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On reciprocality of twisted Alexander invariants

JONATHAN A HILLMAN
DANIEL S SILVER
SUSAN G WILLIAMS

Given a knot and an $\mathrm{SL}_n\mathbb{F}$ representation of its group that is conjugate to its dual, the representation that replaces each matrix with its inverse-transpose, the associated twisted Reidemeister torsion is reciprocal. An example is given of a knot group and $\mathrm{SL}_3\mathbb{Z}$ representation for which the twisted Reidemeister torsion is not reciprocal.

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1 Introduction

Let R be a ring. A Laurent polynomial $f(t) \in R[t^{\pm 1}]$ is *reciprocal* if $f(t) = uf(t^{-1})$, for some unit $u \in R[t^{\pm 1}]$. If R has no zero divisors, the condition is equivalent to $f(t^{-1}) = \pm t^i f(t)$, for some $i \in \mathbb{Z}$.

The Alexander polynomial $\Delta(t)$ of a knot k can be computed from a diagram of k or from a presentation of the knot group (see Kawauchi [5], for example). It is an integral Laurent polynomial, well defined up to multiplication by units, and usually normalized to be a polynomial with nonzero constant coefficient. It is well known that $\Delta(t)$ is reciprocal. This is a consequence of Poincaré duality of the knot exterior (see Torres and Fox [15] for an alternative approach based on duality in the knot group).

In 1990 X S Lin introduced a more sensitive knot invariant by using information from representations of the knot group [10]. Later, refinements were described by M Wada [16] and others including P Kirk and C Livingston [6], and J Cha [1]. These twisted Alexander invariants have proven to be useful for a variety of questions about knots including questions about concordance [6], knot symmetry (see Hillman, Livingston and Naik [4]), and fibrations (see Friedl and Vidussi [2]). See Friedl and Vidussi [3] for a survey.

We briefly review the definition of perhaps the best-known twisted Alexander invariant. Let k be a knot with exterior X, endowed with the structure of a CW complex. We fix a Wirtinger presentation $\langle x_0, x_1, \ldots, x_k \mid r_1, \ldots, r_k \rangle$ for the knot group $\pi = \pi_1(X)$.

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Let $\phi\colon F_k \to \pi$ be the associated projection of the free group $F_k = \langle x_0, x_1, \dots, x_k \mid \rangle$ to π. It induces a ring homomorphism $\widetilde{\phi}\colon \mathbb{Z}[F_k] \to \mathbb{Z}[\pi]$.

Let $\epsilon\colon \pi \to H_1(X;\mathbb{Z}) \cong \langle t \mid \rangle$ be the abelianization mapping each x_i to t. It induces a ring homomorphism $\widetilde{\epsilon}\colon \mathbb{Z}[\pi] \to \mathbb{Z}[t^{\pm 1}]$.

Let R be a Noetherian unique factorization domain. Assume that $\gamma\colon \pi \to \mathrm{GL}_n R$ is a linear representation. Let $\widetilde{\gamma}\colon \mathbb{Z}[\pi] \to M_n(R)$ be the associated ring homomorphism to the algebra of $n \times n$ matrices over R. We obtain a homomorphism

$$\widetilde{\gamma} \otimes \widetilde{\epsilon}\colon \mathbb{Z}[\pi] \to M_n(R[t^{\pm 1}]), \tag{1-1}$$

mapping g to $\epsilon(g)\gamma(g)$, that we denote more simply by Φ.

Let $M_{\gamma\otimes\epsilon}$ denote the $k \times (k+1)$ matrix with (i,j)–component equal to the $n \times n$ matrix $\Phi\big(\frac{\partial r_i}{\partial x_j}\big) \in M_n(R[t^{\pm 1}])$. Here $\frac{\partial r_i}{\partial x_j}$ denotes Fox partial derivative. Let $M^0_{\gamma\otimes\epsilon}$ denote the $k \times k$ matrix obtained by deleting the column corresponding to x_0. We regard $M^0_{\gamma\otimes\epsilon}$ as a $kn \times kn$ matrix with coefficients in $R[t^{\pm 1}]$.

Definition 1.1 The *Wada invariant* $W_\gamma(t)$ is

$$\frac{\det M^0_{\gamma\otimes\epsilon}}{\det \Phi(x_0 - 1)}.$$

When γ is the trivial 1-dimensional representation, $M^0_{\gamma\otimes\epsilon}$ is a matrix $M(t)$ that we call the *Alexander matrix* of k. (This terminology is used, for example, by Rolfsen [13], but it is not standard.) The determinant of $M(t)$ is the (untwisted) Alexander polynomial $\Delta(t)$ of k.

Remark 1.2 Although the rational function $W_\gamma(t)$ is often a polynomial, it need not be. However, in general it is well defined up to multiplication by $(-t)^{ni}$. See Wada [16].

Let $\widetilde{X}$ denote the universal cover of X, with the structure of a CW complex that is lifted from X. The matrix $M_{\gamma\otimes\epsilon}$ represents a boundary homomorphism for a twisted chain complex

$$C_*(X; V[t^{\pm 1}]_\gamma) = (R[t^{\pm 1}] \otimes_R V) \otimes_\gamma C_*(\widetilde{X}). \tag{1-2}$$

Here $V = R^n$ is a free module on which π acts via γ, while $C_*(\widetilde{X})$ denotes the cellular chain complex of $\widetilde{X}$ with coefficients in R. The group ring $R[\pi]$ acts on the left via deck transformations. On the other hand, $R[t^{\pm 1}] \otimes_R V$ has the structure of of a right $R[\pi]$–module via

$$(p \otimes v) \cdot g = (\epsilon(g) p) \otimes (v \gamma(g)), \text{ for } \gamma \in \pi.$$

The homology group $H_1(X; V[t^{\pm 1}])$ of the chain complex (1-2) is a finitely generated $R[t^{\pm 1}]$–module. Its 0th elementary divisor $\Delta_\gamma(t)$, which is well defined up to multiplication by units in $R[t^{\pm 1}]$, lately competes with $W_\gamma(t)$ for the name "twisted Alexander polynomial." In many cases they are equal (up to multiplication by units); generally, $\Delta_\gamma(t)$ is $\det M^0_{\gamma \otimes \epsilon}$ divided by a factor of $\det \Phi(x_0 - 1)$. See Kirk and Livingston [6] or Silver and Williams [14] for details.

The representation γ induces a representation $\gamma\colon \pi \to GL_n(\mathbb{F}(t))$, where $\mathbb{F}(t)$ is the field of fractions of $R[t^{\pm 1}]$. When $\det M^0_{\gamma \otimes \epsilon} \neq 0$, the chain complex

$$C_*(X; V(t)_\gamma) = (\mathbb{F}(t)) \otimes_R V) \otimes_\gamma C_*(\widetilde{X}) \tag{1-3}$$

is acyclic (see Kitano [7]), and hence the Reidemeister torsion $\tau_\gamma(t)$ is defined. It coincides with the Wada invariant (see [7] and also [6]).

Remark 1.3 (1) Conjugating the representation γ corresponds to a change of basis for V. It is well known that the invariants $\Delta_\gamma(t)$ and $\tau_\gamma(t)$ are unchanged.

(2) The indeterminacy of sign in the definition of $\tau_\gamma(t)$ can be removed (see Kitayama [8]).

T Kitano used Poincaré duality to prove in [7] that for orthogonal representations $\gamma\colon \pi \to \mathrm{SO}_n(\mathbb{R})$, the torsion $\tau_\gamma(t)$ is reciprocal, where reciprocality for rational functions is defined as for Laurent polynomials. (In fact Kitano shows that $\tau_\gamma(t^{-1})$ and $\tau_\gamma(t)$ are equal up to multiplication by $\pm t^{ni}$.) He asked whether reciprocality holds for more general representations.

For representations $\gamma\colon \pi \to \mathrm{GL}_n(\mathbb{C})$, "reciprocality" can have another meaning. One can require that $\tau_\gamma(t)$ be equal up to multiplication by a unit to the expression obtained by inverting t *and also taking complex conjugates of coefficients.* Kirk and Livingston showed in [6] that $\tau_\gamma(t)$ satisfies such a condition whenever γ is unitary.

It is not difficult to find representations $\gamma\colon \pi \to \mathrm{GL}_n\mathbb{F}$ such that $\tau_\gamma(t)$ is non-reciprocal. For example, consider the Wirtinger presentation

$$\langle x_0, x_1, x_2 \mid x_0 x_1 = x_2 x_0, x_1 x_2 = x_0 x_1 \rangle$$

of the trefoil knot group π. The assignment $x_i \mapsto X_i \in \mathrm{GL}_1\mathbb{F}$, such that $X_i = \big(2\big)$, $i = 0, 1, 2,$ yields the non-reciprocal invariant

$$\tau_\gamma(t) = \frac{4t^2 - 2t + 1}{2t - 1}.$$

(This simple example was suggested to us by S Friedl.) The question of reciprocality for representations in $\mathrm{SL}_n\mathbb{F}$ is more subtle. The question was proposed by Kitano [7]; it appeared recently in [3].

In Section 2 we show that reciprocality need not hold for general representations in $\mathrm{SL}_n\mathbb{F}$. The representations γ that we consider have the property that the dual representation $\bar{\gamma}$, obtained by replacing each matrix $\gamma(g), g \in \pi$, by its inverse-transpose, is not conjugate to γ.

In Section 3 we prove that if a representation $\gamma\colon \pi \to \mathrm{GL}_n\mathbb{F}$ is conjugate to its dual, then the torsion $\tau_\gamma(t)$ is reciprocal.

2 Examples

Any reciprocal even-degree integral polynomial $\Delta(t)$ such that $\Delta(1) = \pm 1$ arises as the Alexander polynomial of a knot (see Kawauchi [5], for example). Let $f(t)$ be any monic integral polynomial with constant coefficient -1 and $f(1) = \pm 1$. Choose a knot k with Alexander polynomial $\Delta(t) = f(t)f(t^{-1})$.

Let C be the companion matrix of $(t-1)f(t)$. Then $C \in \mathrm{SL}_n\mathbb{Z}$, where $\deg f = n-1$. Consider the cyclic representation $\gamma\colon \pi \to \mathrm{SL}_n\mathbb{Z}$ sending each generator $x_0, x_1, \dots, x_k$ of a Wirtinger presentation of π to C. We have

$$\tau_\gamma(t) = \frac{\det M^0_{\gamma\otimes\epsilon}}{\det \Phi(x_0 - 1)} = \frac{\det M^0_{\gamma\otimes\epsilon}}{f(t^{-1})(t-1)}. \tag{2-1}$$

The matrix $M^0_{\gamma\otimes\epsilon}$ can be obtained from the $(k \times k)$ Alexander matrix $M(t)$ by replacing each polynomial entry $\sum a_i t^i$ with the $(n \times n)$ block matrix $\sum a_i (tC)^i$. Since the $n \times n$ blocks commute,

$$\det M^0_{\gamma\otimes\epsilon} = \prod_\lambda \det M(t\lambda),$$

where λ ranges over the eigenvalues of C, that is, the roots of $(t-1)f(t)$ (see Kovacs, Silver and Williams [9] for details). Hence

$$\det M^0_{\gamma\otimes\epsilon} = \prod_\lambda \Delta(t\lambda) \;=\; \Delta(t) \prod_{\lambda : f(\lambda)=0} f(t\lambda)f(t^{-1}\lambda^{-1}).$$

Since $\Delta(t)$ and $\det M^0_{\gamma\otimes\epsilon}(t)$ are integral polynomials, so is

$$g(t) = \prod_{\lambda : f(\lambda)=0} f(t\lambda) f(t^{-1}\lambda^{-1}).$$

Lemma 2.1 *If* $\deg f = 2$*, then* $g(t)$ *is reciprocal.*

Proof Our assumptions about $f(t)$ imply that its roots have the form $\lambda, -\lambda^{-1}$, for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then $g(t) = f(t\lambda) f(t^{-1}\lambda^{-1}) f(-t\lambda^{-1}) f(-t^{-1}\lambda)$ while $g(t^{-1}) = f(t^{-1}\lambda) f(t\lambda^{-1}) f(-t^{-1}\lambda^{-1}) f(-t\lambda)$. Observe that $g(t)$ and $g(t^{-1})$ have the same roots:

- $f(t\lambda)$ and $f(-t^{-1}\lambda^{-1})$ have roots: $t = 1, -\lambda^{-2}$;
- $f(t^{-1}\lambda^{-1})$ and $f(-t\lambda)$ have roots: $t = -1, \lambda^{-2}$;
- $f(-t\lambda^{-1})$ and $f(t^{-1}\lambda)$ have roots: $t = 1, -\lambda^{2}$;
- $f(-t^{-1}\lambda)$ and $f(t\lambda^{-1})$ have roots: $t = -1, \lambda^{2}$.

It follows that $g(t^{-1}) = \alpha g(t)$, for some $\alpha \in \mathbb{C} \setminus \{0\}$. Letting $t = 1$, we see that $\alpha = 1$. Hence $g(t^{-1}) = g(t)$. □

Remark 2.2 The numerator $\det M^0_{\gamma\otimes\epsilon}$ of Definition 1.1 is a polynomial invariant $D_\gamma(t)$ of k, well defined up to multiplication by units in $\mathbb{C}[t^{\pm 1}]$ (see Silver and Williams [14]). Since $\Delta(t)$ is reciprocal, Lemma 2.1 implies that $D_\gamma(t)$ is reciprocal whenever $\deg f = 2$. Example 2.5 below shows that this conclusion need not hold when $\deg f > 2$.

Proposition 2.3 *Let* $f(t)$ *be a polynomial as above with degree* 2*. If* $f(t)$ *is non-reciprocal, then* $\tau_\gamma(t)$ *is a non-reciprocal integral polynomial of the form* $(t-1)h(t)$*.*

Proof From equation (2-1),

$$\tau_\gamma(t) = \frac{f(t) f(t^{-1}) g(t)}{f(t^{-1})(t-1)} = \frac{f(t) g(t)}{t-1}. \tag{2-2}$$

Since $g(t)$ and $t-1$ are reciprocal but $f(t)$ is not, $\tau_\gamma(t)$ is non-reciprocal. To see that $\tau_\gamma(t)$ has the desired form, note that $(t-1)^2$ divides $g(t)$ since both factors $f(t\lambda), f(-t\lambda^{-1})$ of $g(t)$ vanish when $t = 1$. □

Example 2.4 Let $f(t) = t^2 - t - 1$. Then

$$C = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Computation shows that $g(t) = (t-1)^2(t+1)^2(t^2-3t+1)(t^2+3t+1)$. By equation (2-2),

$$\tau_\gamma(t) = (t^2 - t - 1)(t-1)(t+1)^2(t^2 - 3t + 1)(t^2 + 3t + 1),$$

which is non-reciprocal.

Example 2.5 Let $f(t) = t^3 - t - 1$. Then

$$C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Computation shows that $g(t) = (t-1)^3(t^3 - t - 1)^2(t^3 - t^2 + 2t - 1)(t^6 + 3t^5 + 5t^4 + 5t^3 + 5t^2 + 3t + 1)$. The polynomial $f(t)f(t^{-1})g(t)$ is the numerator $D_\gamma(t)$ of Wada's invariant (see Definition 1.1). It is non-reciprocal.

It is not difficult to see that for any cyclic representation, $D_\gamma(t) = \Delta_\gamma(t)$ (see [14, Section 3]) Hence this example shows that $\Delta_\gamma(t)$ can also be non-reciprocal.

3 Sufficient condition for reciprocality

If $\gamma\colon G \to \mathrm{GL}_n\mathbb{F}$ is a linear representation, then the *dual* (or *contragredient*) representation $\overline{\gamma}$ is defined by

$$\overline{\gamma}(g) = {}^t\gamma(g)^{-1},$$

where ^{t} denotes transpose.

The following elementary lemma is included for the reader's convenience.

Lemma 3.1 *A representation $\gamma\colon G \to \mathrm{GL}_n\mathbb{F}$ is conjugate to its dual if and only if there exists a nondegenerate bilinear form $(v, w) \mapsto \{v, w\} \in \mathbb{F}$ on V such that $\{v \cdot g, w \cdot g\} = \{v, w\}$ for all $v, w \in V$ and $g \in G$.*

Proof Assume that $\overline{\gamma}$ is conjugate to γ. Then there exists a matrix $A \in \mathrm{GL}_n\mathbb{F}$ such that $A^{-1}\gamma(g)A = {}^t\gamma(g)^{-1}$, for all $g \in G$. Define $\{v, w\} = vA\,{}^t w$. Since A is invertible, the bilinear form is nondegenerate. It is easy to check that $\{v \cdot g, w \cdot g\} = \{v, w\}$ for all $v, w \in V$.

Conversely, assume that γ preserves a nondegenerate bilinear form $(v, w) \mapsto \{v, w\}$. There exists an invertible matrix $A \in \mathrm{GL}_n\mathbb{F}$ such that $\{v, w\} = vA\ {}^t w$. Since γ preserves the form, we have $v\gamma(g)A\ {}^t\gamma(g)\ {}^t w = \{v \cdot g, w \cdot g\} = \{v, w\} = vA\ {}^t w$, for all $v, w \in V, g \in G$. It follows that $\gamma(g)A\ {}^t\gamma(g) = A$ for all $g \in G$. Hence $A^{-1}\gamma(g)A = {}^t\gamma(g)^{-1}$, and so $\overline{\gamma}$ is conjugate to γ. □

As before, let k be a knot with group π. Assume that $\gamma\colon \pi \to \mathrm{GL}_n\mathbb{F}$ is a representation, where $\mathbb{F}$ is an arbitrary field. As above, $V = \mathbb{F}^n$ is a right $\mathbb{Z}[\pi]$–module via $v \cdot g = v\gamma(g)$, for all $v \in V$ and $\gamma \in \pi$. Let $W = \mathbb{F}^n$ with the dual $\mathbb{Z}[\pi]$–module structure given by $w \cdot g = w\ {}^t\gamma(t)^{-1}$.

Theorem 3.2 *Assume that* $\det M^0_{\gamma\otimes\epsilon} \neq 0$. *If* γ *is conjugate to its dual representation* $\overline{\gamma}$, *then both* $\tau_\gamma(t)$ *and* $\Delta_\gamma(t)$ *are reciprocal.*

Proof The following argument is similar to those of Kitano [7] and of Kirk and Livingston [6].

Recall that X is the exterior of k, endowed with a CW cell structure. Let X' be the same space but with the dual cell structure. Let $\overline{}\colon \mathbb{F}(t) \to \mathbb{F}(t)$ be the involution induced by $t \mapsto t^{-1}$.

Assume that $\gamma\colon \pi \to \mathrm{GL}_n\mathbb{F}$ is a representation that is conjugate to its dual. By Lemma 3.1 there exists a nondegenerate bilinear form $(v, w) \mapsto \{v, w\}$ such that $\{v \cdot g, w \cdot g\} = \{v, w\}$ for all $v, w \in V, g \in \pi$. Consider the twisted chain complexes

$$\mathcal{C}_* = (\mathbb{F}(t) \otimes_{\mathbb{F}} V) \otimes_\gamma C_*(\widetilde{X}),\ \mathcal{D}_* = (\mathbb{F}(t) \otimes_{\mathbb{F}} W) \otimes_{\overline{\gamma}} C_*(\widetilde{X}', \partial\widetilde{X}'),$$

where $\widetilde{X}$ and $\widetilde{X}'$ denote universal covering spaces of X and X', respectively. We abbreviate these by $V_{\gamma\otimes\epsilon} \otimes C_*(\widetilde{X})$ and $V_{\overline{\gamma}\otimes\epsilon} \otimes C_*(\widetilde{X})$, respectively.

Define a bilinear pairing $\mathcal{C}_q \times \mathcal{D}_{3-q} \to \mathbb{F}(t)$ by

$$\langle p \otimes v \otimes z_1, q \otimes w \otimes z_2\rangle = \sum_{g\in\pi}(z_1 \cdot gz_2)p\overline{q}\{v \cdot g, w\}, \tag{3-1}$$

where $z_1 \cdot gz_2$ is the algebraic intersection number of cells z_1 and gz_2. We extend linearly.

The pairing induces a $\mathbb{F}(t)$–module isomorphism $\mathcal{D}_{3-q} \to \overline{\mathrm{Hom}}(\mathcal{C}_q, \mathbb{F}(t))$, where $\overline{\mathrm{Hom}}$ denotes the dual space with $(q \cdot h)(z) = \overline{q}(h(z))$, for all $q \in \mathbb{F}(t), z \in \mathcal{C}_q$. Consequently, there exists a nondegenerate pairing $H_q(X; V(t)) \times H_{3-q}(S', \partial X'; W(t)) \to \mathbb{F}(t)$. Since the torsion of $\mathcal{C}_*$ is defined, by our hypothesis, the torsion of $\mathcal{D}_*$ is too.

Choose a basis $\{v_i\}$ over $\mathbb{F}$ for V and lifts to $\widetilde{X}$ of simplices of X to get a preferred $\mathbb{F}(t)$–basis for $\mathcal{C}_*$. Basis elements have the form $1 \otimes v_i \otimes z_j$. Then $\mathcal{D}_*$ has a natural dual basis over $\mathbb{F}(t)$ obtained by picking a basis for W that is dual to the basis for V with respect to $\{,\}$, and dual cells in $\widetilde{X}'$ of the fixed lifts of simplices of X. As observed by Kirk and Livingston [6], the bases for $\mathcal{C}_*$ and $\mathcal{D}_*$ that we build are dual with respect to bilinear form (3-1).

Let $\tau(X; V_{\gamma\otimes\epsilon})$ denote the torsion of $\mathcal{C}_*$. Similarly, let $\tau(X', \partial X'; V_{\overline{\gamma}\otimes\epsilon})$ denote the torsion of $\mathcal{D}_*$. Then $\tau(X; V_{\gamma\otimes\epsilon}) = \tau(X', \partial X'; V_{\overline{\gamma}\otimes\overline{\epsilon}})$ by Milnor [12, Theorem 1']. Futhermore,

$$\begin{aligned}
\tau(X', \partial X'; V_{\overline{\gamma}\otimes\overline{\epsilon}}) &= \tau(X, \partial X; V_{\overline{\gamma}\otimes\overline{\epsilon}}) && \text{(by subdivision)}\\
&= \tau(X, \partial X; V_{\gamma\otimes\overline{\epsilon}}) && \text{(since } \gamma \text{ is conjugate to } \overline{\gamma})\\
&= \overline{\tau}(X, \partial X; V_{\gamma\otimes\epsilon})\\
&= \overline{\tau}(X; V_{\gamma\otimes\epsilon}),
\end{aligned}$$

using Milnor [11, Lemma 2] and $\tau(\partial X; V_{\gamma\otimes\epsilon}) = 1$ (see Kirk and Livingston [6]). Hence

$$\tau_\gamma(t) = \tau(X; V_{\gamma\otimes\epsilon}) = \overline{\tau}(X; V_{\gamma\otimes\epsilon}) = \overline{\tau}_\gamma(t).$$

In order to show that $\Delta_\gamma(t)$ is also reciprocal, we need the fact from [6] that $\Delta_\gamma(t)$ is equal to $\tau_\gamma(t)$ times the 0th elementary divisor of $H_0(X; V[t^{\pm 1}])$, computed using the chain complex (1-2). We observe that $H_0(X; V[t^{\pm 1}])$ is the cokernel of the boundary homomorphism ∂_1. For any $g \in \pi$, the set of eigenvalues of $\gamma(g)$ is closed under inversion, since γ is conjugate to its dual. It follows that the 0th elementary of $H_0(X; V[t^{\pm 1}])$ is reciprocal. Hence so is $\Delta_\gamma(t)$. □

Remark 3.3 If $\mathbb{F} = \mathbb{R}$, and the bilinear form in Lemma 3.1 is positive-definite, then by considering a basis for V that is orthonormal with respect to the form, we see that A is the identity matrix. In this case, $\gamma(g) = {}^t\gamma(g)^{-1}$ for all $g \in G$, and hence γ is conjugate to an orthogonal representation. Similarly, if $\mathbb{F} = \mathbb{C}$ and the bilinear form is hermitian and positive-definite, γ is conjugate to a unitary representation.

Corollary 3.4 *If $\gamma\colon \pi \to \mathrm{Sp}_{2n}\mathbb{F}$ is a symplectic representation, then $\tau_\gamma(t)$ is reciprocal.*

Proof The representation preserves the bilinear form given by $A = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$. □

Since $\mathrm{Sp}_2\mathbb{F} = \mathrm{SL}_2\mathbb{F}$, the following is immediate.

Corollary 3.5 *If γ is any representation of π in $\mathrm{SL}_2\mathbb{F}$, then $\tau_\gamma(t)$ is reciprocal.*

Corollary 3.5 shows that Example 2.4 is, in a sense, the simplest possible.

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JAH: *School of Mathematics and Statistics F07, University of Sydney*
Sydney, NSW 2006, Australia

DSS, SGW: *Department of Mathematics and Statistics, University of South Alabama*
Mobile AL 36688, USA

jonathan.hillman@sydney.edu.au, silver@jaguar1.usouthal.edu,
swilliam@jaguar1.usouthal.edu

http://www.maths.usyd.edu.au/u/jonh/,
http://www.southalabama.edu/mathstat/personal_pages/silver/,
http://www.southalabama.edu/mathstat/personal_pages/williams/

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