The Whitehead group and the lower algebraic K–theory of braid groups on $\mathbb{S}^2$ and $\mathbb{R}P^2$

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Let $M = \mathbb{S}^2$ or $\mathbb{R}P^2$. Let $PB_n(M)$ and $B_n(M)$ be the pure and the full braid groups of M respectively. If Γ is any of these groups, we show that Γ satisfies the Farrell–Jones Fibered Isomorphism Conjecture and use this fact to compute the lower algebraic K–theory of the integral group ring $\mathbb{Z}\Gamma$, for $\Gamma = PB_n(M)$. The main results are that for $\Gamma = PB_n(\mathbb{S}^2)$, the Whitehead group of Γ, $\widetilde{K}_0(\mathbb{Z}\Gamma)$ and $K_i(\mathbb{Z}\Gamma)$ vanish for $i \leq -1$ and $n > 0$. For $\Gamma = PB_n(\mathbb{R}P^2)$, the Whitehead group of Γ vanishes for all $n > 0$, $\widetilde{K}_0(\mathbb{Z}\Gamma)$ vanishes for all $n > 0$ except for the cases $n = 2, 3$ and $K_i(\mathbb{Z}\Gamma)$ vanishes for all $i \leq -1$.

1 Introduction

Aravinda, Farrell, and Roushon in [2] showed that if Γ is the pure braid group on any compact (connected) surface, except for the 2–sphere $\mathbb{S}^2$ and the real projective plane $\mathbb{R}P^2$, then the Whitehead group $\mathrm{Wh}(\Gamma)$ of Γ vanishes. Later on, in [14], Farrell and Roushon extended this result to the full braid groups. They showed that if Γ is the full braid group on any compact (connected) surface, except for the 2–sphere and the 2–projective plane, then $\mathrm{Wh}(\Gamma)$ also vanishes. A natural question is what happens to the Whitehead group of the pure and full braid groups in the two remaining cases?

The main tool to answer this question is the Fibered Isomorphism Conjecture of Farrell and Jones [13]. This conjecture has been verified for several groups, for instance, for discrete cocompact subgroups of virtually connected Lie groups by Farrell and Jones [13], for finitely generated Fuchsian groups by Berkove, Juan-Pineda and Pearson [5] and for some mapping class groups by Berkove, Juan-Pineda and Lu in [4].

Let $M = \mathbb{S}^2$ or $\mathbb{R}P^2$. Let $PB_n(M)$ and $B_n(M)$ be the pure and the full braid groups on M respectively. In this paper we recall from [18; 19] that $PB_n(M)$ and $B_n(M)$ satisfy the Farrell–Jones isomorphism conjecture and use this fact to compute the Whitehead group of $PB_n(M)$ and the lower algebraic K–groups for the integral group

ring $\mathbb{Z} PB_n(M)$. This conjecture states that given a group Γ, the algebraic K–groups for $\mathbb{Z}\Gamma$ should be computed from the algebraic K–groups of the integral group rings $\mathbb{Z} H$, where H runs over the family of virtually cyclic subgroups of Γ, and from homological information. We prove the following results in this paper:

Theorem *Let $\Gamma = PB_n(\mathbb{S}^2)$ be the pure braid group on n strands ($n > 0$) on the 2–sphere. Then* $\mathrm{Wh}(\Gamma)$, $\widetilde{K}_0(\mathbb{Z}\Gamma)$ *and* $K_i(\mathbb{Z}\Gamma)$ *vanish for all* $i \leq -1$.

Theorem *Let $\Gamma = PB_n(\mathbb{R}P^2)$ be the pure braid group on n strands ($n > 0$) on the projective plane. Then we have that* $\mathrm{Wh}(\Gamma) = 0$ *and* $K_i(\mathbb{Z}\Gamma) = 0$ *for* $i < 0$. *Moreover, when $n \neq 2, 3$ we have* $\widetilde{K}_0(\mathbb{Z}\Gamma) = 0$ *and in the remaining two cases $n = 2, 3$, we have that* $\widetilde{K}_0(\mathbb{Z}\Gamma) = \mathbb{Z}_2$.

This paper is divided into four sections. In Section 2 we state the Fibered Farrell–Jones Isomorphism Conjecture. In Section 3, we recall that $PB_n(M)$ and $B_n(M)$ satisfy this conjecture. Lastly, in Section 4 we state and prove the above results.

2 Background

2.1 The Fibered Isomorphism Conjecture

The computations of the Whitehead group of a discrete group Γ as well as its lower algebraic K–theory, is in general a difficult task. The Fibered Isomorphism Conjecture provides a tool which helps with these computations. We state a short version of this conjecture.

Let $\mathcal{S}: TOP \to \Omega-SPECTRA$ be a covariant homotopy functor. Let $\mathbf{F}$ be the category of continuous surjective maps: objects in $\mathbf{F}$ are continuous surjective maps $p: E \to B$, where E, B are objects in TOP, and morphisms between $p_1: E_1 \to B_1$ and $p_2: E_2 \to B_2$ consist of continuous maps $f: E_1 \to E_2$ and $g: B_1 \to B_2$ making

the following diagram commute:

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\ f\ } & E_2 \\
{\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow \\
B_1 & \xrightarrow{\ g\ } & B_2
\end{array}
$$

In this setup, Quinn [24] constructs a functor from **F** to Ω–*SPECTRA*. The value of this Ω–spectrum at $p\colon E \to B$ is denoted by

$$
\mathbb{H}(B; \mathcal{S}(p)),
$$

and its value at the object $E \to *$ is $\mathcal{S}(E)$. The map of spectra $\mathbb{A}$ associated to

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\ f\ } & E_2 \\
{\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow \\
B_1 & \xrightarrow{\ g\ } & B_2
\end{array}
$$

is known as the *Quinn* assembly map.

Definition 2.1 A group is called virtually cyclic if it contains a cyclic subgroup of finite index.

In particular all finite groups are virtually cyclic and there is a description of infinite virtually cyclic groups thanks to the work of $C\,T\,C$ Wall [27]. This description is provided in the following theorem.

Theorem 2.2 [27] *An infinite virtually cyclic group* Γ *fits into one of the two following short exact sequences, with* F *a finite group*

$$
1 \longrightarrow F \longrightarrow \Gamma \longrightarrow \mathbb{Z} \longrightarrow 1
$$

or
$$
1 \longrightarrow F \longrightarrow \Gamma \longrightarrow D_\infty \longrightarrow 1.
$$

In the first case Γ *is the semidirect product* $F \rtimes \mathbb{Z}$. *In the latter case,* Γ *is the amalgamated product* $H_1 *_F H_2$, *where* H_1 *and* H_2 *are finite and* F *is an index two subgroup in both* H_1 *and* H_2.

We now state the *Fibered Isomorphism Conjecture*.

Given a discrete group Γ, let $E_{\mathcal{VC}}\Gamma$ be a universal Γ–space for the family of virtually cyclic subgroups of Γ [13, Appendix], denote by $\mathcal{B}_{\mathcal{VC}}\Gamma$ the orbit space $E_{\mathcal{VC}}\Gamma/\Gamma$,

and let X be any free and properly discontinuous Γ–space. If (f, g) is the following morphism in **F**

$$
\begin{array}{ccc}
E_{\mathcal{VC}}\Gamma \times_{\Gamma} X & \xrightarrow{\ f\ } & X/\Gamma \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\
\mathcal{B}_{\mathcal{VC}}\Gamma & \xrightarrow{\ g\ } & *
\end{array}
$$

then the *Fibered Isomorphism Conjecture* for the functor $\mathcal{S}$, the space X, and the group Γ is the assertion that

$$\mathbb{A}\colon \mathbb{H}(\mathcal{B}_{\mathcal{VC}}\Gamma; \mathcal{S}(p_1)) \longrightarrow \mathcal{S}(X/\Gamma)$$

is a homotopy equivalence and hence the induced map

$$\mathbb{A}_*\colon \pi_n(\mathbb{H}(\mathcal{B}_{\mathcal{VC}}\Gamma; \mathcal{S}(p_1)) \longrightarrow \pi_n(\mathcal{S}(X/\Gamma))$$

is an isomorphism for all $n \in \mathbb{Z}$.

This conjecture was stated in [13] for the functors $\mathcal{S} = \mathcal{P}_*(\ ), \mathcal{K}(\ )$, and $\mathcal{L}^{-\infty}$, the pseudoisotopy, algebraic K–theory and $\mathcal{L}^{-\infty}$–theory functors. Throughout this work we will use the functor $\mathcal{S} = \mathcal{P}_*(\ )$.

The relation between $\mathcal{P}_*(\ )$ and the lower algebraic K–theory is described in the work of Anderson and Hsiang in [1], who proved the following:

$$
\pi_j(\mathcal{P}_*(\mathbf{X})) = \begin{cases} \mathrm{Wh}(\mathbb{Z}\pi_1(\mathbf{X})) & j = -1, \\ \widetilde{K}_0(\mathbb{Z}\pi_1(\mathbf{X})) & j = -2, \\ K_{j+2}(\mathbb{Z}\pi_1(\mathbf{X})) & j \le -3. \end{cases}
$$

For the rest of this work we will use FIC to denote the Fibered Isomorphism Conjecture in the manner described above.

2.2 Braid groups and strongly poly-free groups

Let $Q_m = \{q_1, \ldots, q_m\}$ a fixed set of m distinct points of a compact (connected) surface X. Define

$$F(X - Q_m, n) = \{(x_1, \ldots, x_n) \mid x_i \in X - Q_m, \ x_i \ne x_j, \text{ if } i \ne j\}.$$

Observe that when $m = 0$, the above definition reduces to

$$F(X, n) = \{(x_1, \ldots, x_n) \mid x_i \in X, \ x_i \ne x_j, \text{ if } i \ne j\}.$$

Also, we observe that S_n, the symmetric group on n symbols, acts naturally on $F(X, n)$ by permuting the coordinates.

Definition 2.3 [2] The pure braid group, $PB_n(X)$, of n strands on X is the fundamental group of $F(X - \partial X, n)$.

Definition 2.4 The full braid group, $B_n(X)$, of n strands on X is the fundamental group of $F(X - \partial X, n)/S_n$.

Strongly poly-free groups were introduced by Aravinda, Farrell and Roushon [2] to compute the Whitehead group of braid groups on the plane and on surfaces. It was proven that the Whitehead group of braid groups on the plane and on compact (connected) surfaces, except for the 2–sphere and the 2–projective plane, vanish. Strongly poly-free groups were fundamental for the proof of these results. Inspired by this work, we revisit the definition of strongly poly-free group.

Definition 2.5 A discrete group Γ is called *strongly poly-free* if there exists a finite filtration by subgroups $1 = \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_n = \Gamma$ such that the following conditions are satisfied:

  (1) Γ_i is normal in Γ for each i.

  (2) Γ_{i+1}/Γ_i is a finitely generated free group for all i.

  (3) For each $\gamma \in \Gamma$ there is a compact surface F and a diffeomorphism $f\colon F \to F$ such that the induced homomorphism $f_\#$ on $\pi_1(F)$ is equal to c_γ in $\mathrm{Out}(\pi_1(F))$, where c_γ is the action of γ on Γ_{i+1}/Γ_i by conjugation and $\pi_1(F)$ is identified with Γ_{i+1}/Γ_i via a suitable isomorphism.

The third condition says that the algebraic action of γ on Γ_{i+1}/Γ_i can be *geometrically realized*, ie $f_\# = \phi^{-1} \circ C_\gamma \circ \phi$ from the following diagram:

$$\begin{array}{ccc}
\pi_1(F, f_0) & \xrightarrow{\ f_\#\ } & \pi_1(F, f_0) \\
{\scriptstyle \phi}\big\downarrow & & \big\uparrow{\scriptstyle \phi^{-1}} \\
\Gamma_{i+1}/\Gamma_i & \xrightarrow{\ C_\gamma\ } & \Gamma_{i+1}/\Gamma_i
\end{array}$$

Examples of strongly poly-free groups are finitely generated free groups and braid groups on the plane; see Aravinda, Farrell and Roushon [2] and Farrell and Roushon [14]. For the rest of this work we will abbreviate strongly poly-free by spf.

3 The braid groups on $\mathbb{S}^2$ and $\mathbb{R}P^2$ satisfy the FIC

In [18; 19], it was proved that the braid groups on $\mathbb{S}^2$ and on $\mathbb{R}P^2$ satisfy the FIC. In this section we provide a different proof for the case when $M = \mathbb{S}^2$. The proof for the $\mathbb{R}P^2$ case can be reviewed in [19].

Theorem 3.1 *The pure braid group* $PB_n(\mathbb{S}^2)$ *of* n *strands on the sphere* $\mathbb{S}^2$ *satisfies the FIC for all* $n > 0$.

Proof For $n = 1, 2, 3$, $PB_n(\mathbb{S}^2)$ is known to be finite (see [18; 6]), and so satisfies the FIC. For $n > 3$, we use the following locally trivial fiber bundle [11]:

$$F(\mathbb{S}^2, n) \xrightarrow{P} F(\mathbb{S}^2, 3)$$
$$(x_1, x_2, \dots, x_n) \mapsto (x_1, x_2, x_3)$$

with fiber $F(\mathbb{S}^2 - Q_3, n - 3)$. Applying the homotopy long exact sequence to this fibration and using that $\pi_2(F(\mathbb{S}^2, 3)) = 0$ (see [12]) and $\pi_0(F(\mathbb{S}^2 - Q_3, n - 3)) = 0$, we obtain the short exact sequence

$$(1) \qquad 1 \longrightarrow \pi_1(F(\mathbb{S}^2 - Q_3, n - 3)) \longrightarrow \pi_1(F(\mathbb{S}^2, n)) \longrightarrow \pi_1(F(\mathbb{S}^2, 3)) \longrightarrow 1.$$

Claim 3.2 $\Gamma = \pi_1(F(\mathbb{S}^2 - Q_3, n - 3))$ *is a strongly poly-free group.*

Before verifying the claim, observe that $\mathbb{S}^2 - Q_3$ is homeomorphic to $\mathbb{R}^2 - Q_2$. Thus, $\pi_1(F(\mathbb{S}^2 - Q_3, n - 3)) \cong \pi_1(F(\mathbb{R}^2 - Q_2, n - 3))$ for all $n > 3$. For simplicity let $i = n - 3$ to reduce the problem to showing that $\pi_1(F(\mathbb{R}^2 - Q_2, i))$ is a strongly poly-free group for all $i > 0$.

Proof of Claim 3.2 Let $\Gamma_k = \pi_1(F(\mathbb{R}^2 - Q_{i+2-k}, k))$ with $0 < k < i + 2$. Note that Γ_k describes a filtration (in the sense of an spf group; see [2]), for the classical $(i + 2)-$pure braid group, $PB_{i+2}(\mathbb{R}^2) = \pi_1(F(\mathbb{R}^2, i + 2))$ and that $\Gamma_i = \pi_1(F(\mathbb{R}^2 - Q_2, i))$ is a subgroup in this filtration. In [2] it was proven that this group is strongly poly-free and it is not difficult to observe that every subgroup in the filtration of a strongly poly-free group is also strongly poly-free. Hence $\Gamma_i = \pi_1(F(\mathbb{R}^2 - Q_2, i))$ is spf. $\square$

Since $\pi_1(F(\mathbb{S}^2, 3)) \cong \mathbb{Z}_2$ [8], Equation (1) reduces to

$$(2) \qquad\qquad\qquad 1 \longrightarrow \Gamma \longrightarrow PB_n(\mathbb{S}^2) \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

where Γ is spf. To conclude, we now apply the main theorem from [14] and so the pure braid group of $\mathbb{S}^2$ also satisfies the FIC for all $n > 3$. $\square$

Let X be a topological space. Since S_n acts freely and properly discontinuous on $F(X, n)$, the projection map

$$F(X, n) \xrightarrow{p} F(X, n) / S_n$$

is a regular covering map with fiber S_n. Hence

$$p_\#(\pi_1(F(X,n))) \trianglelefteq \pi_1(F(X,n)/S_n)$$

and we have the following short exact sequence

$$1 \longrightarrow \pi_1(F(X,n)) \longrightarrow \pi_1(F(X,n)/S_n) \longrightarrow \pi_1(F(X,n)/S_n)/\pi_1(F(X,n)) \longrightarrow 1,$$

which in the language of braid groups translates into

(3) $$1 \longrightarrow PB_n(X) \longrightarrow B_n(X) \longrightarrow S_n \longrightarrow 1.$$

Theorem 3.3 *The full braid group* $B_n(\mathbb{S}^2)$ *of* n *strands on* $\mathbb{S}^2$ *satisfies the FIC for all* $n > 0$.

Proof Let $X = \mathbb{S}^2$ in the above remark to obtain the following short exact sequence

(4) $$1 \longrightarrow PB_n(\mathbb{S}^2) \longrightarrow B_n(\mathbb{S}^2) \longrightarrow S_n \longrightarrow 1$$

and recall the following facts:

- The center of $B_n(\mathbb{S}^2)$ is isomorphic to $\mathbb{Z}_2$.
- $B_n(\mathbb{S}^2)$ contains a unique element of order 2. See Fadell and van Buskirk [12].
- $PB_n(\mathbb{S}^2) \cong \mathbb{Z}_2 \times \pi_1(F(\mathbb{R}^2 - Q_2, n-3))$. See Cohen and Pakianathan [8].

From the above $\pi_1(F(\mathbb{R}^2 - Q_2, n-3)) = \Gamma$ is an spf group. Since $\mathbb{Z}_2$ is the center of $B_n(\mathbb{S}^2)$ we have that $\mathbb{Z}_2 \trianglelefteq PB_n(\mathbb{S}^2)$ and $\mathbb{Z}_2 \trianglelefteq B_n(\mathbb{S}^2)$. Using this normality we obtain that

$$PB_n(\mathbb{S}^2)/\mathbb{Z}_2 \trianglelefteq B_n(\mathbb{S}^2)/\mathbb{Z}_2,$$

$$(B_n(\mathbb{S}^2)/\mathbb{Z}_2)/(PB_n(\mathbb{S}^2)/\mathbb{Z}_2) \cong B_n(\mathbb{S}^2)/PB_n(\mathbb{S}^2) \cong S_n.$$

This gives the short exact sequence

(5) $$1 \longrightarrow PB_n(\mathbb{S}^2)/\mathbb{Z}_2 \longrightarrow B_n(\mathbb{S}^2)/\mathbb{Z}_2 \longrightarrow S_n \longrightarrow 1.$$

Using the fact that $PB_n(\mathbb{S}^2)/\mathbb{Z}_2 \cong \Gamma$, (5) can be written as

(6) $$1 \longrightarrow \Gamma \longrightarrow B_n(\mathbb{S}^2)/\mathbb{Z}_2 \longrightarrow S_n \longrightarrow 1$$

and the main theorem of [14] we have that $B_n(\mathbb{S}^2)/\mathbb{Z}_2$ satisfies FIC.

Now, we look at the short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow B_n(\mathbb{S}^2) \longrightarrow B_n(\mathbb{S}^2)/\mathbb{Z}_2 \longrightarrow 1$$

and apply [4, lemma 2.9] to show that $B_n(\mathbb{S}^2)$ satisfies the FIC. $\qquad\square$

4 The Whitehead group and the lower algebraic K–theory

4.1 The Whitehead group of $PB_n(\mathbb{S}^2)$

Let $\Gamma = PB_n(\mathbb{S}^2)$ and recall the structure of $PB_n(\mathbb{S}^2)$; see the notes of F Cohen and J Pakianathan [8]:

$$PB_n(\mathbb{S}^2) = \begin{cases} 1 & n = 1, 2, \\ \mathbb{Z}_2 & n = 3, \\ \mathbb{Z}_2 \times PB_{n-3}(\mathbb{R}^2 - Q_2) & n > 3. \end{cases}$$

Our first step will be to describe the virtually cyclic subgroups of Γ. To do this, we must describe all the finite subgroups and use Theorem 2.2. With this in mind and the structure of $PB_n(\mathbb{S}^2)$ we have the following:

Proposition 4.1 Let $\Gamma = PB_n(\mathbb{S}^2)$ be the pure braid group of n strands on the 2–sphere. Then, the virtually cyclic subgroups of Γ are isomorphic to one of the following groups: $\{1\}, \mathbb{Z}_2, \mathbb{Z}$, or $\mathbb{Z}_2 \times \mathbb{Z}$.

Proof For $n > 3$, the result is immediate from the fact that $PB_{n-3}(\mathbb{R}^2 - Q_2)$ is torsion-free, along with the above description.   □

Thus, for $PB_n(\mathbb{S}^2)$ we have the following:

(1) A finite subgroup of Γ is either trivial or isomorphic to $\mathbb{Z}_2$.

(2) A virtually cyclic subgroups of Γ is isomorphic to one of the following groups:

$$\{1\}, \quad \mathbb{Z}_2, \quad \mathbb{Z}, \quad \mathbb{Z}_2 \times \mathbb{Z}.$$

In particular, the infinite virtually cyclic subgroups of $PB_n(\mathbb{S}^2)$ are all isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2 \times \mathbb{Z}$.

Theorem 4.2 Let $\Gamma = PB_n(\mathbb{S}^2)$ be the pure braid group of n strands on the 2–sphere. Then $\mathrm{Wh}(\Gamma)$, $\widetilde{K}_0(\mathbb{Z}\Gamma)$ and $K_i(\mathbb{Z}\Gamma)$ vanish for all $i \leq -1$ and $n > 0$.

Proof The fact that Γ satisfies the FIC reduces the problem to compute the homotopy groups $\pi_k(\mathbb{H}(\mathcal{B}_{\mathcal{VC}}\Gamma; \mathcal{P}_*(F_x)))$ for the family of virtually cyclic subgroups of Γ. We would like to reduce our computations to the family of finite subgroups. For this, by the transitivity principle [13, A.10], we need to verify that for any infinite virtually cyclic subgroup, H, of Γ the relative assembly map

$$(\mathbb{A}_*)_{\mathfrak{F}_H, \mathcal{ALL}} \colon \pi_q(\mathbb{H}(\mathcal{B}_{\mathfrak{F}_H} H; \mathcal{P}_*(p))) \longrightarrow \pi_q(\mathcal{P}_*(E\Gamma/H)))$$

is an isomorphism for all $q \leq k$. It suffices to show that the corresponding Nil-groups for the infinite virtually cyclic subgroups $\mathbb{Z}, \mathbb{Z}_2 \times \mathbb{Z}$ vanish. This follows for $\mathbb{Z}$ as it is a regular ring and by [3, page 677] we have $NK_i(\mathbb{Z}[C_2]) = 0$ for all $i \leq 1$. Hence our relative assembly map is an isomorphism for all $q \leq -1$ and the relative assembly map

$$(\mathbb{A}_*)_{\mathcal{FIN},\mathcal{VC}} \colon \pi_k(\mathbb{H}(\mathcal{B}_{\mathcal{FIN}}\Gamma; \mathcal{P}_*(F_x))) \longrightarrow \pi_k(\mathbb{H}(\mathcal{B}_{\mathcal{VC}}\Gamma; \mathcal{P}_*(F_x)))$$

is also an isomorphism for all $k \leq -1$. Thus, the composition map

$$(\mathbb{A}_*)_{\mathcal{FIN},\mathcal{ALL}} = (\mathbb{A}_*)_{\mathcal{VC},\mathcal{ALL}} \circ (\mathbb{A}_*)_{\mathcal{FIN},\mathcal{VC}}$$

is also an isomorphism which reduces the problem to compute the homotopy groups $\pi_k(\mathbb{H}(\mathcal{B}_{\mathcal{FIN}}\Gamma; \mathcal{P}_*(F_x)))$ for $k \leq -1$.

By Quinn [24] (see also Davis and Lück [10] and Farrell and Jones [13]) there is an Atiyah–Hirzebruch–Quinn spectral sequence with E^2–term

$$E^2_{p,q} = H_p(\mathcal{B}_{\mathcal{FIN}}\Gamma; \{\pi_q(\mathcal{P}_*(F_x))\})$$

for the family $\mathcal{FIN}$, that converges to $\pi_{p+q}(\mathbb{H}(\mathcal{B}_{\mathcal{FIN}}\Gamma; \mathcal{P}_*(F_x)))$. The local coefficients $\{\pi_q(\mathcal{P}_*(F_x))\}$, are described as follows:

$$\pi_q(\mathcal{P}_*(F_x)) = \begin{cases} \mathrm{Wh}(\pi_1(F_x)) & q = -1, \\ \tilde{K}_0(\mathbb{Z}\pi_1(F_x)) & q = -2, \\ K_{q+2}(\mathbb{Z}\pi_1(F_x)) & q \leq -3, \end{cases}$$

with $\pi_1(F_x) \in \mathcal{FIN}$.

But if $\pi_1(F_x)$ is either $\{1\}$ or $\mathbb{Z}_2$, it is well known that $\pi_q(\mathcal{P}_*(F_x)) = 0$ for all $q \leq -1$. Hence, the spectral sequence $E^2_{p,q}$ collapses and the homotopy groups

$$\pi_k(\mathbb{H}(\mathcal{B}_{\mathcal{FIN}}\Gamma; \mathcal{P}_*(F_x)))$$

vanish for all $k \leq -1$. $\qquad\qquad\qquad\square$

4.2 The Whitehead group of $PB_n(\mathbb{R}P^2)$

Recall from Gonçalves and Guaschi [15] the description for the group $PB_n(\mathbb{R}P^2)$

$$PB_n(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}_2 & n = 1, \\ \mathbf{Q}_8 & n = 2, \\ F_2 \rtimes \mathbf{Q}_8 & n = 3, \end{cases}$$

where $\mathbf{Q}_8$ is the quaternion group of order 8 and F_2 the free group on two generators x, y. From this work we have that $PB_3(\mathbb{R}P^2)$ has *generators* $\{x, y, a, b\}$, where x

and y generate a free group and a, b generate the $\mathbf{Q}_8$ subgroup (and can be identified with the unit quaternionic numbers i, j), and *relations*:

$$a^2 = b^2 \qquad ax = ya$$
$$a^4 = b^4 = 1 \qquad ay = xa$$
$$a^{-1}ba = b^{-1} \qquad bx = y^{-1}b$$
$$by = x^{-1}b$$

For $n > 3$, see Gonçalves and Guaschi [16] for a presentation of the group. Next, we recall the classification of the virtually cyclic subgroups of $PB_n(\mathbb{R}P^2)$ described by Gonçalves and Guaschi in [17].

Proposition 4.3 [17] *Let $n \geq 3$ and $\Gamma_n = PB_n(\mathbb{R}P^2)$, the pure braid group of n strands on the projective plane. Then the virtually cyclic subgroups of Γ_n are isomorphic to one of the following groups:*

(1) $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbf{Q}_8$, $\mathbb{Z}$, $\mathbb{Z}_2 \times \mathbb{Z}$, $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$ *if $n = 3$.*

(2) $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}$, $\mathbb{Z}_2 \times \mathbb{Z}$, $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$ *if $n > 3$.*

A universal $\mathcal{FIN}$ space for $F_2 \rtimes \mathbf{Q}_8$ One of the main ingredients for our computations is to find a suitable universal space for the family of virtually cyclic subgroups of our groups. It turns out that in our case, it suffices to find such a space for the family of finite subgroups. Let F_2 be the free group on two generators x and y and Ca be the Cayley graph of F_2. Recall that Ca is the universal cover of the figure eight; see Figure 1. Where p is the natural projection map from Ca to Ca/Γ. The vertices of the graph are the elements of the free group and the natural action of F_2 on the graph is by left multiplication on the vertex set. Recall that we have an edge in Ca from g to gu where $u \in \{x^{\pm 1}, y^{\pm 1}\}$ and $g \in F_2$, hence left multiplication by elements of F_2 naturally extends to the edges to give an action of F_2 on Ca.

Define the space $\widehat{\text{Ca}}$ to be the Cayley graph equipped with the barycentric subdivision. Note that $\widehat{\text{Ca}}$ is a one dimensional simplicial complex and F_2 also acts on it. Using this action and the action of $\mathbf{Q}_8$ on F_2 described above we have:

Proposition 4.4 *Let $\Gamma = F_2 \rtimes \mathbf{Q}_8$ and let $\widehat{\text{Ca}}$ be the space described above. Then $\widehat{\text{Ca}}$ is a universal Γ–space for the family of finite subgroups of Γ.*

Proof Define the action of Γ on $\widehat{\text{Ca}}$ by the group homomorphism $\psi \colon F_2 \rtimes \mathbf{Q}_8 \to \text{homeo}(\widehat{\text{Ca}})$, as follows: elements of F_2 act by left multiplication and the generator a acts by $\psi(a)$ which sends a word w to $\hat{w}$ where $\hat{w}$ is obtained from w by replacing x
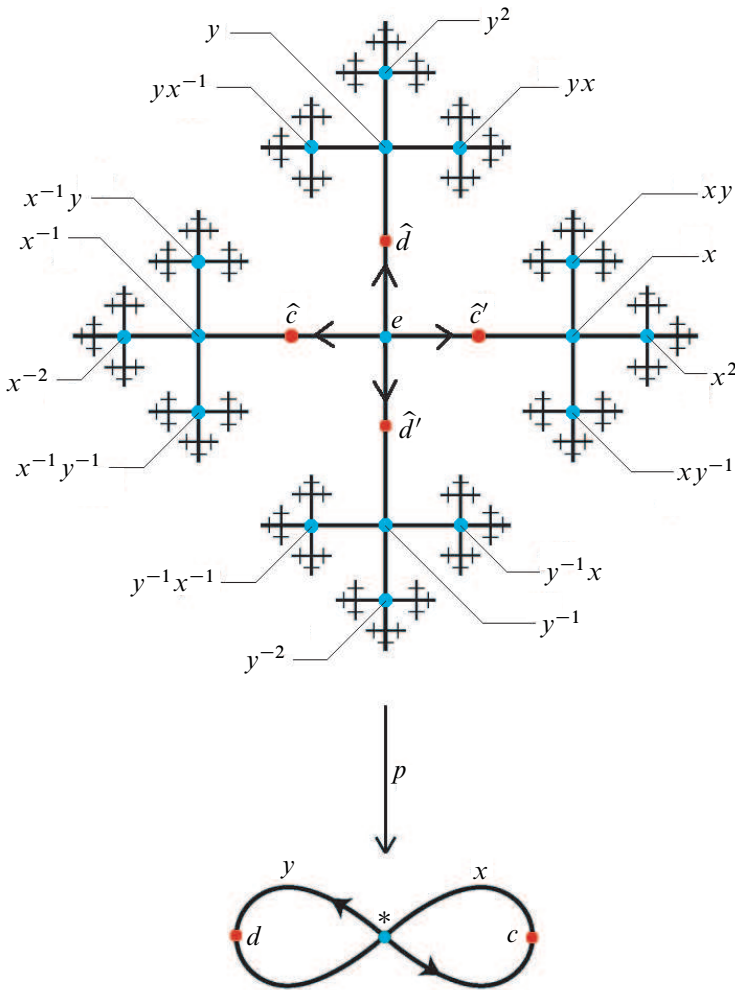
Figure 1: Universal cover for the figure eight

by y and y by x. It is clear that if one has an edge from g to gu in Ca, it follows that $\widehat{gu}$ is of the form $\hat{g}\hat{u}$ with $\hat{u} \in \{x^{\pm 1}, y^{\pm 1}\}$, hence it is also an edge. Similarly, the action by the element b is defined by $\psi(b)$ where $\psi(b)$ applied to a word w is the word $\bar{w}$ where the latter is obtained from w by replacing x by y^{-1} and y by x^{-1}. It is immediate that this extends to an action on Ca.

To verify that ψ is a homomorphism one may check that it satisfies the relations for the group Γ. As an example, we show that $\psi(ax) = \psi(ya)$: let w be a vertex in Ca, then $\psi(ax)(w) = (\widehat{xw}) = y\hat{w}$ and on the other hand $\psi(ya)(w) = y\hat{w}$. The others can be verified similarly.

As $\widehat{\mathrm{Ca}}$ is a tree and the action is cellular, we only need to check that the fixed points are determined by the finite subgroups of Γ. We recall that up to isomorphism, the finite subgroups of $PB(\mathbb{R}P^2)$ are just $\mathbb{Z}_2, \mathbb{Z}_4$, and $\mathbf{Q}_8$.

Now, we need to check that for any nontrivial finite subgroup H of Γ, $(\widehat{\mathrm{Ca}})^H \neq \varnothing$ and it is contractible. This follows from the general theory of groups acting on trees. Let H be any finite group acting by automorphisms on a tree T. By the work of Serre [26, Theorem 15, page 58], we have that T^H is nonempty. Recall that in a tree, there is a unique shortest simplicial path joining any pair of points, thus T^H is path connected: Let u, v be two points in T^H and let γ be the shortest path joining u and v, then for $h \in H$ the path $h\gamma$ joins $hu = u$ to $hv = v$, so by uniqueness it follows that $h\gamma = \gamma$. Furthermore, as a nonempty path connected subset of a tree is itself a tree it follows that T^H is a tree, hence contractible.

For any infinite group H we have $(\widehat{\mathrm{Ca}})^H = \varnothing$: Let H be a subgroup of Γ, if $H \cap F_2 = \{1\}$, from the semidirect product description of Γ, it follows that H is finite. Hence $H \cap F_2 \neq \{1\}$ for any *infinite* subgroup H of Γ. Let $g \in H \cap F_2$ be a nonidentity element. We will show that g cannot fix any vertex: if v is a vertex in Ca we have that $gv = v$ in F_2, if and only if $g = 1$, as the action is by left multiplication. Now, if v is a vertex in $\widehat{\mathrm{Ca}}$ that was not a vertex in Ca, then g would fix this vertex if and only if g leaves invariant the corresponding edge e in the original Cayley graph. It follows that g would interchange the vertices of e, hence $g^2 \in F_2$ would fix the edge e and hence a vertex in the original Cayley graph. It follows that $g^2 = 1$, but $g \in F_2$ which is a torsion free group, hence $g = 1$ which is a contradiction to the choice of g. It follows that $g \in H$ does not fix any vertex in $\widehat{\mathrm{Ca}}$, and hence $(\widehat{\mathrm{Ca}})^H = \varnothing$. $\square$

As F_2 is a normal subgroup of $F_2 \rtimes \mathbf{Q}_8$, we have that the orbit space $(\widehat{\mathrm{Ca}}/F_2 \rtimes \mathbf{Q}_8) \approx (\widehat{\mathrm{Ca}}/F_2)/\mathbf{Q}_8$. As we know $(\widehat{\mathrm{Ca}}/F_2)$ is the figure eight with vertex point $*$ and has two extra vertices c, d in each leaf (from the barycenters of the intervals of Ca; see Figure 1). Now observe that the action of $\mathbf{Q}_8$ identifies c and d and there is a single orbit for the edges. Hence, the orbit space $\widehat{\mathrm{Ca}}/F_2 \rtimes \mathbf{Q}_8$ is the 1–simplex described by the two vertices $*$ and c and the edge e_{c*} (see Figure 2). Note that the vertex $*$ is
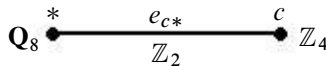
$$\mathbf{Q}_8 \;\overset{\displaystyle *}{\bullet}\!\!\overline{\underset{\displaystyle \mathbb{Z}_2}{\quad e_{c*} \quad}}\!\!\overset{\displaystyle c}{\bullet}\; \mathbb{Z}_4$$

Figure 2: The orbit space $\mathcal{B}_{\mathcal{FIN}}\Gamma$

stabilized by $\mathbf{Q}_8$, while the vertex c is stabilized by $\mathbb{Z}_4$ and the edge e_{c*} is stabilized by $\mathbb{Z}_2$. Thus, we have a full description of the orbit space for the family of finite subgroups of Γ, $\mathcal{B}_{\mathcal{FIN}}\Gamma$.

Theorem 4.5   *Let* $\Gamma = PB_n(\mathbb{R}P^2)$ *be the pure braid group on* n *strands* $(n > 0)$ *on the projective plane. Then we have that* $\mathrm{Wh}(\Gamma) = 0$ *and* $K_i(\mathbb{Z}\Gamma) = 0$ *for* $i < 0$. *Moreover, when* $n \neq 2, 3$ *we have* $\widetilde{K}_0(\mathbb{Z}\Gamma) = 0$ *and in the remaining two cases* $n = 2, 3$, *we have that* $\widetilde{K}_0(\mathbb{Z}\Gamma) = \mathbb{Z}_2$.

Proof   For the cases $n = 1$, and 2 we have that

$$PB_n(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}_2 & n = 1, \\ \mathbf{Q}_8 & n = 2. \end{cases}$$

The Whitehead group of Γ and the lower algebraic K–groups for the integral group ring $\mathbb{Z}\Gamma$ are well known for $\Gamma = \mathbb{Z}_2$. For $\mathbf{Q}_8$ the Whitehead group can be found in Keating [20], the reduced K_0 in Martinet [22], and K_{-1} can be computed using Carter's formula [7, Theorem 1]:

$$K_{-1}(\mathbb{Z}[\mathbf{Q}_8]) = \mathbb{Z}^r \oplus (\mathbb{Z}_2)^s,$$

where $r = 1 - \kappa + \kappa_2 - r_2$ and κ, κ_2 and r_2 are the number of irreducible classes of $\mathbb{Q}, \mathbb{Q}_2$, and $\mathbb{F}_2$ representations respectively. In this case, $\kappa = \kappa_2$ as both correspond to the number of conjugacy classes of cyclic subgroups of $\mathbf{Q}_8$ and $r_2 = 1$ as this is a 2–group [25, Chapters 12 and 15]. Hence $r = 0$. On the other hand s is the number of simple components of $\mathbb{Q}[\mathbf{Q}_8]$ which have even Schur index m but have odd local Schur indices m_Q at every finite prime Q of the center which divides 8. From [9, page 740], we have that

$$\mathbb{Q}[\mathbf{Q}_8] \cong \mathbb{Q}^4 \oplus \mathbb{H}_4.$$

The Schur index of the first term above is 1 and that of $\mathbb{H}_4$ (the rational quaternion algebra) is 2, thus the local Schur index of $\mathbb{H}_4$ at the prime 2 is also 2. This gives $s = 0$, hence $K_{-1}(\mathbb{Q}[\mathbf{Q}_8]) = 0$. Summarizing, the results are

$$\mathrm{Wh}(\mathbb{Z}_2) = 0,$$
$$\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_2]) = 0,$$
$$K_i(\mathbb{Z}[\mathbb{Z}_2]) = 0 \qquad i < 0,$$
$$\mathrm{Wh}(\mathbf{Q}_8) = 0,$$
$$\widetilde{K}_0(\mathbb{Z}[\mathbf{Q}_8]) = \mathbb{Z}_2.$$

For $n > 3$, the isomorphism types of virtually cyclic subgroups of Γ are precisely

$$\{e\}, \ \mathbb{Z}_2, \ \mathbb{Z}_4, \ \mathbb{Z}, \ \mathbb{Z}_2 \times \mathbb{Z}, \ \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4.$$

As before, we now need to compute the homotopy groups $\pi_k(\mathbb{H}(\mathcal{B}_{\mathcal{VC}}\Gamma; \mathcal{P}_*(F_x)))$ for the family of virtually cyclic subgroups of Γ. As for the case $PB_n(\mathbb{S}^2)$, we reduce

the computations to the family of finite groups by showing that the corresponding Nil-groups for the infinite virtually cyclic subgroups of Γ vanish.

The first cases were treated before and the only new case to analyze is the Nil groups of $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$. From work of Lafont and Ortiz [21] it follows that these Nil groups also vanish for all $i \leq 1$. Hence, the relative assembly map

$$(\mathbb{A}_*)_{\mathcal{FIN},\mathcal{VC}}\colon \pi_q(\mathbb{H}(\mathcal{B}_{\mathcal{FIN}}\Gamma; \mathcal{P}_*(F_x))) \longrightarrow \pi_q(\mathbb{H}(\mathcal{B}_{\mathcal{VC}}\Gamma; \mathcal{P}_*(F_x)))$$

is an isomorphism for all $q \leq -1$. As before, we now concentrate on the computation of the homotopy groups $\pi_k(\mathbb{H}(\mathcal{B}_{\mathcal{FIN}}\Gamma; \mathcal{P}_*(F_x)))$ for the family of finite subgroups of Γ. We analyze the Atiyah–Hirzebruch–Quinn spectral sequence $E_{p,q}^2 = H_p(\mathcal{B}_{\mathcal{FIN}}\Gamma; \{\pi_q(\mathcal{P}_*(F_x))\})$ for the family of finite subgroups, $\mathcal{FIN}$. The family $\mathcal{FIN}$ for this case consists of subgroups isomorphic to

$$\{1\}, \quad \mathbb{Z}_2, \quad \mathbb{Z}_4$$

and the local coefficients $\{\pi_q(\mathcal{P}_*(F_x))\}$ are described as

$$\pi_q(\mathcal{P}_*(F_x)) = \begin{cases} \mathrm{Wh}(\pi_1(F_x)) & q = -1, \\ \tilde{K}_0(\mathbb{Z}\pi_1(F_x)) & q = -2, \\ K_{q+2}(\mathbb{Z}\pi_1(F_x)) & q \leq -3, \end{cases}$$

with $\pi_1(F_x) \in \mathcal{FIN}$.

However, if $\pi_1(F_x) = \{1\}$, $\mathbb{Z}_2$ or $\mathbb{Z}_4$, it is well known that the local coefficients $\pi_q(\mathcal{P}_*(F_x)) = 0$ for all $q \leq -1$. For $\mathbb{Z}_4$, see Oliver [23, Theorem 14.2, Example 14.4] for the Whitehead group, see Curtis and Reiner [9, Corollary 50.17] for reduced K_0 and Carter's formula [7, Theorem 1] is immediate. Hence, the spectral sequence $E_{p,q}^2$ collapses and the homotopy groups

$$\pi_k(\mathbb{H}(\mathcal{B}_{\mathcal{FIN}}\Gamma; \mathcal{P}_*(F_x)))$$

vanish for all $k \leq -1$. Therefore, $\mathrm{Wh}(\Gamma)$, $\tilde{K}_0(\mathbb{Z}\Gamma)$ and $K_i(\mathbb{Z}\Gamma)$ vanish for all $n > 3$ and $i \leq -1$.

To complete the proof of our theorem we are left with studying the case $n = 3$. Recall that the virtually cyclic subgroups of Γ in this case are

$$\{e\}, \quad \mathbb{Z}_2, \quad \mathbb{Z}_4, \quad \mathbf{Q}_8, \quad \mathbb{Z}, \quad \mathbb{Z}_2 \times \mathbb{Z}, \quad \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4,$$

so $\mathcal{FIN} = \{\{e\}, \mathbb{Z}_2, \mathbb{Z}_4, \mathbf{Q}_8\}$ and the infinite virtually cyclic subgroups are $\mathcal{VC}-\mathcal{FIN} = \{\mathbb{Z}, \mathbb{Z}_2 \times \mathbb{Z}, \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4\}$.

Observe that the difference with the case $n > 3$ is the presence of the finite group $\mathbf{Q}_8$. Thus, the only local coefficient that may change our results are the ones corresponding to the group $\mathbf{Q}_8$.

We begin by computing $\mathrm{Wh}(\Gamma)$. To do this, look at the spectral sequence

$$E_{p,q}^2 = H_p(\mathcal{B}_{\mathcal{FIN}}\Gamma; \{\pi_q(\mathcal{P}_*(F_x))\})$$

where the orbit space $\mathcal{B}_{\mathcal{FIN}}\Gamma$ is the 1–simplex described earlier. Therefore, the only terms of the spectral sequence that contribute to compute $\mathrm{Wh}(\Gamma)$ are $E_{0,-1}^2$, and $E_{1,-2}^2$.

The term $E_{0,-1}^2$ is $H_0(\mathcal{B}_{\mathcal{FIN}}\Gamma; \{\pi_{-1}(\mathcal{P}_*(F_x))\})$. The local coefficients are

$$\{\pi_{-1}(\mathcal{P}_*(F_x))\} = \{\mathrm{Wh}(\pi_1(F_x))\}_{\pi_1(F_x)\in\mathcal{FIN}}.$$

We already know that $\mathrm{Wh}(\pi_1(F_x)) = 0$ for all $\pi_1(F_x) \in \mathcal{FIN}$. Hence, $E_{0,-1}^2 = 0$.

The term $E_{1,-2}^2$ is $H_1(\mathcal{B}_{\mathcal{FIN}}\Gamma; \{\pi_{-2}(\mathcal{P}_*(F_x))\})$ and the local coefficients are

$$\{\pi_{-2}(\mathcal{P}_*(F_x))\} = \{\tilde{K}_0(\mathbb{Z}\pi_1(F_x)\}_{\pi_1(F_x)\in\mathcal{FIN}}.$$

The chain complex that provides this homology is the following:

$$0 \longrightarrow \tilde{K}_0(\mathbb{Z}[\mathbb{Z}_2]) \longrightarrow \tilde{K}_0(\mathbb{Z}[\mathbf{Q}_8]) \oplus \tilde{K}_0(\mathbb{Z}[\mathbb{Z}_4]) \longrightarrow 0$$

Hence, $H_1(\mathcal{B}_{\mathcal{FIN}}\Gamma; \{\pi_{-2}(\mathcal{P}_*(F_x))\}) = 0$ and $\mathrm{Wh}(\Gamma) = 0$.

Now, to compute $\tilde{K}_0(\mathbb{Z}\Gamma)$ the only terms that contribute are $E_{0,-2}^2$, and $E_{1,-3}^2$. Using the above chain complex we have that

$$E_{0,-2}^2 = H_0(\mathcal{B}_{\mathcal{FIN}}\Gamma; \{\pi_{-2}(\mathcal{P}_*(F_x))\}) = \mathbb{Z}_2.$$

As
$$E_{1,-3}^2 = H_1(\mathcal{B}_{\mathcal{FIN}}\Gamma; \{\pi_{-3}(\mathcal{P}_*(F_x))\}) = 0,$$

since the local coefficients

$$\{\pi_{-3}(\mathcal{P}_*(F_x))\} = \{K_{-1}(\mathbb{Z}\pi_1(F_x)\}_{\pi_1(F_x)\in\mathcal{FIN}}$$

and $K_{-1}(\mathbb{Z}\pi_1(F_x)) = 0$ for all $\pi_1(F_x) \in \mathcal{FIN}$, we obtain that $\tilde{K}_0(\mathbb{Z}\Gamma) = \mathbb{Z}_2$.

Lastly, note that the local coefficients for $q < -3$

$$\{\pi_q(\mathcal{P}_*(F_x))\} = \{K_{q+2}(\mathbb{Z}\pi_1(F_x)\}_{\pi_1(F_x)\in\mathcal{FIN}}$$

are all zero. Thus, the spectral sequence collapses for all p and for all $q < -3$. Hence, the lower algebraic K–groups for the integral group ring $\mathbb{Z}\Gamma$ vanish. $\qquad\square$

Final remarks   Similar computations may be carried out for the groups $B_n(\mathbb{S}^2)$ and $B_n(\mathbb{R}P^2)$. New ingredients appear: the infinite virtually cyclic groups are more complex and there are many cases where the computations cannot be reduced to the family of finite groups.

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