Commensurability classes containing three knot complements

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This paper exhibits an infinite family of hyperbolic knot complements that have three knot complements in their respective commensurability classes.

57M10, 57M25

1 Introduction

The study of the commensurability classes of hyperbolic knot complements that contain other knot complements has attracted some recent interest (see Boileau, Boyer and Walsh [4], Calegari and Dunfield [6], Goodman, Heard and Hodgson [13] Hoste and Shanahan [14], Macasieb and Mattman [15], Neumann and Reid [19], Reid [20] and Reid and Walsh [21]). A particularly interesting set of examples results from cyclic surgeries on hyperbolic knot complements, since the cyclic surgeries give rise to cyclic covers by other knot complements (see González-Acuña and Whitten [11]). Moreover, the Cyclic Surgery Theorem of Culler, Gordon, Luecke and Shalen [8] shows that there are at most two nontrivial cyclic surgeries on a hyperbolic knot complement and so a hyperbolic knot complement has at most two nontrivial, finite sheeted covers which are other knot complements. Similarly, if a hyperbolic knot complement, $S^3 - k_1$ is covered by another knot complement, $S^3 - k_2$, then $S^3 - k_1$ admits a cyclic surgery. There are known examples of hyperbolic knot complements with exactly three knot complements in their commensurability classes. For example, the $(-2, 3, 7)$ pretzel knot famously admits two nontrivial cyclic surgeries by Fintushel and Stern [10] and is therefore covered by two other hyperbolic knot complements. Indeed, one can show that these three knot complements are the only knot complements in their commensurability class (see Section 4.1, case $n = 1$, $m = 0$, Section 2.1 and Reid and Walsh [21, Corollary 5.4]).

An infinite family of pairs of commensurable hyperbolic knot complements was constructed by Walter Neumann. For a discussion of this construction, see Goodman, Heard and Hodgson [13].

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Finally, two hyperbolic knot complements can be commensurable if they both have hidden symmetries. This property is equivalent to both knot complements nonnormally covering the same orbifold (see Section 2.1). The dodecahedral knots of Aitchison and Rubinstein [2] admit the only known examples of nonarithmetic knot complements with hidden symmetries (see Neumann and Reid [19]) and the figure 8 knot complement is the only arithmetic knot complement (see Reid [20]).

This discussion motivates the following conjecture of Reid and Walsh [21, Conjecture 5.2].

Conjecture *Let $S^3 - K$ be a hyperbolic knot complement. There are at most two other knot complements in its commensurability class.*

It has been announced by Boileau, Boyer and Walsh [4, Theorem 1.3] that the conjecture holds for knot complements without hidden symmetries. In their paper, they show that if a hyperbolic knot complement does not admit hidden symmetries, then any commensurable hyperbolic knot complement will cyclically cover a common orbifold. Furthermore, this orbifold admits a finite cyclic surgery for each knot complement that covers it. This paper presents a family of such orbifolds that are covered by exactly three hyperbolic knot complements. Specifically, the main theorem of this paper is the following (see Section 2 for definitions):

Theorem 1.1 *Let $n \geq 1$ and $(n, 7) = 1$. For all but at most finitely many pairs of integers (n, m), the result of (n, m) Dehn surgery on the unknotted cusp of the Berge manifold is a hyperbolic orbifold with exactly three knot complements in its commensurability classes.*

The infinite family of orbifolds described by Theorem 1.1 which we refer to as $\beta_{n,m}$ (see Section 2) also has the property that for $n \neq 1$, each knot complement covering $\beta_{n,m}$ admits an n–fold symmetry which does not fix any point on the cusp. In particular, even when $n = 2$, this symmetry is not a strong involution. By [25], such a knot complement cannot admit a lens space surgery and so, by the above discussion, is not covered by any other knot complement.

The paper is organized as follows. In addition to some background material and definitions, in Section 2 we prove a lemma about possible orbifold quotients of the Berge manifold. In Section 3, we show that the orbifolds $\beta_{n,m}$ admit three cyclic surgeries, and the proof of the main theorem is contained in Section 4. In Section 5, we provide a partial classification of commensurability classes containing exactly three knot complements.

2 Preliminaries

Two subgroups Γ_1 and Γ_2 of $\mathrm{PSL}(2,\mathbb{C})$ are said to be *commensurable* if they share a common finite index subgroup. Two hyperbolic 3–orbifolds, $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$, are said to be *commensurable* if they share a common finite sheeted cover. In terms of their groups, this means that $\exists g \in \mathrm{PSL}(2,\mathbb{C})$ such that Γ_1 and $g\Gamma_2 g^{-1}$ are commensurable.

Let $\mathrm{Comm}^+(\Gamma) = \{g \in \mathrm{PSL}(2,\mathbb{C}) \mid \Gamma \text{ and } g\Gamma g^{-1} \text{ are commensurable}\}$. Let $N^+(\Gamma)$ be the normalizer of Γ in $\mathrm{PSL}(2,\mathbb{C})$. We say that a group Γ has *hidden symmetries* if $[\mathrm{Comm}^+(\Gamma) : N^+(\Gamma)] > 1$. A hyperbolic orbifold, Q, has *hidden symmetries* if $\pi_1^{\mathrm{orb}}(Q)$ has hidden symmetries. For this discussion, we consider only orientable manifolds and orbifolds.

2.1 Cusp properties

When a hyperbolic knot group has hidden symmetries, the associated knot complement nonnormally covers some orbifold with a *rigid cusp* ie the cusp is $C \times [0,\infty)$ where C is $S^2(2,3,6)$, $S^2(3,3,3)$ or $S^2(2,4,4)$ (see Reid [20, Lemma 4]).

By [19, Proposition 2.7], the cusp field of a hyperbolic orbifold is a subfield of the invariant trace field. Thus, if a hyperbolic orbifold has a $S^2(3,3,3)$ or $S^2(2,3,6)$ cusp, $\mathbb{Q}(\sqrt{-3})$ must be a subfield of the orbifold's invariant trace field and if the cusp is $S^2(2,4,4)$, $\mathbb{Q}(i)$ must be a subfield of the orbifold's invariant trace field (see [19, Proof of Theorem 5.1(iv)]).

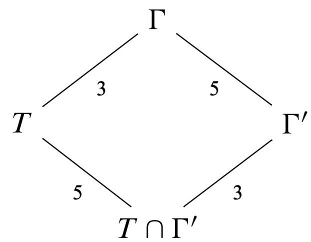
Like the torus, the orbifolds $S^2(3,3,3)$, $S^2(2,4,4)$, and $S^2(2,3,6)$, can admit self-covers. Since the groups of deck transformations for the above orbifolds are of the form $(\mathbb{Z}\times\mathbb{Z}) \rtimes_f \mathbb{Z}/n\mathbb{Z}$ (where n is 3, 4, or 6 respectively), there exists a degree m self-cover of one of these orbifolds for each index m subgroup of $\mathbb{Z}\times\mathbb{Z}$ preserved by f, an outer automorphism of $\mathbb{Z}\times\mathbb{Z}$. The following proposition shows that there are certain degrees such that $S^2(3,3,3)$ does not admit a self-cover.

Proposition 2.1 *Let $p\colon O' \to O$ be a covering map. If $O \cong S^2(3,3,3)$, then degree of p is not 2 or 5.*

Proof Denote by $\Gamma = \pi_1^{\mathrm{orb}}(O)$ and note that the abelianization of Γ is $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$. In particular, Γ has no index 2 subgroups.

Assume that $\Gamma' \subset \Gamma$ has index 5. The abelianization of Γ excludes the case that $\Gamma' \lhd \Gamma$.

Note Γ has a torsion-free subgroup $T \cong \mathbb{Z}\times\mathbb{Z}$ of index 3. Also, $[\Gamma : \Gamma'\cdot T][\Gamma'\cdot T : T] = 3$ and since Γ' has torsion elements, $[\Gamma : \Gamma'\cdot T] = 1$. Thus, we get the following lattice of subgroups:



Note that $T \triangleleft \Gamma$. Hence, we have that $T \cap \Gamma' \triangleleft \Gamma'$ by the Second Isomorphism Theorem. Also, $T \cap \Gamma' \triangleleft T$ since T is abelian. Using $\Gamma = \Gamma' \cdot T$, we obtain that $T \cap \Gamma' \triangleleft \Gamma$. Thus, $\Gamma / T \cap \Gamma'$ is isomorphic to a cyclic group of order 15, which is a contradiction to abelianization of Γ being $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. This completes the proof. □

2.2 Surgeries on the Berge manifold

For $n \geq 1$ and $(n, 7) = 1$, let $\beta_{n,m}$ be the orbifold obtained by (n, m) Dehn surgery on the unknotted cusp of the Berge manifold (see Figure 1) using a standard framing on the cusps of this link complement as in [22].

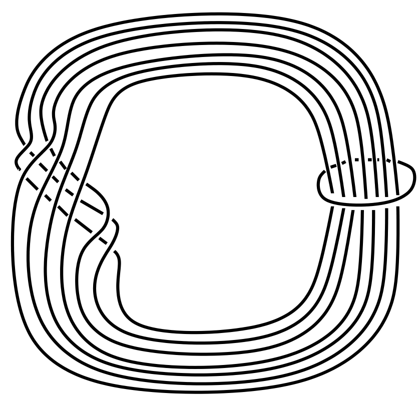


Figure 1: The Berge manifold is the complement of this link.

The Berge manifold admits several surgery slopes of interest. First, if we perform Dehn surgery along the $(1, 0)$ slope of the unknotted cusp of the Berge manifold, we will obtain the $(-2, 3, 7)$ pretzel knot (see Fintushel and Stern [10]). Also, if we drill out a solid torus along the unknotted cusp of the manifold we would obtain the one knot in the solid torus (defined up to homeomorphism of the solid torus) that admits three $D^2 \times S^1$ fillings (see [3, Corollary 2.9]). Furthermore, if we perform Dehn surgery along the $(1, r)$ slope and then drill along the core of the surgered torus, we would also obtain a knot complement in $D^2 \times S^1$ that admits three $D^2 \times S^1$ surgeries. In fact, by the above mentioned corollary, these are the only knots in solid tori with this property.

The above construction shows that Dehn surgery along a $(1,r)$ slope of the unknotted cusp of the Berge manifold produces knot complements that produce three lens space surgeries. In fact, it is well known that the $(1,0)$, $(18,1)$ and $(19,1)$ surgery slopes on the $(-2,3,7)$ pretzel knot admit lens space surgeries [10]. By drilling out the unknotted cusp of the Berge manifold, these are also the surgery slopes that produce a solid torus filling. Since the linking number of the knotted cusp and the unknotted cusp is 7, the longitude gets sent to the curve $(49r,1)$ after $(1,r)$ Dehn surgery on the unknotted cusp while the meridian $(1,0)$ remains fixed [22, Section 9.H]. So the $(1,0)$, $(18,1)$, and $(19,1)$ surgery parameters get sent to $(1,0)$, $(49r+18,1)$, and $(49r+19,1)$ respectively after $(1,r)$ Dehn surgery on the unknotted cusp. Therefore, these fillings produce solid tori in the new coordinates. Furthermore, we can use the surgery parameters to compute the homology of the manifolds resulting from lens space surgeries on the knot complements. In fact, we see that for these knots we obtain S^3 and two lens spaces – one with fundamental group of order $|49r+18|$ and another of order $|49r+19|$.

More generally, if we allow Dehn surgery along any (p,q) slope of the unknotted cusp of the Berge manifold where (p,q)=1, and either $(1,0)$, $(18,1)$, or $(19,1)$ Dehn surgery on the knotted cusp, we will also get lens spaces. Again, by [22, Sect 9.H], we see that the $(1,0)$ surgery slope corresponds to a lens space of order $|p|$, $(18,1)$ surgery slope corresponds to a lens space of order $|49q+18p|$, and $(19,1)$ surgery slope corresponds to a lens space of order $|49q+19p|$.

2.3 Commensurability class of the Berge manifold

Denote $v_0 \approx 1.01494146$ as the volume of the regular ideal tetrahedron. The Berge manifold is comprised of four such tetrahedra and therefore its volume is $4v_0$. Denote by Γ_L the fundamental group of the Berge manifold. Since the complement of the Berge manifold is comprised of four regular ideal tetrahedra, $\Gamma_L \subset \mathrm{Isom}^+(\mathbb{T}) \cong \mathrm{PGL}(2,\mathbb{O}_3)$, where $\mathbb{T}$ is a tessellation of $\mathbb{H}^3$ by regular ideal tetrahedra and $\mathbb{O}_3$ is the ring of integers in $\mathbb{Q}(\sqrt{-3})$. Since $[\mathrm{PGL}(2,\mathbb{O}_3):\Gamma_L]=48$ and $[\mathrm{PGL}(2,\mathbb{O}_3):\mathrm{PSL}(2,\mathbb{O}_3)]=2$, Γ_L is commensurable with $\mathrm{PSL}(2,\mathbb{O}_3)$. Therefore, Γ_L is arithmetic and so the Berge manifold is arithmetic as well (see Maclachlan and Reid [16, Definition 8.2.1]).

The proof of the following lemma takes advantage of the fact that the Berge manifold has relatively low volume in order to show that it cannot cover an orbifold with a torus cusp and a rigid cusp. It is worth mentioning that $\mathrm{PGL}(2,\mathbb{O}_3)$ is the orbifold fundamental group of hyperbolic orbifold with a single $S^2(2,3,6)$ cusp. Additionally, the following proof will consider $\mathrm{PSL}(2,\mathbb{O}_3)$, which is an index 2 subgroup of $\mathrm{PGL}(2,\mathbb{O}_3)$, and so $\mathbb{H}^3/\mathrm{PSL}(2,\mathbb{O}_3)$ is a two-fold cover of the orbifold $\mathbb{H}^3/\mathrm{PSL}(2,\mathbb{O}_3)$ [19]. Hence for this paper, we consider $\mathrm{PGL}(2,\mathbb{O}_3)$ under the image of its representation into $\mathrm{PSL}(2,\mathbb{C})$. Also, we will consider all other groups as subgroups of $\mathrm{PSL}(2,\mathbb{C})$ where necessary.

Lemma 2.2 *The Berge manifold does not cover an orbifold with a torus cusp and a rigid cusp.*

Proof of Lemma 2.2 Assume Q_T is an orbifold with a torus cusp and a rigid cusp covered by the Berge manifold. Since the invariant trace field of the Berge manifold is $\mathbb{Q}(\sqrt{-3})$, the rigid cusp of Q_T must be either $S^2(3,3,3)$ or $S^2(2,3,6)$. In either case, consideration of the unknotted torus cusp of the Berge manifold covering the rigid cusp shows the degree of such a cover is $3k$ for some integer $k \geq 1$. Also, since the Berge manifold is arithmetic and the class number of $\mathbb{Q}(\sqrt{-3})$ is 1, it follows from [7, Theorem 1.1] that any maximal group commensurable with the Berge manifold has exactly one cusp. Thus, there exists a one-cusped orbifold Q_M covered by Q_T.

Denote the Berge manifold by B. By consideration of a torus cusp of B covering the rigid cusp of Q_T, we see that $p_1\colon B \to Q_T$ is a covering map of degree $3k$ ($k \geq 1$). Also, by consideration of the torus cusp of Q_T covering the rigid cusp of Q_M, $p_2\colon Q_T \to Q_M$ is a covering map of degree at least 3. If Q_M has a $S^2(3,3,3)$ cusp, we use the fact that $\mathrm{vol}(Q_M) \leq 4v_0/9$ to show that it must be one of the orbifolds described by Adams in [1, Theorem 4.2]. However, as pointed out in the comment before this theorem, each of the orbifolds is a double cover of an orbifold with a $S^2(2,3,6)$ cusp. Since we may assume Q_M is corresponds to a maximal subgroup of $\mathrm{PSL}(2,\mathbb{C})$, and is therefore not a cover of any smaller volume orbifold, we only have to consider to the case that Q_M has a $S^2(2,3,6)$ cusp. In this case, Q_M has a $S^2(2,3,6)$ cusp and the degree of p_2 is at least 7. Since the torus cusp of Q_T is at least a six fold-cover of the rigid cusp of Q_M and the rigid cusp of Q_T is at least a one-fold cover of the rigid cusp of Q_M. Hence, $\mathrm{vol}(Q_M) \leq 2v_0/9$ and Q_M is described by Adams in [1, Theorem 3.3] (with arithmetic information of the corresponding orbifold groups described in [19]). Furthermore, since $\mathrm{vol}(B) = 4v_0$, $\mathrm{vol}(Q_M)$ is either $v_0/6$ or $v_0/12$. Thus, the covering of Q_M by the Berge manifold is of order 24 or 48, respectively. We will consider these two cases separately by further analyzing the covering maps p_1 and p_2.

Case 1 Q_M has volume $v_0/6$ and the degree of the cover $p\colon B \to Q_M$ is 24.

By noting that $\pi_1^{\mathrm{orb}}(Q_M)$ has an index 2 subgroup

$$\Gamma = \langle x, y, z \mid x^2, y^2, z^3, (yz^{-1})^2, (zx^{-1})^6, (xy^{-1})^3 \rangle$$

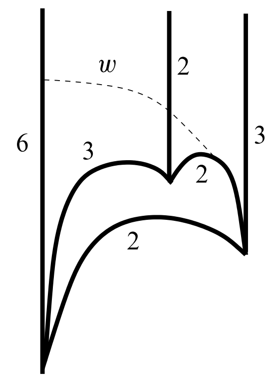


Figure 2: The fundamental domain for Γ together with the involution w

and $\pi_1^{\text{orb}}(Q_M) = \langle \Gamma, w \rangle$ where w is the order 2 rotation on the fundamental domain of Γ, we obtain a presentation for $\pi_1^{\text{orb}}(Q_M)$ (see Neumann and Reid [19], Maclachlan and Reid [16, page 144, formula 4.7 and Figure 4.4] and Figure 2).

Thus, we obtain the following presentation:

$$\pi_1^{\text{orb}}(Q_M) = \langle w, x, y, z \mid x^2, y^2, z^3, w^2, (yz^{-1})^2, (zx^{-1})^6, (xy^{-1})^3, (wx)^2, wywyz^{-1} \rangle.$$

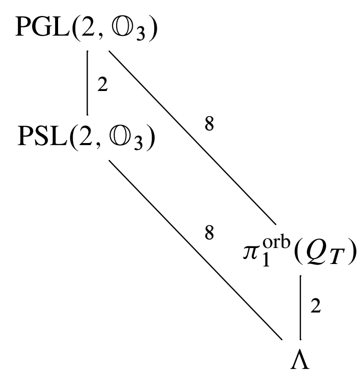
By the constraints mentioned above, the degree $p_1\colon B \to Q_T$ must be 3 and the degree of $p_2\colon Q_T \to Q_M$ must be 8. However, using GAP, the above group $\pi_1^{\text{orb}}(Q_M)$ does not have any index 8 subgroups. Thus, there can be no orbifold Q_T.

Case 2 Q_M has volume $v_0/12$ and the degree of the cover $p\colon B \to Q_M$ is 48.

In this case, $Q_M \cong \mathbb{H}^3/\text{PGL}(2, \mathbb{O}_3)$ and so we will consider the group picture. Here, $[\text{PGL}(2, \mathbb{O}_3) : \pi_1^{\text{orb}}(Q_T)] = 8$ or 16, since degree of $p_2 \geq 7$ and the degree of $p_1 = 3k$ $(k \geq 1)$.

First, assume $[\text{PGL}(2, \mathbb{O}_3) : \pi_1^{\text{orb}}(Q_T)] = 8$. If $\pi_1^{\text{orb}}(Q_T) \subset \text{PSL}(2, \mathbb{O}_3)$, $[\text{PSL}(2, \mathbb{O}_3) : \pi_1^{\text{orb}}(Q_T)] = 4$. Using GAP, there is a unique index 4 subgroup G of $\text{PSL}(2, \mathbb{O}_3)$. However, G has finite abelianization, and therefore cannot be the orbifold group of Q_T.

Thus, we may assume that $\pi_1^{\text{orb}}(Q_T) \not\subset \text{PSL}(2, \mathbb{O}_3)$ and deduce that there is a unique subgroup Λ of index 2 in $\pi_1^{\text{orb}}(Q_T)$ such that $\Lambda \subset \text{PSL}(2, \mathbb{O}_3)$. By covolume considerations Λ has index 8 in $\text{PSL}(2, \mathbb{O}_3)$. Also, $\mathbb{H}^3/\Lambda$ has a torus cusp and a $S^2(3,3,3)$ cusp. Since $\mathbb{H}^3/\text{PSL}(2, \mathbb{O}_3)$ has a $S^2(3,3,3)$ cusp, the degree of the covering $p\colon \mathbb{H}^3/\Lambda \to \mathbb{H}^3/\mathrm{PSL}(2,\mathbb{O}_3)$ has to be $3l+m$. However, $m \neq 2,5$ (see Proposition 2.1), a contradiction.



Now, assume that $[\mathrm{PGL}(2,\mathbb{O}_3) : \pi_1^{\mathrm{orb}}(Q_T)] = 16$. We know that $p_1\colon B \to Q_T$ is of degree 3 and therefore, Q_T has a $S^2(3,3,3)$ cusp and a torus cusp. Thus, $\pi_1^{\mathrm{orb}}(Q_T) \subset \mathrm{PSL}(2,\mathbb{O}_3)$ and $[\mathrm{PSL}(2,\mathbb{O}_3) : \pi_1^{\mathrm{orb}}(Q_T)] = 8$, giving us the same contradiction as in above paragraph.

This completes the proof. □

3 Cyclic surgeries on $\beta_{n,m}$

In this section, we show that for fixed n and m, $\beta_{n,m}$ admits three finite cyclic surgeries. We also show directly it is covered by three knot complements if $n \neq 7$.

Lemma 3.1 *The orbifolds $\beta_{n,m}$ are covered by three knot complements. Furthermore, the degrees of the corresponding covering maps are distinct.*

Proof For a fixed $\beta_{n,m}$, let $r=(n,m)$ and consider $\beta_{n,m}$ as the union of the complement of a knot in a solid torus, T_1 and a solid torus with core a singular locus of order r, T_2 (see Figure 3).

By [3, Corollary 2.9], T_1 admits three Dehn surgeries that result in a solid torus. Thus, $\beta_{n,m}$ admits three Dehn surgeries that are homeomorphic to T_2 and a solid torus glued together along their boundaries. Each orbifold O_j $(j \in \{1,2,3\})$ resulting from one of these Dehn surgeries has underlying space a lens space with $\pi_1^{\mathrm{orb}}(O_j)$ finite cyclic.

In fact, $|\pi_1^{\mathrm{orb}}(O_j)|$ is distinct for each choice of j. To see this, we observe, as noted above, that O_j is an orbifold with underlying space a lens space. Moreover, this underlying space is a lens space with fundamental group of order either $\frac{n}{r}$, $|49\frac{m}{r}+18\frac{n}{r}|$,

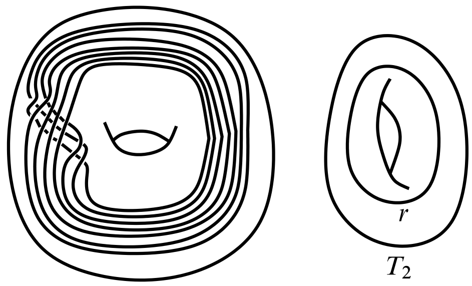


Figure 3: The decomposition of a surgered $\beta_{n,m}$ along a torus

or $|49\frac{m}{r} + 19\frac{n}{r}|$ depending on the choice of surgery on T_1 (see Section 2). Splitting O_j into a solid torus coming from the Dehn surgery on T_1 and T_2 the solid torus core a singular curve, we can compute $\pi_1^{\text{orb}}(O_j)$ using van Kampen's theorem. Thus, the orders of the each fundamental group increase by a factor of r and $|\pi_1^{\text{orb}}(O_j)|$ is either n, $r \cdot |49\frac{m}{r} + 18\frac{n}{r}|$, or $r \cdot |49\frac{m}{r} + 19\frac{n}{r}|$ which take on three distinct values for fixed n, m and r.

In addition, by the Orbifold Theorem (see [5, Theorem 2]) and the above argument that $\pi_1^{\text{orb}}(O_j)$ is finite cyclic, each O_j has S^3 as its universal cover. Denote this covering map by $\phi_j \colon S^3 \to O_j$. We may view O_j as the union of the solid torus coming from the cusp Dehn filling of $\beta_{n,m}$ and the complement of this solid torus, which we denote by B. Hence, $\phi_j^{-1}(B)$ is a knot or link exterior in S^3. Since $(n, 7) = 1$ and the singular set of T_2 has linking number 7 with the knotted cusp of $\beta_{n,m}$, the boundary of $\phi_j^{-1}(B)$ is connected. Hence, if $(n, 7) = 1$, $\beta_{n,m}$ will be covered by three knot complements in S^3. Also, since the orders of $|\pi_1^{\text{orb}}(O_j)|$ are distinct, the covering degree of ϕ_j will take on a distinct value for each j. □

Remark When $n = 1$, the classification of exceptional Dehn surgeries in [17, Table A.1, Remark A.3] shows that $\beta_{n,m}$ is hyperbolic. Hence, $\beta_{1,m}$ is a hyperbolic knot complement that admits three cyclic surgeries.

4 Proof of the main theorem

In this section, we prove Theorem 1.1. Also for this section, we consider $\Omega_{n,m}$, $\Delta_{n,m}$, and Ω_L as subgroups of $\mathrm{PSL}(2, \mathbb{C})$.

Proof of Theorem 1.1 Using Lemma 3.1, each $\beta_{n,m}$ is covered by three knot complements such that the covers are of distinct degrees. Also, Thurston's Hyperbolic Dehn

Surgery Theorem [24, Theorem 5.8.2] shows that all but at most finitely many of the $\beta_{n,m}$ are hyperbolic. For the rest of the proof we only consider those $\beta_{n,m}$ that are hyperbolic. Given this condition, each $\beta_{n,m}$ we consider is covered by three distinct knot complements. By [4, Theorem 1.3], to prove Theorem 1.1 it suffices to show that the knot complements covering $\beta_{n,m}$ do not have hidden symmetries.

Suppose an infinite number of the hyperbolic knot complements that cover $\beta_{n,m}$ admit hidden symmetries. By the discussion in Section 2.1, every such knot complement will nonnormally cover an orbifold $Q_{n,m}$ with a rigid cusp. Furthermore, on passage to a subset of the $\beta_{n,m}$, we can assume that the orbifolds $Q_{n,m}$ have the same type of rigid cusp, C. Let $\Omega_{n,m} = \pi_1^{\text{orb}}(\beta_{n,m})$, $\Delta_{n,m} = \pi_1^{\text{orb}}(Q_{n,m})$ and let $P \subset \mathrm{PSL}(2,\mathbb{C})$ be the peripheral subgroup of $\Delta_{n,m}$. We may assume that each $\Omega_{n,m}$ is conjugated so that P has a fixed representation in $\mathrm{PSL}(2,\mathbb{C})$. Since $\beta_{n,m}$ has one cusp, notice that $\Delta_{n,m} = P \cdot \Omega_{n,m}$.

By Thurston's Hyperbolic Dehn Surgery Theorem [24, Theorem 5.8.2], the volumes of the $\beta_{n,m}$ are bounded from above by the volume of the Berge manifold. In addition, the minimum volume of a noncompact oriented hyperbolic 3–orbifold is $v_0/12$ [18]. Hence, $\mathrm{vol}(Q_{n,m}) \geq v_0/12$. Thus, we can further subsequence to arrange that $\beta_{n,m}$ covers $Q_{n,m}$, that the $Q_{n,m}$'s have the same type of rigid cusp, and that the covering degree is fixed, say d.

Since $\beta_{n,m}$ is obtained by Dehn surgery on the Berge manifold, the $\Omega_{n,m}$ will converge algebraically and geometrically to Ω_L, the fundamental group of the Berge manifold (see [24, Theorem 5.8.2]). As P is a fixed group in our construction, $\Delta_{n,r}$ also converges algebraically and geometrically to $P \cdot \Omega_L$.

We have the following diagram:

$$\begin{array}{ccc} \Delta_{n,m} & \xrightarrow{(n,m)\to\infty} & P \cdot \Omega_L \\ \uparrow d & & \uparrow d \\ \Omega_{n,m} & \xrightarrow{(n,m)\to\infty} & \Omega_L \end{array}$$

Note, $[P \cdot \Omega_L : \Omega_L] = d < \infty$. Let $Q_T = \mathbb{H}^3 / P \cdot \Omega_L$. Q_T has two cusps: a torus cusp, corresponding to the cusp created by geometric convergence from Dehn surgery, and a rigid cusp, corresponding to the cusp with peripheral group P.

However by Lemma 2.2, such a limiting Q_L cannot exist. Hence, at most finitely many of the $\beta_{n,m}$ have hidden symmetries. □

4.1 Computations and examples

To find explicit examples of hyperbolic knot complements with three knot complements in the commensurability class, we can use the computer program Snap [12] to show directly that there are no hidden symmetries. Specifically, for $m = 0$ and $n = 1, 2, 3, 4, 5, 6, 7$, the manifold $\beta_{n,m}$ is hyperbolic and Snap shows us that $\beta_{n,m}$ has an invariant trace field with real embeddings. These fields cannot contain $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$ as subfields. Thus, the knot complements covering such $\beta_{n,m}$ do not have hidden symmetries (recall Section 2.1) and there are exactly three knot complements in each of these commensurability classes.

5 Remarks

The following theorem provides a partial classification of hyperbolic orbifolds covered by three knot complements. It can be seen as a direct corollary to a result of [4], however we provide a proof for completeness.

Theorem 5.1 *Let O be a closed 3–orbifold and let K be a knot in O that is disjoint from the singular locus of O. If $O - K$ is*

(1) *hyperbolic,*

(2) *covered by 3 knot complements,*

(3) *does not admit hidden symmetries, and*

(4) *O has nonempty singular locus,*

then $O - K \cong \beta_{n,m}$ for some pair (n, m).

Proof Let γ be the singular locus of O. Denote $|O|$ the underlying space of O. By [4, Theorem 1.2] and the assumptions above, we know that $|O|$ is a lens space, γ is a nonempty subset of the cores of a genus 1 Heegaard splitting of $|O|$, and if $S^3 - K$ covers $O - K$, then it does so cyclically and corresponds to a finite cyclic filling of $O - K$. Finally, denote $M = O - \gamma - K$.

First assume γ has one component. Each of the three knot complements covering $O - K$ corresponds to M admitting a $S^1 \times D^2$ filling along its knotted cusp. Again, we appeal to the fact that there is a unique family of knots in solid tori that admits 3 nontrivial $S^1 \times D^2$ fillings (see [3, Corollary 9.1]). Hence, M is obtained by performing $(1, m)$ surgery on the unknotted cusp of the Berge manifold then drilling out the core of the surgered torus. Gluing back in the neighborhood of the fixed point set of $\langle\gamma\rangle$ gives us $\beta_{n,m}$ for some n, m.

Now, assume that γ has two components γ_1 and γ_2. $M = T^2 \times I - K'$, where K' is a knot. Each cyclic filling on $O - K$ corresponds to M admitting a $T^2 \times I$ filling. Hence, Dehn filling along the cusp corresponding to γ_1 will produce a knot complement in $D^2 \times S^1$ with three $D^2 \times S^1$ fillings.

Denote l_1 to be the linking number of γ_1 and K' and l_2 to be the linking number of γ_2 and K'. If l_1 is zero, K' would be a knot in a solid torus that is not a 1–braid after $(1,0)$ on γ_2 but has two nontrivial $S^1 \times D^2$ fillings. This contradicts [3, Corollary 9.1]. Hence, we may assume $l_1 \neq 0$ and $l_2 \neq 0$.

Also, $(1,n)$ surgery on γ_2 will produce a knot K'' in a solid torus that has linking number $l_2 + n \cdot l_1$ with γ_2. In particular for large enough n, $l_2 + n \cdot l_1 \neq 7$. Hence, it cannot be in the family of knots that admit two nontrivial $S^1 \times D^2$ fillings. □

One might hope to relax condition (4) above. However, Brandy Guntel pointed out to the author that the $k(2,2,0,2)$ knot complement (see Figure 4) is hyperbolic and admits two nontrivial cyclic surgeries. In fact, John Berge first showed that this knot complement produced two lens space surgeries in unpublished work. Additionally, Mario Eudave-Muñoz [9] gave a construction of the two nontrivial lens space surgeries of this knot complement. Let $M(r)$ denote Dehn filling the torus cusp with respect to the slope r on the cusp torus. Furthermore, we will observe the convention that $1/0$ is the meridian and $0/1$ the homologically determined longitude. From the discussion following [9, Proposition 5.4], we obtain that k(2,2,0,2)$(32/1)$ and k(2,2,0,2)$(31/1)$ are lens space surgeries where the fundamental groups of these lens spaces are of orders 32 and 31 respectively (see [9, Proposition 5.3]). By our original discussion in Section 2.2, knot complements obtained by Dehn surgery on the unknotted cusp of the Berge manifold have lens spaces of order $|49r - 18|$ and $|49r - 19|$, none of which can be 32. Hence, the $k(2,2,0,2)$ complement is not one of the $\beta_{n,m}$. However, since the invariant trace field of the $k(2,2,0,2)$ is an odd degree extension of $\mathbb{Q}$, we see that this knot complement does not admit hidden symmetries and the $k(2,2,0,2)$ has exactly three knot complements in its commensurability class (see [21, Corollary 5.4]).

As mentioned above $(1,m)$ surgery on the unknotted cusp of the Berge manifold produces Berge knots. It seems natural to ask if any hyperbolic Berge knots can have hidden symmetries. More generally, we might ask if any hyperbolic knot complements can have hidden symmetries and admit nontrivial lens space surgeries. As discussed in Section 1, there are three hyperbolic knot complements known to have hidden symmetries: the complements of the two dodecahedral knots of Aitchison and Rubinstein [2],

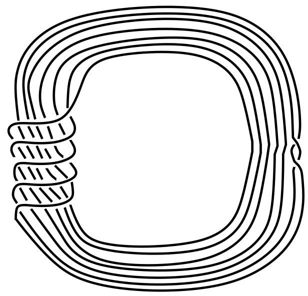


Figure 4: The k(2,2,0,2) knot

and the figure eight knot complement [19]. Using SnapPea [26], one can see that both dodecahedral knots are amphichiral. Thus, by [8, Corollary 4] they cannot admit a lens space surgery. Additionally, it is well known that the figure eight knot complement does not admit a lens space surgery (see Takahashi [23] for example).

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References

[1] **C C Adams**, *Noncompact hyperbolic 3–orbifolds of small volume*, from: "Topology '90 (Columbus, OH, 1990)", (B Apanasov, W D Neumann, A W Reid, L Siebenmann, editors), Ohio State Univ. Math. Res. Inst. Publ. 1, de Gruyter, Berlin (1992) 1–15 MR1184398

[2] **I R Aitchison**, **J H Rubinstein**, *Combinatorial cubings, cusps, and the dodecahedral knots*, from: "Topology '90 (Columbus, OH, 1990)", (B Apanasov, W D Neumann, A W Reid, L Siebenmann, editors), Ohio State Univ. Math. Res. Inst. Publ. 1, de Gruyter, Berlin (1992) 17–26 MR1184399

[3] **J Berge**, *The knots in $D^2 \times S^1$ which have nontrivial Dehn surgeries that yield $D^2 \times S^1$*, Topology Appl. 38 (1991) 1–19 MR1093862

[4] **M Boileau**, **S Boyer**, **G Walsh**, *On commensurability of knot complements*, Preprint Available at `http://www.tufts.edu/~gwalsh01`

[5] **M Boileau**, **J Porti**, *Geometrization of 3–orbifolds of cyclic type*, Astérisque 272 (2001) MR1844891 Appendix A by M Heusener and Porti

[6] **D Calegari**, **N M Dunfield**, *Commensurability of 1–cusped hyperbolic 3–manifolds*, Trans. Amer. Math. Soc. 354 (2002) 2955–2969 MR1895211

[7] **T Chinburg**, **D Long**, **A W Reid**, *Cusps of minimal non-compact arithmetic hyperbolic 3–orbifolds*, Pure Appl. Math. Q. 4 (2008) 1013–1031 MR2440250

[8] **M Culler**, **C M Gordon**, **J Luecke**, **P B Shalen**, *Dehn surgery on knots*, Ann. of Math. (2) 125 (1987) 237–300 MR881270

[9] **M Eudave-Muñoz**, *On hyperbolic knots with Seifert fibered Dehn surgeries*, from: "Proceedings of the First Joint Japan-Mexico Meeting in Topology (Morelia, 1999)", volume 121 (2002) 119–141 MR1903687

[10] **R Fintushel**, **R J Stern**, *Constructing lens spaces by surgery on knots*, Math. Z. 175 (1980) 33–51 MR595630

[11] **F González-Acuña**, **W C Whitten**, *Imbeddings of three-manifold groups*, Mem. Amer. Math. Soc. 99 (1992) viii+55 MR1117167

[12] **O Goodman**, *Snap: a computer program for studying arithmetic invariants of hyperbolic 3–manifolds* Available at `http://www.ms.unimelb.edu.au/~snap/`

[13] **O Goodman**, **D Heard**, **C Hodgson**, *Commensurators of cusped hyperbolic manifolds*, Experiment. Math. 17 (2008) 283–306 MR2455701

[14] **J Hoste**, **P D Shanahan**, *Commensurability classes of twist knots*, J. Knot Theory Ramifications 14 (2005) 91–100 MR2124555

[15] **M L Macasieb**, **T W Mattman**, *Commensurability classes of $(-2, 3, n)$ pretzel knot complements*, Algebr. Geom. Topol. 8 (2008) 1833–1853 MR2448875

[16] **C Maclachlan**, **A W Reid**, *The arithmetic of hyperbolic 3–manifolds*, Graduate Texts in Math. 219, Springer, New York (2003) MR1937957

[17] **B Martelli**, **C Petronio**, *Dehn filling of the "magic" 3–manifold*, Comm. Anal. Geom. 14 (2006) 969–1026 MR2287152

[18] **R Meyerhoff**, *The cusped hyperbolic 3–orbifold of minimum volume*, Bull. Amer. Math. Soc. (N.S.) 13 (1985) 154–156 MR799800

[19] **W D Neumann**, **A W Reid**, *Arithmetic of hyperbolic manifolds*, from: "Topology '90 (Columbus, OH, 1990)", (B Apanasov, W D Neumann, A W Reid, L Siebenmann, editors), Ohio State Univ. Math. Res. Inst. Publ. 1, de Gruyter, Berlin (1992) 273–310 MR1184416

[20] **A W Reid**, *Arithmeticity of knot complements*, J. London Math. Soc. (2) 43 (1991) 171–184 MR1099096

[21] **A W Reid**, **G S Walsh**, *Commensurability classes of 2–bridge knot complements*, Algebr. Geom. Topol. 8 (2008) 1031–1057 MR2443107

[22] **D Rolfsen**, *Knots and links*, Math. Lecture Ser. 7, Publish or Perish, Berkeley, CA (1976) MR0515288

[23] **M-o Takahashi**, *Two-bridge knots have property* P, Mem. Amer. Math. Soc. 29 (1981) iii+104 MR597092

[24] **W P Thurston**, *The geometry and topology of three-manifolds*, Princeton Univ. Math. Dept. Lecture Notes (1979) Available at `http://msri.org/publications/books/gt3m/`

[25] **S C Wang**, **Q Zhou**, *Symmetry of knots and cyclic surgery*, Trans. Amer. Math. Soc. 330 (1992) 665–676 MR1031244

[26] **J Weeks**, *Snappea: A computer program for creatingand studying hyperbolic 3 manifolds* Available at `http://www.geometrygames.org/SnapPea`

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