_\phi) = T_1 \sqcup T_2 \sqcup \cdots \sqcup T_{(n+1)/2}.$$

The torus T_1 is marked as shown in Figure 12 on the left. Under the restriction of the covering map $\widetilde{X}_\rho \to X$ to this boundary component, T_1 is a one-sheeted covering of $\partial N(K)$. The other tori, T_i where i runs from 2 to $(n+1)/2$, are marked as shown in Figure 12 on the right. These tori give two-sheeted coverings of $\partial N(K)$.

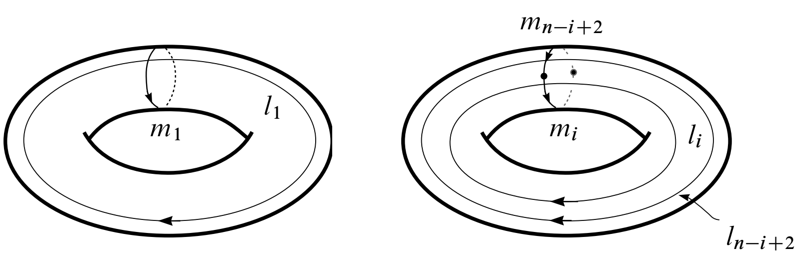


Figure 12: The torus T_1 (on the left) and a torus T_i for some $2 \le i \le (n+1)/2$ (on the right), together with their markings

The *branched* irregular dihedral covering space M, together with the covering link $\{\widetilde{K}_i\}_{i=1}^{(n+1)/2}$, is obtained from $\widetilde{X}_\rho$ by:

(1) Gluing a copy of $N(K)$ into T_1 so as to match m to m_1 and l to l_1.

(2) For each i such that $2 \le i \le (n+1)/2$, gluing a copy of $N(K)$ into T_i so as to match m to the curve $m_i m_{n-i+2}$, and l to either l_i or l_{n-i+2}.

This completes the construction of M.

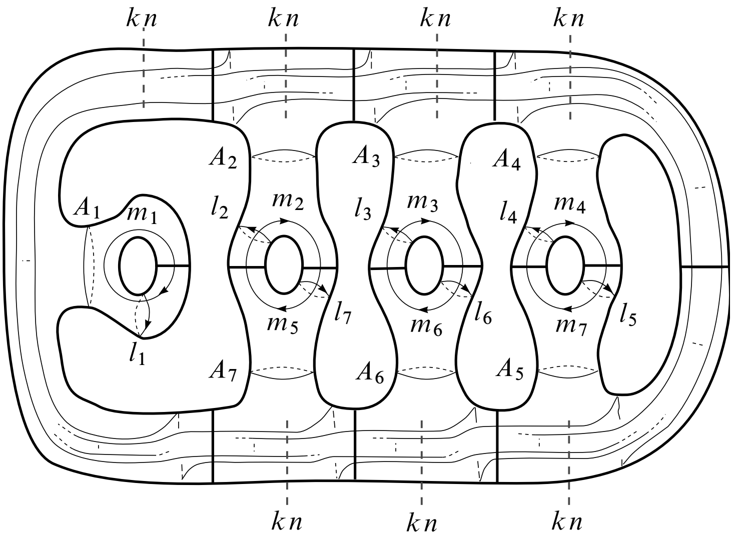
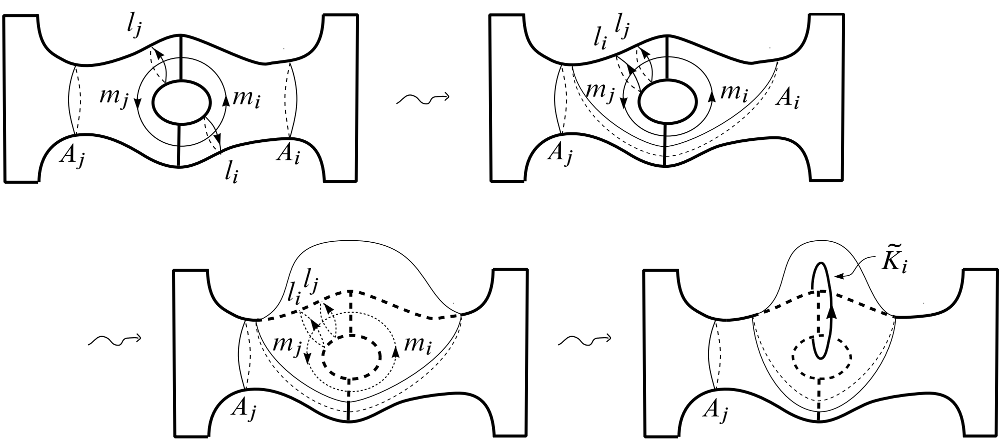


Figure 13: The lifts of the attaching circles and peripheral markings to $\widetilde{H}_\rho$, in the case that $n = 7$

Our task is to turn the construction we have just detailed into a surgery presentation for M. Consider the sequence below, where the index i runs from 2 to $(n+1)/2$, and $j = n - i + 2$.



The first move is to slide the attaching circle A_i over the attaching circle A_j. Before we do that, we'll get the longitude marking l_i out of the way by sliding it over A_j first. Next we attach a 2–handle to A_i. Observe that the result can be embedded in S^3, and that the torus T_i is now embedded in this diagram. Glue a copy of $N(K)$ into T_i in the required way (matching m to $m_i m_j$). In a similar way we can immediately attach a 2–handle to A_1 and glue a copy of $N(K)$ into T_1. These are the three steps which we carried out in the sequence above.

After the above sequence, if we attach 2–handles to the circles $A_{(n+3)/2}$ through A_n and another 2–handle to B_1, then the boundary of the space is a copy of S^2 (it is connected and of genus 0). Call this S^2 boundary Y. Plugging this boundary Y with a 3–ball right away is the same as attaching 2–handles to the attaching circles $B_2, \dots, B_n \subset Y$ and then plugging each of the resulting S^2 boundary components with 3–balls (all the extra 2–handles and 3–balls pair up into canceling pairs of complementary handles—see eg Rourke and Sanderson [24, Lemma 6.4]). In other words, we can discard $B_2, \dots, B_n$ without changing the result.

In the same way, we can add extra attaching circles for 2–handles into Y without changing the result. Let's then attach 2–handles into Y to cut the complement in S^3 of the handlebody into solid tori, in the way indicated in Figure 14. The attaching circles of the extra 2–handles are labeled $E_1, \dots, E_{(n+1)/2}$ in the figure.

We are done. The space constructed is in the complement of a $(n+1)/2$ component unlink in the three-sphere, and attaching the remaining 2–handles and balls is equivalent to doing surgery on that unlink, in precisely the way detailed in Theorem 2.

To illustrate with an example, the surgery presentation for the dihedral branched covering space and covering link for the D_{14}–coloured 5_2 knot considered in Section 3.1.4 is as given in Figure 15.

4 Band projection approach

In this approach we obtain a surgery presentation of a D_{2n}–coloured knot (K, ρ) as a link L consisting of ± 1–framed *unknotted* surgery components in $\ker \rho$ which live in the complement of an element in a complete set of base-knots in S^3. We then lift this presentation to a surgery presentation of the branched dihedral cover M. A by-product of this approach is a proof of a conjecture that two D_{2n}–coloured knots are ρ–equivalent if and only if they have the same coloured untying invariant.

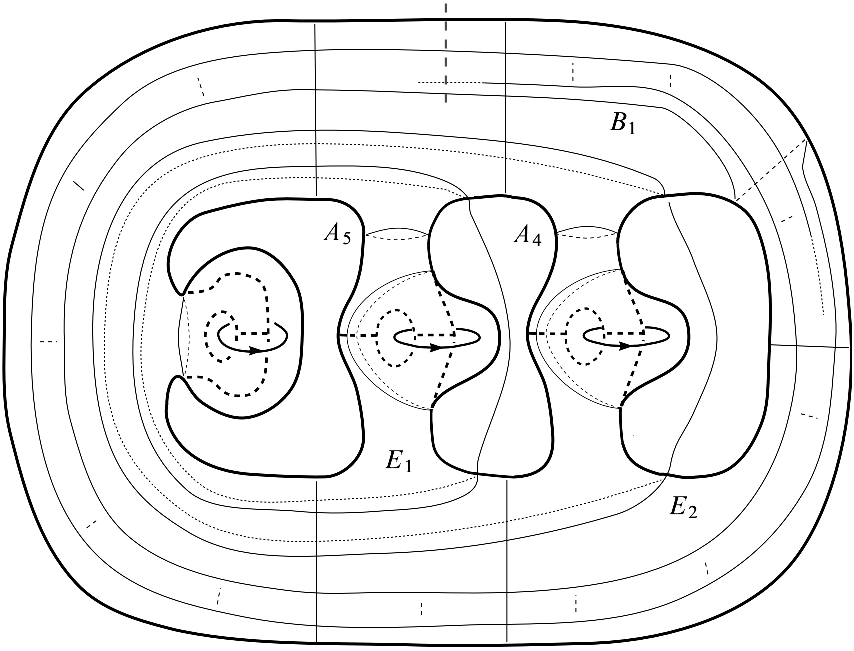


Figure 14: The final diagram, after we have discarded B_2 through B_n and attached extra 2–handles E_1 thorough $E_{(n+1)/2}$ so as to cut the complement of the handlebody into solid tori

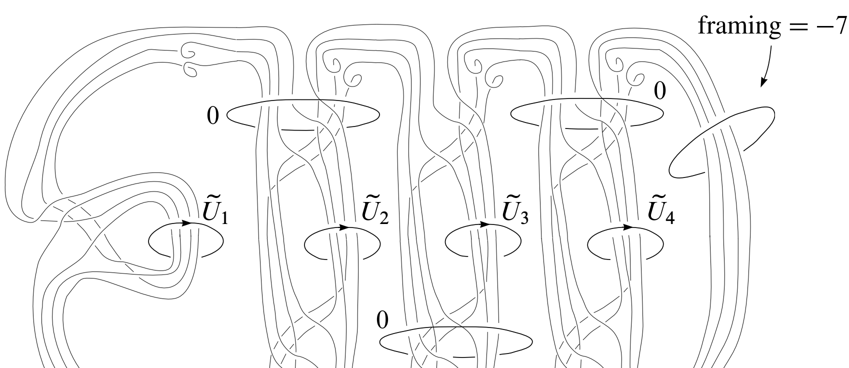


Figure 15: A surgery presentation for the dihedral covering space and covering link of the D_{14}–coloured 5_2 knot of Section 3.1.4

4.1 Preliminaries

In Section 4.1.1, given a D_{2n}–coloured knot (K,ρ), for each choice of Seifert surface F for K and for each choice of basis for $H_1(F)$ we introduce a pair $(S,\vec{v})$ consisting of a Seifert matrix S for K and a vector $\vec{v}$ which (loosely speaking) represents the D_{2n}–colouring ρ restricted to the complement of F. The precise correspondence between ρ and $\vec{v}$ is given in Proposition 8. The pair $(S,\vec{v})$ is called *surface data* for (K,ρ), and may be thought of as an analogue of the Seifert matrix for coloured knots. The material in this section is somewhat standard, but doesn't appear in the literature in precisely the form we need, so for the purpose of completeness we describe it in some detail.

In Section 4.1.2 we recall the coloured untying invariant, which is an obstruction to ρ–equivalence given in terms of surface data.

Following this, in Section 4.1.3, we restrict to the case that our chosen basis for $H_1(F)$ is a "symplectic" basis induced by a band projection for K. The action of the mapping class group of F on surface data for (K,ρ) with respect to such a basis for $H_1(F)$ can be interpreted in terms of sliding bands. Some of these band slides are investigated in Section 4.1.4.

4.1.1 The surface data Let (K,ρ) be a D_{2n}–coloured knot. Let $F\subset\overline{S^3-N(K)}$ be a Seifert surface for K, and let $F\times[-1,1]\subset\overline{S^3-N(K)}$ be a bicollar for F. Thus $(\partial F)\times[-1,1]$ is an annulus embedded in $\partial N(K)$. Set $N(F)$ to be $N(K)\cup(F\times[-1,1])$. Choose a basepoint $\star$ in $\overline{S^3-N(F)}$. See Figure 16 for an illustration. Denote by ι the inclusion mapping $\iota\colon \overline{S^3-N(F)}\hookrightarrow\overline{S^3-N(K)}$.

The image of the induced map $\iota_*\colon \pi_1(\overline{S^3-N(F)})\to\pi_1(\overline{S^3-N(K)})$ lies in the commutator subgroup of the knot group $\pi_1(\overline{S^3-N(K)})$. This is because the abelianization map Ab: $\pi_1(\overline{S^3-N(K)})\twoheadrightarrow\mathbb{Z}$ is given by $\mathrm{Ab}(x)=\mathrm{Link}(x,K)$, while any $\star$–based loop in the complement of F has linking zero with K (in fact $\mathrm{Link}(x,K)$ equals the algebraic intersection number of x with F). It follows that the image of the composition $\rho\circ\iota_*$ is contained in the normal subgroup of rotations:

$$\mathrm{Im}\,(\rho\circ\iota_*)\subset\mathcal{C}_n\lhd D_{2n}.$$

Let the induced map from the first homology of the Seifert surface complement to $\mathcal{C}_n$ be denoted $\bar{\rho}$:

$$\bar{\rho}\colon H_1(\overline{S^3-N(F)})\to\mathcal{C}_n.$$

Taking some small liberties, we will refer to this map as the restriction of the representation ρ to the complement of the Seifert surface.

Definition 5 Let $x_1, \dots, x_{2g}$ be a basis of $H_1(F)$ and let $\xi_1, \dots, \xi_{2g}$ be the associated basis for $H_1(S^3 - N(F))$ which is uniquely characterized by the condition that $\mathrm{Link}(x_i, \xi_j) = \delta_{ij}$ (see eg Burde and Zieschang [2, Definition 13.2]). The *colouring vector* associated to this Seifert surface F and choice of basis is defined to be the vector

$$\vec{v} := (v_1, \dots, v_{2g})^T := (\overline{\rho}(\xi_1), \dots, \overline{\rho}(\xi_{2g}))^T \in (\mathcal{C}_n)^{2g}.$$

We'll assume the "+" side of the bicollar is the side pointed to by the normal vector to F determined by the orientation of K. Let the *push-off maps* $\tau^{\pm}\colon F \to \overline{S^3 - N(F)}$ be the maps which take $x \in F$ to $(x, \pm 1) \in F \times \{\pm 1\} \subset \overline{S^3 - N(F)}$. The *Seifert matrix* is the matrix which describes these maps with respect to the chosen dual bases $x_1, \dots, x_{2g}$ and $\xi_1, \dots \xi_{2g}$. Namely, if coefficients S_{ij} are determined by

$$\tau_*^-(x_i) = \sum_{j=1}^{2g} S_{ij}\xi_j\,,$$

then we set the Seifert matrix S to be: $S = (S_{ij})$. The following formula follows from this definition:

$$S_{ij} = \mathrm{Link}\left(\tau_*^-(x_i), x_j\right)\,.$$

The map corresponding to the push-off in the positive direction is given by the transpose of the Seifert matrix:

$$\tau_*^+(x_i) = \sum_{j=1}^{2g} (S^T)_{ij}\xi_j\,.$$

Below we will need the following, which follows directly from our definitions. An integer vector $\widetilde{\vec{v}} = (z_1, \dots, z_n)^T \in \mathbb{Z}^{2g}$ will be called an *integral lift* of a colouring vector $\vec{v} := (v_1, \dots, v_n)^T \in (\mathcal{C}_n)^{2g}$ if $v_i = s^{z_i}$ for $i = 1, \dots, n$. Then, if $a = \sum_{i=1}^{2g} a_i x_i \in H_1(F)$:

(4-1) $$\overline{\rho}\left(\tau_*^+(a)\right) = s^{(a_1,\dots,a_{2g})S\widetilde{\vec{v}}}.$$

Similarly:

(4-2) $$\overline{\rho}\left(\tau_*^-(a)\right) = s^{(a_1,\dots,a_{2g})S^T\widetilde{\vec{v}}}.$$

Definition 6 The pair $(S, \vec{v})$ is called the *surface data* corresponding to the D_{2n}–coloured knot (K, ρ), a Seifert surface F, and a choice of basis $x_1, \dots, x_{2g}$ of $H_1(F)$.

The following theorem will summarize the relationships between the representation ρ, the restricted representation $\overline{\rho}$, and the colouring vector $\vec{v}$. The statement of the theorem will use the notion of *rotation-equivalent representations*. We will say that two representations

$$\rho_1, \rho_2\colon \pi_1(\overline{S^3 - N(K)}) \to D_{2n}$$

are rotation-equivalent precisely when there is a fixed integer j with the property that for all $g \in \pi_1(\overline{S^3 - N(K)})$,

$$\rho_2(g) = s^j \rho_1(g) s^{-j} \in D_{2n}.$$

Note the immediate fact that if two representations are rotation-equivalent $\rho_1 \sim \rho_2$ then their restrictions are equal:

$$\overline{\rho}_1 = \overline{\rho}_2.$$

Proposition 8 *Let K be an oriented knot, let F be a Seifert surface for K, and let $x_1, \dots, x_{2g}$ be a basis for $H_1(F)$. Corresponding to this data, there are bijections between three sets:*

(1) *The set of epimorphisms $\{\rho\colon \pi_1(\overline{S^3 - N(K)}) \twoheadrightarrow D_{2n}\}$ modulo rotation-equivalence.*

(2) *The set of epimorphisms $\{\psi\colon H_1(\overline{S^3 - N(F)}) \twoheadrightarrow \mathcal{C}_n\}$ satisfying the condition that for every $a \in H_1(F)$, $\psi\left(\tau_*^+(a)\right) = \psi\left(\tau_*^-(a)\right)^{-1}$.*

(3) *The set of vectors $\{\vec{v} = (v_1, \dots, v_{2g}) \in \mathcal{C}_n^{2g}\}$ satisfying the two conditions that:*
   (a) *The elements $\{v_1, \dots, v_{2g}\}$ together generate $\mathcal{C}_n$.*
   (b) *For any integral lift $\tilde{\vec{v}}$ of $\vec{v}$:*

$$(S + S^T)\,\tilde{\vec{v}} = \vec{0} \bmod n.$$

The map from the first set to the second set is restriction of a representation

$$\rho\colon \pi_1(\overline{S^3 - N(K)}) \to D_{2n}$$

to a representation $\overline{\rho}\colon H_1(\overline{S^3 - N(F)}) \to \mathcal{C}_n$. The map from the second set to the third is to take the colouring vector corresponding to the given choice of basis for $H_1(F)$.

To establish these bijections we'll employ a presentation of the fundamental group $\pi_1(\overline{S^3 - N(K)})$ that results from a Seifert–van Kampen calculation based on the Seifert surface F. The presentation is stated in the following lemma. Our description will be somewhat broad because the details are standard (see eg Rotman [23, Chapter 11, pages 407–410]).

The presentation depends on a choice of a special curve γ. This curve will be chosen as follows. Let $\star_F$ be a base-point on F. Choose a simple path γ^+ in $\overline{S^3 - N(F)}$ from the basepoint $\star$ to $(\star_F, 1)$, recalling that we defined $N(F)$ at the start of this section to be $N(K) \cup (F \times [-1,1])$. Similarly, choose a second simple path γ^- in $S^3 - (F \times [-1,1])$, disjoint from the first γ^+, from $\star$ to $(\star_F, -1)$. Let γ^0 denote the natural path in $F \times [-1,1]$ from $(\star_F, 1)$ to $(\star_F, -1)$ (the path whose projection to F is constantly $\star_F$). Now set $\gamma = \gamma^+ \gamma^0 (\gamma^-)^{-1}$. Make these choices so that γ is a simple closed PL curve in general position with respect to F. We can choose γ to be homotopic to a meridian of K. See Figure 16.

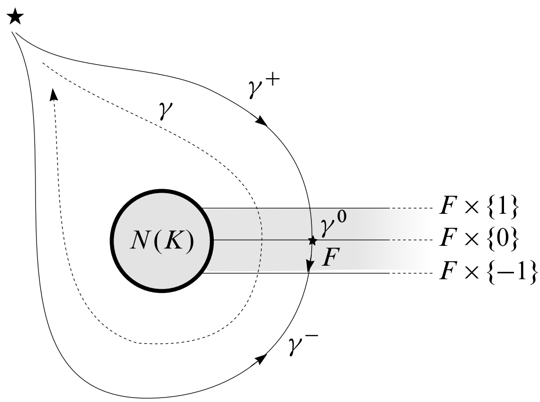


Figure 16: A schematic picture of a cross-section of F

The presentation employs two homomorphisms

$$\mu^{\pm}\colon\ \pi_1(F, \star_F) \to \pi_1(\overline{S^3 - N(F)}, \star).$$

The definition is: take α, some $\star_F$–based curve in F, representing some element $[\alpha] \in \pi_1(F, \star_F)$. Then $\mu^{\pm}([\alpha])$ is represented by the curve

$$\mu^{\pm}([\alpha]) = [\gamma^{\pm} \cdot (\alpha \times \pm 1) \cdot (\gamma^{\pm})^{-1}].$$

Lemma 9 *Consider the following free product, where $\langle m \rangle$ denotes the infinite cyclic group generated by m:*

$$\pi_1(\overline{S^3 - N(F)}, \star) * \langle m \rangle.$$

Let $\mathcal{N}$ denote the smallest normal subgroup containing the elements

$$\left\{ m^{-1} \mu^+(z) m (\mu^-(z))^{-1},\ z \in \pi_1(F, \star_F) \right\}.$$

There is an isomorphism

$$\varphi\colon\ \frac{\pi_1(\overline{S^3 - N(F)}, \star) * \langle m \rangle}{\mathcal{N}} \xrightarrow{\ \simeq\ } \pi_1(\overline{S^3 - N(K)}, \star),$$

where the subgroup $\pi_1(\overline{S^3 - N(F)}, \star)$ maps by the inclusion map ι, and the element m is mapped to $[\gamma] \in \pi_1(\overline{S^3 - N(K)}, \star)$.

Proof This is a standard Seifert–van Kampen calculation. See eg Rotman [23, Chapter 11, pages 407–410]. □

Proof of Proposition 8 We'll address the two bijections in turn, working directly with the presentation in the lemma.

Between sets (1) and (2) In this case we are taking a surjective representation

$$\rho\colon\ \frac{\pi_1(\overline{S^3-N(F)},\star)*\langle m\rangle}{\mathcal{N}} \twoheadrightarrow D_{2n},$$

satisfying the property that $\rho(\pi_1(\overline{S^3-N(F)},\star))\subset \mathcal{C}_n$, and considering the induced map (its "restriction")

$$\overline{\rho}\colon\ H_1(\overline{S^3-N(F)})\to \mathcal{C}_n.$$

We'll check each part of the statement in turn.

- ($\overline{\rho}$ satisfies the stated algebraic condition.) First we'll explain why the restriction satisfies the condition that for every $a\in H_1(F)$,

  $$\overline{\rho}(\tau_*^+(a)) = \overline{\rho}(\tau_*^-(a))^{-1}\in \mathcal{C}_n.$$

  The reason is that ρ must be well-defined modulo the normal subgroup $\mathcal{N}$. So for every $z\in\pi_1(F,\star_F)$, the element

  $$\rho(m)^{-1}\rho(\mu^+(z))\rho(m)\rho(\mu^-(z))^{-1}\in D_{2n}$$

  must be trivial. Now in order for ρ to be surjective, it must map m to a reflection. So the above element of D_{2n} will equal $\rho(\mu^+(z))^{-1}\rho(\mu^-(z))^{-1}$, which is the same as $\overline{\rho}(\tau_*^+(\tilde{z}))^{-1}\overline{\rho}(\tau_*^-(\tilde{z}))^{-1}$ where $\tilde{z}$ denotes the image of z in $H_1(F)$. This explains the required condition.

- ($\overline{\rho}$ is surjective.) Next we'll ask: Why is the restriction $\overline{\rho}$ surjective onto $\mathcal{C}_n$ if ρ is surjective onto D_{2n}? Our task is to find some element in $H_1(\overline{S^3-N(F)})$ which $\overline{\rho}$ maps to $s\in\mathcal{C}_n$. Well, if ρ is surjective, then there is some word

  $$w := x_1m^{n_1}x_2m^{n_2}\cdots x_km^{n_k},$$

  with each $x_i\in\pi_1(\overline{S^3-N(F)},\star)$, such that $\rho(w)=s\in D_{2n}$. Because $\rho(x_1),\ldots,\rho(x_k)$ are rotations, while $\rho(m)$ is a reflection, $\rho(x_im)=\rho(mx_i^{-1})$ for all $1\le i\le k$. Thus we may push all powers of m to the head of the word, and $\rho(w)=\rho(m^{n_1+\cdots+n_k})\rho(x_1)^{\pm1}\cdots\rho(x_k)^{\pm1}$ for certain choices of signs. On the other hand, because $\rho(w)=s$ it follows that $\rho(m^{n_1+\cdots+n_k})$ vanishes. Therefore

  $$\overline{\rho}(\pm\tilde{x}_1\cdots\pm\tilde{x}_k)=s.$$

- (Surjectivity of the "restriction" correspondence) Now we'll check that every ψ satisfying the stated conditions is the restriction of a surjective representation ρ. Choose some reflection $T = ts^j$, then define

  $$\rho\colon\ \frac{\pi_1(\overline{S^3 - N(F)}, \star) * \langle m\rangle}{\mathcal{N}} \twoheadrightarrow D_{2n}$$

  by

  (1) $\rho(x) = \psi(\tilde{x})$ if $x \in \pi_1(\overline{S^3 - N(F)}, \star)$,
  (2) $\rho(m) = T$.

  The condition on ψ ensures that $\mathcal{N}$ is sent to the identity. It is also clear that ρ is surjective onto D_{2n}.

- (Injectivity of the "restriction" correspondence) If two representations ρ_1 and ρ_2 have the same restriction, then they only differ in which reflection m is sent to. But any two such choices will be related by a rotation-equivalence.

Between sets (2) and (3) It is clear that the set of homomorphisms $H_1(\overline{S^3 - N(F)}) \to \mathcal{C}_n$ corresponds bijectively with $\mathcal{C}_n^{2g}$: the set of vectors of images of the basis elements ξ_1 up to ξ_{2g}. What we have to check is that the two extra conditions imposed on the homomorphisms are equivalent to the algebraic conditions imposed on the vectors. We'll check these conditions in turn.

- (Surjectivity of $\bar{\rho}$) It is clear that $\bar{\rho}$ is surjective precisely when it is possible to construct s from the images of the base elements.

- (The pushoff condition) Choose $z_i \in \mathbb{Z}$ so that $\bar{\rho}(\xi_i) = s^{z_i}$. Following Equations (4-1) and (4-2), if

  $$a = \sum_{i=1}^{2g} a_i x_i \in H_1(F)$$

  then $$\bar{\rho}\left(\tau_*^+(a)\right)\bar{\rho}\left(\tau_*^-(a)\right) = s^{(a_1,\ldots,a_{2g})(S+S^T)(z_1,\ldots,z_{2g})^T}.$$

  It follows that the conditions imposed on the sets (2) and (3) are equivalent. □

Remark The vectors $\tilde{\tilde{v}}$ and $\tilde{\tilde{v}}$ mod n are called the p–colouring vector in [14] and in [16], respectively.

4.1.2 The coloured untying invariant Let (K,ρ) be a D_{2n}–coloured knot. We have just explained how a choice of Seifert surface F and choice of basis for $H_1(F)$ determines surface data $(S,\vec{v})$. In [16, Section 6] it was shown that the expression

$$\mathrm{cu}(K,\rho) = \frac{2(\tilde{\vec{v}}^T \cdot S \cdot \tilde{\vec{v}})}{n} \bmod n, \tag{4-3}$$

where $\tilde{\vec{v}}$ is an integral lift of $\vec{v}$, depends neither on the choice of Seifert surface F nor on the choice of basis for $H_1(F)$. Hence it is an invariant of D_{2n}–coloured knots. It is also shown that this is a nontrivial $\mathbb{Z}_n$–valued invariant of D_{2n}–coloured knots in S^3 which is constant on ρ–equivalence classes. A homological version of this invariant provides a generalization to D_{2n}–coloured knots in more general 3–manifolds [14; 16].

The culmination of this section is to show that *two knots are ρ–equivalent if and only if they have the same coloured untying invariant*.

4.1.3 Band projection Any knot has a *band projection* (see for instance Burde and Zieschang [2, Proposition 8.2]). This is a projection of form shown in Figure 17. Pairs of bands B_{2i-1} and B_{2i} for $i=1,\dots,g$ will be called *twin bands*.

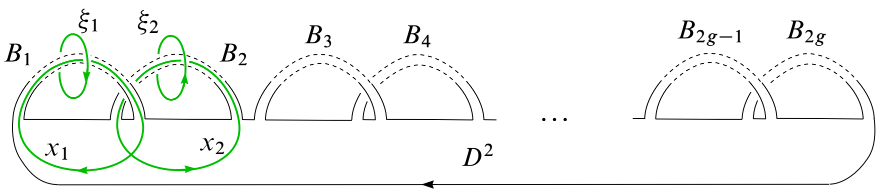


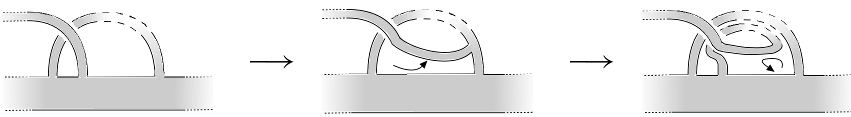
Figure 17: A band projection of a knot, in which we have indicated the orientation of the knot, a basis $x_1,\dots,x_{2g}$ for $H_1(F)$, and the associated basis $\xi_1,\dots,\xi_{2g}$ for $H_1(\overline{S^3 - N(F)})$

A knot in band projection comes equipped with a canonical choice of a Seifert surface F and a choice of basis for $H_1(F)$: let $x_1,\dots,x_{2g}$ be elements of $H_1(F)$ such that for each $1\le i\le 2g$ the class x_i is represented by a curve in F which threads once through the band B_i, with orientations as determined by Figure 17. Recall that the associated basis $\xi_1,\dots,\xi_{2g}$ for $H_1(\overline{S^3 - N(F)})$ is determined by the condition $\mathrm{Link}(x_i,\xi_j)=\delta_{ij}$. In this case the class ξ_i is represented by the appropriately oriented boundary of a small disc which the band intersects the interior of transversely as shown in Figure 17.

The surface data of a knot in band projection refers to the Seifert matrix and colouring vector for this canonical choice of basis.

4.1.4 Band slides At the heart of this approach are moves which allow us to realize algebraic manipulations of the surface data by ambient isotopy, modifying the choice of band projection of a fixed D_{2n}–coloured knot.

We say that some band projection is obtained from another by doing a *band slide* of band B_{2i-1} counterclockwise over band B_{2i} if it is obtained by the following sequence of ambient isotopies:



Similarly we can slide B_{2i-1} clockwise over B_{2i}, and we can slide B_{2i} over B_{2i-1} both clockwise and counterclockwise.

These moves fix F but change the choice of basis for $H_1(F)$, and so will change the surface data. The effect on the choice of basis is:

- Sliding B_{2i-1} counterclockwise (respectively clockwise) over B_{2i}:
$$\left(x_1,\dots,x_{2i-1},x_{2i},\dots,x_{2g}\right)\mapsto\left(x_1,\dots,x_{2i-1}\pm x_{2i},x_{2i},\dots,x_{2g}\right)$$
- Sliding B_{2i} counterclockwise (respectively clockwise) over B_{2i-1}:
$$\left(x_1,\dots,x_{2i-1},x_{2i},\dots,x_{2g}\right)\mapsto\left(x_1,\dots,x_{2i-1},x_{2i}\pm x_{2i-1},\dots,x_{2g}\right)$$

And the corresponding effect on the colouring vector is as follows:

- Sliding B_{2i-1} counterclockwise (respectively clockwise) over B_{2i}:
$$\left(v_1,\dots,v_{2i-1},v_{2i},\dots,v_{2g}\right)\mapsto\left(v_1,\dots,v_{2i-1},v_{2i}\cdot v_{2i-1}^{\mp1},\dots,v_{2g}\right)$$
- Sliding B_{2i} counterclockwise (respectively clockwise) over B_{2i-1}:
$$\left(v_1,\dots,v_{2i-1},v_{2i},\dots,v_{2g}\right)\mapsto\left(v_1,\dots,v_{2i-1}\cdot v_{2i}^{\mp1},v_{2i},\dots,v_{2g}\right)$$

The corresponding effects on the Seifert matrix are $S\mapsto(P^{\pm}_{(2i-1,2i)})S(P^{\pm}_{(2i-1,2i)})^T$ and $S\mapsto(P^{\pm}_{(2i,2i-1)})S(P^{\pm}_{(2i,2i-1)})^T$ for $P^{\pm}_{j,k}:=I\pm E_{j,k}$.

Example 2 Let (K,ρ) be a D_{2n}–coloured genus one knot for which

$$(S,\vec{v})=\left(\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix},\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)\tag{4-4}$$

with respect to a basis of $H_1(F)$ determined by a band projection. The effect of band sliding B_1 over B_2 counterclockwise is as follows:

$$(S,\vec{v}) \mapsto \left(\begin{pmatrix} a_{11}+a_{12}+a_{21}+a_{22} & a_{12}+a_{22} \\ a_{21}+a_{22} & a_{22} \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \cdot v_1^{-1} \end{pmatrix} \right) \tag{4-5}$$

The following two lemmas are crucial in this approach. They show how much freedom band slides give us to engineer the colouring vector.

Lemma 10 *For twin bands B_{2i-1} and B_{2i} for which either v_{2i-1} or v_{2i} generates $\mathcal{C}_n$, band slides allow us to transform the pair (v_{2i-1}, v_{2i}) to any other pair (v'_{2i-1}, v'_{2i}) for which either v'_{2i-1} or v'_{2i} generates $\mathcal{C}_n$.*

Proof Assume without the limitation of generality that v_{2i-1} generates $\mathcal{C}_n$. Then by sliding B_{2i-1} over B_{2i} an appropriate number of times, we can transform v_{2i} into a generator of $\mathcal{C}_n$ (in fact into any element of $\mathcal{C}_n$). We can therefore assume that both v_{2i-1} and v_{2i} generate $\mathcal{C}_n$. Symmetrically, we can assume that both v'_{2i-1} and v'_{2i} generate $\mathcal{C}_n$. Slide B_{2i-1} over B_{2i} until the corresponding entry in the colouring vector becomes v'_{2i} and then sliding B_{2i} over B_{2i-1} until the corresponding entry in the colouring vector becomes v'_{2i-1}. □

Lemma 11 *For any pair of twin bands B_{2i-1} and B_{2i}, by band slides we can obtain a band projection which induces a colouring vector such that either v_{2i-1} vanishes, or v_{2i} vanishes, as desired.*

Proof Totally order $\mathcal{C}_n$ as $\{s^0, s^1, \dots, s^{n-1}\}$, and for each $i = 1, \dots, g$ slide B_{2i} over B_{2i-1} if $v_{2i-1} \geq v_{2i}$, or B_{2i-1} over B_{2i} otherwise. We obtain a pair (v'_{2i-1}, v'_{2i}) which is smaller than (v_{2i-1}, v_{2i}) in the lexicographical ordering. Repeat until we kill v_{2i} (in which case we're finished) or v_{2i-1}.

Now that we have obtained a colouring vector with $v_{2i-1} = s^0$, if we want a colouring vector with $v_{2i} = s^0$, exchange the $(2i)$–th and the $(2i-1)$–st entries in the colouring vector by a sequence of band slides corresponding to the following operations on entries of the colouring vector:

$$(s^0, v_{2i}) \to (v_{2i}, v_{2i}) \to (v_{2i}, s^0)$$

Analogously, if $v_{2i} = s^0$ and we want v_{2i-1} to vanish, we can reverse the sequence of band slides above. □

4.2 Reduction of genus

The goal of this section is to show that any D_{2n}–coloured knot (K,ρ) of genus g is ρ–equivalent to a D_{2n}–coloured knot of genus 1. The proof consists of three steps. We first show that for any (K,ρ) we may choose a band projection such that the induced colouring vector has its first entry equal to s. The second step is to arrange every other entry to be 1. Having prepared such a band projection the final step is to reduce genus by ρ–equivalences.

4.2.1 Step 1: Engineer a band projection such that $v_1 = s$. If n is prime, engineering a band projection such that $v_1 = s$ is straightforward (Lemma 10), and one may proceed directly to Step 2. If n is composite, however, a more involved argument may be required. Our strategy is to construct the desired band projection directly, by finding an appropriate *cut system*. For the purpose of this section's discussion we'll formalize a few terms.

Definition 7 A *cut* on some Seifert surface for some knot K is a simple nonseparating oriented curve lying on the surface whose two boundary points lie on K.

Definition 8 Consider some cut C on some Seifert surface F with boundary K. Recall that $N(F)$ was defined to be $N(K)\cup(F\times[-1,1])$, where F is regarded as occupying the 0–slice of $F\times[-1,1]$. The *ring around* C is a particular simple closed oriented curve in $S^3 - N(F)$ constructed in the following way. Let the boundary points of C be C_0 and C_1 (so that C runs from C_0 to C_1). The ring around C is now the loop which starts at $C_0\times\{1\}$, follows the curve $C\times 1$ to $C_1\times\{1\}$, loops around K to $C_1\times\{-1\}$ via the path γ_1 shown in Figure 18, returns along $C\times\{-1\}$ to $C_0\times\{-1\}$, then loops back around K to its starting point using the obvious path γ_0.

So given a cut on a Seifert surface, we may take the ring around it, which now evaluates in the representation ρ to give a well-defined element of $\mathcal{C}_n$.

The constructions which resolve this step can now be described by the following two lemmas.

Lemma 12 *Consider a D_{2n}–coloured knot (K,ρ), and a Seifert surface F for K. If there exists a cut C on the surface whose corresponding ring evaluates to s, then the knot has a band projection whose corresponding colouring vector has its first entry, v_1, equal to s.*

Lemma 13 *Every Seifert surface F of a D_{2n}–coloured knot (K,ρ) has a cut on it whose corresponding ring evaluates to s.*

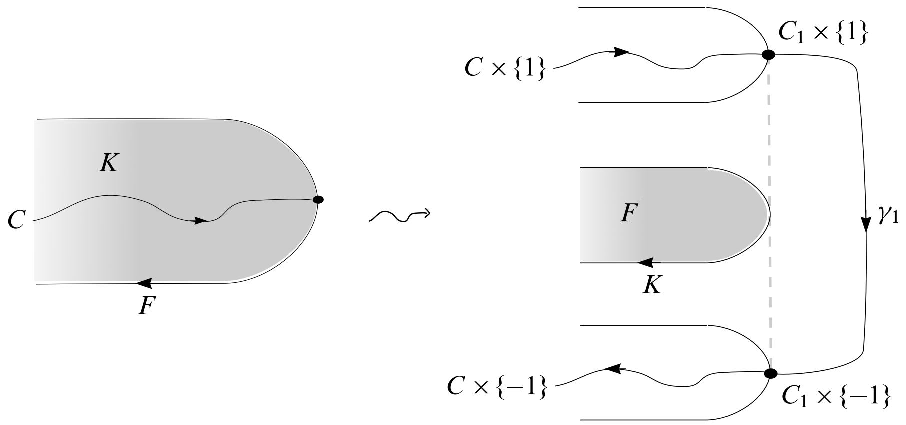


Figure 18: The path γ_1

We'll explain the proofs of these lemmas in turn.

Proof of Lemma 12 This proof is essentially a rereading of the standard manipulations that show that every Seifert surface has a band projection (see eg Seifert and Threfall [26, Chapter 6]).

A system of cuts C_1 through C_{2g} on F is called a *cut system* if when we remove the bands coming from the regular neighbourhoods of the cuts, we are left with a disc. If we have a cut system on F, then the disc that remains after we remove the bands from it has its boundary marked with $2g$ pairs of intervals, corresponding to the two sides that are created when an arc is cut open. Label these intervals using B_1 through B_{2g}, say, depending on which cut an interval came from. (So, in particular, each label will appear twice.) If we have chosen our cuts so that these labels appear in the usual "product of commutators" order, then an ambient isotopy which takes this disc into a standard unknotted disc position will carry the original Seifert surface into standard band-projection position. Furthermore, that ambient isotopy will also carry the rings around the cuts to the rings around the "standard" cuts of a knot in band projection (see Figure 19), which are the usual ξ_i's.

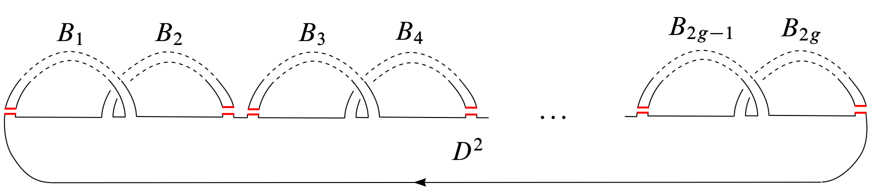


Figure 19: The "standard" cuts of a band projection

So our only task is to show that any given cut C_1 may be completed to a cut system, C_1 through C_{2g}, marking the disc in the desired "product of commutators" order. This is a standard manipulation. □

Proof of Lemma 13 Begin with any band projection of the given D_{2n}–coloured knot. Using Lemma 11 kill odd numbered entries in the colouring vector by band slides.

Next we'll introduce the collection of cuts amongst which we'll find our desired cut. To every vector $(a_1, \ldots, a_g) \in \mathbb{Z}^g$ associate a cut in the way illustrated by Figure 20.

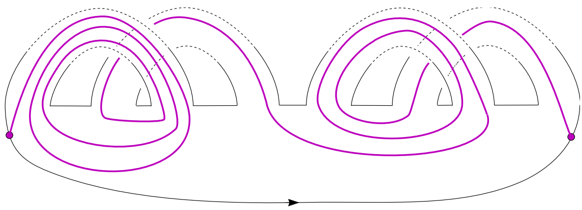


Figure 20: The cut for $g = 2$ and $(a_1, a_2) = (3, -2)$

We claim that we can choose the vector $(a_1, \ldots, a_g)$ so that the ring around the corresponding cut evaluates under ρ to s.

Our next task, then, is to determine how the image under ρ of the ring around one of these cuts depends on the given vector. Well, observe that this ring is homologous in $H_1(\overline{S^3 - N(F)})$ to

$$\sum_{i=1}^{g} \left(a_i(\tau^+ x_{2i-1} - \tau^- x_{2i-1}) + (\tau^- x_{2i} - \tau^+ x_{2i})\right),$$

where, recall, $\tau^{\pm} x$ denotes the push-off from the Seifert surface of a curve x in the positive (resp. negative) direction. Furthermore, note for all i that $\tau^+ x_{2i-1} - \tau^- x_{2i-1}$ is homologous to ξ_{2i} and $\tau^- x_{2i} - \tau^+ x_{2i}$ is homologous to ξ_{2i-1} (see Figure 21). Thus the ring around the cut corresponding to the vector $(a_1, \ldots, a_g)$ evaluates under ρ to

$$(v_2)^{a_1}(v_4)^{a_2} \cdots (v_{2g})^{a_g}.$$

To finish the proof we ask: can we choose the vector $(a_1, \ldots, a_g)$ so that this expression evaluates to s? The answer is yes, because ρ is surjective. Therefore, by Proposition 8, its restriction $\overline{\rho}$ is also surjective. Thus $\overline{\rho}$ maps some homology class $\psi \in H_1(\overline{S^3 - N(F)})$ to $s \in \mathcal{C}_n$. So we can write ψ as a product

$$(\xi_1)^{k_1}(\xi_2)^{k_2} \cdots (\xi_{2g})^{k_{2g}}.$$

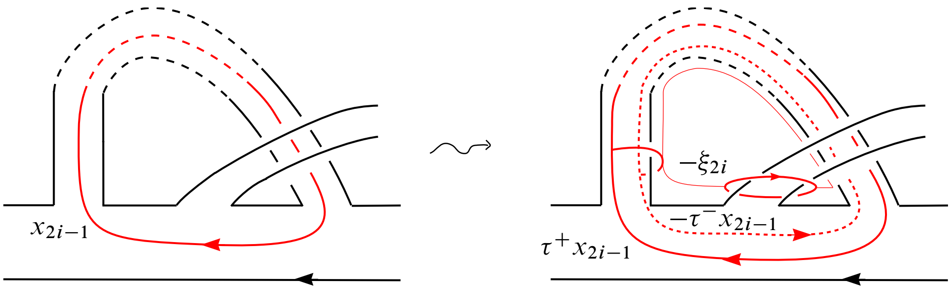


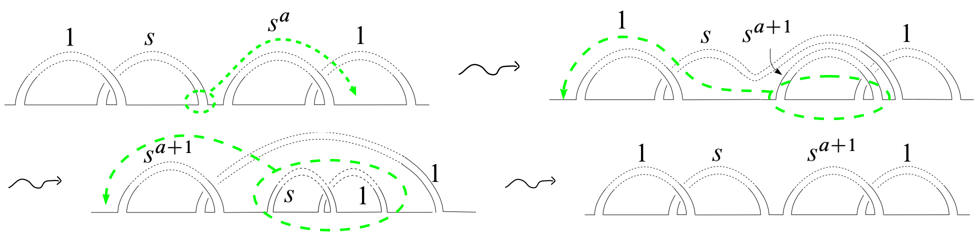
Figure 21: The class $\tau^+ x_{2i-1} - \tau^- x_{2i-1}$ is homologous to ξ_{2i} for all $i = 1, 2, \dots, g$.

As $v_{2i-1} = 1$ for all i, this is mapped via $\overline{\rho}$ to the expression:

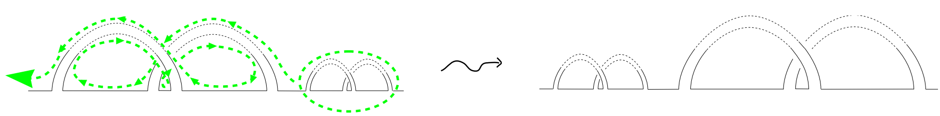
$$\overline{\rho}(\psi) = (v_2)^{k_2} (v_4)^{k_4} \cdots (v_{2g})^{k_{2g}},$$

which equals s by the choice of ψ. Therefore a representative $\widetilde{\psi} \in \pi_1(\overline{S^3 - N(K)})$ of ψ is mapped to s by ρ. This gives the desired expression as $(a_1, \dots, a_g) = (k_2, \dots, k_{2g})$. □

4.2.2 Step 2: Kill v_i for $i > 1$. First, kill v_{2i} for $i = 1, \dots, g$ by Lemma 11 (note that this leaves v_1 untouched because s generates $\mathcal{C}_n$). If $v_3 = s^a$ with $a \neq 0 \bmod n$, first exchange v_1 and v_2 by band slides using Lemma 10, then slide bands as follows:



Repeat the above sequence of band-slides $n - a$ times to kill v_3 without altering the rest of the colouring vector. Now slide B_3 and B_4 over B_5 and B_6:



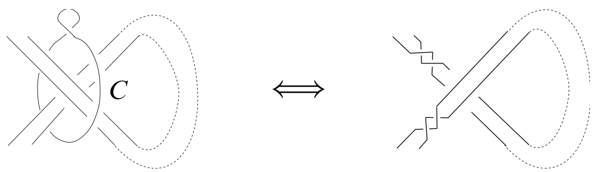
and repeat the argument above to kill v_5.

Repeat all steps above to kill v_{2i-1} for all $i = 2, 3, \dots, g$, and the colouring vector becomes $\vec{v} = (s, 1, \dots, 1)^T$ as required.

4.2.3 Step 3: Reduction of genus via ρ–equivalence After Step 2 we have a D_{2n}–coloured knot (K,ρ) in band projection whose colouring vector is $\vec{v}=(s,1,\dots,1)^T$. In this section we will show that (K,ρ) is ρ–equivalent to a genus 1 knot.

So assume that the genus of K is $g>1$. We will now show that (K,ρ) is ρ–equivalent to a knot with Seifert surface of genus $g-1$. We'll present this argument as a sequence of 8 observations and ρ–equivalences.

(1) Note that crossings of a band with itself can always be changed by ρ–equivalence:

(4-6) 

Note that the introduced surgery component C lies in $\ker\rho$, so that this is indeed a ρ–equivalence. The first step is to change crossings of B_{2g}, the rightmost band, with itself, so as to untie it and to put the knot into the position shown in Figure 22.

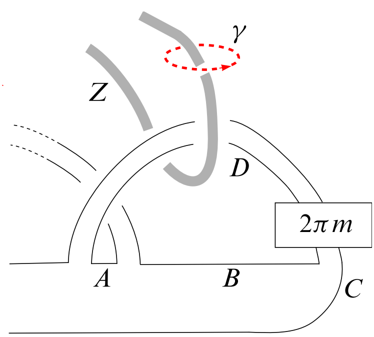


Figure 22: The local picture for the genus reduction step. The box labeled $2\pi m$ denotes m full clockwise twists in B_{2g}. Inside this local picture, linkage of $B_1,\dots,B_{2g-1}$ with B_{2g} can be bunched together into the inside of a cylinder Z as shown.

(2) Recall that the arcs A and B come labeled with reflections. Because v_{2g-1}, the colour corresponding to B_{2g-1}, has been prepared to be $s^0=1$, the labels on A and B must be the same: ts^a for some $a\in\mathbb{Z}_n$.

(3) Continuing, because v_{2g} is also trivial, the label on arc C must also be ts^a.

(4) Arc D will also be labeled ts^a. To see this we'll determine the labels on the arcs in the box of twists from the bottom up. We have determined that the two lowermost

arcs B and C are both labeled ts^a. When an arc labeled ts^a passes under another arc labeled ts^a, the label on the resulting arc is again ts^a. So working from the bottom up, we see that all of the arcs in the box of twists, including the arc D, will be labeled ts^a.

(5) The next step is to determine $\rho(\gamma)$, where γ is a path which rings Z once as shown in Figure 22. First observe that because γ lies in the complement of the Seifert surface, $\rho(\gamma)$ will be of the form s^j for some $j \in \mathbb{Z}_n$. The index j is determined by the condition, clear from the diagram, that this element will conjugate the label of A to the label of D. Because $ts^a = s^{-j} \cdot ts^a \cdot s^j = ts^{a+2j}$ and because n is odd, j must be zero.

(6) Therefore, the surgery curve C in Figure 23 will also lie in $\ker \rho$. So surgery on C is a ρ–equivalence, and it unlinks everything from B_{2g} (while adding a 2π twist to B_{2g}).

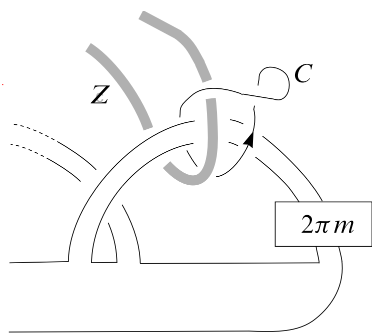
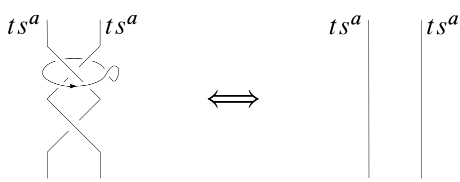


Figure 23: The ρ–equivalence which reduces genus

(7) Using the fact that $v_{2g} = 1$, untwist B_{2g} by the following ρ–equivalence:

(4-7)



(8) The resulting diagram represents a D_{2n}–coloured knot with Seifert surface of genus $g-1$, because the bands B_{2g} and B_{2g-1} come unraveled (see Figure 24).

4.3 Genus one knots

For $k = 0, \ldots, n-1$, denote the D_{2n}–coloured pretzel knots $p\left(2kn+1, -1, -n\right)$ drawn in Figure 3 by $\{(\mathcal{B}_0, \rho_0), \ldots, (\mathcal{B}_{n-1}, \rho_{n-1})\}$ correspondingly.

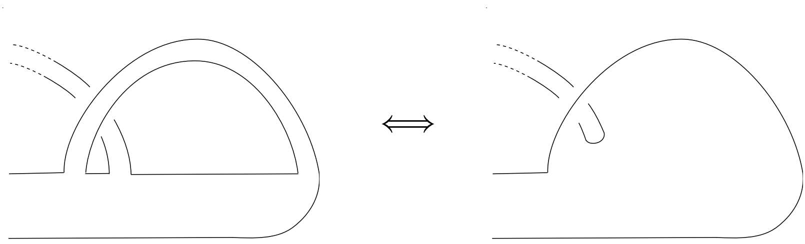


Figure 24: The local picture after untwisting B_{2g}, untying it and unlinking everything from it

In the last section we saw that any D_{2n}–coloured knot (K,ρ) is ρ–equivalent to a D_{2n}–coloured genus one knot (K',ρ'). The goal of this section is to complete the proof of Theorem 3 by showing that (K',ρ') is ρ–equivalent to $(\mathcal{B}_k,\rho_k)$ for some $k=0,\dots,n-1$. We prove this assertion in two steps. First, we show that (K',ρ') is ρ–equivalent to a D_{2n}–coloured knot (K'',ρ'') which has the same surface data as $(\mathcal{B}_k,\rho_k)$ for some $k=0,\dots,n-1$. We then show that in fact (K'',ρ'') and $(\mathcal{B}_k,\rho_k)$ are ρ–equivalent.

4.3.1 Step 1: Modifying the surface data With respect to a given band projection, write the surface data of (K',ρ') as

(4-8) $$(S,\vec{v}) := \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{12}+1 & a_{22}\end{pmatrix}, \begin{pmatrix} v_1 \\ v_2\end{pmatrix}\right).$$

The goal of this section is to show that (K',ρ') is ρ–equivalent to a D_{2n}–coloured knot (K'',ρ'') with surface data

(4-9) $$(S_k,\vec{v}_k) = \left(\begin{pmatrix} kn+\frac{n+1}{2} & 0 \\ 1 & \frac{1-n}{2}\end{pmatrix}, \begin{pmatrix} s \\ s^{-1}\end{pmatrix}\right),$$

for some $0\le k<n$, which coincides with the surface data of the D_{2n}–coloured knot $(\mathcal{B}_k,\rho_k)$ with respect to the obvious band projection. Performing a single band slide, it is sufficient then to show that (K',ρ') is ρ–equivalent to a D_{2n}–coloured knot whose surface data with respect to some band projection is:

(4-10) $$(S,\vec{v}) = \left(\begin{pmatrix} kn & \frac{n-1}{2} \\ \frac{n+1}{2} & \frac{1-n}{2}\end{pmatrix}, \begin{pmatrix} s \\ 1\end{pmatrix}\right).$$

The proof comprises of the following 5 observations and ρ–equivalences:

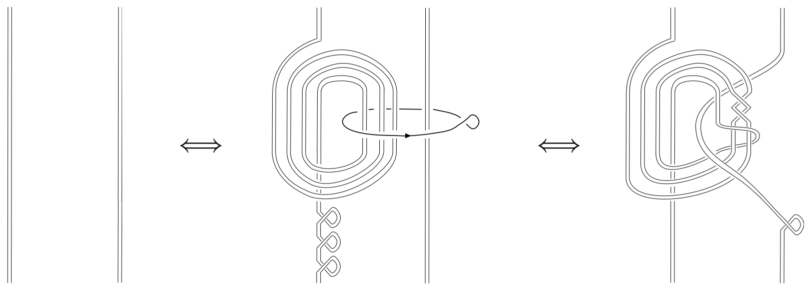
(1) By Lemma 10, there exists a band projection for (K',ρ') with respect to which its colouring vector is $\binom{v_1}{v_2} = \binom{s}{1}$.

(2) By Proposition 8, we know that

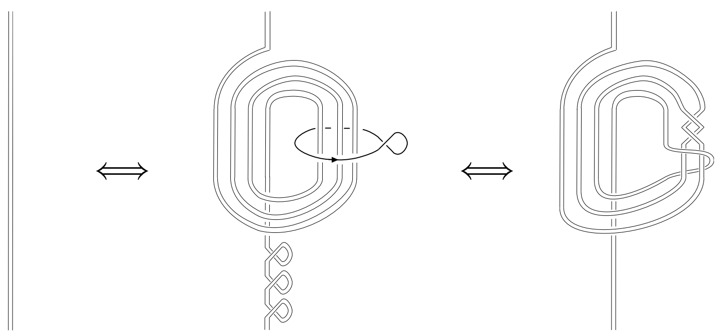
$$\begin{pmatrix} 2a_{11} & 2a_{12}+1 \\ 2a_{12}+1 & 2a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, because n is odd, $a_{11} = 0 \bmod n$ and $2a_{12}+1 = 0 \bmod n$.

(3) The fact that $2a_{12}+1 = 0 \bmod n$ means that $a_{12} = j((n-1)/2)$ for some integer j. Below, we show how to add or subtract n from a_{12} by ρ–equivalence. Thus j may be chosen to be 1. First, coil B_1 into $2n$ parallel strands, then add a ± 1–framed unknotted surgery component around those strands and B_2. (Choose $+1$ to increase a_{12} by n, and -1 to decrease a_{12} by n.) This is clearly a ρ–equivalence. After this move, we will have changed a_{12} by n, and we will also have changed a_{11} by n^2 and changed a_{22} by 1. The move is illustrated below in the case $n = 3$.

(4-11) 

(4) In Step (2) we saw that $a_{11} = kn$ for some integer k. Below, we show how to add or subtract n^2 from a_{11} by ρ–equivalence without changing the rest of the surface data (compare with (3-5) and (3-6)). It follows that k may be chosen so that $0 \le k < n$. First, coil B_1 into $2n$ parallel strands, then add a ± 1–framed unknotted surgery component around them. (Choose $+1$ to increase a_{11} by n^2, and -1 to decrease a_{11} by n^2.) This is a ρ–equivalence. The move is illustrated below in the case $n = 3$, where it is used to subtract 9 from a_{11}.

(4-12) 

(5) Using the fact that $v_2 = 1$, twist or untwist B_2 as in (4-7) to set a_{22} to $(1-n)/2$. The rest of the surface data remains unchanged.

Having set a_{11}, a_{12}, a_{22}, v_1, and v_2 to their required values, we have shown that (K', ρ') is ρ–equivalent to a D_{2n}–coloured knot with the same surface data as one of the knots $\{(\mathcal{B}_0, \rho_0), \ldots, (\mathcal{B}_{n-1}, \rho_{n-1})\}$ with respect to some band projection.

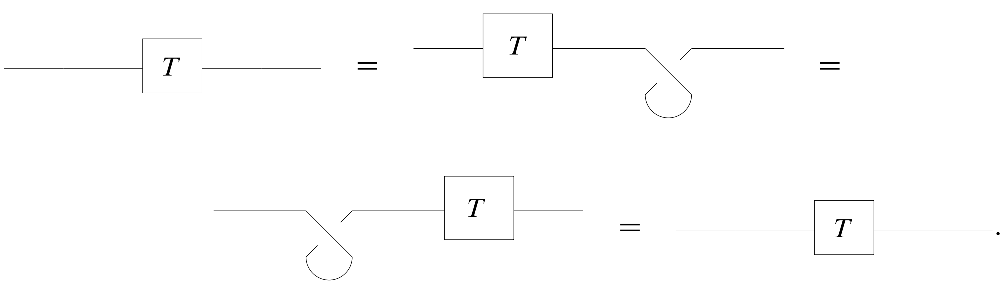
4.3.2 Step 2: Genus one knots with the same surface data are ρ–equivalent At this point we have shown that any D_{2n}–coloured knot (K, ρ) is ρ–equivalent to a D_{2n}–coloured knot (K'', ρ'') which shares the same surface data as one of the D_{2n}–coloured genus one knots $\{(\mathcal{B}_0, \rho_0), \ldots, (\mathcal{B}_{n-1}, \rho_{n-1})\}$ in Figure 3. It remains to show that two genus one knots which share the same surface data are ρ–equivalent.

We begin by fixing the D_{2n}–colouring ρ'' on a knot diagram D of K'' using ambient isotopy. The proof is then completed by Lemma 15.

Lemma 14 *Let ρ_1 and ρ_2 be D_{2n}–colourings of a knot K such that $\overline{\rho}_1 = \overline{\rho}_2$. Then (K, ρ_1) and (K, ρ_2) are ambient isotopic.*

Proof By Proposition 8, $\overline{\rho}_1 = \overline{\rho}_2$ implies that there exists $a \in \mathcal{C}_n$ such that for all $g \in \pi_1(S^3 - N(K))$, we have $\rho_2(g) = a^{-1}\rho_1(g)a$. If a vanishes there is nothing to prove, so we assume $a \neq s^0$. Fix a knot diagram D for K. Because ρ_1 is surjective, there exist nonzero integers $a_1, \ldots, a_n$ such that $a = \rho_1(w_n)^{a_n} \cdots \rho_1(w_1)^{a_1}$, where $w_1, \ldots, w_n$ are Wirtinger generators of the knot group, corresponding to arcs $A_1, \ldots, A_n$ in D. We exhibit a sequence of Reidemeister moves converting (K, ρ_1) to (K, ρ_2).

Perform a Reidemeister I move on A_1, and push the entire knot though it, then undo the Reidemeister I move as follows:

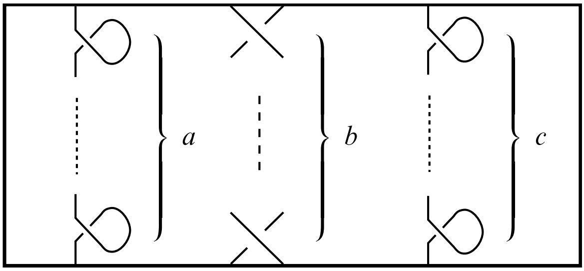


Here T denotes the $(1,1)$–tangle whose closure is K.

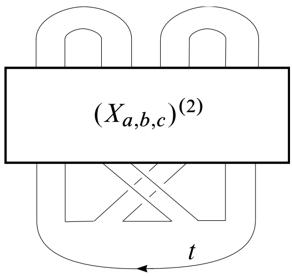
The effect of the above sequence is to conjugate ρ_1 by $\rho_1(w_1)$, without changing D. Repeat a_1 times, then perform the above sequence a_2 times on A_2, and so on until, after repeating the above sequence a_n times on A_n, we will have conjugated ρ_1 by a, concluding the proof. □

In this section, if T denotes a framed tangle, then $T^{(2)}$ will denote the unframed tangle that results from doubling each component of T using the framing.

Let $X_{a,b,c}$ denote the following blackboard framed $(4,4)$–tangle:



Let $K_{a,b,c}$ denote the following knot:

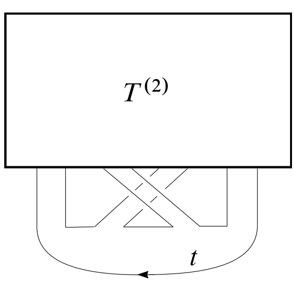


Lemma 15 *Let (K,ρ) be a D_{2n}–coloured knot presented by a genus 1 band projection. If the corresponding surface data is of the form*

$$\left(\begin{bmatrix} a & b \\ b+1 & c \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right)$$

then (K,ρ) is ρ–equivalent to the D_{2n}–coloured knot $(K_{a,b,c},\rho_0)$, where ρ_0 is determined by the colouring vector $(v_1,v_2)^T$.

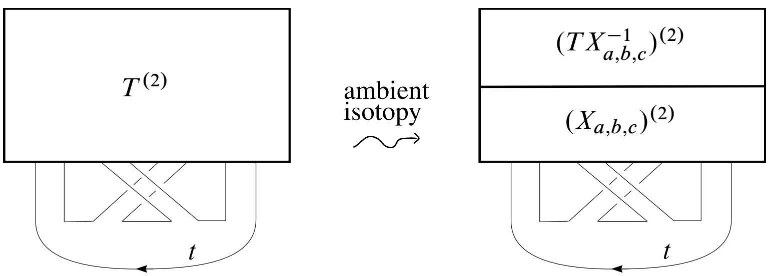
Proof Begin with a diagram for the assumed genus 1 band projection of the following form:



Here T denotes a framed $(0,4)$–tangle representing the position of the bands. This framed tangle has two components, and satisfies:

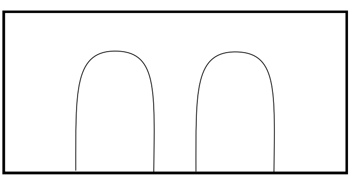
- The framing of the first band is a. (In other words, the linking number between the two boundary components of the first band, when co-oriented, is a.)
- The framing of the second band is c.
- The linking number between the two bands (the signed sum of crossings when the two bands are oriented according to our convention, from the outside in) is b.

The first step in our manipulations will be to introduce a cancelling pair of copies of $X_{a,b,c}$:



Let Y denote the framed $(0,4)$–tangle $TX^{-1}_{a,b,c}$. Observe that each of the two components of Y have framing zero, and that their mutual linking number is zero. For the discussion below, let L denote the left component, and let R denote the right component.

Recall that by a single ρ–equivalence we can change a crossing of a band with itself in a way that does not change the framing of the band. So it remains for us to show that we can perform crossing changes of the components of Y with themselves in order to trivialize Y, so that it becomes:



Because the framings of the components of Y are not affected by these moves, we can ignore them for the remainder of this proof.

There are three steps:

(1) The first step is to change crossings of L with itself, and to use ambient isotopy, in order to trivialize L and put it into the obvious standard position. This step does not require further comment.

(2) The second step is to change crossings of R with itself, and use ambient isotopy, in order to obtain a diagram in which there are no crossings between the two components.

(3) The final step is to trivialize R.

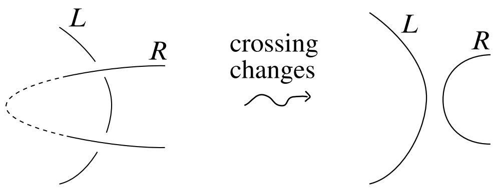
It remains for us to explain the details of the second step above, which echoes the proof that the fundamental group of S^1 is $\mathbb{Z}$.

So assume that the first step is complete, so that L has been trivialized. Obtain a general position diagram for the resulting tangle (ignoring framings). Orient the two components according to our convention: from the outside in.

From this diagram construct a word in the symbols $+$ and $-$ by following R along its whole length in the direction of its orientation and, every time that R crosses L, writing down the sign of that crossing. Observe that, because the mutual linking number of these two components is zero, this word must contain an equal number of $+$'s and $-$'s. We will regard the length of this word as the complexity of the diagram.

To reduce complexity: find a place in this word where a $+$ is adjacent to a $-$, and consider the crossings they represent in the diagram. Either R crossed over L in both crossings, or it crossed under L in the both crossings.

If over: consider the arc A of R that goes between these two crossings. Change crossings between this arc and other arcs of R so that, traveling along A in the direction of its chosen orientation, we always overcross. After doing this, we can do an ambient isotopy to reduce complexity:



If under: the corresponding moves are obvious. □

4.3.3 Section summary After having shown in Section 4.2 that any D_{2n}–coloured knot (K, ρ) is ρ–equivalent to a D_{2n}–coloured knot (K', ρ') of genus one, in this section we have shown that (K', ρ') may be chosen to have the same surface data as one of the knots $\{(\mathcal{B}_0, \rho_0), \ldots, (\mathcal{B}_{n-1}, \rho_{n-1})\}$ (Equation (4-9)). We then showed that any two genus one D_{2n}–coloured knots with the same surface data are ρ–equivalent. Combining these facts proves Theorem 3.

4.4 Constructing the cover

In the previous section we proved that for any D_{2n}–coloured knot (K, ρ) with coloured untying invariant $0 \leq k < n$ there exists a ± 1–framed link L in the complement of the D_{2n}–coloured pretzel knot $(\mathcal{B}_k, \rho_k)$ of Figure 25, whose components are unknotted and in $\ker \rho$, and surgery along which recovers (K, ρ). The goal of this section is to construct the branched dihedral covering space and covering link corresponding to this data, and to lift the surgery information to this cover.

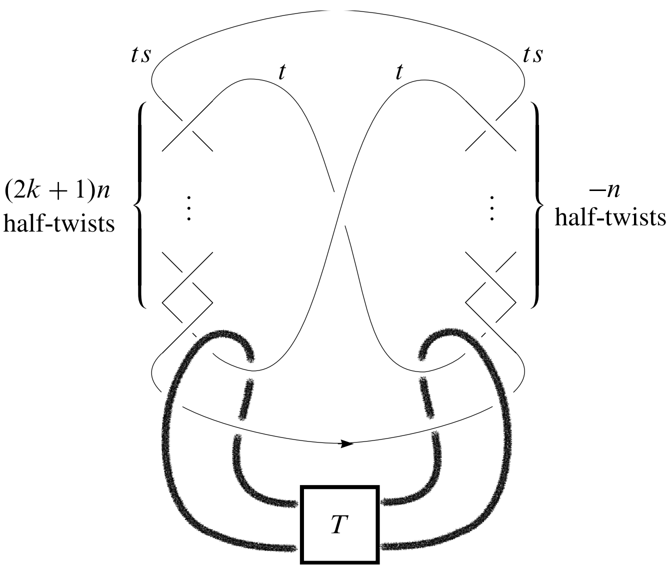


Figure 25: The D_{2n}–coloured knot $(\mathcal{B}_k, \rho_k)$ with the surgery link in its complement

4.4.1 Language and notation Coordinates in $\mathbb{R}^3 \subset S^3$ will be employed to explicitly describe configurations of objects in 3–space. Denote by $\Sigma \subset \mathbb{R}^2$ the surface arising from a sufficiently large disc when small discs centred at the points $(-2, 0)$, $(-1, 0)$, $(1, 0)$ and $(2, 0)$ are removed. The surgery link L will lie inside $\Sigma \times [0, 1] \subset \mathbb{R}^3$. We will think of L as being the closure of a ± 1 framed tangle T such that diagram of L arising from the projection onto $\Sigma \times \{0\}$ is as pictured in Figure 26.

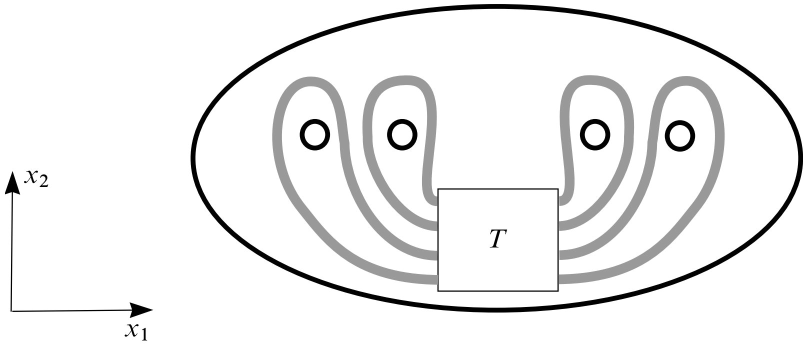


Figure 26: A diagram of the surgery link $L \subset \Sigma \times [0,1]$

The knot $\mathcal{B}_k$ over which we'll be taking a branched dihedral cover can be assumed to live in $\mathbb{R}^3 - (\Sigma \times [0,1])$, as pictured in Figure 27 (using the convention that the coordinate x_2 increases *into* the page).

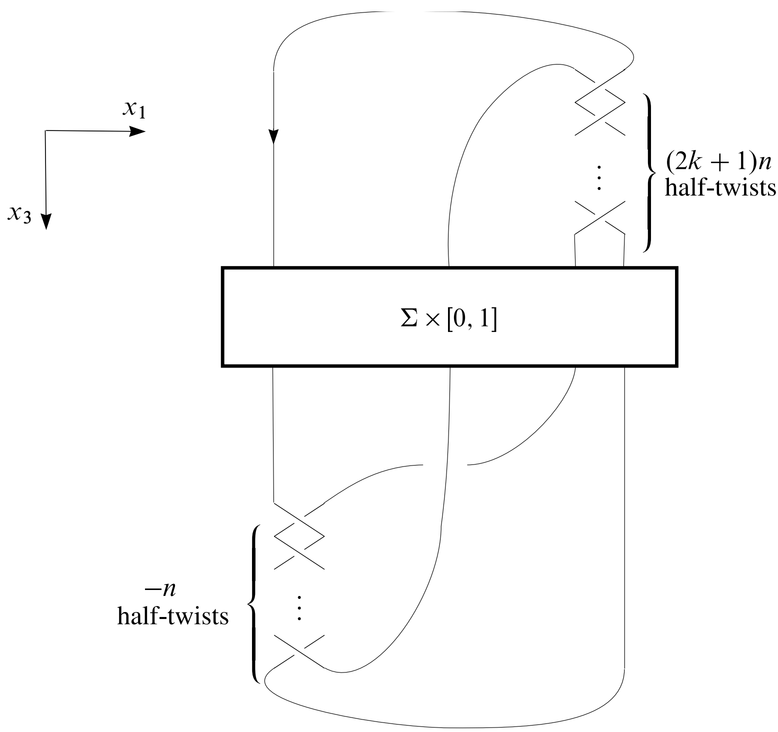


Figure 27: The cylinder $\Sigma \times [0,1]$ sitting inside the knot complement $\mathbb{R}^3 - \mathcal{B}_k$

The D_{2n}–colouring ρ induces a representation from $\pi_1(\Sigma \times [0,1])$ into D_{2n}, which we shall also call ρ by abuse of notation. To describe this representation, choose a base point for $\Sigma \times [0,1]$ lying on the surface $\Sigma \times \{0\}$, and specify the images of generators of $\pi_1(\Sigma \times [0,1])$ with respect to this basepoint as shown in Figure 28. The two "outer" generators map to ts, while the two "inner" generators map to t.

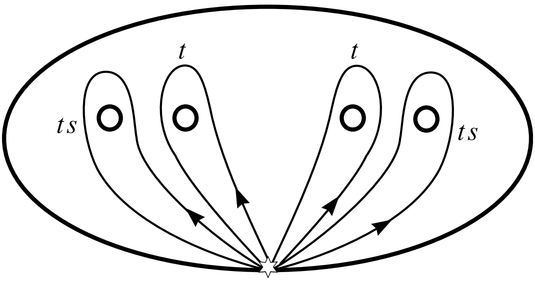


Figure 28: Generators for $\pi_1(\Sigma \times [0,1])$ and their images in D_{2n}

4.4.2 Constructing $\widetilde{\Sigma\times[0,1]}$ We now have language and notation which is sufficiently explicit to describe the construction of $\widetilde{\Sigma\times[0,1]}$, the dihedral cover of $\Sigma \times [0,1]$ with respect to ρ, and its embedding in the branched dihedral covering over $(\mathcal{B}_k, \rho_k)$, which is S^3 because $\mathcal{B}_k$ is a 2–bridge knot (see eg [1]).

We build $\widetilde{\Sigma\times[0,1]}$ embedded in $\mathbb{R}^2 \times [0,1] \subset \mathbb{R}^3$ by slotting together copies of $\Sigma \times [0,1]$ cut open along planes, as shown in Figure 29. These are our "lego blocks", which we can bend, stretch, and shrink. To construct $\widetilde{\Sigma\times[0,1]}$ (embedded in $\mathbb{R}^3$) we take n blocks denoted $X_1, \dots, X_n$ (copies of the cut-open $\Sigma \times [0,1]$ of Figure 29) and slot them together in the usual way: always matching an A to an A', and so on, and using the representation to decide which copy of the 3–cell one passes to when crossing a cut.

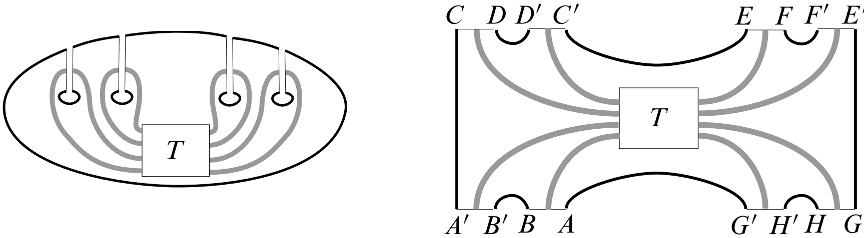


Figure 29: A block X_i obtained by cutting open $\Sigma \times [0,1]$ along planes and X_i after being "opened out" by isomorphism

To see that combinatorially the blocks end up lined up in a line, consider the graph with n vertices labeled $1,\dots,n$ and an arc connecting vertices i and j if and only if X_i and X_j are incident in $\widetilde{\Sigma\times[0,1]}$, ie if and only if we slot X_i and X_j together, which is if and only if $t(i)=j$ or $ts(i)=j$ where D_{2n} is acting on $1,\dots,n$ by symmetries of the regular n–gon (remember that when crossing a cut labeled t we are going to be crossing from X_i to $X_{t(i)}$, and similarly for ts). When $n=7$ the graph is given in Figure 30. Because $t(1)=1$ and $ts((n+3)/2)=(n+3)/2$, the graph will consist of two loops and a path from 1 to $(n+3)/2$.

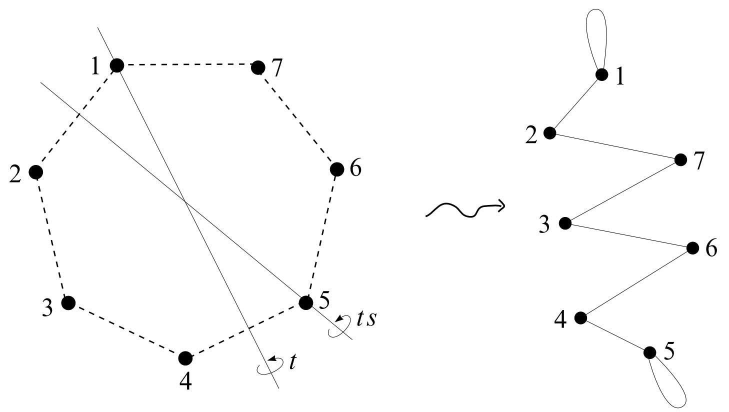


Figure 30: The graph showing in which order the blocks X_i slot together to build $\widetilde{\Sigma\times[0,1]}$

Now that we know what $\widetilde{\Sigma\times[0,1]}$ looks like combinatorially, we describe its embedding in $\mathbb{R}^3$ which we will use in the presentation of the final result. For this purpose it is useful to notice that the construction of $\widetilde{\Sigma\times[0,1]}$ defines a permutation τ of $1,\dots,n$, taking i (representing X_i) to the position of X_i on the path from 1 to $(n+3)/2$ (one plus its distance on the graph from the vertex labeled 1). Thus for Figure 30:

$$\tau=\begin{pmatrix}1&2&3&4&5&6&7\\1&2&4&6&7&5&3\end{pmatrix}$$

Now, for each $i=1,\dots,n$, if $\tau(i)$ is even, bend the arms of the dumbbell (Figure 29) down and place the block X_i in $\mathbb{R}^3$ the position shown in Figure 31. If $\tau(i)$ is even, bend the arms up and place the result in the position shown in Figure 32. The reader can observe that the resulting identifications are exactly those determined by the representation. Finish the construction by gluing up the four remaining pairs of cuts—the cuts next to each other around the points $(-n-1,0,0)$, $(-1,0,0)$, $(1,0,0)$ and $(n+1,0,0)$.

For example, the result for $n=3$ is displayed in Figure 33.

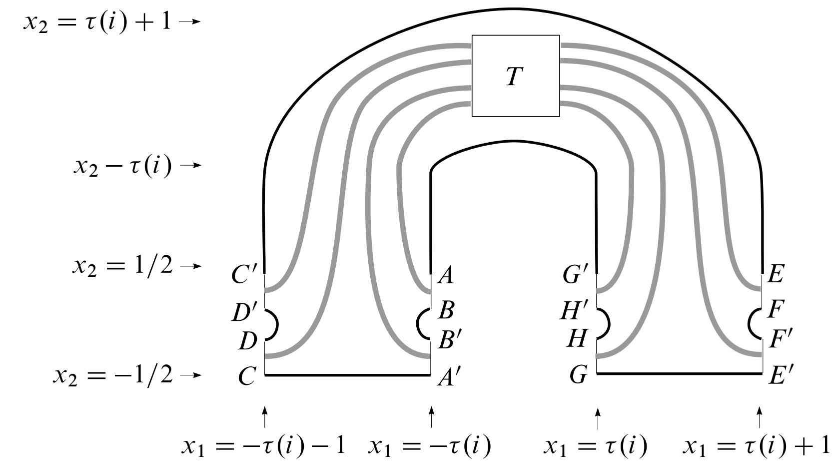


Figure 31: X_i with arms bent down

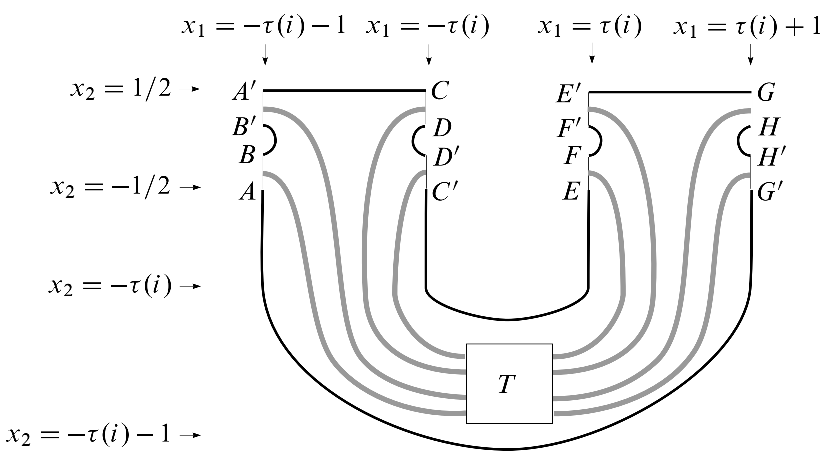


Figure 32: X_i with arms bent up

4.5 The branching set

We began this section with a framed link in $\Sigma \times [0, 1]$ in the complement of a coloured knot $\mathcal{B}_k$. Since $\mathcal{B}_k$ happens to be a 2–bridge knot, its dihedral covering space is S^3 with an $((n+1)/2)$–component covering link embedded in it (see eg Birman [1]). It remains for us to describe this link, and show how $\widetilde{\Sigma \times [0, 1]}$ embeds into its complement.

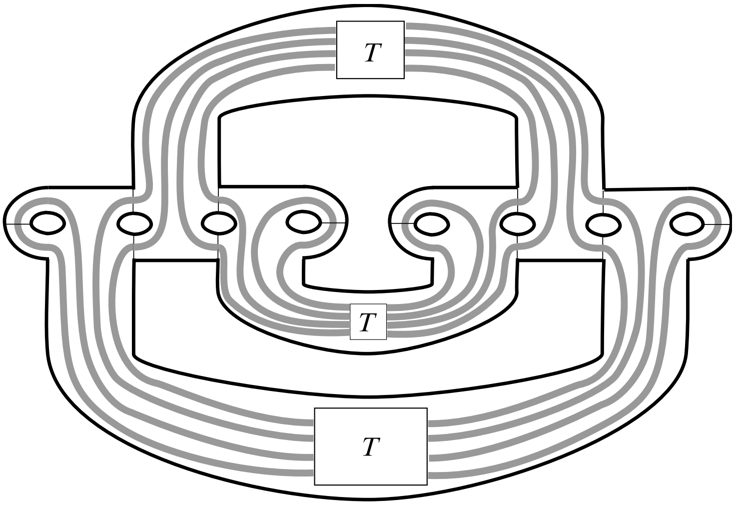


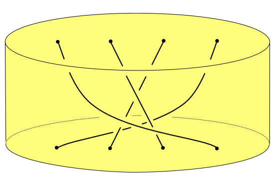
Figure 33: $\widetilde{\Sigma\times[0,1]}$ embedded in $\mathbb{R}^3$ for $n=3$

To present the result we need to introduce some additional notation. The result will use certain braids on $2(n+1)$ strands. The strands of the braids will be indexed by the set

$$I_n = \{-n-1 \le i \le n+1\,,\ i \in \mathbb{Z}/\{0\}\}.$$

The coordinate x_3 will be the vertical coordinate of the braid, and the projections of the endpoints of the strands to the (x_1, x_2)–plane will be the points $\{(x,0), x \in I_n\}$. (Note that these are precisely the coordinates of the "holes" in the construction we just gave of $\widetilde{\Sigma\times[0,1]}$.)

Let $i < j$ be indices from I_n. Let $\mathfrak{X}[i,j]$ denote the braid you get by putting a clockwise half-twist into the group of strands starting with the strand at position i, up to the strand at position j. For example, if $n=4$, then:

$\mathfrak{X}[-2,2] =$ 

We can now state the result.

Theorem 16 *Take the construction given earlier of* $\widetilde{\Sigma\times[0,1]}$ *as a subset of* $\mathbb{R}^3$. *The branching set over* $\mathcal{B}_k$ *lies in its complement as shown in Figure 34, where* B *denotes the braid:*

$$\mathfrak{X}[-n,n]\cdot\mathfrak{X}[-n+1,n-1]\cdots\mathfrak{X}[-2,2]\cdot\mathfrak{X}[-1,1].$$

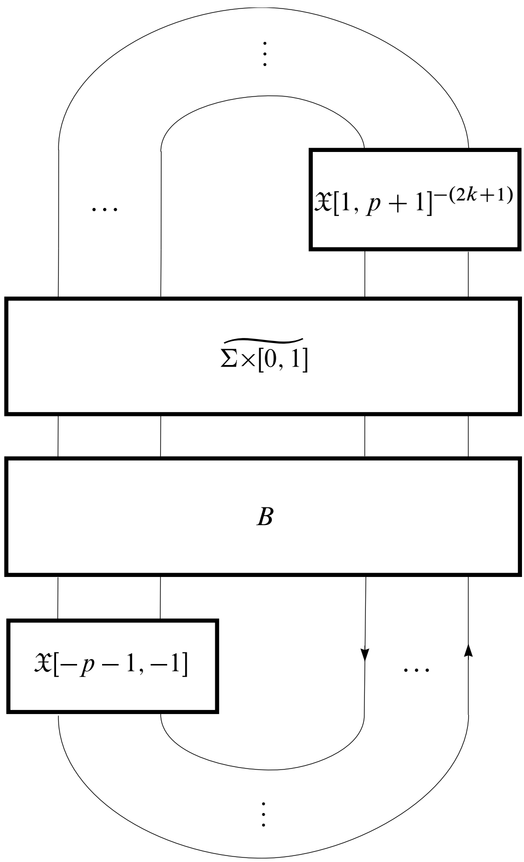


Figure 34: The lifted picture

5 Odds and ends

In this section we consider several corollaries to the constructions given in the previous sections. In Section 5.1 we list some different choices of complete sets of base knots which we might end up with via the band projection approach. In Section 5.2 we show how one of these choices leads to a proof that closed 3–manifolds with D_{2n}–symmetry have surgery presentations with D_{2n}–symmetry.

5.1 Different choices for a complete set of base-knots

Our choice of $(\mathcal{B}_k, \rho_k)$ as a complete set of base-knots was made because we have an explicit algorithm to reduce any D_{2n}–coloured knot to one of them by surgery, and because in addition we know how to explicitly find their branched dihedral covering spaces, covering links, and the lifts of the surgery presentations. This set was found by trial and error. Other complete sets of base-knots are possible of course, and some of these have advantages over $(\mathcal{B}_k, \rho_k)$.

Our starting point is a genus one knot with unknotted bands and with the surface data given by Equation (4-10), repeated here for the reader's convenience:

$$(S,\vec{v}) = \left(\begin{pmatrix} kn & \frac{n-1}{2} \\ \frac{n+1}{2} & \frac{1-n}{2} \end{pmatrix}, \begin{pmatrix} s \\ 1 \end{pmatrix} \right).$$

5.1.1 Linking number zero with the distinguished component In this section we prove that we may choose a separated dihedral surgery presentation such that the curves in $\ker\rho$ all have linking number zero with the distinguished surgery component. First perform the band slide we did in order to obtain $(\mathcal{B}_k, \rho_k)$:

$$(S,\vec{v}) = \left(\begin{pmatrix} kn & \frac{n-1}{2} \\ \frac{n+1}{2} & \frac{1-n}{2} \end{pmatrix}, \begin{pmatrix} s \\ 1 \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} kn+\frac{n+1}{2} & 0 \\ 1 & \frac{1-n}{2} \end{pmatrix}, \begin{pmatrix} s \\ s^{-1} \end{pmatrix} \right)$$

Perform $(n+1)/2$ additional surgeries between the bands:

$$(S,\vec{v}) \mapsto \left(\begin{pmatrix} (k+1)n+1 & \frac{n+1}{2} \\ \frac{n+3}{2} & 1 \end{pmatrix}, \begin{pmatrix} s \\ s^{-1} \end{pmatrix} \right)$$

Slide B_2 over B_1 repeatedly $(n+1)/2$ times:

$$(S,\vec{v}) \mapsto \left(\begin{pmatrix} (k+1)n+m & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} s \\ s^{-\frac{n+3}{2}} \end{pmatrix} \right)$$

where $k = 0,\dots,n-1$ and $m = (n+1)/2 - 2\sum_{i=1}^{(n+1)/2} i - 1$. If $(n+1)/2$ is even, then $m = 1-(n+1)^2/4$, while if $(n+1)/2$ is odd then $m = 2-(n^2+1)/2$. This is the twist knot with $(k+1)n+m$ twists. Untie this knot by a single surgery as shown in Figure 35. Put this into a separated dihedral surgery presentation by untying the distinguished surgery component by surgery in $\ker\rho$. We obtain a separated dihedral surgery presentation where all surgery components in $\ker\rho$ have linking number zero not only with the knot, but also with the distinguished surgery component.

5.1.2 Torus knot presentation By constructing complete sets of base knots with cardinality n in previous sections, we proved Corollary 4 which states that two knots are ρ–equivalent if and only if they have the same coloured untying invariant. As calculated by Moskovich [16] (see also Litherland and Wallace [14]), the left-hand $((2k+1)n, 2)$–torus knots of Figure 5, with $k = 0,\dots,n-1$, have coloured untying invariant $k \bmod n$ and thus realize all possible values of the coloured untying invariant. Thus we have:

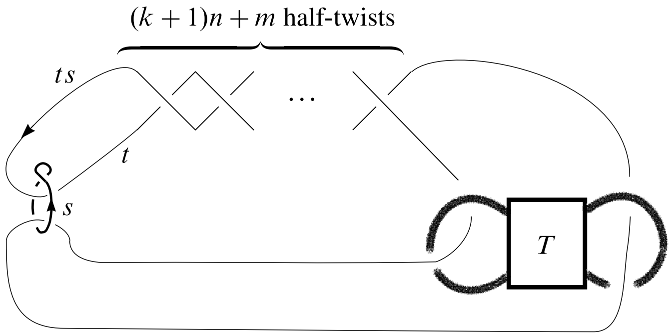
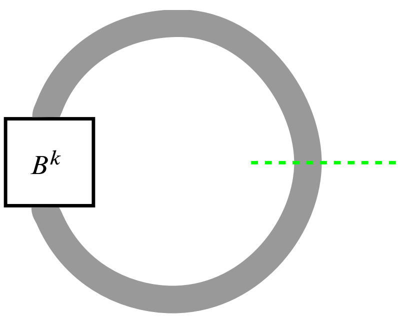


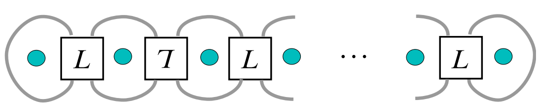
Figure 35: Untying the twist knot

Corollary 17 *The knots depicted in Figure 5 (the left-hand* $((2k+1)n, 2)$*–torus knots with the given colouring for* $k = 0, \ldots, n-1$*) comprise a complete set of base-knots for* D_{2n}*.*

The surgery presentation of the branched dihedral covering and of the covering link which this picture gives is:



with the thick line denoting $n+1$ parallel strands and with



being the lift of the covering link, slotted into the lift of the torus knot at the dotted line, where the strands of the covering link of the torus knot thread up out of the page through the holes indicated.

5.1.3 One knot, different colourings We can choose a complete set of base-knots as a fixed knot K whose colouring varies. Using the triviality of v_2, kill a_22 by a series of ρ–equivalences of the form shown in Equation (4-7). Then for $a_{11} = kn$, slide B_2 over B_1 counterclockwise repeatedly k times. We obtain:

$$(S, \vec{v}) \mapsto \left(\begin{pmatrix} 0 & \frac{n-1}{2} \\ \frac{n+1}{2} & 0 \end{pmatrix}, \begin{pmatrix} s \\ s^k \end{pmatrix} \right) \tag{5-1}$$

Because a D_{2n}–coloured knot modulo ρ–equivalence is uniquely characterized by its surface data, we obtain the minimal complete set of base-knots shown in Figure 6.

5.2 Visualizing dihedral actions on manifolds

The following section deals with an observation due to Makoto Sakuma, that Corollary 17 implies a visualization theorem for dihedral actions on manifolds. We summarize his argument, essentially contained in [25].

Let D_{2n} act on a closed oriented connected smooth 3–manifold M via orientation preserving diffeomorphisms $\mathbf{f} := (f_t, f_s)$ where $f_t^2 = f_s^n = 1$, and $f_t f_s f_t = f_s^{n-1}$. In fact the assumption that f_t and f_s are smooth may be replaced by the weaker assumption that they be locally linear [25, Remark 2.3]. Viewing the 3–sphere as a one point compactification of $\mathbb{R}^3$, the claim is then that M has a surgery presentation $L \subset S^3$ such that L is invariant under $2\pi/n$ rotation around the Z–axis and under π rotation around the X–axis as a framed link.

The proof is by taking the quotient smooth orbifold $\mathfrak{O} := M/D_{2n}$ (see eg Cooper, Hodgson and Kerckhoff [6, Section 2.1]), with singular set Σ. So pr: $M \twoheadrightarrow \mathfrak{O}$ is a $2n$–fold regular dihedral branched covering space (see eg Rolfsen [22]) with monodromy given by a representation $\psi\colon \pi_1(\mathfrak{O} - \Sigma) \twoheadrightarrow D_{2n}$ induced by the action of $\mathbf{f}$. The idea is to construct an integral framed surgery link $\mathcal{L}$ to make the following diagram commute:

$$\begin{array}{ccccc} & M & \xleftarrow{\operatorname{surg}(\widetilde{\mathcal{L}})} & S^3 & \supset \widetilde{\mathcal{L}} \\ & {\scriptstyle \mathrm{pr}_\psi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_{\rho_t}} & \\ \Sigma \subset & \mathfrak{O} & \xleftarrow[\operatorname{surg}(\mathcal{L})]{} & S^3 & \supset \mathcal{L} \cup \mathfrak{t}((2k+1)n, 2) \end{array} \tag{5-2}$$

where surg$(-)$ performs surgery by its argument (note that this is not a map), and Σ and $\mathfrak{t}((2k+1)n,2)$ are the covering loci. The lifted link $\widetilde{\mathcal{L}}$ will then have the required dihedral symmetry by construction, inherited from the dihedral symmetry of $\mathfrak{t}((2k+1)n,2)$ lying symmetrically along a torus.

The link $\mathcal{L}$ is constructed as the combination of two framed links $\mathcal{L}_1 \cup \mathcal{L}_2$ such that

(1) The sublink $\mathcal{L}_1$ is in $\ker\rho$, its components are ±1–framed and are unknotted, and $\operatorname{surg}(\mathcal{L}_1)\colon S^3 \longrightarrow S^3$ takes $(\mathfrak{t}((2k+1)n, 2), \rho_{\mathfrak{t}})$ to some D_{2n}–coloured knot (K', ρ').

(2) For the integral framed sublink $\mathcal{L}_2$, the procedure $\operatorname{surg}(\mathcal{L}_2)\colon S^3 \longrightarrow \mathfrak{O}$ takes (K', ρ') to (Σ, ψ).

The sublink $\mathcal{L}_1$ is given to us by Corollary 17, while $\mathcal{L}_2$ may be constructed in complete analogy with [25, pages 383–384 and Section 4].

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