Knots which admit a surgery with simple knot Floer homology groups

EAMAN EFTEKHARY

We show that if a positive integral surgery on a knot K inside a homology sphere X results in an induced knot $K_n \subset X_n(K) = Y$ which has simple Floer homology then $n \geq 2g(K)$. Moreover, for $X = S^3$ the three-manifold Y is an L–space, and the Heegaard Floer homology groups of K are determined by its Alexander polynomial.

57M27; 57R58

1 Introduction

The importance of studying knots inside rational homology spheres which have simple knot Floer homology came up in the study of the Berge conjecture using techniques from Heegaard Floer homology by Hedden [7], Rasmussen [14] and Baker, Grigsby and Hedden [1]. By definition, a knot K inside a rational homology sphere X has simple knot Floer homology if the rank of $\widehat{\mathrm{HFK}}(X, K)$ is equal to the rank of $\widehat{\mathrm{HF}}(X)$ (see Oszváth and Szabó [11; 10] and Rasmussen [15] for the background on Heegaard Floer homology and knot Floer homology). The Berge conjecture concerning knots in S^3 which admit a lens space surgery may almost be reduced to showing that a knot inside a lens space with simple knot Floer homology is simple.

For knots inside arbitrary rational homology spheres, it is not clear what the topological implications of having simple knot Floer homology are. With more restrictions on the ambient three-manifold, however, certain conclusions may be made. In particular, if the ambient manifold X is an integer homology sphere, the author has shown [4] that the only knot with simple knot Floer homology is the trivial knot. In this paper, we prove two more theorems in this direction. The first theorem is about the knots obtained by small surgery from knots inside homology spheres. If $K \subset X$ is a null-homologous knot inside a three-manifold X, we may remove a tubular neighborhood of K and glue it back in a different way so that the resulting manifold $X_n(K)$ is the three-manifold obtained from X by n–surgery on K. In this situation the core of the new solid torus will determine a knot $K_n \subset X_n(K)$. We show:

Published: 4 May 2011 DOI: 10.2140/agt.2011.11.1243

Theorem 1.1 *Suppose that $K \subset X$ is a knot inside a homology sphere X of Seifert genus $g(K)$. Suppose that $0 < n < 2g(K)$ is a given integer and $K_n \subset X_n(K)$ is the knot obtained from K by n–surgery. Then K_n can not have simple knot Floer homology, ie*

$$\operatorname{rk}\big(\widehat{\mathrm{HFK}}(X_n(K), K_n)\big) > \operatorname{rk}\big(\widehat{\mathrm{HF}}(X_n(K))\big). \tag{1}$$

When the surgery coefficient n is greater than or equal to $2g(K)$, there is more freedom for choosing K so that K_n has simple knot Floer homology. In particular, a necessary and sufficient condition may be given when X is a L–space. In order to state the precise theorem, for a knot K inside the homology sphere L–space X let $\mathbb{B} = \mathbb{B}(K)$ denote the vector space $\widehat{\mathrm{HFK}}(X, K; \mathbb{Z}/2\mathbb{Z})$ and let $d_{\mathbb{B}}\colon \mathbb{B} \to \mathbb{B}$ denote the differential obtained by counting the disks passing through the second marked point in a doubly pointed Heegaard diagram associated with the pair (X, K). The homology group $\mathrm{H}_*(\mathbb{B}(K), d_{\mathbb{B}})$ is then equal to $\widehat{\mathrm{HF}}(X; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Let $d(X)$ denote the homological degree of the generator of this latter vector space.

Theorem 1.2 *Let $K \subset X$ be a knot in a homology sphere L–space X of Seifert genus $g(K)$. If for an integer $n \geq 2g(K)$ the knot $K_n \subset X_n(K)$ has simple knot Floer homology, $X_n(K)$ is an L–space and there is an increasing sequence of integers*

$$-g(K) = n_{-k} < n_{1-k} < \cdots < n_k = g(K)$$

with $n_i = -n_{-i}$ for which the following is true. For $i \in \mathbb{Z}$ with $|i| \leq g(K)$ define

$$\delta_i = \begin{cases} 0 + d(X) & \text{if } i = g(K), \\ \delta_{i+1} + 2(n_{i+1} - n_i) - 1 & \text{if } i < g(K) \text{ and } g(K) - i \equiv 1 \pmod 2, \\ \delta_{i+1} + 1 & \text{if } i < g(K) \text{ and } g(K) - i \equiv 0 \pmod 2. \end{cases}$$

In this situation, $\widehat{\mathrm{HFK}}(X, K, \mathfrak{s}) = 0$ unless $\mathfrak{s} = n_i$ for some $-k \leq i \leq k$, in which case $\widehat{\mathrm{HFK}}(X, K, \mathfrak{s}) = \mathbb{Z}/2\mathbb{Z}$ and it is supported entirely in homological degree δ_i. If the generator of $\widehat{\mathrm{HFK}}(X, K, n_i)$ is denoted by $\mathbf{x}_i$, the filtered chain complex $(\mathbb{B}(K), d_{\mathbb{B}})$ may be described (up to quasi-isomorphism) as $\mathbb{B} = \langle \mathbf{x}_{-k}, \ldots, \mathbf{x}_k \rangle$, where the differential is given by

$$d_{\mathbb{B}}(\mathbf{x}_i) = \begin{cases} 0 & \text{if } i = k \text{ or } 0 < k - i \text{ is odd}, \\ \mathbf{x}_{i+1} & \text{otherwise.} \end{cases}$$

Moreover, if for a knot K inside a homology sphere L–space X the filtered chain complex $(\mathbb{B}(K), d_{\mathbb{B}})$ has the above form, then for any integer $n \geq 2g(K)$, $X_n(K)$ is an L–space and $K_n \subset X_n(K)$ has simple knot Floer homology.

This gives a complete classification of the knot Floer homology groups of the knots in homology sphere L–spaces upon which integer surgery yields a knot with simple knot Floer homology. In particular, if n–surgery on a knot $K \subset S^3$ has simple knot Floer homology, then $n \geq 2g(K)$ and all the coefficients of the symmetrized Alexander polynomial associated with K are equal to 1 (see Ozsváth and Szabó [12], where this was first proved).

We hope that the techniques used here are useful in the study of general knots with simple knot Floer homology, although understanding such knots in full generality requires significant breakthroughs at this point. In particular, the Berge conjecture is wide open from this perspective.

Acknowledgements I would like to thank Matt Hedden for bringing up this problem and helpful discussions during a visit to MSRI. I would also like to thank MSRI for providing us with an opportunity for such interactions.

2 Knot Floer homology background

2.1 Relative Spin^c structures and rationally null-homologous knots

Let X be a homology sphere and K be an oriented knot inside X. Consider a tubular neighborhood $\mathrm{nd}(K)$ of K and let T be the torus boundary of this neighborhood. Let $\mu \subset T$ be an oriented meridian for K, ie μ is the oriented boundary of an oriented disk D_μ in $\mathrm{nd}(K)$ so that the intersection number of K with D_μ is 1. Let $\lambda \subset T$ be a zero framed longitude for K, ie a curve that is isotopic to K in $\mathrm{nd}(K)$ and bounds a Seifert surface S for K in $X - \mathrm{nd}(K)$. The curve λ inherits an orientation from K in a natural way. We may assume that λ and μ intersect each other in a single transverse point. Having fixed these two curves, by (p,q)–surgery on K we mean removing $\mathrm{nd}(K)$ and replacing it with a solid torus so that the simple closed curve $p\mu + q\lambda$ bounds a disk in the new solid torus. Denote the resulting three-manifold by $X_{p/q}(K)$. We denote by $K_{p/q} \subset X_{p/q}(K)$ the core of the new solid torus in $X_{p/q}(K)$, ie

$$K_{p/q} = \{0\} \times S^1 \subset D^2 \times S^1 \subset X_{p/q}.$$

Let $\mathbb{H}_{p/q}(K)$ be the Heegaard Floer homology group

$$\widehat{\mathrm{HFK}}(X_{p/q}(K), K_{p/q}; \mathbb{Z}/2\mathbb{Z}).$$

In particular $\mathbb{H}_\infty(K) = \mathbb{H}_{1/0}(K)$ is the knot Floer homology group (hat theory) $\widehat{\mathrm{HFK}}(X, K; \mathbb{Z}/2\mathbb{Z})$ defined in [10] and $\mathbb{H}_0(K) = \mathbb{H}_{0/1}(K)$ is the longitude Floer homology group $\widehat{\mathrm{HFL}}(X, K; \mathbb{Z}/2\mathbb{Z})$ defined in [5].

Note that relative Spin^c structures on $X-\mathrm{nd}(K)$ may be regarded as equivalence classes of no-where vanishing vector fields on $X-\mathrm{nd}(K)$ which restrict to a translation invariant vector field on the boundary of this three-manifold. Two such vector fields

$$V_0, V_1 \in \Gamma(X-\mathrm{nd}(K), T_X)$$

are defined equivalent or *homologous* if there is a ball $B \subset X-\overline{\mathrm{nd}(K)}$ and a continuous isotopy $\{V_t\}_{t\in[0,1]}$ of no-where vanishing vector fields on $X-\mathrm{nd}(K)-B$ connecting V_0 to V_1, such that each V_t restricts to a translation invariant vector field on the boundary of $\mathrm{nd}(K)$ (see Ni [8] for more details on this interpretation of relative Spin^c structures). The set of relative Spin^c structures form an affine space $\underline{\mathrm{Spin}}^c(X,K)$ over $\mathrm{H}^2(X,K;\mathbb{Z})$. Since $X-\mathrm{nd}(K)$ is the same as $X_{p/q}-\mathrm{nd}(K_{p/q})$, it is clear that

$$\underline{\mathrm{Spin}}^c(X,K) = \underline{\mathrm{Spin}}^c(X_{p/q}(K), K_{p/q})$$

in a natural way. The group $\mathbb{H}_{p/q}(K)$ is decomposed into subgroups associated with relative Spin^c structures:

$$\mathbb{H}_{p/q}(K) = \bigoplus_{\mathfrak{s}\in\underline{\mathrm{Spin}}^c(X,K)} \mathbb{H}_{p/q}(K,\mathfrak{s}). \tag{2}$$

There is a natural involution

$$J\colon \underline{\mathrm{Spin}}^c(X,K) \longrightarrow \underline{\mathrm{Spin}}^c(X,K) \tag{3}$$

which takes a Spin^c class $\mathfrak{s}$ represented by a nowhere vanishing vector field V on $X-\mathrm{nd}(K)$, to the Spin^c class $J(\mathfrak{s})$ represented by $-V$. The difference $\mathfrak{s}-J(\mathfrak{s}) \in \mathrm{H}^2(X,K;\mathbb{Z})$ is usually denoted by $c_1(\mathfrak{s})$. There is a symmetry in knot Floer homology which may be described by the formula

$$\widehat{\mathrm{HFK}}(X,K,\mathfrak{s}) \simeq \widehat{\mathrm{HFK}}(X,K,J(\mathfrak{s})+\mathrm{PD}[\mu]). \tag{4}$$

Since X is a homology sphere, the cohomology group $\mathrm{H}^2(X,K;\mathbb{Z})$ is generated by the class $\mathrm{PD}[\mu]$ and we may thus naturally identify $\mathrm{H}^2(X,K;\mathbb{Z})$ with $\mathbb{Z}$. We then have a map

$$\begin{aligned} \mathfrak{h}\colon \underline{\mathrm{Spin}}^c(X,K) &\longrightarrow \mathbb{Z} = \mathrm{H}^2(X,K;\mathbb{Z}) \\ \mathfrak{h}(\mathfrak{s}) &:= \frac{c_1(\mathfrak{s})-\mathrm{PD}[\mu]}{2} \end{aligned} \tag{5}$$

which satisfies $\mathfrak{h}(\mathfrak{s}) = -\mathfrak{h}(J(\mathfrak{s})+\mathrm{PD}[\mu])$ for all relative Spin^c classes $\mathfrak{s}\in\underline{\mathrm{Spin}}^c(X,K)$. Using this map $\underline{\mathrm{Spin}}^c(X,K)$ may also be identified with $\mathbb{Z}$ in a natural way.

The following theorem of Ozsváth and Szabó [9], generalized by Ni [8] to rationally null-homologous knots, allows us compute the genus of a knot $K \subset X$, using Heegaard Floer homology:

Theorem 2.1 *If $K \subset X$ is a knot inside a homology sphere X as above, the Seifert genus $g(K)$ of K may be computed from*

$$g(K) = \max \left\{ \mathfrak{s} \in \underline{\mathrm{Spin}}^c(X, K) = \mathbb{Z} \;\middle|\; \widehat{\mathrm{HFK}}(X, K; \mathfrak{s}) \neq 0 \right\}. \tag{6}$$

2.2 Surgery formulas

Suppose that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v)$ is a Heegaard diagram for the knot K, such that $\boldsymbol{\beta} = \boldsymbol{\beta}_0 \cup \{\mu = \beta_g\}$, $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_0)$ is a Heegaard diagram for $X - \mathrm{nd}(K)$, while $\mu = \beta_g$ represents the meridian of K and the two marked points u and v are placed on the two sides of β_g. Think of the vector space $\mathbb{B} = \widehat{\mathrm{HFK}}(X, K; \mathbb{Z}/2\mathbb{Z}) = \mathbb{H}_\infty(K) = \mathbb{H}_{1/0}(K)$ as a vector space computed as $\mathbb{B} = \widehat{\mathrm{HF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v)$. Then the differential of the complex $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u)$, which is defined by letting holomorphic disks pass through the marked point v in the Heegaard diagram, induces a map $d_{\mathbb{B}}\colon \mathbb{B} \to \mathbb{B}$, which is a filtered differential on the filtered vector space $\mathbb{B}$, with the filtration induced by relative Spin^c structures. The homology of the complex $(\mathbb{B}, d_{\mathbb{B}})$ gives $H_*(\mathbb{B}, d_{\mathbb{B}}) = \widehat{\mathrm{HF}}(X; \mathbb{Z}/2\mathbb{Z})$. Note that the restriction of $d_{\mathbb{B}}$ to each $\mathbb{B}(\mathfrak{s})$ is trivial by definition.

For a relative Spin^c class $\mathfrak{s} \in \underline{\mathrm{Spin}}^c(X, K) = \mathbb{Z}$ we set

$$\mathbb{B}\{\geq \mathfrak{s}\} = \bigoplus_{\substack{\mathfrak{t} \in \underline{\mathrm{Spin}}^c(X,K) = \mathbb{Z} \\ \mathfrak{t} \geq \mathfrak{s}}} \mathbb{B}(\mathfrak{t}) \quad \text{and} \quad \mathbb{B}\{> \mathfrak{s}\} = \bigoplus_{\substack{\mathfrak{t} \in \underline{\mathrm{Spin}}^c(X,K) = \mathbb{Z} \\ \mathfrak{t} > \mathfrak{s}}} \mathbb{B}(\mathfrak{t}).$$

Then the subspaces $\mathbb{B}\{\geq \mathfrak{s}\}$ and $\mathbb{B}\{> \mathfrak{s}\}$ of $\mathbb{B}$ are mapped to themselves by the differential $d_{\mathbb{B}}$ of $\mathbb{B}$. Furthermore, let $\imath_{\mathfrak{s}}\colon \mathbb{B}\{\geq \mathfrak{s}\} \to \mathbb{B}$ be the inclusion map, and denote the homology of $(\mathbb{B}, d_{\mathbb{B}})$ by $\mathbb{H}$ and the homology of the subcomplexes $\mathbb{B}\{\geq \mathfrak{s}\}$ and $\mathbb{B}\{> \mathfrak{s}\}$ by $\mathbb{H}\{\geq \mathfrak{s}\}$ and $\mathbb{H}\{> \mathfrak{s}\}$ respectively. Similarly, we may denote the homology of the quotient complexes

$$\mathbb{B}\{< \mathfrak{s}\} = \frac{\mathbb{B}}{\mathbb{B}\{\geq \mathfrak{s}\}} \quad \text{and} \quad \mathbb{B}\{\leq \mathfrak{s}\} = \frac{\mathbb{B}}{\mathbb{B}\{> \mathfrak{s}\}}$$

by $\mathbb{H}\{< \mathfrak{s}\}$ and $\mathbb{H}\{\leq \mathfrak{s}\}$ respectively.

Remark 2.2 Our convention for the orientation of the holomorphic disks connecting two intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ in a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v)$ is opposite to the one used by Ozsváth and Szabó in their original papers [11; 10]. Thus, in the above setup, $\mathbb{B}\{\geq \mathfrak{s}\}$ becomes a *subcomplex* of $(\mathbb{B}, d_{\mathbb{B}})$ (as opposed to [10], where it is a *quotient complex* of $(\mathbb{B}, d_{\mathbb{B}})$) and $\mathbb{B}\{\leq\mathfrak{s}\}$ becomes a *quotient complex* of $(\mathbb{B}, d_{\mathbb{B}})$ (as opposed to [10] where it is a *subcomplex* of $(\mathbb{B}, d_{\mathbb{B}})$). We chose the current convention to stay compatible with [4], which provides us with our main technical tool.

The following theorem which gives an explicit formula for the groups $\mathbb{H}_n(K,\mathfrak{s})$ is proved in [4]:

Theorem 2.3 *Suppose that $K \subset X$ is a knot inside a homology sphere X. With the above notation fixed, the group $\widehat{\mathrm{HFK}}(X_n(K), K_n, \mathfrak{s}; \mathbb{Z}/2\mathbb{Z})$ may be computed as the homology of the complex*

$$C_n(\mathfrak{s}) = \mathbb{B}\{\geq\mathfrak{s}\} \oplus \mathbb{B}\{\geq n+1-\mathfrak{s}\} \oplus \mathbb{B}$$

which is equipped with a differential $d_n\colon C_n(\mathfrak{s}) \to C_n(\mathfrak{s})$ defined by

$$d_n(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \big(d_{\mathbb{B}}(\mathbf{x}), d_{\mathbb{B}}(\mathbf{y}), d_{\mathbb{B}}(\mathbf{z}) + \imath_{\mathfrak{s}}(\mathbf{x}) + \imath_{n+1-\mathfrak{s}}(\mathbf{y})\big).$$

Note that instead of the complex $C_n(\mathfrak{s})$ we may consider a complex

$$\widetilde{C}_n(\mathfrak{s}) = \mathbb{H}\{\geq\mathfrak{s}\} \oplus \mathbb{H}\{\geq n+1-\mathfrak{s}\} \oplus \mathbb{H}$$

which is equipped with a differential $\widetilde{d}_n\colon \widetilde{C}_n(\mathfrak{s}) \to \widetilde{C}_n(\mathfrak{s})$ defined by

$$\widetilde{d}_n(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \big(0, 0, (\imath_{\mathfrak{s}})_*(\mathbf{x}) + (\imath_{n+1-\mathfrak{s}})_*(\mathbf{y})\big).$$

Lemma 2.4 *The homology groups $H_*(\widetilde{C}_n(\mathfrak{s}), \widetilde{d}_n)$ and $H_*(C_n(\mathfrak{s}), d_n)$ are isomorphic as vector spaces over $\mathbb{Z}/2\mathbb{Z}$. In particular, these homology groups have the same rank.*

Proof Let $\mathbb{A}$ be the direct sum of the complexes $\mathbb{B}\{\geq\mathfrak{s}\}$ and $\mathbb{B}\{\geq n+1-s\}$, $d_{\mathbb{A}}$ be the differential of $\mathbb{A}$, and let $f\colon (\mathbb{A}, d_{\mathbb{A}}) \to (\mathbb{B}, d_{\mathbb{B}})$ be the map $\imath_{\mathfrak{s}}$ on the first summand and $\imath_{n+1-\mathfrak{s}}$ on the second summand. Thus f is a chain map. $C_n(\mathfrak{s})$ is then the mapping cone of f and it suffices to show that for any chain map $f\colon (\mathbb{A}, d_{\mathbb{A}}) \to (\mathbb{B}, d_{\mathbb{B}})$, the homology of the mapping cone of f is isomorphic to the homology of the mapping cone of

$$f_*\colon H_*(\mathbb{A}, d_{\mathbb{A}}) \to H_*(\mathbb{B}, d_{\mathbb{B}}).$$

In appropriate decompositions $\mathbb{A} = Z_{\mathbb{A}} \oplus H_{\mathbb{A}} \oplus Z_{\mathbb{A}}$ and $\mathbb{B} = Z_{\mathbb{B}} \oplus H_{\mathbb{B}} \oplus Z_{\mathbb{B}}$ for the vector spaces $\mathbb{A}$ and $\mathbb{B}$ to direct sum of subcomplexes (with $H_{\mathbb{A}} \simeq H_*(\mathbb{A}, d_{\mathbb{A}})$ and $H_{\mathbb{B}} \simeq H_*(\mathbb{B}, d_B)$), $d_{\mathbb{A}}$, $d_{\mathbb{B}}$ and f will have the block forms

$$d_{\mathbb{A}} = \begin{pmatrix} 0 & 0 & \mathrm{Id}_{Z_{\mathbb{A}}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad d_{\mathbb{B}} = \begin{pmatrix} 0 & 0 & \mathrm{Id}_{Z_{\mathbb{B}}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} A(f) & B(f) & C(f) \\ 0 & f_* & D(f) \\ 0 & 0 & A(f) \end{pmatrix}.$$

Thus the rank of the differential of $C_n(\mathfrak{s})$, which has the block form

$$d_n = \begin{pmatrix} d_{\mathbb{A}} & f \\ 0 & d_{\mathbb{B}} \end{pmatrix}$$

equals $|Z_{\mathbb{A}}|+|Z_{\mathbb{B}}|+\mathrm{rk}(f_*)$. The rank of the homology of $C_n(\mathfrak{s})$ is therefore equal to

$$\begin{aligned}
|H_*(C_n(\mathfrak{s}),d_n)| &= |\mathbb{A}|+|\mathbb{B}|-2\,\mathrm{rk}(d_{C_n(\mathfrak{s})}) \\
&= 2|Z_{\mathbb{A}}|+|H_{\mathbb{A}}|+2|Z_{\mathbb{B}}|+|H_{\mathbb{B}}|-2(|Z_{\mathbb{A}}|+|Z_{\mathbb{B}}|+\mathrm{rk}(f_*)) \\
&= |H_{\mathbb{A}}|+|H_{\mathbb{B}}|-2\,\mathrm{rk}(f_*) \\
&= |H_*(\mathbb{A},d_{\mathbb{A}})|+|H_*(\mathbb{B},d_{\mathbb{B}})|-2\,\mathrm{rk}(f_*) \\
&= |\widetilde{C}_n(\mathfrak{s})|-2\,\mathrm{rk}(\widetilde{d}_n) = |H_*(\widetilde{C}_n(\mathfrak{s}),\widetilde{d}_n)|.
\end{aligned}$$

Therefore the sizes of the homologies of the two complexes are equal, and they are isomorphic as $\mathbb{Z}/2\mathbb{Z}$ vector spaces. □

We now recall a similar formula for the Heegaard Floer homology groups associated with the three-manifold $X_n(K)$ which is due to Ozsváth and Szabó [13]. Alternative surgery formulas for $\widehat{\mathrm{HF}}(X_n(K))$, based on the results of [3] and [2], may be found in [6]. We slightly modify the statement of Ozsváth and Szabó's theorem from [13] so that it looks more compatible with the notation of Theorem 2.3. To state the theorem, let $\pi_{\mathfrak{s}}\colon \mathbb{B}\{\geq\mathfrak{s}\}\to\mathbb{B}\{\mathfrak{s}\}=\mathbb{B}\{\geq\mathfrak{s}\}/\mathbb{B}\{>\mathfrak{s}\}$ be the projection map,

$$\Xi_{\mathfrak{s}}\colon \mathbb{B}\{\mathfrak{s}\}=\widehat{\mathrm{HFK}}(X,K,\mathfrak{s})\to\widehat{\mathrm{HFK}}(X,K,-\mathfrak{s})=\mathbb{B}\{-\mathfrak{s}\}$$

be the duality isomorphism, and $\jmath_{\mathfrak{s}}\colon \mathbb{B}\{\mathfrak{s}\}\to\mathbb{B}$ be the inclusion map. Associated with any Spin^c class $[\mathfrak{s}]\in\mathrm{Spin}^c(X_n(K))=\mathbb{Z}/n\mathbb{Z}$ we may construct a chain complex

$$F_n[\mathfrak{s}] = \bigoplus_{\mathfrak{t}\in[\mathfrak{s}]\subset\mathbb{Z}} C_n(\mathfrak{t})$$

which is equipped with a differential $f_n\colon F_n[\mathfrak{s}]\to F_n[\mathfrak{s}]$ defined as follows on a generator $(\mathbf{x},\mathbf{y},\mathbf{z})\in C_n(\mathfrak{s})$:

$$f_n(\mathbf{x},\mathbf{y},\mathbf{z}) = (\boldsymbol{\alpha}_{\mathfrak{t}})_{\mathfrak{t}\in[\mathfrak{s}]\subset\mathbb{Z}},\quad \boldsymbol{\alpha}_{\mathfrak{t}}\in C_n(\mathfrak{t}),$$

$$\boldsymbol{\alpha}_{\mathfrak{t}} = \begin{cases} d_n(\mathbf{x},\mathbf{y},\mathbf{z}) & \text{if } \mathfrak{t}=\mathfrak{s}, \\ \Upsilon_{\mathfrak{s}}(\mathbf{x},\mathbf{y},\mathbf{z}) & \text{if } \mathfrak{t}=\mathfrak{s}+n, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where $\quad \Upsilon_{\mathfrak{s}}(\mathbf{x},\mathbf{y},\mathbf{z}) := \big(0,(d_{\mathbb{B}}\circ\Xi_{\mathfrak{s}}\circ\pi_{\mathfrak{s}})(\mathbf{x}),(\jmath_{\mathfrak{s}}\circ\Xi_{\mathfrak{s}}\circ\pi_{\mathfrak{s}})(\mathbf{x})\big).$

It is not hard to see that $\Upsilon_{\mathfrak{s}}\colon C_n(\mathfrak{s})\to C_n(\mathfrak{s}+n)$ is a chain map and $\Upsilon_{\mathfrak{s}+n}\circ\Upsilon_{\mathfrak{s}}=0$. Consequently, $f_n\colon F_n[\mathfrak{s}]\to F_n[\mathfrak{s}]$ defines a differential on this vector space.

Theorem 2.5 *Suppose that $K \subset X$ is a knot inside a homology sphere X of genus $g = g(K)$. For a class $[\mathfrak{s}] \in \mathrm{Spin}^c(X_n(K)) = \mathbb{Z}/n\mathbb{Z}$, let the complex $(F_n[\mathfrak{s}], f_n)$ be defined as before. The Heegaard Floer homology group $\widehat{\mathrm{HF}}(X_n(K), [\mathfrak{s}])$ may then be computed as the homology of the chain complex $(F_n[\mathfrak{s}], f_n)$:*

$$\widehat{\mathrm{HF}}(X_n(K), [\mathfrak{s}]) = H_*(F_n[\mathfrak{s}], f_n) \tag{8}$$

Note that $\pi_{\mathfrak{s}}$ is trivial if $|\mathfrak{s}| > g(K)$, and that the homology of the chain complex $(C_n(\mathfrak{s}), d_n)$ is zero unless $-g(K) < \mathfrak{s} \leq n + g(K)$. This implies that the homology of the chain complex $(F_n[\mathfrak{s}], f_n)$ is the same as the homology of the chain complex

$$\begin{aligned} G_n[\mathfrak{s}] &= \bigoplus_{\substack{\mathfrak{t}\in[\mathfrak{s}]\subset\mathbb{Z} \\ -g(K)<\mathfrak{t}\leq n+g(K)}} C_n(\mathfrak{t}) \\ g_n\colon G_n[\mathfrak{s}] &\to G_n[\mathfrak{s}], \quad g_n = f_n|_{G_n[\mathfrak{s}]}. \end{aligned} \tag{9}$$

Corollary 2.6 *Suppose that $K \subset X$ is a knot inside a homology sphere X and $n > 0$ is a given integer. The induced knot $K_n \subset X_n(K)$ has simple knot Floer homology if and only if the maps induced in homology by the chain maps*

$$\Upsilon_{\mathfrak{s}}\colon C_n(\mathfrak{s}) \longrightarrow C_n(\mathfrak{s}+n) \tag{10}$$

vanish for all $\mathfrak{s} \in \underline{\mathrm{Spin}}^c(X, K)$.

Proof Immediate corollary of a comparison between the above two surgery formulas. □

3 Small surgery on a knot

In this section, we will assume that the surgery coefficient n is small.

Theorem 3.1 *Suppose that $K \subset X$ is a knot inside a homology sphere X of Seifert genus $g(K)$. Suppose that $0 < n < 2g(K)$ is a given integer and $K_n \subset X_n(K)$ is the knot obtained from K by n–surgery. Then K_n can not have simple knot Floer homology, ie*

$$\mathrm{rk}\left(\widehat{\mathrm{HFK}}(X_n(K), K_n)\right) > \mathrm{rk}\left(\widehat{\mathrm{HF}}(X_n(K))\right). \tag{11}$$

Proof Let us use Corollary 2.6 for $\mathfrak{s} = g(K)$. Note that $\widehat{\mathrm{HFK}}(X_n(K), K_n; g(K))$ is isomorphic to the homology of the mapping cone

$$C_n(g(K)) = \mathbb{B}\{g\} \oplus \mathbb{B}\{>n - g(K)\} \oplus \mathbb{B},$$

which is equipped with the differential d_n as before. Furthermore,

$$C_n(g(K)+n) = \mathbb{B}\{\geq g(K)+n\} \oplus \mathbb{B}\{>-g(K)\} \oplus \mathbb{B} = 0 \oplus \mathbb{B}\{>-g(K)\} \oplus \mathbb{B}$$

and its homology is thus isomorphic to $\mathbb{H}\{-g(K)\} = \mathbb{H}\{g(K)\}$. The map

$$(\Upsilon_{g(K)})_*\colon \mathrm{H}_*(C_n(g(K)), d_n) \to \mathbb{H}\{g(K)\}$$

is therefore given by sending

$$\mathbb{H}\{g(K)\} = \mathbb{B}\{g(K)\} \subset C_n(g(K)) = \mathbb{B}\{g(K)\} \oplus \mathbb{B}\{>n-g(K)\} \oplus \mathbb{B}$$

to $\mathbb{H}\{g(K)\}$ in the target vector space by the identity map. Note that the differential of $\mathbb{B}$ is trivial on $\mathbb{B}\{g(K)\}$ since $d_{\mathbb{B}}(\mathbb{B}(g(K)))$ is in

$$\mathbb{B}\{>g(K)\} = \bigoplus_{\substack{\mathfrak{s}\in\underline{\mathrm{Spin}}^c(X,K)\\ \mathfrak{s}>g(K)}} \widehat{\mathrm{HFK}}(X,K;\mathfrak{s}),$$

which is trivial by Theorem 2.1. In order for the map $(\Upsilon)_*$ to be trivial, we therefore need the map on homology induced by the inclusion of $\mathbb{H}\{g(K)\}$ in $C_n(g(K))$ to be trivial. In other words, the map $(\imath_{g(K)})_*\colon \mathbb{H}\{g(K)\} \to \mathbb{H}$ must be injective and its image should be disjoint from the image of the map

$$(\imath_{n+1-\mathfrak{s}})_*\colon \mathbb{H}\{>n-g(K)\} \to \mathbb{H}.$$

Let $\mathbf{x}$ be a generator of $\mathbb{H}\{g(K)\}$ which is mapped to a nontrivial element $[\mathbf{x}] \in \mathbb{H}$. Thus, $\mathbf{x}$ is not in the image of $d_{\mathbb{B}} \to d_{\mathbb{B}}$. If $n < 2g(K)$, $n-g(K) \leq g(K)-1$ and $\mathbb{B}\{>n-g(K)\}$ contains $\mathbf{x}$ as a closed generator. Since $\mathbf{x}$ is not in the image of $d_{\mathbb{B}}$, it survives in $\mathbb{H}\{>n-g(K)\}$ and is mapped to the same class $[\mathbf{x}] \in \mathbb{H}$ by $(\imath_{n+1-g(K)})_*$. Thus, $(\mathbf{x},\mathbf{x},0)$ is a nontrivial element in $\mathrm{H}_*(C_n(g(K)), d_n)$ which is mapped to $\mathbf{x} \in \mathbb{H}\{g(K)\}$ by $(\Upsilon_{g(K)})_*$. This contradiction proves the theorem. □

4 Large surgery on a knot

Once again, let $K \subset X$ be a knot inside a homology sphere L–space. Suppose now that $n \geq 2g(K)$ and $K_n \subset X_n$ has simple knot Floer homology. Then, as before, all maps $(\Upsilon_{\mathfrak{s}})_*$ must vanish. If the relative Spin^c structure $\mathfrak{s} \in \underline{\mathrm{Spin}}^c(X,K) = \mathbb{Z}$ satisfies $-g(K) < \mathfrak{s} \leq g(K)$, then $n-\mathfrak{s} \geq g(K)$ and $n+\mathfrak{s} > g(K)$, which imply that $\mathbb{H}\{>n-\mathfrak{s}\}$ and $\mathbb{H}\{\geq \mathfrak{s}+n\}$ are trivial. Thus $C_n(\mathfrak{s}) = \mathbb{H}\{\geq \mathfrak{s}\} \oplus 0 \oplus \mathbb{H}$ and $C_n(\mathfrak{s}+n) = 0 \oplus \mathbb{H}\{>-\mathfrak{s}\} \oplus \mathbb{H}$. This implies that

$$\begin{aligned} H_*(C_n(\mathfrak{s}), d_n) &= \mathbb{H}\{<\mathfrak{s}\}, \\ H_*(C_n(\mathfrak{s}+n), d_n) &= \mathbb{H}\{\leq -\mathfrak{s}\}. \end{aligned} \tag{12}$$

Under the above identifications the map $(\Upsilon_{\mathfrak{s}})_*$ which will be denoted by $\epsilon_{\mathfrak{s}}\colon \mathbb{H}\{<\mathfrak{s}\}\to \mathbb{H}\{\leq -\mathfrak{s}\}$ may be described as follows. It is the map obtained by first taking $\mathbb{H}\{<\mathfrak{s}\}$ to $\mathbb{H}\{\mathfrak{s}\}$ using the map $\tau_{\mathfrak{s}}$ induced by the differential $d_{\mathbb{B}}$ of the complex $\mathbb{B}$, then using the duality map $\Xi_{\mathfrak{s}}$ to take $\mathbb{H}\{\mathfrak{s}\}$ to $\mathbb{H}\{-\mathfrak{s}\}$, and finally going from $\mathbb{H}\{-\mathfrak{s}\}$ to $\mathbb{H}\{\leq -\mathfrak{s}\}$ using the induced map on homology of the inclusion of the first vector space in the quotient complex $\mathbb{B}\{\leq -\mathfrak{s}\}$ of $\mathbb{B}$. The assumption that $K_n\subset X_n(K)$ has simple knot Floer homology then implies that all the maps $\epsilon_{\mathfrak{s}}$ should vanish.

We may now prove the following theorem, which should be compared with Theorem 1.2 of [12] and Theorem 2.4 from [7].

Theorem 4.1 *Let $K\subset X$ be a knot in a homology sphere L–space X of Seifert genus $g(K)$, and fix the above notation. If for all $-g(K)<\mathfrak{s}\leq g(K)$ the maps $\epsilon_{\mathfrak{s}}$ vanish, there is an increasing sequence of integers*

$$-g(K)=n_{-k}<n_{1-k}<\cdots<n_k=g(K)$$

with $n_i=-n_{-i}$ for which the following is true. For $i\in\mathbb{Z}$ with $|i|\leq g(K)$ define

$$\delta_i=\begin{cases}0+d(X) & \text{if } i=g(K),\\ \delta_{i+1}+2(n_{i+1}-n_i)-1 & \text{if } i<g(K) \text{ and } g(K)-i\equiv 1 \pmod 2,\\ \delta_{i+1}+1 & \text{if } i<g(K) \text{ and } g(K)-i\equiv 0 \pmod 2.\end{cases}$$

In this situation, $\widehat{\mathrm{HFK}}(X,K,\mathfrak{s})=0$ unless $\mathfrak{s}=n_i$ for some $-k\leq i\leq k$, in which case $\widehat{\mathrm{HFK}}(X,K,\mathfrak{s})=\mathbb{Z}/2\mathbb{Z}$ and it is supported entirely in homological degree δ_i. If the generator of $\widehat{\mathrm{HFK}}(X,K,n_i)$ is denoted by $\mathbf{x}_i$, the filtered chain complex $(\mathbb{B}(K),d_{\mathbb{B}})$ may be described as $\mathbb{B}=\langle \mathbf{x}_{-k},\dots,\mathbf{x}_k\rangle$, where the differential is given by

$$d_{\mathbb{B}}(\mathbf{x}_i)=\begin{cases}0 & \text{if } i=k \text{ or } 0<k-i \text{ is odd},\\ \mathbf{x}_{i+1} & \text{otherwise.}\end{cases}$$

Proof Let us assume that

$$-g(K)=n_{-k}<n_{1-k}<\cdots<n_k=g(K)$$

are the values for $\mathfrak{s}$ so that the associated vector space $\mathbb{H}\{\mathfrak{s}\}$ is nontrivial. From the duality isomorphism of knot Floer homology groups we thus know that $n_{-i}=-n_i$ for each i. Let us denote the generators of $\mathbb{H}\{n_i\}$ by $\mathbf{y}_i^1,\mathbf{y}_i^2,\dots,\mathbf{y}_i^{\ell_i}$. We may assume that under the duality map $\Xi(\mathbf{y}_{-i}^j)=\mathbf{y}_i^j$. From the computation of $C_n(g(K)+n)$ in the beginning of the proof for Theorem 3.1, we know that $\ell_k=1$, and we may thus denote $\mathbf{y}_k^1$ by $\mathbf{x}_k$ and $\mathbf{y}_{-k}^1$ by $\mathbf{x}_{-k}$. We also know that $\mathbf{x}_k$ survives in the homology of $\mathbb{B}(K)$, ie it is not in the image of $d_{\mathbb{B}}$.

We will prove the following claim by an induction on i, which implies the above theorem.

Claim *In the above situation, $\ell_k = \ell_{k-1} = \cdots = \ell_{k-i} = 1$. Furthermore, we may choose $\mathbf{x}_{k-j} \in \mathbb{B}\{\geq n_{k-j}\}$ and $\mathbf{x}_{j-k} \in \mathbb{B}\{\geq n_{j-k}\}$ so that they survive in the homology of quotient complexes $\mathbb{H}\{n_{k-j}\} = \mathbb{B}\{\geq n_{k-j}\}/\mathbb{B}\{> n_{k-j}\}$ and $\mathbb{H}\{n_{j-k}\} = \mathbb{B}\{\geq n_{j-k}\}/\mathbb{B}\{> n_{j-k}\}$ respectively, so that*

$$\begin{aligned} d_{\mathbb{B}}(\mathbf{x}_{k-j}) &= \begin{cases} 0 & \text{if } j=0 \text{ or } j \text{ is odd,} \\ \mathbf{x}_{k-j+1} & \text{otherwise,} \end{cases} && j = 0,1,\ldots,i, \\ d_{\mathbb{B}}(\mathbf{x}_{j-k}) &= \begin{cases} 0 & \text{if } j \text{ is odd,} \\ \mathbf{x}_{j-k+1} & \text{otherwise,} \end{cases} && j = 0,1,\ldots,i-1. \end{aligned} \tag{13}$$

From the above considerations, the case $i=0$ is already proved. Now, assume that the claim is true for i. We will prove that it will also follow for $i+1$. We will need to consider two cases depending on the parity of i.

First, assume that i is even. For any value of $\mathfrak{s}$, we have $\epsilon_{\mathfrak{s}} = q_{-\mathfrak{s}} \circ \Xi_{\mathfrak{s}} \circ \tau_{\mathfrak{s}} = 0$. Note that the map $\tau_{\mathfrak{s}}\colon \mathbb{H}\{<\mathfrak{s}\} \to \mathbb{H}\{\mathfrak{s}\}$ is an isomorphism if and only if the homology group $\mathbb{H}\{\leq\mathfrak{s}\}$ is trivial. However, the differential $d_{\mathbb{B}}$ of the complex $\mathbb{B}$ induces a map $p_{\mathfrak{s}}\colon \mathbb{H}\{\leq\mathfrak{s}\} \to \mathbb{H}\{>\mathfrak{s}\}$ and its image does not cover the class of $\mathbf{x}_k$ in the target. Since the homology of the mapping cone of $p_{\mathfrak{s}}$ is $\mathbb{H}$, this means that $p_{\mathfrak{s}}$ is injective and that the rank of $\mathbb{H}\{\leq\mathfrak{s}\}$ is one less than the rank of $\mathbb{H}\{>\mathfrak{s}\}$. If we set $\mathfrak{s} = n_{k-i-1}$, since there are no generators associated with relative Spin^c classes $n_{k-i-1} < \mathfrak{t} < n_{k-i}$, we have $\mathbb{H}\{>\mathfrak{s}\} = \mathbb{H}\{\geq n_{k-i}\} = \mathbb{Z}/2\mathbb{Z}$ by the induction hypothesis. The above observation thus implies that $\mathbb{H}\{\leq n_{k-i-1}\}$ is trivial. Thus the map $\tau_{n_{k-i-1}}$ is an isomorphism. Since for $\mathfrak{s} = n_{k-i-1}$, $\tau_{\mathfrak{s}}$ and $\Xi_{\mathfrak{s}}$ are isomorphisms and $\epsilon_{\mathfrak{s}}$ is trivial, we may conclude that $q_{-\mathfrak{s}} = 0$. This last assumption is equivalent to the assumption that $\tau_{-\mathfrak{s}}\colon \mathbb{H}\{<-\mathfrak{s}\} \to \mathbb{H}\{-\mathfrak{s}\}$ is surjective. However, for $\mathfrak{s} = n_{k-i-1}$, setting $j = i+1-k$ we have

$$\mathbb{H}\{<-\mathfrak{s}\} = \mathbb{H}\{\leq n_{i-k}\} = \langle \mathbf{x}_{i-k} \rangle \quad \text{and} \quad \mathbb{H}\{-\mathfrak{s}\} = \langle \mathbf{y}_j^1, \ldots, \mathbf{y}_j^{\ell_j} \rangle.$$

Surjectivity of $\tau_{-\mathfrak{s}}$ implies that $\ell_j = 1$ and that $\tau_{-\mathfrak{s}}$ is an isomorphism. Setting $\mathbf{x}_{i+1-k} = \mathbf{y}_{i+1-k}^1$, this means that $d_{\mathbb{B}}(\mathbf{x}_{i-k})$ is equal to $\mathbf{x}_{i+1-k}$ plus terms in higher filtration levels. After a suitable change of basis for the filtered chain complex we may assume that $d_{\mathbb{B}}(\mathbf{x}_{i-k}) = \mathbf{x}_{i+1-k}$. On the other hand, $\mathbb{H}\{n_{k-i-1}\} = \mathbb{H}\{n_{i+1-k}\} = \langle \mathbf{x}_{i+1-k} \rangle$ is generated by a generator $\mathbf{x}_{k-i-1} = \Xi(\mathbf{x}_{i+1-k})$. Since the homology group $\mathbb{H}\{\geq n_{k-i}\} = \mathbb{H}\{> n_{k-i-1}\}$ is generated by $\mathbf{x}_k$ and the image of $\mathbf{x}_{k-i-1}$ under $d_{\mathbb{B}}$ in $\mathbb{H}\{> n_{k-i-1}\}$ can not be equal to $\mathbf{x}_k$, after another suitable change of basis for the filtered chain complex $\mathbb{B}$ we may assume that $d_{\mathbb{B}}(\mathbf{x}_{k-i-1}) = 0$. This completes the proof of the assertions of the above claim for $i+1$ when i is even.

If i is odd, $\mathbb{H}\{\leq n_{i-k}\}$ is trivial by the induction hypothesis. This implies that $\mathbb{H}\{\leq n_{i+1-k}\}$ is isomorphic to $\mathbb{H}\{n_{i+1-k}\}$ and the isomorphism is given by the map

$$q_{n_{i+1-k}}\colon \mathbb{H}\{n_{i+1-k}\} \longrightarrow \mathbb{H}\{\leq n_{i+1-k}\}.$$

Since $\epsilon_{n_{k-i-1}}$ is trivial, we may conclude that $\tau_{n_{k-i-1}}$ is trivial. In other words, $\mathbb{H}\{n_{k-i-1}\}$ is not in the image of $d_{\mathbb{B}}$. As a result, the map $r_{n_{k-i-1}}$ from $\mathbb{H}\{n_{k-i-1}\}$ to $\mathbb{H}\{> n_{k-i-1}\} = \mathbb{H}\{\geq n_{k-i}\}$, induced by the differential $d_{\mathbb{B}}$, is injective. From induction hypothesis we may conclude that $\mathbb{H}\{\geq n_{k-i}\} = \langle \mathbf{x}_{k-i}, \mathbf{x}_k\rangle$, and $\mathbf{x}_k$ is not in the image of $d_{\mathbb{B}}$. Thus $\mathbb{H}\{n_{k-i-1}\}$ is generated by a single generator $\mathbf{x}_{k-i-1} = \mathbf{y}^1_{k-i-1}$ and we may assume that $d_{\mathbb{B}}(\mathbf{x}_{k-i-1}) = \mathbf{x}_{k-i}$. Since $\mathbb{H}\{< n_{i+1-k}\} = \mathbb{H}\{\leq n_{i-k}\}$ is trivial, $\mathbf{x}_{i+1-k} = \Xi(\mathbf{x}_{k-i-1})$ is not in the image of $d_{\mathbb{B}}$ and the assertions of the above claim are thus satisfied for $i+1$ if i is odd. This completes the induction and proves the above claim.

In order to complete the proof of the theorem, let us denote the homological degree of the generator $\mathbf{x}_j$ by $\delta_j = \delta(\mathbf{x}_j)$. Since $\mathbb{H}$ is generated by $\mathbf{x}_k$, it is clear that $\delta_k = d(X)$ and since for even values of i, $d_{\mathbb{B}}(\mathbf{x}_{k-i}) = \mathbf{x}_{k-i+1}$, we have $\delta_{k-i} = \delta_{k-i+1} + 1$. Also, note that $\delta_{-j} = 2n_j + \delta_j$. Thus for any odd value of i we have

$$\begin{aligned}\delta_{k-i} = -2n_{k-i} + \delta_{i-k} &= -2n_{k-i} + \delta(d_{\mathbb{B}}(\mathbf{x}_{i-k-1}))\\ &= -2n_{k-i} + (\delta_{i-k-1} - 1)\\ &= -2n_{k-i} + 2n_{k-i+1} + \delta_{k-i+1} - 1.\end{aligned} \tag{14}$$

This completes the proof of the theorem. □

We are almost done with the proof of our main theorem, Theorem 1.2.

Proof of Theorem 1.2 Suppose that the knot $K \subset X$ in the homology sphere L–space X has Seifert genus $g(K)$. We first show that if $n \geq 2g(K)$ is an integer so that K_n has simple knot Floer homology, then $X_n(K)$ is an L–space and that the knot Floer homology of K has the structure described in Theorems 1.2 and 4.1. We have seen in the beginning of Theorem 4.1 that this assumption (ie that K_n has simple knot Floer homology) implies that all the maps

$$\epsilon_{\mathfrak{s}}\colon \mathbb{H}\{< \mathfrak{s}\} \to \mathbb{H}\{\leq -\mathfrak{s}\}$$

are trivial. Therefore, the hypothesis of Theorem 4.1 are satisfied for the knot $K \subset X$, and the Floer homology of K has the structure described in Theorems 1.2 and 4.1, ie the filtered chain complex $(\mathbb{B}(K), d_{\mathbb{B}})$ has the special form described in Theorem 4.1.

Therefore, for any given relative Spin^c structure $\mathfrak{s}$ satisfying $-g(K) < \mathfrak{s} \le g(K)$, precisely one of the two groups $\mathbb{H}\{<\mathfrak{s}\}$ and $\mathbb{H}\{\le -\mathfrak{s}\}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and the other one is trivial. Thus $\widehat{\mathrm{HF}}(X_n(K),[\mathfrak{s}]) = \mathbb{Z}/2\mathbb{Z}$ by our previous considerations. The isomorphism $\widehat{\mathrm{HF}}(X_n(K),[\mathfrak{s}]) = \widehat{\mathrm{HF}}(X) = \mathbb{Z}/2\mathbb{Z}$ is clear for other Spin^c structures. This shows that if $n \ge 2g(K)$ and K_n has simple knot Floer homology, $X_n(K)$ is an L–space and completes the proof in one direction.

For the other direction, suppose that $(\mathbb{B}(K), d_{\mathbb{B}})$ has the structure described in Theorems 1.2 and 4.1. Theorem 2.3 then gives a description of the Floer homology groups $\widehat{\mathrm{HFK}}(X_n(K), K_n; \mathfrak{s})$ for different values of $\mathfrak{s} \in \underline{\mathrm{Spin}}^c(X,K) = \mathbb{Z}$ as the homology of the mapping cone of a map

$$h_n\colon \mathbb{H}\{\ge \mathfrak{s}\} \oplus \mathbb{H}\{> n-\mathfrak{s}\} \to \mathbb{H} = \mathbb{Z}/2\mathbb{Z}. \tag{15}$$

If $\mathfrak{s} \le g(K)$, $n-\mathfrak{s} \ge n-g(K) \ge g(K)$ and $\mathbb{H}\{> n-g(K)\} = 0$. In this situation $\widehat{\mathrm{HFK}}(X_n(K), K_n; \mathfrak{s}) = \mathbb{H}\{<\mathfrak{s}\}$ and

$$\mathbb{H}\{<\mathfrak{s}\} = \begin{cases} 0 & \text{if } n_{k-i-1} < \mathfrak{s} \le n_{k-i} \text{ and } i \text{ is even,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n_{k-i-1} < \mathfrak{s} \le n_{k-i} \text{ and } i \text{ is odd,} \\ 0 & \text{if } \mathfrak{s} \le n_{-k} = -g(K). \end{cases}$$

A similar argument shows that for $\mathfrak{s} > g(K)$ we have $\widehat{\mathrm{HFK}}(X_n(K), K_n; \mathfrak{s}) = \mathbb{H}\{\le n-\mathfrak{s}\}$ and

$$\mathbb{H}\{\le n-\mathfrak{s}\} = \begin{cases} 0 & \text{if } n-n_{k-i} < \mathfrak{s} \le n-n_{k-i-1} \text{ and } i \text{ is even,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n-n_{k-i} < \mathfrak{s} \le n-n_{k-i-1} \text{ and } i \text{ is odd,} \\ 0 & \text{if } \mathfrak{s} > n-n_{-k} = n+g(K). \end{cases}$$

For a relative Spin^c structure $\mathfrak{s} \in \underline{\mathrm{Spin}}^c(X,K) = \mathbb{Z}$ the nontriviality assumption $\widehat{\mathrm{HFK}}(X_n(K), K_n; \mathfrak{s}) \ne 0$ thus implies that $-g(K) < \mathfrak{s} \le n+g(K)$. If for two relative Spin^c structures $\mathfrak{s}$ and $\mathfrak{t}$ both $\widehat{\mathrm{HFK}}(X_n(K), K_n; \mathfrak{s})$ and $\widehat{\mathrm{HFK}}(X_n(K), K_n; \mathfrak{t})$ are nontrivial and $\mathfrak{t}-\mathfrak{s}$ is a positive multiple of n, we should have $-g(K) < \mathfrak{s} \le g(K)$ and $\mathfrak{t} = n+\mathfrak{s}$. Furthermore, $n_{k-i-1} < \mathfrak{s} \le n_{k-i}$ for an even integer i. Thus

$$\begin{aligned} n - n_{k-(2k-i-1)} = n + n_{k-i-1} < \mathfrak{t} = \mathfrak{s} + n \\ \le n + n_{k-i} = n - n_{k-(2k-i-1)-1}, \end{aligned} \tag{16}$$

where $j = 2k-i-1$ is an odd integer. From the above description we have $\widehat{\mathrm{HFK}}(X_n(K), K_n; \mathfrak{t}) = 0$, which is a contradiction. Thus for all relative Spin^c classes in $\underline{\mathrm{Spin}}^c(X,K)$ in the congruence class of $\mathfrak{s}$ modulo n, at most one of them, say $\mathfrak{t}$, has the property that $\widehat{\mathrm{HFK}}(X_n(K), K_n; \mathfrak{t}) \ne 0$. This clearly implies that K_n has simple knot Floer homology, completing the proof. □

References

[1] **K L Baker**, **J E Grigsby**, **M Hedden**, *Grid diagrams for lens spaces and combinatorial knot Floer homology*, Int. Math. Res. Not. (2008) Art. ID rnm024, 39 MR2429242

[2] **E Eftekhary**, *Floer homology and existence of incompressible tori in homology spheres* arXiv:0807.2326

[3] **E Eftekhary**, *Floer homology and splicing knot complements* arXiv:0802.2874

[4] **E Eftekhary**, *Knots inside homology spheres with simple knot Floer homology are trivial* arXiv:1003.0737

[5] **E Eftekhary**, *Longitude Floer homology and the Whitehead double*, Algebr. Geom. Topol. 5 (2005) 1389–1418 MR2171814

[6] **E Eftekhary**, *A combinatorial approach to surgery formulas in Heegaard Floer homology*, Algebr. Geom. Topol. 9 (2009) 2225–2246 MR2558310

[7] **M Hedden**, *On Floer homology and the Berge conjecture on knots admitting lens space surgeries*, Trans. Amer. Math. Soc. 363 (2011) 949–968 MR2728591

[8] **Y Ni**, *Link Floer homology detects the Thurston norm*, Geom. Topol. 13 (2009) 2991–3019 MR2546619

[9] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and genus bounds*, Geom. Topol. 8 (2004) 311–334 MR2023281

[10] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and knot invariants*, Adv. Math. 186 (2004) 58–116 MR2065507

[11] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) 159 (2004) 1027–1158 MR2113019

[12] **P Ozsváth**, **Z Szabó**, *On knot Floer homology and lens space surgeries*, Topology 44 (2005) 1281–1300 MR2168576

[13] **P Ozsváth**, **Z Szabó**, *Knot Floer homology and integer surgeries*, Algebr. Geom. Topol. 8 (2008) 101–153 MR2377279

[14] **J Rasmussen**, *Lens space surgeries and L–space homology spheres* arXiv:0710.2531

[15] **J Rasmussen**, *Floer homology and knot complements*, PhD thesis, Harvard University (2003) arXiv:math.GT/0306378

School of Mathematics, Institute for Research in Fundamental Sciences (IPM)
PO Box 19395-5746, Tehran 19395, Iran

eaman@ipm.ir

Received: 19 March 2010 Revised: 31 August 2010