4–fold symmetric quandle invariants of 3–manifolds

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The paper introduces 4–fold symmetric quandles and 4–fold symmetric quandle homotopy invariants of 3–manifolds. We classify 4–fold symmetric quandles and investigate their properties. When the quandle is finite, we explicitly determine a presentation of its inner automorphism group. We calculate the container of the 4–fold symmetric quandle homotopy invariant. We also discuss symmetric quandle cocycle invariants and coloring polynomials of 4–fold symmetric quandles.

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1 Introduction

A quandle is a set with a certain binary operation satisfying a self-distribute law. Quandles are adapted to the oriented link theory. For an oriented link $L \subset S^3$, Joyce [16] defined the link quandle Q_L as an analog of the fundamental group $\pi_1(S^3 \setminus L)$. For a quandle X, a quandle homomorphism $Q_L \to X$ is called an X–coloring of L. From algebraic topology, given a quandle X, Fenn, Rourke and Sanderson [9] defined the rack space analogous to the classifying space of groups. They [10; 11] show that the second homotopy group is isomorphic to a bordism group consisting of all "framed X–colorings". Then, a quandle homotopy invariant of oriented links can be defined by an invariant valued in the group ring $\mathbb{Z}[\pi_2(BX)]$, where the space BX is a certain modification of the rack space. On the other hand, quandle cocycle invariants of oriented links introduced by Carter et al [3] using 2–cocycles of $H^2(BX; A)$ are computable and practical; they can, however, be derived from the quandle homotopy invariant (see, eg, Carter, Kamada and Saito [4] and Fenn and Rourke [8]).

In another direction, Hatakenaka [13] reformulated certain Dijkgraaf–Witten invariants of 3–manifolds [5] as quandle cocycle invariants. To see this, she made use of the fact that any 3–manifold is a 4–fold simple branched covering of the 3–sphere branched along a link L. Then the associated simple monodromy representation onto $\mathfrak{S}_4$ can be regarded as an $\mathcal{S}$–coloring of L, which we call a labeled link. Here $\mathcal{S} := \{(ij) \in \mathfrak{S}_4\}$ is a quandle with the conjugate operation. Hence, we may consider any 3–manifold to be a labeled link. Further, it is known (see Apostolakis [1] and Bobtcheva and Piergallini [2]) that homeomorphism classes of 3–manifolds are in 1–1 correspondence

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with the set of labeled links modulo some “MI and MII moves” (see Figure 3). The key point is that, using these facts, she presented the Dijkgraaf–Witten invariants of 3–manifolds as some invariants of labeled links.

In this paper, our purpose is to construct and study an invariant of 3–manifolds obtained from the quandle homotopy invariant of labeled links. The idea behind the construction is simple: since any 3–manifold can be regarded as a labeled link of L, we define a quandle X over $\mathcal{S}$, and consider X–colorings obtained by lifting the $\mathcal{S}$–coloring to be an invariant of the 3–manifold. For this, noting that the monodromies are unrelated to orientations of links, we focus on symmetric quandles introduced by Kamada [17] which are suitable for *un*oriented links. Then we obtain an axiomatization necessary to construct invariants of 3–manifolds in terms of symmetric quandles, and define a 4–fold symmetric quandle to be a symmetric quandle which is unchanged under MI and MII moves mentioned above (Section 3.1). Further, for a finite 4–fold symmetric quandle X, we construct a 4–fold symmetric homotopy invariant valued in a group ring $\mathbb{Z}[\Pi_{2,\rho}^{4\mathrm{f}}(X)]$. Here, the group $\Pi_{2,\rho}^{4\mathrm{f}}(X)$ is defined as a certain link cobordance group which is invariant under MI, MII moves (Definition 3.3), and turns out to be a quotient group of $\pi_2(BX)$ (see Section 6.1).

Although we have obtained the invariant of 3–manifolds, the definition seems teleological and abstract. Particularly, it is a problem to study what the container $\Pi_{2,\rho}^{4\mathrm{f}}(X)$ is. To deal with this, our next step is to resolve 4–fold symmetric homotopy invariants into concrete objects.

We first classify the 4–fold symmetric quandles as follows. We define a cored group to be a pair of a group G and a central element $c \in Z(G)$ satisfying $c^2 = e$. A cored group (G, c) gives rise to a 4–fold symmetric quandle denoted by $\widetilde{G}_c$ (Example 4.1), which is a slight generalization of quandles considered by Hatakenaka [13]. Roughly speaking, $\widetilde{G}_c$ is like to be a principal G–bundle over $\mathcal{S}$ with an involution. Conversely, given a 4–fold symmetric quandle X, we find a cored group (G, c) related to X by a 4–fold symmetric quandle isomorphism $X \cong \widetilde{G}_c$ (Theorem 4.2). As a corollary, we obtain a category equivalence between a category of cored groups and a category of (two-pointed) 4–fold symmetric quandles (Corollary 4.3). The corollary says that the symmetric structure introduced by Kamada [17] makes our work of 3–manifold invariants broader. In conclusion, as a result of Theorem 4.2, we may consider only 4–fold symmetric quandles of the forms $\widetilde{G}_c$ later.

We next investigate some properties of $\widetilde{G}_c$. In general, any “connected” quandle X is known to be determined by the inner automorphism group $\mathrm{Inn}(X)$ (see Section 5.1). We show the quandle $\widetilde{G}_c$ is connected and of type 4 (Lemmas 3.5, 3.7). Further, for a finite cored group (G, c), we explicitly determine $\mathrm{Inn}(\widetilde{G}_c)$ using a wreath product

$G^4 \rtimes \mathfrak{S}_4$ (Theorem 5.4). More precisely, putting the commutator subgroup $[G,G]$ of G, $\mathrm{Inn}(\widetilde{G}_c)$ is isomorphic to a quotient group $I_{G,c}/Z_{G,c}$, where

$$I_{G,c} = \{\, (x,y,z,w;\sigma) \in G^4 \rtimes \mathfrak{S}_4 \mid c^{(\mathrm{sgn}(\sigma)-1)/2} xyzw \in [G,G] \,\},$$
$$Z_{G,c} = \{\, (z,z,z,z;e) \in G^4 \rtimes \mathfrak{S}_4 \mid z^4 \in [G,G],\ z \in Z(G) \,\}.$$

Following the theory of Eisermann [7; 6], the explicit presentation of $\mathrm{Inn}(\widetilde{G}_e)$ helps later a computation of "quandle cocycle invariants" explained below.

Next, we give two approaches to estimate the container $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ of our invariant. One shows that for a finite cored group (G,c), $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ is a finite abelian group whose elements are annihilated by $2^{12} \cdot 3^4 \cdot |G|^{12} |[G,G]|^4$ (Theorem 6.2). To prove this, we take a viewpoint that $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ is a quotient of the homotopy group $\pi_2(BX)$, and use the author's results on $\pi_2(BX)$ in [21]. Also, this viewpoint concretely enable us to compute $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ with $(G,c) = (\mathbb{Z}/2\mathbb{Z}, 0)$ and $(\mathbb{Z}/2\mathbb{Z}, 1)$ in Section 6.3. In another direction, we discuss the 4–fold symmetric quandle homotopy invariant of 2–fold and 3–fold simple branched covering spaces of S^3 (Section 6.2). As an application, we obtain a combinatorial estimate whether a 3–manifold is a double branched covering of S^3 or not (Proposition 7.4), although we find no such examples by using the estimate (Problem 7.5).

However, it is difficult to compute explicitly the group $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ and our invariant. Our purpose in Sections 7–8 is to reduce the invariant to other computable invariants. For an abelian group A, we define a 4–fold symmetric quandle 2–cocycle of $\widetilde{G}_c$ with (local) coefficients A. This cocycle is a modification of the (symmetric) quandle cocycles given in Kamada et al [3; 18]. Using a 4–fold symmetric quandle cocycle, we define a 4–fold symmetric quandle cocycle invariant of 3–manifolds. Similar to quandle cocycle invariants of links, any 4–fold symmetric quandle *cocycle* invariant of $\widetilde{G}_c$ is derived from the 4–fold symmetric quandle *homotopy* invariant (Proposition 7.3). As a corollary of the above estimate of $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$, if $A \otimes \mathbb{Z}/(6|G|)\mathbb{Z} \cong 0$ (eg, $A = \mathbb{Q}$), the associated 4–fold symmetric quandle cocycle invariants are trivial (Remark 5). Therefore, for a discovery of a new invariant, we shall assume $A \otimes \mathbb{Z}/(6|G|)\mathbb{Z} \neq 0$.

A benefit of the 4–fold symmetric quandle cocycle invariants is computable given a presentation of a 4–fold quandle symmetric 2–cocycle. However, in practice it is not easy to find such 2–cocycles; hence, it is not easy to calculate the cocycle invariants either in general. Then, as a simple case, we confine ourselves to quandle cocycles of $\widetilde{G}_e$ with trivial coefficients, when $c = e$. To begin, as to $\widetilde{G}_e$, we show that any symmetric quandle cocycle introduced by Kamada and Oshiro [18] is 4–fold, if the coefficient group is annihilated by 2 (Proposition 8.1). Furthermore, we obtain a calculation of 4–fold symmetric quandle cocycle invariants without having been given the presentations of 4–fold symmetric quandle cocycles as follows. To see this, we focus here on the coloring polynomial introduced by Eisermann [7]; he showed that the coloring polynomial is the universal among cocycle invariants of knots, and is computable without knowing an explicit presentation of quandle cocycles. However, the problem is that, in order to study the coloring polynomial, we have to determine the container of the polynomial. Under the influence of his work, we then modify the coloring polynomial as an invariant of 3–manifolds. Further, we show that the polynomial produces some 4–fold symmetric quandle cocycle invariants with trivial coefficients (see Proposition 8.3). In addition, we determine an explicit presentation of the container (Proposition 8.4). As examples, we concretely compute the containers of some groups (Examples 8.6, 8.7, 8.8). Consequently, we obtain a method to calculate 4–fold symmetric quandle cocycle invariants with trivial coefficients. However, unfortunately, we have not been able to find examples of nontrivial invariants yet (Problem 8.9).

This paper approaches the 4–fold symmetric quandle homotopy invariant in an algebraic context. In the next paper [14], using results in this paper, Hatakenaka and the author will give some topological approaches and applications of our invariant, and compare our invariant with the Dijkgraaf–Witten invariant and with the Chern–Simons invariant. It is true that the definition of our invariant seems a little abstract and universal, but studying universal objects is useful to relate other objects in mathematics.

This paper is organized as follows. In Section 2, we review the definitions of symmetric quandles and 4–fold branched covering spaces. In Section 3, we introduce a 4–fold symmetric quandle homotopy invariant, and investigate properties of 4–fold symmetric quandles. In Section 4, we classify 4–fold symmetric quandles. In Section 5, we determine the inner automorphism group of any finite 4–fold symmetric quandle. In Section 6, we estimate the group $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$. In Section 7, we introduce and study the 4–fold symmetric quandle cocycle invariant. In Section 8, we discuss coloring polynomials. Section 9 is an Appendix: we compare the symmetric quandle homotopy invariant of 3–manifolds with that of links.

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2 Review of symmetric quandles and covering presentations

We begin reviewing some of notation on the symmetric quandles introduced by Kamada [17, Section 2.1], and a relation between labeled diagrams and 3–manifolds in Section 2.2.

2.1 Symmetric quandles and symmetric colorings

A *quandle* is a set X with a binary operation $(x, y) \to x * y$ satisfying the following:

(Q1) For any $x \in X, x * x = x$.

(Q2) For any $x, y \in X$, there exists a unique element $z \in X$ such that $z * y = x$.

(Q3) For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

A quandle X is of *type* n, if it satisfies $(\cdots(x * y) * y \cdots) * y = x$ (star n–times on the right with y) for any $x, y \in X$. For example, any group G is a quandle by an operation $a * b := ba^{-1}b$ for $a, b \in G$, whose type is 2.

For a quandle X, an X*–coloring* of an oriented link diagram D_0 is a map

$$C\colon \{\text{arcs of } D_0\} \to X$$

satisfying the condition shown in Figure 1 at each crossing of D_0. We denote the set of X–colorings of D_0 by $\mathrm{Col}_X(D_0)$. It is well-known that if oriented link diagrams D_0 and $D'_{0'}$ are related by Reidemeister moves, then there exists a canonical bijection between $\mathrm{Col}_X(D_0)$ and $\mathrm{Col}_X(D'_{0'})$ (see, eg, [8]). Therefore, for a finite quandle X, the cardinality of $\mathrm{Col}_X(D_0)$ is an invariant of oriented links.

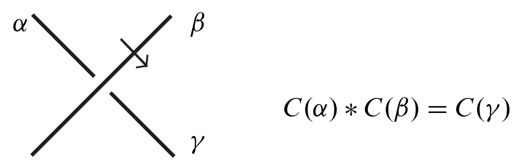


Figure 1: The condition of a coloring at each crossing

Next, we review symmetric quandles and symmetric colorings introduced by Kamada [17]. For a quandle X, a map $\rho\colon X \to X$ is a *good involution*, if it is an involution (ie, $\rho \circ \rho = \mathrm{id}_X$) such that

(Q4) For any $x, y \in X$, it satisfies $\rho(x * y) = \rho(x) * y$ and $(x * y) * \rho(y) = x$.

Such a pair (X, ρ) is called a *symmetric quandle*.

Let D be an unoriented link diagram on $\mathbb{R}^2$. Divide over-arcs at crossings of D. We call the resulting arcs *semiarcs* of D. Put the two normal orientations on each semiarc such as α_1, α_2 in Figure 2. For a symmetric quandle (X, ρ), an X_ρ*–coloring* of D is a map C: {the two orientations on semiarcs of $D\} \to X$ satisfying the following two conditions:

(X1) For the two normal orientations α_1, α_2 of the same semiarc as shown in Figure 2, the colors satisfy $C(\alpha_1) = \rho\big(C(\alpha_2)\big)$. (Since the color of an arc with one orientation determines the another, we will later draw the only one color of the two.)

(X2) For the four semiarcs coming from over-arcs of D at each crossing illustrated in Figure 2, the four orientations satisfy $C(\delta) = C(\alpha) * C(\beta)$ and $C(\beta) = C(\gamma)$.

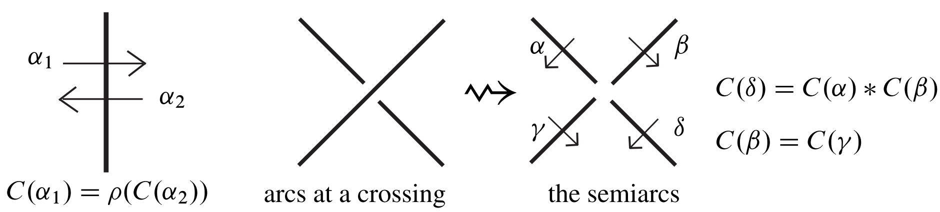


Figure 2: The condition of a symmetric coloring on semiarcs and at each crossings

Note that, by the axiom (Q4), the second condition (X2) is well-defined from the choice of those orientations. Let $\mathrm{Col}_{X,\rho}(D)$ denote the set of all X_ρ–colorings of D. If two unoriented link diagrams D_1 and D_2 are related by Reidemeister moves, then we have a natural bijection between $\mathrm{Col}_{X,\rho}(D_1)$ and $\mathrm{Col}_{X,\rho}(D_2)$ (see Kamada and Oshiro [18, Proposition 6.2]).

We explain a bijection below (1). Denote by $D_\circ$ the unoriented link diagram D equipped with an orientation. For a symmetric quandle (X, ρ), an X–coloring of $D_\circ$ is naturally extended to an X_ρ–coloring of D, using the above conditions (X1)–(X2). We thus have a canonical map

$$\mathfrak{P}_X\colon \mathrm{Col}_X(D_\circ) \longrightarrow \mathrm{Col}_{X,\rho}(D). \tag{1}$$

Conversely, given an X_ρ–coloring of D, if we restrict the X_ρ–coloring to the orientations of $D_\circ$, then we have an X–coloring of $D_\circ$ as an inverse of the map (1). Hence the map (1) turns out to be bijective (see [18, Theorem 6.7] for detail).

Let us review homomorphisms of quandles. A map $f\colon X \to Y$ between two quandles is called a *quandle homomorphism* if $f(a * b) = f(a) * f(b)$ for any $a, b \in X$. A quandle homomorphism $f\colon X \to Y$ between symmetric quandles (X, ρ_X) and (Y, ρ_Y) is *symmetric*, if $f(\rho_X(x)) = \rho_Y(f(x))$ for any $x \in X$. By elementary calculation, the preimage of a symmetric subquandle of Y is also a symmetric subquandle of X.

Finally, we give some remarks on a specified quandle denoted by $\mathcal{S}$, which plays a key role in this paper. The quandle $\mathcal{S}$ is defined to be elements of the symmetric group $\mathfrak{S}_4$ on two letters with the conjugation: $(ab) * (cd) = (cd)^{-1}(ab)(cd)$ for any $(ab),\ (cd) \in \mathfrak{S}_4$. $\mathcal{S}$ has a unique symmetric quandle structure with $\rho = \mathrm{id}_{\mathcal{S}}$ (see also Corollary 3.6 in the case $X = \mathcal{S}$). Note that $\mathcal{S} = \{(ij) \in \mathfrak{S}_4\}$, and that $\mathcal{S}$ is of order 6 and of type 2. Then, for an $\mathcal{S}_{\mathrm{id}}$–coloring, the two orientations of each semiarcs have the same color. So we often draw $\mathcal{S}_{\mathrm{id}}$–colorings such as a picture of semiarcs without orientation, similar to Figure 3.

2.2 Covering presentations

In this section, we briefly review covering presentations of 3–manifolds. We consider a d–fold simple covering of S^3 branched over a link L. Throughout this paper, the word "3–manifold" will always mean a connected, compact, oriented 3–dimensional manifold without boundary, and d–fold branched coverings are assume to be simple (in the sense of Hilden [15]). It is shown by Hilden [15] and Montesinos [20] that any 3–manifold M is a 3–fold branched covering space of S^3 along a knot. However, for the purpose to construct invariants of 3–manifolds, in this paper we mainly deal with 4–fold branched coverings.

Given a 4–fold covering of S^3 branched over a link $L \subset S^3$, we have the associated simple monodromy representation $\phi\colon \pi_1(S^3 \setminus L) \to \mathfrak{S}_4$ (see, eg, Prasolov and Sossinsky [22] or Rolfsen [23]). Here a homomorphism $\phi\colon \pi_1(S^3 \setminus L) \to \mathfrak{S}_4$ is said to be *simple*, if this is surjective and sends each meridian to a transposition in $\mathfrak{S}_4$. Put a link diagram D of L. An $\mathcal{S}_{\mathrm{id}}$–coloring of D whose image ($\subset \mathcal{S}$) generates $\mathfrak{S}_4$ will be called a *labeled diagram*. By Wirtinger presentation of $\pi_1(S^3 \setminus L)$, simple homomorphisms $\pi_1(S^3 \setminus L) \to \mathfrak{S}_4$ naturally correspond to labeled diagrams of D (see, eg, [22, Section 24]). In summary, any 3–manifold can be regarded as a labeled diagram. We often denote a labeled diagram by D_ϕ with respect to D and ϕ in this paper. Conversely, given a labeled diagram D_ϕ of a link L, since $\mathfrak{S}_4$ is generated by $\mathcal{S} \subset \mathfrak{S}_4$, we obtain a simple monodromy representation $\pi_1(S^3 \setminus L) \to \mathfrak{S}_4$ (see Rolfsen [23] for detail). Then we have a 3–manifold M as the resulting 4–fold branched covering of the link L.

It is known (see, eg, [22, Section 24.5]) that MI and MII moves of labeled diagrams, shown in Figure 3, do not change the topological type of the branched covering spaces. Conversely, Apostolakis [1] and Bobtcheva and Piergallini [2] showed:

Theorem 2.1 ([1], a special case of [2, Theorem 3]) *Two 4–fold simple coverings of S^3 branched over links represent the same 3–manifold up to homeomorphic, if and only if their associated labeled diagrams can be related by a finite sequence of MI, MII and Reidemeister moves on $\mathbb{R}^2$.*

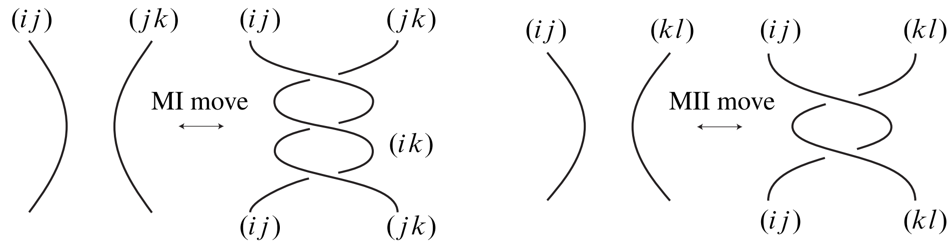


Figure 3: MI, II moves of labeled diagrams

In this paper, constructions of invariants of 3–manifolds are based on Theorem 2.1.

Finally, we give some remarks on 2–fold and 3–fold branched coverings of S^3. A labeled diagram D_ϕ is said to be 2*–fold* if $D_\phi = D_{12} \sqcup U_{23} \sqcup U_{34}$ forms, as shown in Figure 4 satisfying that its subdiagram D_{12} is labeled by $(12) \in \mathcal{S}$, where U_{23} (resp. U_{34}) is a trivial knot diagram labeled by $(23) \in \mathcal{S}$ (resp. $(34) \in \mathcal{S}$). One notices that if M is a double covering of S^3 branched over a link L, then we can choose a 2–fold labeled diagram of L which presents M. On the other hand, a labeled diagram D_ϕ is said to be 3*–fold*, if $D_\phi = D_{\mathcal{R}_3} \sqcup U_{34}$ as shown in Figure 4 satisfying that $D_{\mathcal{R}_3}$ is labeled by $\{(12),\ (23),\ (13)\} \subset \mathcal{S}$. Since any 3–manifold M is a 3–fold branched covering of a knot, M is represented by a 3–fold labeled diagram.

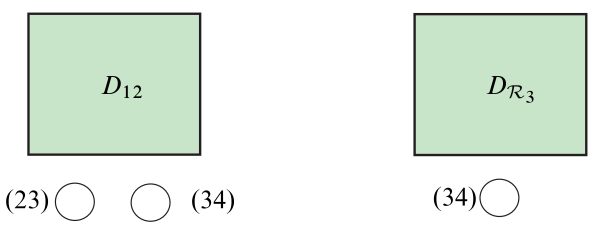


Figure 4: A 2–fold labeled diagram and a 3–fold labeled diagram

3 4–Fold symmetric quandle homotopy invariant

In Section 3.1, we introduce a 4–fold symmetric quandle, and define a 4–fold symmetric quandle homotopy invariant. In Section 3.2, we investigate some properties of 4–fold symmetric quandles.

3.1 Definitions

In this section, we first introduce 4–fold symmetric quandles. Further, we will define a group $\Pi^{4\mathrm{f}}_{2,\rho}(X)$, and a 4–fold symmetric quandle homotopy invariant valued in the group ring $\mathbb{Z}[\Pi^{4\mathrm{f}}_{2,\rho}(X)]$.

Definition 3.1 A 4–*fold symmetric quandle* is a triple (X, p_X, ρ) satisfying

(F1) (X, ρ) is a symmetric quandle.

(F2) The map $p_X\colon X \to \mathcal{S}$ is a symmetric quandle epimorphism. For $(ij) \in \mathcal{S}$, let us denote the preimage $p_X^{-1}(ij) \subset X$ by X_{ij}.

(F3) If i, j, k are distinct, then for any $x_{ij} \in X_{ij}$, $y_{jk} \in X_{jk}$, we have $x_{ij} * y_{jk} = \rho(y_{jk}) * x_{ij}$.

(F4) If i, j, k, l are distinct, then for any $z_{ij} \in X_{ij}$, $w_{kl} \in X_{kl}$, we have $z_{ij} * w_{kl} = z_{ij}$.

Throughout this paper, the symbols $1 \leq i, j, k, l \leq 4$ mean distinct indices. For simplicity, we often denote by X a 4–fold symmetric quandle.

We define colorings of labeled diagrams. For a 4–fold symmetric quandle X, we note that the quandle homomorphism $p_X\colon X \to \mathcal{S}$ induces $(p_X)_*\colon \mathrm{Col}_{X,\rho}(D) \to \mathrm{Col}_{\mathcal{S},\mathrm{id}}(D)$. Then, for a labeled diagram $D_\phi \in \mathrm{Col}_{\mathcal{S},\mathrm{id}}(D)$, we denote the preimage $(p_X)_*^{-1}(D_\phi)$ by $\mathrm{Col}_{X,\rho}(D_\phi)$, and call an element of $\mathrm{Col}_{X,\rho}(D_\phi)$ an X_ρ–*coloring of* D_ϕ.

The following proposition indicates that the axioms (F3) and (F4) correspond to MI and MII moves, respectively.

Proposition 3.2 *Let (X, p_X, ρ) be a 4–fold symmetric quandle. If two labeled diagrams are related by a finite sequence of MI, MII and Reidemeister moves on $\mathbb{R}^2$, then there is a bijection between the sets of X_ρ–colorings of the labeled diagrams. In particular, for a 3–manifold M presented by a labeled diagram D_ϕ, if X is finite, then the cardinality of the X_ρ–colorings $|\mathrm{Col}_{X,\rho}(D_\phi)| < \infty$ is a topological invariant of M.*

Proof If two link labeled diagrams are related by Reidemeister moves on $\mathbb{R}^2$, then we obtain the required bijection by the routine argument.

Then it is sufficient to give the proof for a single MI or MII move. First, let us consider the case where a labeled diagram D_ϕ changes into another one $D'_{\phi'}$ by a single MII move. Put an X_ρ–coloring of $D'_{\phi'}$ such as the right of Figure 5. It follows from the axiom (F4) that the right (or left) top and bottom arcs have the same X_ρ–color. Then, we have an X_ρ–coloring of D_ϕ corresponding to the X_ρ–coloring of $D'_{\phi'}$. Conversely,

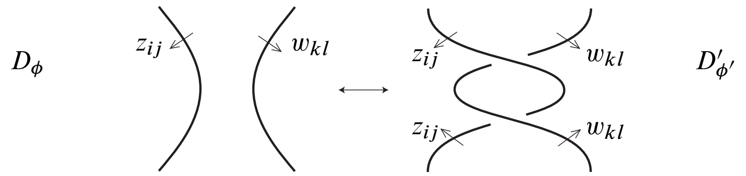


Figure 5: X_ρ–colorings of D_ϕ and $D'_{\phi'}$ related by a single MII move

given an X_ρ–coloring of D_ϕ, we can obtain an X_ρ–coloring of $D'_{\phi'}$ in a similar manner. Therefore this allows us to obtain a bijection $\mathrm{Col}_{X,\rho}(D_\phi) \simeq \mathrm{Col}_{X,\rho}(D'_{\phi'})$.

Finally, assume that D_ϕ changes into $D'_{\phi'}$ by a single MI move. Put an X_ρ–coloring of $D'_{\phi'}$ such as the right of Figure 6 below. If the left and right top arcs are colored

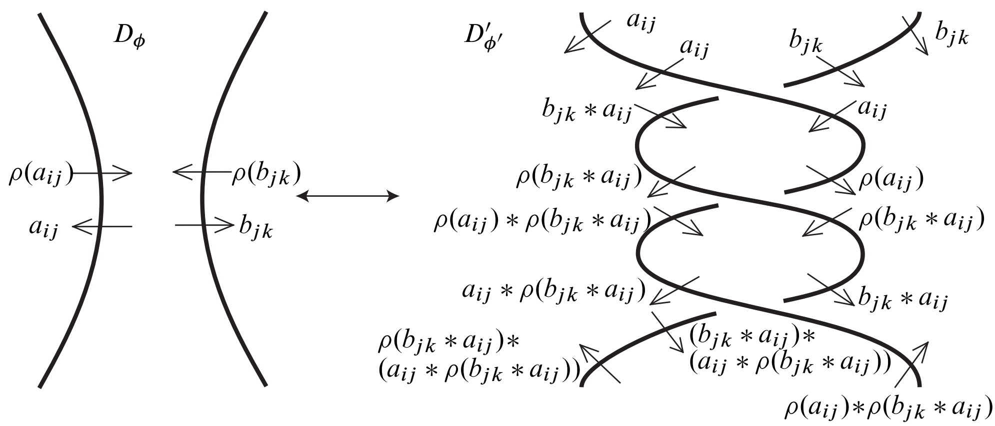


Figure 6: X_ρ–colorings of D_ϕ and $D'_{\phi'}$ related by a single MI move

by $a_{ij} \in X_{ij}$ and $b_{jk} \in X_{jk}$, respectively, then by the rule of X_ρ–colorings the left and right bottom arcs are colored by $\rho(b_{jk} * a_{ij}) * (a_{ij} * \rho(\, b_{jk} * a_{ij}))$ and $\rho(a_{ij}) * \rho(b_{jk} * a_{ij})$, respectively. By the axiom (F3) we have

$$\rho(a_{ij}) * \rho(b_{jk} * a_{ij}) = (b_{jk} * a_{ij}) * \rho(a_{ij}) = b_{jk}.$$

Further, it follows from this equality and (F3) that

$$\rho(b_{jk} * a_{ij}) * (a_{ij} * \rho(b_{jk} * a_{ij})) = \rho(\rho(a_{ij}) * b_{jk}) * \rho(b_{jk}) = (a_{ij} * b_{jk}) * \rho(b_{jk}) = a_{ij}.$$

Hence, since the right (or left) top and bottom arcs have the same X_ρ–color, we obtain an X_ρ–coloring of D_ϕ on the left in Figure 6. In summary, given an X_ρ–coloring of $D'_{\phi'}$, we have obtained that of D_ϕ, and vice versa. Similarly, the correspondence gives rise to a bijection $\mathrm{Col}_{X,\rho}(D_\phi) \simeq \mathrm{Col}_{X,\rho}(D'_{\phi'})$. □

In addition, we will provide the invariant $\mathrm{Col}_{X,\rho}(D_\phi)$ with a grading by an abelian group $\Pi^{4\mathrm{f}}_{2,\rho}(X)$ as follows. For this, for a symmetric quandle (X,ρ), we first define a group $\Pi_{2,\rho}(X)$, modifying a certain group introduced by Fenn, Rourke and Sanderson [9; 10] denoted by $\mathcal{D}(2,BX)$. $\Pi_{2,\rho}(X)$ is defined to be the set of all X_ρ–colorings of all link diagrams in $\mathbb{R}^2$ modulo Reidemeister moves and symmetric concordance relations, where the *symmetric concordance relations* are local moves shown in Figure 7. The set $\Pi_{2,\rho}(X)$ has a binary operation $\Pi_{2,\rho}(X)\times\Pi_{2,\rho}(X)\to\Pi_{2,\rho}(X)$ given by disjoint union. Precisely, given X_ρ–colorings C_1 and C_2, choose copies in disjoint half spaces, then define $C_1\cdot C_2$ to be $C_1\sqcup C_2$. This operation is well-defined and makes $\Pi_{2,\rho}(X)$ into an abelian group. Here, the inverse element of a representative X_ρ–coloring C is the mirror image of C, and the identity element of $\Pi_{2,\rho}(X)$ is the empty set. From the definition of $\Pi_{2,\rho}(X)$, we have a natural map

$$\Xi_X(D;\bullet)\colon \mathrm{Col}_{X_\rho}(D)\longrightarrow \Pi_{2,\rho}(X), \tag{2}$$

that is, $\Xi_X(D;\bullet)$ maps an X_ρ–coloring C to the canonical class $[C]\in\Pi_{2,\rho}(X)$.

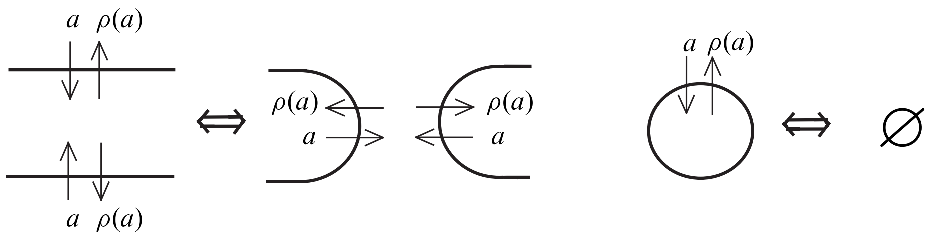


Figure 7: The symmetric concordance relations

Next, let X be a 4–fold symmetric quandle. We put a subgroup of $\Pi_{2,\rho}(X)$ generated by all X_ρ–colorings of all trefoils and of all Hopf links shown in Figure 8, where indices i,j,k,l run over all distinct natural numbers ≤ 4 and $x_{ij}, y_{jk}, z_{ij}, w_{kl}$ run over X_{ij}, X_{jk}, X_{ij}, X_{kl}, respectively. Let $\Pi^{4\mathrm{f}}_{2,\rho}(X)$ denote the quotient group modulo the subgroup. Then we put the natural projection $p^{4\mathrm{f}}\colon \Pi_{2,\rho}(X)\to\Pi^{4\mathrm{f}}_{2,\rho}(X)$, and consider a composite map

$$\Xi^{4\mathrm{f}}_X(D_\phi;\bullet)\colon \mathrm{Col}_{X,\rho}(D_\phi)\xrightarrow{\Xi_X(D_\phi;\bullet)}\Pi_{2,\rho}(X)\xrightarrow{p^{4\mathrm{f}}}\Pi^{4\mathrm{f}}_{2,\rho}(X), \tag{3}$$

where we denote by $\Xi_X(D_\phi;\bullet)$ the restriction of $\Xi_X(D;\bullet)$ to $\mathrm{Col}_{X,\rho}(D_\phi)$.

Definition 3.3 Let X be a finite 4–fold symmetric quandle. Let D_ϕ be a labeled diagram. Then a 4*–fold symmetric quandle homotopy invariant* of D_ϕ is the expression

$$\Xi^{4\mathrm{f}}_X(D_\phi):=\sum_{C\in\mathrm{Col}_{X,\rho}(D_\phi)}\Xi^{4\mathrm{f}}_X(D_\phi;C)\in\mathbb{Z}[\Pi^{4\mathrm{f}}_{2,\rho}(X)].$$

Figure 8: X_ρ–colorings of the trefoil and Hopf link

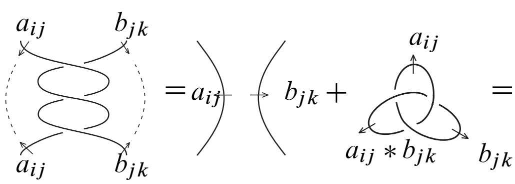
This is an invariant of 3–manifolds as follows.

Theorem 3.4 *Let D_ϕ and $D'_{\phi'}$ be labeled diagrams related by a finite sequences of MI, MII and Reidemeister moves. For a finite 4–fold symmetric quandle X, $\Xi^{4\mathrm{f}}_X(D_\phi) = \Xi^{4\mathrm{f}}_X(D'_{\phi'}) \in \mathbb{Z}[\Pi^{4\mathrm{f}}_{2,\rho}(X)]$. In particular, for a 3–manifold M presented by D_ϕ, the 4–fold symmetric quandle homotopy invariant $\Xi^{4\mathrm{f}}_X(D_\phi) \in \mathbb{Z}[\Pi^{4\mathrm{f}}_{2,\rho}(X)]$ is a topological invariant of M.*

Proof Recall the natural bijection $\mathfrak{B}\colon \mathrm{Col}_{X,\rho}(D_\phi) \to \mathrm{Col}_{X,\rho}(D'_{\phi'})$ of Proposition 3.2. Then, it suffices to show that $\Xi^{4\mathrm{f}}_X(D_\phi; C) = \Xi^{4\mathrm{f}}_X(D'_{\phi'}; \mathfrak{B}(C)) \in \Pi^{4\mathrm{f}}_{2,\rho}(X)$ for any $C \in \mathrm{Col}_{X,\rho}(D_\phi)$.

First, for two X_ρ–colorings related by Reidemeister moves on $\mathbb{R}^2$, the required equality follows from the definition of $\Pi_{2,\rho}(X)$. Next, if the two X_ρ–colorings are related by a single MI move as shown in Figure 6, we obtain

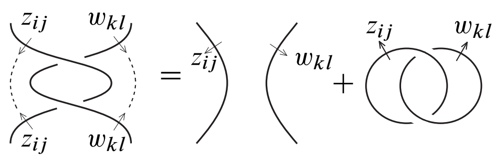
$$\Xi^{4\mathrm{f}}_X(D_\phi; C) =$$



$$= \Xi^{4\mathrm{f}}_X(D'_{\phi'}; \mathfrak{B}(C)) \in \Pi^{4\mathrm{f}}_{2,\rho}(X),$$

where we use symmetric concordance relations along the dashed lines in the second equality. Finally, if two X_ρ–colorings are related by a single MII move illustrated in Figure 5, then we similarly conclude

$$\Xi^{4\mathrm{f}}_X(D_\phi; C) =$$



$$= \Xi^{4\mathrm{f}}_X(D'_{\phi'}; \mathfrak{B}(C)) \in \Pi^{4\mathrm{f}}_{2,\rho}(X),$$

which completes the proof. □

Therefore we often denote the invariant of a 3–manifold M by $\Xi_X^{4\mathrm{f}}(M)$. Speaking of the invariants, it is important to calculate the container $\Pi_{2,\rho}^{4\mathrm{f}}(X)$. We will estimate or calculate $\Pi_{2,\rho}^{4\mathrm{f}}(X)$ in Section 6.

3.2 Properties of 4–fold symmetric quandles

In this section, we discuss some properties of 4–fold symmetric quandles (X, p_X, ρ). As before, we prepare a bijection (4) below. By the axiom (Q2), for $q_{ki} \in X_{ki}$, we have a bijection $(\bullet * q_{ki})\colon X \to X$ which sends x to $x * q_{ki}$. Since the projection $p_X\colon X \to \mathcal{S}$ is a quandle homomorphism, p_X commutes with the right operation $(\bullet * q_{ki})$. Hence the restriction on X_{jk} of $(\bullet * q_{ki})$

$$\big(\bullet * q_{ki}\big)\colon X_{jk} \longrightarrow X_{ij} \tag{4}$$

is bijective. Further, by the axioms (Q3) and (Q4), this map (4) is a symmetric quandle isomorphism. Therefore, for any $(ij) \in \mathcal{S}$, each symmetric subquandles X_{ij} are symmetric quandle isomorphic one another.

Lemma 3.5 *Let X be a 4–fold symmetric quandle. Fix $x_{ij} \in X_{ij}$.*

(i) *For $k \neq i, j$ and $y_{jk} \in X_{jk}$, $(x_{ij} * y_{jk}) * y_{jk} = \rho(x_{ij})$.*

(ii) *For any $z_{ij} \in X_{ij}$, $(x_{ij} * z_{ij}) * z_{ij} = x_{ij}$. The subquandle $X_{ij} \subset X$ is, particularly, of type 2.*

(iii) *X is a quandle of type 4. Namely, for any $a, b \in X$, $(((a * b) * b) * b) * b = a$.*

Proof (i) By the axiom (F3), we have

$$(x_{ij} * y_{jk}) * y_{jk} = \big(\rho(y_{jk}) * x_{ij}\big) * y_{jk} = \big(\rho(x_{ij}) * \rho(y_{jk})\big) * y_{jk} = \rho(x_{ij}),$$

where the last equality is obtained from the axiom (Q4).

(ii) By the bijection (4), there exists $p_{jk} \in X_{jk}$ such that $x_{ij} = p_{jk} * q_{ki}$. Then we obtain

$$\begin{aligned}(x_{ij} * z_{ij}) * z_{ij} &= \big((p_{jk} * q_{ki}) * z_{ij}\big) * z_{ij} = \big((p_{jk} * z_{ij}) * z_{ij}\big) * \big((q_{ki} * z_{ij}) * z_{ij}\big) \\ &= \rho(p_{jk}) * \rho(q_{ki}) = \rho(q_{ki}) * p_{jk} = p_{jk} * q_{ki} = x_{ij},\end{aligned}$$

where the second equality is obtained from (Q3), the third is obtained from (i) and the last line follows from the axiom (F3).

(iii) Fix $a = x_{ij} \in X_{ij}$. When $b \in X_{jk}$, by (i) we have

$$\big(((a * b) * b) * b\big) * b = \rho((a * b) * b) = (\rho \circ \rho)(a) = a.$$

Next, if $b \in X_{ij}$, then it follows from (ii) that $(((a*b)*b)*b)*b = (a*b)*b = a$. Finally, when $b \in X_{kl}$, it is clear that $(((a*b)*b)*b)*b = a$ by the axiom (F4). To summarize, for any $a, b \in X$, they satisfy the required conditions. □

Corollary 3.6 *A 4–fold symmetric quandle X is of type 2 if and only if the good involution ρ is the identity map of X.*

Proof When $\rho \neq \mathrm{id}_X$, X is not of type 2 by Lemma 3.5(i). Conversely, if $\rho = \mathrm{id}_X$, then X is immediately of type 2 from the definition of the good involution ρ. □

Next, we discuss inner automorphism groups of quandles denoted by $\mathrm{Inn}(X)$. Given a quandle X, recall the bijection $(\bullet * z)\colon X \to X$ for any $z \in X$, by the axiom (Q2). Then, the group $\mathrm{Inn}(X)$ is defined by a subgroup of $\mathfrak{S}_{|X|}$ generated by the right actions $(\bullet * z)$ for $z \in X$. A quandle X is said to be *connected*, if the action of $\mathrm{Inn}(X)$ on X is transitive. Also, we consider a natural map

$$\mathfrak{inn}_X\colon X \longrightarrow \mathrm{Inn}(X) \quad \text{given by} \quad \mathfrak{inn}_X(x) = (\bullet * x). \tag{5}$$

In general, $\mathfrak{inn}_X$ is not injective and it is difficult to determine $\mathrm{Inn}(X)$. However, in the case of $X = \mathcal{S}$, one can verify that $\mathrm{Inn}(X) \cong \mathfrak{S}_4$, and that the map $\mathfrak{inn}_X$ coincides with the natural inclusion $\mathcal{S} \hookrightarrow \mathfrak{S}_4$. Thereby the map $\mathfrak{inn}_X$ is injective, and the quandle $\mathcal{S}$ is connected. More generally, these properties hold for 4–fold symmetric quandles:

Lemma 3.7 *Any 4–fold symmetric quandle X is connected.*

Proof Put arbitrary $a, b \in X$. Denote a by $3x_{ij} \in X_{ij}$. Let us connect a to b case by case. First, we consider the case $b = y_{jk} \in X_{jk}$, where $i \neq k$. Then by the axioms (Q4) and (F3), we have a connection between x_{ij} and y_{jk} as follows:

$$y_{jk} = (y_{jk} * \rho(x_{ij})) * x_{ij} = (x_{ij} * y_{jk}) * x_{ij}.$$

Finally we consider the other cases. Namely, $b \in X_{ij} \cup X_{kl}$, where i, j, k, l are distinct. Then, as a result of the previous case, for $z_{jk} \in X_{jk}$, we can connect a with z_{jk}. Iterating the process, since z_{jk} can be connected to b, we have connected between a and b, which has dealt with all cases of $a, b \in X$. □

Lemma 3.8 *For a 4–fold symmetric quandle X, the map $\mathfrak{inn}_X$ is injective.*

Proof Let $a, b \in X$. Assume $x * a = x * b$ for any $x \in X$. We have to show $a = b$. Applying the epimorphism $p_X\colon X \to \mathcal{S}$ to the assumption, we have $p_X(x) * p_X(a) = p_X(x) * p_X(b) \in \mathcal{S}$. Since $\mathfrak{inn}_{\mathcal{S}}$ is injective, we obtain $p_X(a) = p_X(b)$. Without loss of generality, we may assume $p_X(a) = (12) \in \mathcal{S}$. Put $\kappa_{23} \in X_{23}$. By the axioms (Q4) and (F3), we thus conclude

$$\begin{aligned} a = (a * \kappa_{23}) * \rho(\kappa_{23}) &= (\rho(\kappa_{23}) * a) * \rho(\kappa_{23}) \\ &= (\rho(\kappa_{23}) * b) * \rho(\kappa_{23}) = (b * \kappa_{23}) * \rho(\kappa_{23}) = b. \qquad \square \end{aligned}$$

In Section 5 we determine $\mathrm{Inn}(X)$ and $\mathfrak{inn}_X$ for a finite 4–fold symmetric quandle X (Theorem 5.4 and Corollary 5.5).

Lastly, we discuss 4–fold symmetric quandle homomorphisms. For 4–fold symmetric quandles (X, p_X, ρ) and (Y, p_Y, ρ'), a symmetric quandle homomorphism $\psi\colon X \to Y$ is said to be 4*–fold*, if $p_X = p_Y \circ \psi$.

Lemma 3.9 *Let X, Y be 4–fold symmetric quandles. Fix $(\alpha, \beta) \in X_{12} \times X_{23}$, $(p, q) \in Y_{12} \times Y_{23}$.*

(i) *For any pair $(x, y) \in Y_{12} \times Y_{23}$, there exists $S \in \mathrm{Inn}(Y)$ such that $x \cdot S = p$ and $y \cdot S = q$.*

(ii) *Such $S \in \mathrm{Inn}(Y)$ induces a bijection*

$$(\bullet \cdot S)_*\colon \mathrm{Hom}_{4\mathrm{sQnd}}^{(\alpha,\beta)(x,y)}(X, Y) \simeq \mathrm{Hom}_{4\mathrm{sQnd}}^{(\alpha,\beta)(p,q)}(X, Y), \tag{6}$$

where $\mathrm{Hom}_{4\mathrm{sQnd}}^{(\alpha,\beta)(x,y)}(X, Y)$ stands for the set of 4–fold homomorphisms from X to Y which send (α, β) to (x, y).

Proof To prove (i), we fix $\kappa \in Y_{24}$. Define

$$\begin{aligned} T_{y,q} &:= \Big(\bullet * \big((q * y) * \rho'(\kappa)\big)\Big) * (y * \kappa) \in \mathrm{Inn}(Y), \\ U_{x,p} &:= \Big(\bullet * \big((p * x) * \rho'(\kappa)\big)\Big) * (x * \kappa) \in \mathrm{Inn}(Y). \end{aligned}$$

Note $x \cdot T_{y,q} = x$, since $x * \big((q * y) * \rho'(\kappa)\big) = x$ and $x * (y * \kappa) = x$ by (F4). Furthermore, by the axiom (F3) we obtain

$$\begin{aligned} y \cdot T_{y,q} &= \Big(y * \big((q * y) * \rho'(\kappa)\big)\Big) * (y * \kappa) \\ &= \Big(\big((q * y) * \rho'(\kappa)\big) * \rho'(y)\Big) * (y * \kappa) \\ &= \Big(\big((q * y) * \rho'(y)\big) * \big(\rho'(\kappa) * \rho'(y)\big)\Big) * (y * \kappa) \\ &= \Big(q * \big(\rho'(y) * \kappa\big)\Big) * (y * \kappa) = q. \end{aligned}$$

Similarly, we can verify $x \cdot U_{x,p} = p$ and $q \cdot U_{x,p} = q$. Hence, $S := T_{y,q} \cdot U_{x,p} \in \mathrm{Inn}(Y)$ satisfies the desired conditions.

It remains to show (ii). Such $S \in \mathrm{Inn}(Y)$ induces a 4–fold isomorphism $(\bullet \cdot S)\colon Y \cong Y$. Hence, this induces a bijective map $(\bullet \cdot S)_*\colon \mathrm{Hom}_{4\mathrm{sQnd}}(X,Y) \simeq \mathrm{Hom}_{4\mathrm{sQnd}}(X,Y)$. Then the restriction on $\mathrm{Hom}^{(\alpha,\beta)(x,y)}_{4\mathrm{sQnd}}(X,Y)$ is the required bijection (6). □

4 Classification of 4–fold symmetric quandles

In this paper, we mainly deal with a pair of a group G and its central element $c \in Z(G)$ satisfying $c^2 = e$, where $Z(G)$ is the center of G. We call such a pair of (G,c) *a cored group*[1]. For a cored group (G,c), we present an example of 4–fold symmetric quandle (Example 4.1). Further, we classify 4–fold symmetric quandles (Theorem 4.2). As a corollary, we give a category equivalence between a category of two-pointed 4–fold symmetric quandles and a category of cored groups (Corollary 4.3). We give a slight reduction of 4–fold symmetric homotopy invariant (Lemma 4.4).

4.1 Classification and a category equivalence

A 4–fold symmetric quandle arises from a cored group (G,c) in the following manner.

Example 4.1 Let G be a group and $c \in Z(G)$ a central element. Putting $T_{12} := \{(i,j) \in \mathbb{Z}^2 \mid 1 \le i,j \le 4,\ i \neq j\}$, we define $\widetilde{G}_c$ to be a quotient set $G \times T_{12}/\sim$, where the equivalence relation $\sim$ on $(G \times T_{12})$ is defined by $(g,(i,j)) \sim (g^{-1}c,(j,i))$, for any $(i,j) \in T_{12}$ and $g \in G$. Further, we provide $\widetilde{G}_c$ with a binary operation $*\colon \widetilde{G}_c \times \widetilde{G}_c \to \widetilde{G}_c$ defined by Table 1 below. To summarize, by a discussion similar to the proof by Hatakenaka [13, Proposition 3.4.], it can be verified that $\widetilde{G}_c$ is a quandle. Note that if G is finite, then $\widetilde{G}_c$ has order $6|G|$.

Let us assume $c^2 = e$. Put a map $\rho\colon \widetilde{G}_c \to \widetilde{G}_c$ given by $\rho(g,(i,j)) = (gc,(i,j))$ and a natural projection $p_{\widetilde{G}_c}\colon \widetilde{G}_c \to \mathcal{S}$ which sends $(g,(i,j))$ to $(ij) \in \mathcal{S}$. Then the triple $(\widetilde{G}_c, p_{\widetilde{G}_c}, \rho)$ is a 4–fold symmetric quandle.

(g,t)	(g',t')	$(g,t)*(g',t')$
$(g,(i,j))$	$(g',(i,j))$	$(g'g^{-1}g',(i,j))$
$(g,(i,j))$	$(g',(j,k))$	$(gg',(i,k))$
$(g,(i,j))$	$(g',(k,l))$	$(g,(i,j))$

Table 1: The binary operation $*$ in $\widetilde{G}_c$. Here, in each line, i,j,k,l are all distinct and $t,t' \in T_{12}$.

[1] We here name the pair after a shape of an apple with the cored center.

Notice that the quandle $\widetilde{G}_c$ is of type 2 if and only if $c = e$ by Corollary 3.6. We remark that in the case of $c = e$, $\widetilde{G}_e$ is the quandle introduced by Hatakenaka [13, Section 3.2] without good involution. Then, the quandle $\widetilde{G}_c$ is a slight generalization and a symmetric quandle version of [13].

However, any 4–fold symmetric quandle turns out to be one of the type $\widetilde{G}_c$: we classify 4–fold symmetric quandles as follows:

Theorem 4.2 *Let (X, p_X, ρ) be a 4–fold symmetric quandle. Then X_{12} has a group structure G related to X by a 4–fold symmetric quandle isomorphism $X \cong \widetilde{G}_c$ for some central element $c \in G$. As a special case, if the good involution ρ is the identity map, then the 4–fold symmetric quandle (X, p_X, id) reduces to one of the quandle introduced by Hatakenaka [13].*

We defer its proof to the next section. As an application, in order to state Corollary 4.3, we prepare some notation of categories. Let (G, c) and (G', c') be two cored groups. A group homomorphism $f\colon G \to G'$ is said to be *cored*, if $f(c) = c'$. $\boldsymbol{Grp_c}$ denotes the category whose objects are cored groups, and arrows are cored homomorphisms. Restricting to the case of $c = e$, $\boldsymbol{Grp_c}$ contains the category of groups, denoted by $\boldsymbol{Grp}$, as a full subcategory.

On the other hand, a *two-pointed* 4*–fold symmetric quandle* is defined to be a pair of a 4–fold symmetric quandle X and two points $(x_{12}, x_{23}) \in X_{12} \times X_{23}$. For two-pointed 4–fold symmetric quandles (X, x_{12}, x_{23}) and (X', x'_{12}, x'_{23}), a 4–fold homomorphism $\psi\colon X \longrightarrow X'$ is *two-pointed*, if $\psi(x_{12}) = x'_{12}$ and $\psi(x_{23}) = x'_{23}$. When $X = X'$, a two-pointed 4–fold isomorphism $\psi\colon (X, x_{12}, x_{23}) \longrightarrow (X, x'_{12}, x'_{23})$ is said to be *equivalent*, if there exists $S \in \mathrm{Inn}(X)$ such that the restrictions satisfy $\psi|_{X_{12}\cup X_{23}\cup X_{13}} = (\bullet * S)|_{X_{12}\cup X_{23}\cup X_{13}}$. Let $\boldsymbol{Qnd_{4s}}$ denote the category composed of 4–fold symmetric quandles modulo the equivalence relation. Further, we define $\boldsymbol{Qnd_4}$ to be a full subcategory of $\boldsymbol{Qnd_{4s}}$ consisting of 4–fold symmetric quandles whose good involutions are the identity maps.

Corollary 4.3 *The functor $\mathcal{T}$ which takes a cored group (G, c) to $\widetilde{G}_c$ gives an equivalence of categories between $\boldsymbol{Grp_c}$ and $\boldsymbol{Qnd_{4s}}$. Further, the restriction of the functor to $\boldsymbol{Grp}$ induces an equivalence of categories between $\boldsymbol{Grp}$ and $\boldsymbol{Qnd_4}$.*

Proof It follows from Theorem 4.2 and Lemma 3.9 (i) that any two-pointed 4–fold symmetric quandle (X, x_{12}, x_{23}) is equivalent to $\big(\widetilde{G}_c, (e, (1,2)), (e, (2,3))\big)$ for some cored group (G, c). Hence, it suffices to show that the functor $\mathcal{T}$ is full and faithful as follows. For a 4–fold homomorphism $\psi\colon \widetilde{G}_c \longrightarrow \widetilde{G}'_{c'}$, we will construct a cored homomorphism $\Psi\colon G \longrightarrow G'$ satisfying $\mathcal{T}(\Psi) = \psi$ as follows. Since we consider ψ modulo the above equivalence, using Lemma 3.9 (i), we may assume that $\psi(e,(1,2)) = (e',(1,2))$ and $\psi(e,(2,3)) = (e',(2,3))$. From the definition in Table 1, we have

$$\psi(g,(1,2)) * \psi(h,(2,3)) = \psi(gh,(1,3)) \in \widetilde{G}'_{c'}, \tag{7}$$

for any $g, h \in G$. We put a natural projection $\pi\colon \widetilde{G}'_{c'} \to G'$. By applying (7) to $g = e$ (resp. $h = e$), we have $\pi(\psi(h,(2,3))) = \pi(\psi(h,(1,3)))$ (resp. $\pi(\psi(h,(1,2))) = \pi(\psi(h,(1,3)))$). Hence, we obtain

$$\pi\big(\psi(g,(1,2))\big)\cdot\pi\big(\psi(h,(2,3))\big) = \pi\big(\psi(gh,(1,3))\big) \in \widetilde{G}'. \tag{8}$$

Further, notice $\psi(c,(1,2)) = (c',(1,2))$ by the symmetric structure. Therefore a map $\Psi\colon G \longrightarrow G'$ given by $\Psi(g) = \pi\big(\psi(g,(1,2))\big)$ is a cored homomorphism.

Next, we claim that $\mathcal{T}(\Psi)$ is equivalent to ψ. Denote $\pi(e,(3,4)) \in G'$ by ϵ. Define an equivalent 4–fold isomorphism $T_\epsilon\colon \widetilde{G}'_{c'} \to \widetilde{G}'_{c'}$ by

$$T_\epsilon(g',(1,2)) = (g',(1,2)),\ T_\epsilon(g',(2,3)) = (g',(2,3)),\ T_\epsilon(g',(1,3)) = (g',(1,3)),$$
$$T_\epsilon(g',(1,4)) = (g'\epsilon,(1,4)), T_\epsilon(g',(2,4)) = (g'\epsilon,(2,4)), T_\epsilon(g',(3,4)) = (g'\epsilon,(3,4)).$$

Then, $T_\epsilon \circ \mathcal{T}(\Psi) = \psi$ as required, which implies the fullness of $\mathcal{T}$. Also, the discussion easily results the faithfulness of $\mathcal{T}$.

The latter part results from a similar argument as well. □

Remark 1 Let X, Y be 4–fold symmetric quandles. Let $(a,b,c) \in X_{12} \times X_{23} \times X_{34}$ and $(\alpha,\beta,\gamma) \in Y_{12} \times Y_{23} \times Y_{34}$. By this proof, we remark that, the set of the homomorphisms in $\boldsymbol{Qnd_{4s}}$ between X and Y is in 1–1 correspondence with the set of 4–fold homomorphisms $X \to Y$ sending (a,b,c) to (α,β,γ). Namely, $\mathrm{Hom}_{\boldsymbol{Qnd_{4s}}}(X,Y) \simeq \mathrm{Hom}^{(a,b,c),(\alpha,\beta,\gamma)}_{\mathrm{4sQnd}}(X,Y)$.

Remark In general, the category of quandles is bigger than the category of symmetric quandles, since some quandles have no good involution. For example, let M be a connected Alexander quandles, ie, M is a $\mathbb{Z}[T^{\pm}]$–module with a quandle operation $x * y = Tx + (1-T)y$ and satisfies $M = (1-T)M$ as a $\mathbb{Z}[T^{\pm}]$–module. Then it is easy to see that M has no good involution if and only if $T^2M \neq M$.

Then the results can be summarized as follows:

$$\begin{array}{ccccccc}
\left\{\begin{array}{c}\text{4–fold symmetric}\\ \text{quandles of type 2}\end{array}\right\} & \subset & \left\{\begin{array}{c}\text{4–fold symmetric}\\ \text{quandles}\end{array}\right\} & \subset & \left\{\begin{array}{c}\text{symmetric}\\ \text{quandles}\end{array}\right\} & \subset & \left\{\text{quandles}\right\} \\
\wr\Vert & & \wr\Vert & & & & \\
\{\text{groups}\} & \subset & \{\text{cored groups}\} & & & &
\end{array}$$

With the illustration, we give an interpretation of our 4–fold symmetric quandles and a comparison with Hatakenaka's work [13]. She introduced a quandle $\widetilde{G}_e$ in the case $c = e$ without using a good involution. Therefore Theorem 4.2 and Corollary 4.3 suggests that a 4–fold symmetric quandle without good involution is under the category of groups, and that the symmetric quandle structure makes a progress of her construction of 3–manifold invariants wider.

Also, we compare symmetric quandle homotopy invariants of 3–manifolds with those of oriented links. The author [21] studied quandle homotopy invariants of oriented links. The invariant is defined by using only a finite quandle without good involution (see [21, Section 2]). Similarly, for any finite symmetric quandle (X, ρ), we can also define the *symmetric* quandle homotopy invariant of *un*oriented links (see Section 9). However, some quandle has no good involution, and the symmetric homotopy invariant of (X, ρ) is derived from the homotopy invariant of X without good involution (Proposition 9.1). In conclusion, the symmetric structure develops the quandle homotopy invariant of links no wider.

However, Theorem 4.2 raises a limitation of our philosophy. Section 3.1 teleologically develops an axiomatization so as to construct invariants of 3–manifolds M in term of symmetric quandles, and introduces 4–fold symmetric quandles. Then it is important to discuss how broad the class of 4–fold symmetric quandles is. Theorem 4.2 directly suggests that the axioms are limited by cored groups in the categorical context. Hence, for a further invariant, we need to introduce other axioms with various ideas.

Incidentally, we give a slight reduction of the 4–fold symmetric homotopy invariant of $X = \widetilde{G}_c$ (Lemma 4.4). Let D_ϕ be a 3–fold labeled diagram. We fix three arcs $\alpha_{12}, \alpha_{23}, \alpha_{34}$ of D_ϕ labeled by $(12), (23), (34) \in \mathcal{S}$, respectively. For $(x_{12}, x_{23}, x_{34}) \in X_{12} \times X_{23} \times X_{34}$, we define $\mathrm{Col}^{x_{12},x_{23},x_{34}}_{X,\rho}(D_\phi)$ to be the subset of $\mathrm{Col}_{X,\rho}(D_\phi)$ such that the arcs α_{ij} are colored by x_{ij}, respectively.

Lemma 4.4 *Let (G, c) be a cored group, and $\widetilde{G}_c$ the associated 4–fold symmetric quandle. Let D_ϕ and α_{ij} be as above. Then there exists a bijection*

$$\mathrm{Col}_{\widetilde{G}_c,\rho}(D_\phi) \simeq |G|^3 \times \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_\phi),$$

where we denote $(e, (i, j)) \in \widetilde{G}_c$ by e_{ij}. Further, if G is finite, then the 4–fold quandle homotopy invariant of a 3–manifold M is equal to

$$\Xi^{4\mathrm{f}}_{\widetilde{G}_c}(M) = |G|^3 \sum_{C \in \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_\phi)} \Xi^{4\mathrm{f}}_{\widetilde{G}_c}(D_\phi; C). \tag{9}$$

Proof Since the subdiagram of D_ϕ colored by $(34) \in \mathcal{S}$ is a trivial knot, for any $r \in X_{34}$, we have a bijection

$$\mathrm{Col}^{e_{12},e_{23},r}_{\widetilde{G}_c,\rho}(D_\phi) \to \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_\phi).$$

Further, by Lemma 3.9, for any $p \in X_{12}$, $q \in X_{23}$, there exists $S \in \mathrm{Inn}(\widetilde{G}_c)$ such that $p \cdot S = e_{12}$ and $q \cdot S = e_{23}$. Thus we have a bijection

$$\mathrm{Col}^{p,q,r}_{\widetilde{G}_c,\rho}(D_\phi) \to \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_\phi)$$

for any $(p,q,r) \in X_{12} \times X_{23} \times X_{34}$. Hence, we obtain

$$\mathrm{Col}_{\widetilde{G}_c,\rho}(D_\phi) \simeq |G|^3 \times \mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_\phi)$$

as required.

Note for any $\widetilde{G}_c$–coloring C and $S \in \mathrm{Inn}(\widetilde{G}_c)$, we have an equality

$$\Xi^{4\mathrm{f}}_{\widetilde{G}_c,\rho}(D_\phi; C) = \Xi^{4\mathrm{f}}_{\widetilde{G}_c,\rho}(D_\phi; C \cdot S)$$

from the definition of $\Pi_{2,\rho}(\widetilde{G}_c)$ (see also [21, Lemma 5.6; 11, Proposition 5.2]). So

$$\Xi^{4\mathrm{f}}_{\widetilde{G}_c}(M) = \sum_{\substack{(p,q,r)\in \\ X_{12}\times X_{23}\times X_{34}}} \sum_{C\in\mathrm{Col}^{p,q,r}_{\widetilde{G}_c,\rho}(D_\phi)} \Xi^{4\mathrm{f}}_{\widetilde{G}_c}(D_\phi; C) = |G|^3 \sum_{C\in\mathrm{Col}^{e_{12},e_{23},e_{34}}_{\widetilde{G}_c,\rho}(D_\phi)} \Xi^{4\mathrm{f}}_{\widetilde{G}_c}(D_\phi; C). \quad \square$$

4.2 Proof of Theorem 4.2

The hasty reader may skip the proof, since the discussion is an ad hoc method.

Proof We first equip X_{12} with a group structure step by step.

Step 1 To begin with, we construct a symmetric subquandle $\widetilde{K}$ of X. We fix three elements $p_{12} \in X_{12}$, $p_{23} \in X_{23}$, $p_{34} \in X_{34}$. Put $p_{13} = p_{12} * p_{23} \in X_{13}$, $p_{24} = p_{23} * p_{34} \in X_{24}$ and $p_{14} = (p_{12} * p_{23}) * p_{34} \in X_{14}$. Define q_{ij} to be $\rho(p_{ij})$ for any $(ij) \in \mathcal{S}$. Then, by using Lemma 3.5, we can verify that a subset

$$\widetilde{K} := \{p_{ij},\ q_{ij} \mid (ij) \in \mathcal{S}\}$$

is a symmetric subquandle. We remark that if $\rho \neq \mathrm{id}$, then the subquandle is $\widetilde{K} \cong \widetilde{\mathbb{Z}/2\mathbb{Z}}_c$ with $c \neq e$, and that if $\rho = \mathrm{id}$, then $\widetilde{K} \cong \mathcal{S}$, since $p_{ij} = q_{ij}$.

Define a binary operation of X_{12} by a composite

$$\star\colon X_{12} \times X_{12} \xrightarrow{\mathrm{id}_{X_{12}}\times(\bullet * p_{12})} X_{12} \times X_{12} \xrightarrow{\mathrm{id}_{X_{12}}\times(\bullet * q_{13})} X_{12} \times X_{23} \xrightarrow{*} X_{13} \xrightarrow{(\bullet * q_{23})} X_{12},$$

$$x_{12} \star y_{12} = \big(x_{12} * ((y_{12} * p_{12}) * q_{13})\big) * q_{23}, \tag{10}$$

for $x_{12}, y_{12} \in X_{12}$. We claim that the operation satisfies the axioms of a group.

Step 2 We will show the associativity as follows. Notice that, for $\alpha_{12} \in X_{12}$, $\beta_{23} \in X_{23}$, $\gamma_{34} \in X_{34}$, the axiom (F4) says

$$(\alpha_{12} * \beta_{23}) * \gamma_{34} = (\alpha_{12} * \gamma_{34}) * (\beta_{23} * \gamma_{34}) = \alpha_{12} * (\beta_{23} * \gamma_{34}). \tag{11}$$

Operating $(((\bullet * q_{23}) * p_{34}) * q_{23}) \in \mathrm{Inn}(X)$ to this equality, the left hand side equals

$$\big(((\alpha_{12} * \beta_{23}) * q_{23}) * ((\gamma_{34} * q_{23}) * p_{34})\big) * q_{23}. \tag{12}$$

Put $x_{12},\ y_{12},\ z_{12} \in X_{12}$. Applying $\alpha_{12} = x_{12}, \beta_{23} = (y_{12} * p_{12}) * q_{13}$, $\gamma_{34} = (((z_{12} * p_{12}) * q_{13}) * q_{34}) * p_{23}$ to (12), one then obtains

$$\Big(\big((x_{12} * ((y_{12} * p_{12}) * q_{13})\big) * q_{23}) * \big((z_{12} * p_{12}) * q_{13})\Big) * q_{23} = (x_{12} \star y_{12}) \star z_{12}.$$

On the other hand, by applying $(((\bullet * q_{23}) * p_{34}) * q_{23})$ to the right hand side in (11), we have

$$\Big(\big(\alpha_{12} * (\beta_{23} * \gamma_{34})\big) * p_{34}\Big) * q_{23} = \Big(\alpha_{12} * \big((\beta_{23} * p_{34}) * (\gamma_{34} * p_{34})\big)\Big) * q_{23}, \tag{13}$$

where we use (Q3) and (F4). We now calculate $\gamma_{34} * p_{34}$ and $\beta_{23} * p_{34}$. When $\gamma_{34} = (((z_{12} * p_{12}) * q_{13}) * q_{34}) * p_{23}$, we remark that

$$\begin{aligned}
\gamma_{34} * p_{34} &= \Big(\big(((z_{12} * p_{12}) * (p_{23} * p_{12})\big) * q_{34}) * p_{34}\Big) * (p_{23} * p_{34}) \\
&= \big((z_{12} * p_{23}) * p_{12}\big) * p_{24} = (z_{12} * p_{23}) * (p_{12} * p_{24}) = (q_{23} * z_{12}) * p_{14} \\
&= \rho(q_{23} * z_{12}) * \rho(p_{14}) = (p_{23} * z_{12}) * q_{14} = (z_{12} * q_{23}) * q_{14},
\end{aligned}$$

where the first equality follows from (Q4) and $q_{13} = p_{23} * p_{12}$, the third equality is derived from (F4) and other equalities are obtained from (F3).

Furthermore, we easily have $\beta_{23} * p_{34} = (y_{12} * p_{12}) * (q_{13} * p_{34}) = (y_{12} * p_{12}) * q_{14}$. Therefore the right hand side in (13) is equal to

$$\Big(x_{12} * \big(((y_{12} * p_{12}) * (z_{12} * q_{23})) * q_{14}\big)\Big) * q_{23} = \Big(x_{12} * \big((y_{12} * p_{12}) * (z_{12} * q_{23})\big)\Big) * q_{23},$$

where we use $(y_{12} * p_{12}) * (z_{12} * q_{23}) \in X_{23}$, and the axioms (Q4). We claim that the right hand side equals $x_{12} \star (y_{12} \star z_{12})$. Indeed, by definition,

$$\begin{aligned}
x_{12} \star (y_{12} \star z_{12}) &= \Big(x_{12} * \Big(\Big(\big((y_{12} * ((z_{12} * p_{12}) * q_{13})\big) * q_{23}) * p_{12}\Big) * q_{13}\Big)\Big) * q_{23} \\
&= \Big(x_{12} * \Big(\big(y_{12} * ((z_{12} * p_{12}) * q_{13})\big) * p_{12}\Big)\Big) * q_{23} \\
&= \Big(x_{12} * \Big((y_{12} * p_{12}) * \big(((z_{12} * p_{12}) * q_{13}) * p_{12})\Big)\Big) * q_{23} \\
&= \Big(x_{12} * \big((y_{12} * p_{12}) * (z_{12} * q_{23})\big)\Big) * q_{23},
\end{aligned}$$

where the second equality is obtained from Lemma 4.5 (i), the third follows from (Q3) and the last is obtained by an easy calculation as $\big((z_{12} * p_{12}) * q_{13}\big) * p_{12} = z_{12} * q_{23}$.

Step 3 Here we claim that $p_{12} \in X_{12}$ is a right identity element. Indeed, for any $x_{12} \in X_{12}$,

$$x_{12} \star p_{12} = \big(x_{12} * ((p_{12} * p_{12}) * q_{13})\big) * q_{23} = (x_{12} * p_{23}) * q_{23} = x_{12},$$

where the second equality is obtained from $(p_{12} * p_{12}) * q_{13} = p_{23}$.

Step 4 Further, we assert that the right inverse element of $x_{12} \in X_{12}$ is $x_{12} * p_{12} \in X_{12}$. Actually,

$$\begin{aligned} x_{12} \star (x_{12} * p_{12}) &= \Big(x_{12} * \big(((x_{12} * p_{12}) * p_{12}) * q_{13}\big)\Big) * q_{23} \\ &= \big(x_{12} * (x_{12} * q_{13})\big) * q_{23} \\ &= \big((\rho(x_{12}) * q_{13}) * x_{12}\big) * q_{23} \\ &= \Big(\big(p_{13} * \rho(x_{12})\big) * x_{12}\Big) * q_{23} = p_{13} * q_{23} = p_{12}, \end{aligned}$$

where the second equality is obtained from Lemma 3.5 (ii), and the third and forth equalities are obtained from the axiom (F3).

Therefore, the binary map $\star$ provides X_{12} with a group structure. Let G denote a set X_{12} with the binary operation $\star$. Since we can easily check $x_{12} \star q_{12} = \rho(x_{12}) = q_{12} \star x_{12}$ by direct calculation, q_{12} belongs to the center of G. In particular, taking $x_{12} = q_{12}$, we find $q_{12} \star q_{12} = \rho(q_{12}) = p_{12}$. The pair of (G, q_{12}) is therefore a cored group. Denote $q_{12} \in X_{12}$ by c. Then our goal is to construct a symmetric quandle isomorphism $\widetilde{G}_c \cong X$ as follows.

First, notice that, through the equality (10), the restricted operation $*\colon X_{12} \times X_{23} \longrightarrow X_{13}$ is determined by the group operation $\star$ and the right operation from the subquandle $\widetilde{K}$ in Step 1. Furthermore, for any distinct elements i, j and k, the binary operation $*\colon X_{ij} \times X_{jk} \longrightarrow X_{ik}$ is also determined by the operation $\star$ and $\widetilde{K}$. Indeed, for example, in the case of $(i, j, k) = (1, 2, 4)$, the operation $*\colon X_{12} \times X_{24} \longrightarrow X_{14}$ is given by

$$x_{12} * y_{24} = \big((x_{12} * y_{24}) * p_{34}\big) * q_{34} = \big(x_{12} * (y_{24} * p_{34})\big) * q_{34},$$

for any $x_{12} \in X_{12},\ y_{14} \in X_{14}$, noting $y_{24} * p_{34} \in X_{23}$. It also holds for other cases in a similar manner.

Next, we will investigate the subquandle operation $*\colon X_{12}\times X_{12}\longrightarrow X_{12}$. We claim $y_{12}\star x_{12}^{-1}\star y_{12}=x_{12}*y_{12}$ for any $x_{12},y_{12}\in X_{12}$. Indeed,

$$\begin{aligned}(y_{12}\star x_{12}^{-1})\star y_{12}&=\Big(\big(\big((y_{12}*(x_{12}*q_{13})\big)*q_{23}\big)*\big((y_{12}*p_{12})*q_{13}\big)\Big)*q_{23}\\&=\Big(\big(\big((y_{12}*(x_{12}*q_{13})\big)*q_{23}\big)*q_{23}\Big)*\Big(\big((y_{12}*p_{12})*q_{13}\big)*q_{23}\Big)\\&=\big(\rho(y_{12})*(x_{12}*q_{13})\big)*(y_{12}*q_{13})\\&=\big((x_{12}*q_{13})*y_{12}\big)*(p_{13}*y_{12})\\&=\big((x_{12}*y_{12})*(q_{13}*y_{12})\big)*\big(\rho(q_{13})*y_{12}\big)=x_{12}*y_{12},\end{aligned}$$

where the second equality is obtained from (Q3), the third equality is derived from Lemma 4.5 (ii) and Lemma 3.5 (ii), the forth equality is obtained from the axiom (F3). In conclusion, the operation $*\colon X_{12}\times X_{12}\longrightarrow X_{12}$ is determined by the group structure; further, so is the operation $*\colon X_{ij}\times X_{ij}\longrightarrow X_{ij}$ for any $(ij)\in\mathcal{S}$ via the quandle isomorphism (4).

In summary, the binary operation $*$ and the good involution ρ on X is determined by the group structure G and the subquandle $\widetilde{K}$. Further, note that the above discussion and the construction of $\widetilde{G}_c$ are parallel. Hence, it is not hard to construct a symmetric quandle isomorphism $\widetilde{G}_c\cong X$, and is thus left to the reader. □

Lemma 4.5 *Let p_{ij}, $q_{ij}\in X$ be as above. Then we have two equalities:*

(i) $\big((\bullet*q_{23})*p_{12}\big)*q_{13}=\bullet*p_{12}\in\mathrm{Inn}(X)$.

(ii) $\big((\bullet*p_{12})*q_{13}\big)*q_{23}=\bullet*q_{13}\in\mathrm{Inn}(X)$.

Proof By direct calculation with using the axioms (Q3), (Q4) and (F3). For example, (ii) follows from

$$\begin{aligned}\big((\bullet*p_{12})*q_{13}\big)*q_{23}&=\big((\bullet*q_{13})*(p_{12}*q_{13})\big)*q_{23}\\&=\big((\bullet*q_{13})*p_{23}\big)*q_{23}=\bullet*q_{13}.\end{aligned}$$ □

5 Inner automorphism group $\mathrm{Inn}(\widetilde{G}_c)$

Following Theorem 4.2, any 4–fold symmetric quandle is isomorphic to $\widetilde{G}_c$ for some cored group (G,c). We then deal with only $\widetilde{G}_c$ in this section. Our goal is to determine the inner automorphism group $\mathrm{Inn}(\widetilde{G}_c)$ of a finite cored group (G,c) (Theorem 5.4).

5.1 Preliminaries

For this, following Joyce [16], we first review the homomorphism (15) below. Let $K \subset I$ be groups. If $x_0 \in I$ commutes with any elements of K, then the left quotient set $K \backslash I$ has a quandle structure given by

$$[\alpha] * [\beta] := [x_0^{-1} \alpha \beta^{-1} x_0 \beta], \tag{14}$$

for representatives $\alpha, \beta \in I$. Conversely, any connected quandle X conforms to this model as follows. Recall that $\mathrm{Inn}(X)$ transitively acts on X by definition. Fix $x_0 \in X$. Let $Z(x_0)$ be the stabilizer group of x_0: in other word, $Z(x_0)$ is the centralizer subgroup of $\mathfrak{inn}_X(x_0) \in \mathrm{Inn}(X)$. We equip the group $\mathrm{Inn}(X)$ with a quandle operation given by (14) with $I = \mathrm{Inn}(X)$ and $K = \{e\}$. Then, it is shown by Joyce [16, Theorem 7.1] that the natural map

$$\mathrm{Inn}(X) \longrightarrow X \quad \text{given by} \quad g \longmapsto x_0 \cdot g \tag{15}$$

is a quandle homomorphism, and induces a quandle isomorphism $Z(x_0) \backslash \mathrm{Inn}(X) \cong X$. In conclusion, the quandle structure of X is determined by the two groups $\mathrm{Inn}(X)$ and $Z(x_0)$. Hence, it is important for the study of a connected quandle X to compute $\mathrm{Inn}(X)$.

Changing into our work, we let (G, c) be a cored group. Since the associated quandle $\widetilde{G}_c$ is connected by Lemma 3.7, $\widetilde{G}_c$ fits with the above model. To say it more concretely (Proposition 5.3), we prepare some notation. We consider what is called the wreath product $G^4 \rtimes \mathfrak{S}_4$. To be concise, the group operation in $G^4 \rtimes \mathfrak{S}_4$ is defined by

$$(g_1, g_2, g_3, g_4; \sigma) \cdot (g'_1, g'_2, g'_3, g'_4; \sigma') := (g_1 g'_{\sigma(1)}, g_2 g'_{\sigma(2)}, g_3 g'_{\sigma(3)}, g_4 g'_{\sigma(4)}; \sigma\sigma')$$

where $\sigma, \sigma' \in \mathfrak{S}_4$ and $g_i, g'_i \in G$ ($i \in \{1, 2, 3, 4\}$), and we divide the components between G^4 and $\mathfrak{S}_4$ by semicolons ";". We now consider a group homomorphism

$$G^4 \rtimes \mathfrak{S}_4 \longrightarrow G/[G, G] \quad \text{given by} \quad (x, y, z, w; \sigma) \longmapsto c^{(\mathrm{sgn}(\sigma)-1)/2} xyzw,$$

where $[G, G]$ is the commutator subgroup of G. Then the kernel is presented by the formula

$$I_{G,c} := \{(x, y, z, w; \sigma) \in G^4 \rtimes \mathfrak{S}_4 \mid c^{(\mathrm{sgn}(\sigma)-1)/2} xyzw \in [G, G]\}. \tag{16}$$

We denote $(c, e, e, e; (12)) \in I_{G,c}$ by z_0. Let $K_{G,c}$ denote the centralizer subgroup of $z_0 \in I_{G,c}$. Then, by elementary calculation, we can verify that $K_{G,c}$ is given by

$$\begin{aligned} &\{(x, x, z, w; \sigma) \in I_{G,c} \mid \sigma = e \text{ or } (12)(34)\} \\ &\qquad \cup \{(x, cx, z, w; \sigma) \in I_{G,c} \mid \sigma = (12) \text{ or } (34)\}. \end{aligned} \tag{17}$$

Let us provide the left quotient $K_{G,c}\backslash I_{G,c}$ with a quandle structure obtained from (14).

Next, we consider a map $\chi\colon G \times T_{12} \longrightarrow G^4 \rtimes \mathfrak{S}_4$ defined by

$$\chi(g,(i,j)) = (a_1^g, a_2^g, a_3^g, a_4^g; (ij)), \tag{18}$$

where we put $a_\dagger^g = cg$ for $\dagger = i$, $a_\dagger^g = g^{-1}$ for $\dagger = j$, and $a_\dagger^g = e$ otherwise. Then this passes to a map $\chi\colon \widetilde{G}_c \longrightarrow I_{G,c}$. Further, using the map χ, we define a map $\eta_{G,c}\colon \widetilde{G}_c \longrightarrow K_{G,c}\backslash I_{G,c}$ by

$$\begin{gathered}\eta_{G,c}(g,(1,2)) = [(cg,e,g^{-1},e;(12))], \quad \eta_{G,c}(g,(3,4)) = [(e,g^{-1},cg,e;(13)(24))],\\ \eta_{G,c}(g,(2,3)) = [\chi(g,(1,3))], \quad \eta_{G,c}(g,(1,3)) = [\chi(g,(2,3))],\\ \eta_{G,c}(g,(1,4)) = [\chi(g,(2,4))], \quad \eta_{G,c}(g,(2,4)) = [\chi(g,(1,4))].\end{gathered} \tag{19}$$

Noting $z_0 = \chi(e,(1,2))$, it is not hard to see that, for any $(g,(i,j)) \in \widetilde{G}_c$,

$$[\chi(g,(i,j))] = \eta_{G,c}(g,(i,j))^{-1} \cdot [z_0] \cdot \eta_{G,c}(g,(i,j)) \in K_{G,c}\backslash I_{G,c}. \tag{20}$$

Also, we put a subset $\Gamma_{\mathcal{S}} := \{(1,3),(2,3),(1,4),(2,4)\} \subset \mathbb{Z}^2$. Notice that, for any $(g,(i,j)) \in \widetilde{G}_c$ with $(i,j) \in \Gamma_{\mathcal{S}}$, we have

$$\eta_{G,c}(g,(i,j)) = [\chi(g,(i,j))^{-1} \cdot z_0 \cdot \chi(g,(i,j))] \in K_{G,c}\backslash I_{G,c}. \tag{21}$$

5.2 Presentation of $\mathrm{Inn}(\widetilde{G}_c)$

In order to determine $\mathrm{Inn}(\widetilde{G}_c)$, the following is a key proposition.

Proposition 5.1 *Let (G,c) be a cored group. Then the map $\eta_{G,c}$ above is a quandle epimorphism.*

Proof First, using (20) above, we can verify that $\eta_{G,c}$ is a quandle homomorphism by direct calculation. For example, $\eta_{G,c}(g,(1,2)) * \eta_{G,c}(h,(2,3))$ is equal to

$$\begin{aligned}&\big[\big(z_0^{-1} \cdot (cg,e,g^{-1},e;(12))\big) \cdot \big(\eta_{G,c}(h,(2,3))^{-1} \cdot z_0 \cdot \eta_{G,c}(h,(2,3))\big)\big]\\ &= [(e,g,g^{-1},e;e) \cdot \chi(h,(2,3))] = [(e,g,g^{-1},e;e) \cdot (e,ch,h^{-1},e;(23))]\\ &= [(e,cgh,g^{-1}h^{-1},e;(23))] = [(e,e,h^{-1}g^{-1}hg,e;e) \cdot (e,cgh,g^{-1}h^{-1},e;(23))]\\ &= [(e,cgh,(gh)^{-1},e;(23))] = \eta_{G,c}(gh,(1,3)) = \eta_{G,c}\big((g,(1,2)) * (h,(2,3))\big).\end{aligned}$$

For other cases, the verifications can be done by similar calculations, and are thus left to the reader.

It remains to show $\eta_{G,c}$ is a surjection. By Lemma 5.2 below, any $\Upsilon \in I_{G,c}$ is presented by $\Upsilon = \chi(g_1,(i_1,j_1))\cdots\chi(g_n,(i_n,j_n))$, for some $(g_1,(i_1,j_1)),\dots,(g_n,(i_n,j_n)) \in \widetilde{G}_c$ with $(i_1,j_1),\dots,(i_n,j_n) \in \Gamma_{\mathcal{S}}$. In addition, by the equalities (14) and (21), we obtain

$$\begin{aligned}
&\chi(g_1,(i_1,j_1))\cdots\chi(g_n,(i_n,j_n))\\
&\qquad = z_0^{1-n}\cdot\Big(\cdots\big(\chi(g_1,(i_1,j_1)) * \chi(g_2,(\overline{i}_2,j_2))\big)\cdots\Big) * \chi(g_n,(\overline{i}_n,j_n)),
\end{aligned}$$

where we put $\overline{1}=2$ and $\overline{2}=1$. Consider this equality modulo $K_{G,c}$. By (19), $[\Upsilon]$ is derived from the following as required:

$$\begin{aligned}
[\Upsilon] &= \Big[\Big(\big(\cdots\big(\eta_{G,c}(g_1,(\overline{i}_1,j_1)) * \eta_{G,c}(g_2,(i_2,j_2))\big)\cdots\big)\Big) * \eta_{G,c}(g_n,(i_n,j_n))\Big]\\
&= \Big[\eta_{G,c}\Big(\big(\cdots\big((g_1,(\overline{i}_1,j_1)) * (g_2,(i_2,j_2))\big) * \cdots\big) * (g_n,(i_n,j_n))\Big)\Big]\\
&\in K_{G,c}\backslash I_{G,c}.
\end{aligned}$$
□

Lemma 5.2 *Let (G,c) be a cored group. The group $I_{G,c}$ is generated by elements of the forms $\chi(g,(i,j))$ with $g \in G$ and $(i,j) \in \Gamma_{\mathcal{S}}$. In particular, $I_{G,c}$ is generated by the image of $\chi\colon \widetilde{G}_c \to I_{G,c}$ given by* (18).

Proof Put $(x,y,z,w;\sigma) \in I_{G,c}$. By direct calculation, $(x,y,z,w;\sigma)$ is equal to

$$\chi(z^{-1},(1,3))\cdot\chi(w^{-1},(2,4))\cdot\chi(c,(2,4))\cdot\chi(wy,(2,3))\cdot(e,wyzx,e,e;(23)(13)\sigma).$$

Since $c^{(\mathrm{sgn}(\sigma)-1)/2}wyzx \in [G,G]$, there exist $a_1,\dots,a_n,b_1,\dots,b_n \in G$ such that

$$wyzx = c^{(\mathrm{sgn}(\sigma)-1)/2}a_nb_na_n^{-1}b_n^{-1}a_{n-1}b_{n-1}a_{n-1}^{-1}b_{n-1}^{-1}\cdots a_1b_1a_1^{-1}b_1^{-1}.$$

Denote the right side by $A_{n,\sigma}$. Hence, it is enough for the proof to show that any such elements $(e,A_{n,\sigma},e,e;\sigma)$ can be presented by a product of some elements of the forms $\chi(g,(i,j))$ with $g \in G$ and $(i,j) \in \Gamma_{\mathcal{S}}$. We will show this by induction on $n \geq 0$.

To begin, if $n=0$, then we can easily show it by elementary calculation. Let $n \geq 1$. Assume it holds for $A_{n-1,\sigma}$. By direct calculation, we see that $(e,A_{n,\sigma},e,e;\sigma)$ equals

$$\chi\big(a_nb_n,(2,3)\big)\cdot\chi\big(a_n,(2,3)\big)\cdot\chi\big(e,(2,3)\big)\cdot\chi\big(b_n,(2,3)\big)\cdot(e,A_{n-1,\sigma},e,e;\sigma).$$

By assumption, $A_{n,\sigma}$ satisfies the required condition. This completes the proof. □

We consider the map $\eta_{G,c}$ in the case where G is finite.

Proposition 5.3 *Let (G,c) be a finite cored group. Then the map $\eta_{G,c}\colon \widetilde{G}_c \to K_{G,c}\backslash I_{G,c}$ is a quandle isomorphism $\widetilde{G}_c \cong K_{G,c}\backslash I_{G,c}$.*

Proof From the definitions, we immediately notice that

$$|I_{G,c}| = 24 \cdot |G|^3 \cdot |[G,G]|, \quad |K_{G,c}| = 4 \cdot |G|^2 \cdot |[G,G]|, \quad |\widetilde{G}_c| = |K_{G,c} \backslash I_{G,c}| = 6 \cdot |G|.$$

Since the above $\eta_{G,c}$ is a quandle epimorphism, $\eta_{G,c}$ is isomorphic. □

Following Proposition 5.3, we will determine $\mathrm{Inn}(\widetilde{G}_c)$ of a finite cored group (G,c). For this, we now introduce a group defined by

$$Z_{G,c} := \{ (z,z,z,z;e) \in G^4 \rtimes \mathfrak{S}_4 \mid z^4 \in [G,G],\ z \in Z(G) \}.$$

By direct calculation, we see that $Z_{G,c}$ is precisely the center of $I_{G,c}$.

Theorem 5.4 *Let (G,c) be a cored group. If the epimorphism $\eta_{G,c}$ is isomorphic, then there exists a group isomorphism $\mathrm{Inn}(\widetilde{G}_c) \cong I_{G,c}/Z_{G,c}$. In particular, if G is finite, $\mathrm{Inn}(\widetilde{G}_c) \cong I_{G,c}/Z_{G,c}$.*

Proof Let us regard a natural right action of $I_{G,c}$ on $K_{G,c} \backslash I_{G,c}$ as a group homomorphism $I_{G,c} \to \mathrm{Aut}(K_{G,c} \backslash I_{G,c})$ sending $\Upsilon \in I_{G,c}$ to $(\bullet \cdot \Upsilon)$. We claim that the image is contained in $\mathrm{Inn}(K_{G,c} \backslash I_{G,c})$. Put an arbitrary $\Upsilon \in I_{G,c}$. Then by Lemma 5.2 we have $\Upsilon = \chi(g_1,(i_1,j_1)) \cdots \chi(g_n,(i_n,j_n))$ for some $(g_1,(i_1,j_1)), \dots, (g_n,(i_n,j_n)) \in \widetilde{G}_c$ with $(i_1,j_1), \dots, (i_n,j_n) \in \Gamma_S$. Hence, using (14) and (20), for any $\mathcal{X} \in K_{G,c} \backslash I_{G,c}$, we have

$$\begin{aligned}
\mathcal{X} \cdot \Upsilon &= [z_0^{-n+1}] \cdot [\mathcal{X} \cdot \chi(g_1,(i_1,j_1)) \cdots \chi(g_n,(i_n,j_n))] \\
&= \Big[\Big(\big(\cdots \big(\mathcal{X} * \eta_{G,c}(g_1,(i_1,j_1)) \big) * \cdots \big) * \eta_{G,c}(g_n,(i_n,j_n)) \Big) \Big] \in K_{G,c} \backslash I_{G,c},
\end{aligned}$$

noting $z_0 \in K_{G,c}$ by definition. From the definition of $\mathrm{Inn}(K_{G,c} \backslash I_{G,c})$, the above equalities mean $[\bullet \cdot \Upsilon] \in \mathrm{Inn}(K_{G,c} \backslash I_{G,c})$ as required. We claim that the homomorphism $I_{G,c} \to \mathrm{Inn}(K_{G,c} \backslash I_{G,c})$ is surjective. Indeed, for any $\mho \in I_{G,c}$, by definition the equality

$$\bullet * [\mho] = \big[\bullet \cdot (\mho^{-1} z_0 \mho) \big] \in \mathrm{Aut}(K_{G,c} \backslash I_{G,c})$$

means that any generator $\bullet * [\mho]$ of $\mathrm{Inn}(K_{G,c} \backslash I_{G,c})$ is derived from $\mho^{-1} z_0 \mho \in I_{G,c}$.

To complete the proof, it is enough to show that the kernel is $Z_{G,c}$. To see this, note that the right action of $\mathrm{Inn}(\widetilde{G}_c)$ on $\widetilde{G}_c$ is effective by definition. Therefore it is sufficient to show that if an element $\Upsilon \in I_{G,c}$ satisfies $\mathcal{X} \cdot \Upsilon = \mathcal{X}$ for any $\mathcal{X} \in K_{G,c} \backslash I_{G,c}$, then $\Upsilon \in Z_{G,c}$. Further, note that, for any representative $\mho \in I_{G,c}$ of $\mathcal{X} \in K_{G,c} \backslash I_{G,c}$, the above equality $\mathcal{X} \cdot \Upsilon = \mathcal{X}$ means $[\mho \cdot \Upsilon] = [\mho] \in K_{G,c} \backslash I_{G,c}$: in other words,

$$\mho \Upsilon^{-1} \mho^{-1} \in K_{G,c} \quad \text{for any } \mho \in I_{G,c}. \tag{22}$$

Therefore it suffices to show that if $\Upsilon \in I_{G,c}$ satisfies the condition (22), then $\Upsilon \in Z_{G,c}$.

Let us assume (22). When $\mho$ is the identity element of $I_{G,c}$, the condition (22) immediately means $\Upsilon \in K_{G,c}$. Further, we can verify $P_{\mathfrak{S}_4}(\Upsilon) = e \in \mathfrak{S}_4$ by applying $\mho = \chi(e,(2,3))$ and $\mho = \chi(e,(3,4))$ to the condition (22), where $P_{\mathfrak{S}_4}$ is the natural projection $P_{\mathfrak{S}_4}\colon G^4 \rtimes \mathfrak{S}_4 \to \mathfrak{S}_4$. Hence, it follows from the presentation (17) that $\Upsilon \in K_{G,c}$ is of the form

$$\Upsilon = (x,x,z,w;e) \in G^4 \rtimes \mathfrak{S}_4,$$

for some $x,z,w \in G$. Put $g \in G$. Further, by applying $\mho = \chi(g,(2,3))$ to (22),

$$K_{G,c} \ni \chi(g,(2,3)) \cdot \Upsilon^{-1} \cdot \chi(g,(2,3))^{-1} = (x^{-1}, gz^{-1}g^{-1}, g^{-1}x^{-1}g, w^{-1}; e).$$

In particular, we have $x^{-1} = gz^{-1}g^{-1}$. Therefore we obtain $z = x$ and $x \in Z(G)$, since $g \in G$ is arbitrary. Similarly, by applying $\mho = \chi(g,(2,4))$ to (22) again, we obtain $x = w$. In summary, Υ equals to $(x,x,x,x;e) \in Z_{G,c}$ as desired. □

We give a corollary of Theorem 5.4, when G is finite. To begin with, we give a concrete presentation of $\mathfrak{inn}_{\widetilde{G}_c}$ given in (5):

Corollary 5.5 *Let (G,c) be a finite cored group. Then, under the identification $\mathrm{Inn}(\widetilde{G}_c) \cong I_{G,c}/Z_{G,c}$, the map $\mathfrak{inn}_{\widetilde{G}_c}\colon \widetilde{G}_c \to \mathrm{Inn}(\widetilde{G}_c)$ coincides with a composite*

$$\widetilde{G}_c \xrightarrow{\chi} I_{G,c} \xrightarrow{\text{proj}} I_{G,c}/Z_{G,c},$$

where χ is obtained from (18).

Proof By the construction of the isomorphism $\mathrm{Inn}(\widetilde{G}_c) \cong I_{G,c}/Z_{G,c}$. □

Hence, we notice that $\mathfrak{inn}_{\widetilde{G}_c}(x * \mho) = \mho^{-1} \cdot \mathfrak{inn}_{\widetilde{G}_c}(x) \cdot \mho \in \mathrm{Inn}(\widetilde{G}_c)$ for any $x \in \widetilde{G}_c$, $\mho \in \mathrm{Inn}(\widetilde{G}_c)$. Namely, the pair $(\widetilde{G}_c,\ I_{G,c}/Z_{G,c})$ is an "augmented quandle" (see Joyce [16, Section 9]), which is also called a *crossed $\widetilde{G}_c$–set*.

6 Estimate of $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ and of 4–fold symmetric quandle homotopy invariant

In this section, we estimate the group $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ of a finite 4–fold symmetric quandle $\widetilde{G}_c$ (see Section 3.1 for the definition). Further, we give an estimate for the 4–fold symmetric quandle homotopy invariants of double and 3–fold branched covering spaces. Also, we calculate $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ when $|\widetilde{G}_c| = 12$.

6.1 Estimate of $\Pi^{4\mathrm{f}}_{2,\rho}(X)$

To begin, for a quandle X, we will review $\Pi_2(X)$ and the quandle space BX. $\Pi_2(X)$ is defined to be a set of all X–coloring of all diagrams modulo concordance relations and Reidemeister I, II, III moves, where *concordance relations* are shown in Figure 9. $\Pi_2(X)$ has a multiplication given by disjoint union, which turns $\Pi_2(X)$ into an abelian group, similar to $\Pi_{2,\rho}(X)$ in Section 3.1.

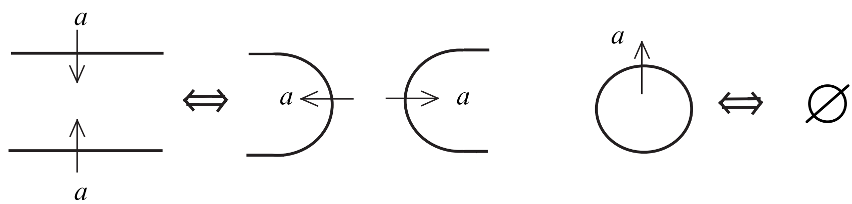


Figure 9: The concordance relations

On the other hand, for a quandle X, Fenn, Rourke and Sanderson [9; 10; 11] introduced the rack space, and examined a relation between its second homotopy group and the concordance relations of X–colorings of framed links (see [11, Theorem 3.9] for more detail). As a modification of *unframed* links, the author [21] used the quandle space BX and showed:

Theorem 6.1 ([21, Theorem A.2]; see also [10, Theorem 4.11]) *There exists an isomorphism $\Pi_2(X) \cong \pi_2(BX)$.*

Using this, for a symmetric quandle (X,ρ), we will address a relation between $\Pi_2(X)$ and $\Pi_{2,\rho}(X)$. Recall the bijection (1) in Section 2. Note that the *symmetric* concordance relations in Section 3.1 are stronger than the concordance relations mentioned above. Therefore, by running over all X– and X_ρ–colorings of all diagrams, the bijections (1) induce

$$\mathcal{S}_{\Pi_2}\colon \Pi_2(X) \longrightarrow \Pi_{2,\rho}(X). \tag{23}$$

Remark that $\mathcal{S}_{\Pi_2}$ is a surjective homomorphism by construction. For a 4–fold symmetric quandle $\widetilde{G}_c$, let us estimate $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$ using Theorem 6.1 and the surjection (23).

Theorem 6.2 *Given a finite cored group (G,c), let $\widetilde{G}_c$ be the associated 4–fold symmetric quandle. Then $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$ is a finite abelian group whose elements are annihilated by $2^{12}\cdot 3^4\cdot |G|^{12}\cdot |[G,G]|^4$. In particular, if $n\in\mathbb{N}$ is prime to $6|G|$, then $\mathbb{Z}/n\mathbb{Z}\otimes \Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)=0$.*

Proof Theorem 6.1 says $\Pi_2(\widetilde{G}_c) \cong \pi_2(B\widetilde{G}_c)$. Since $\widetilde{G}_c$ is a connected quandle by Lemma 3.7, it immediately follows from Nosaka [21, Theorem 3.6(ii)] that $\pi_2(B\widetilde{G}_c)$ is a finite abelian group whose elements are of order $|\mathrm{Inn}(\widetilde{G}_c)|^4$. Recall that $\mathrm{Inn}(\widetilde{G}_c)$ is a quotient subgroup of $I_{G,c}$ and $|I_{G,c}| = 24 \cdot |G|^3 \cdot |[G,G]|$ by Theorem 5.4. Hence, $|\mathrm{Inn}(\widetilde{G}_c)|^4$ is a divisor of $2^{12} \cdot 3^4 \cdot |G|^{12} \cdot |[G,G]|^4$. Therefore $\Pi_2(\widetilde{G}_c)$ is a finite abelian group whose elements are of order $2^{12} \cdot 3^4 \cdot |G|^{12} \cdot |[G,G]|^4$. Hence, $\Pi_{2,\rho}(\widetilde{G}_c)$ inherits this property from $\Pi_2(\widetilde{G}_c)$ through the surjection (23); hence, so the quotient $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$ does. □

Although this theorem shows that $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$ is finite, it is a rough estimate. It is difficult to calculate explicitly $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$ in general.

6.2 Double branched covering and 3–fold branched covering

In this section, we give some approaches of the 4–fold symmetric homotopy invariant of double and 3–fold coverings branched over links L. To begin with, put the associated 2–fold labeled diagram D_ϕ as in Figure 4. For a finite 4–fold symmetric quandle (X, p_X, ρ), recall the subquandle $X_{12} = p_X^{-1}(12) \subset X$. Therefore, any X_ρ–coloring C of D_ϕ is regarded as an X_{12}–coloring by definition. Hence, we may consider that the image of the natural map defined in (2) is contained in $\Pi_{2,\rho}(X_{12})$: that is, the map (2) can be regarded as

$$\Xi_{X_{12}}(D_\phi;\bullet)\colon \mathrm{Col}_{X,\rho}(D_\phi) \longrightarrow \Pi_{2,\rho}(X_{12}), \tag{24}$$

which sends an X_ρ–coloring C of D_ϕ to the canonical class $[C]$. Then, the map $\Xi^{4\mathrm{f}}_X$ given in (3) factors through $\Pi_{2,\rho}(X_{12})$ as follows:

$$\Xi^{4\mathrm{f}}_X(D_\phi;\bullet)\colon \mathrm{Col}_{X,\rho}(D_\phi) \xrightarrow{\Xi_{X_{12}}(D_\phi;\bullet)} \Pi_{2,\rho}(X_{12}) \xrightarrow{(i_{12})_*} \Pi_{2,\rho}(X) \xrightarrow{p^{4\mathrm{f}}} \Pi^{4\mathrm{f}}_{2,\rho}(X), \tag{25}$$

where $(i_{12})_*$ is the induced map from the inclusion $i_{12}\colon X_{12} \hookrightarrow X$. Therefore, for the research of a 4–fold symmetric homotopy invariant of a double branched covering of S^3, it is important to study $\Pi_{2,\rho}(X_{12})$. For example, we consider the case of $G = \mathbb{Z}/m\mathbb{Z}$.

Proposition 6.3 *Let m be an odd number, and $(G,e) = (\mathbb{Z}/m\mathbb{Z}, 0)$. For the 4–fold symmetric quandle $X = \widetilde{G}_e$, the group $\Pi_{2,\rho}(X_{12})$ is a quotient of $\mathbb{Z}/m\mathbb{Z}$.*

Proof Note that the quandle operation of $X_{12} = \mathbb{Z}/m\mathbb{Z}$ is $x * y = 2y - x$. Since it is shown by the author [21, Remark 4.4] that $\Pi_2(X_{12})$ is a quotient of $\mathbb{Z}/m\mathbb{Z}$, it follows from the epimorphism (23) that $\Pi_{2,\rho}(X_{12})$ is also a quotient of $\mathbb{Z}/m\mathbb{Z}$. □

Remark Hatakenaka and the author [14, Section 5.3] will construct an epimorphism $\Pi_{2,\rho}(X_{12}) \to \mathbb{Z}/m\mathbb{Z}$, which implies $\Pi_{2,\rho}(X_{12}) \cong \mathbb{Z}/m\mathbb{Z}$. Further, by the construction of the epimorphism, we can show that the homotopy invariant of the double branched coverings is equal to a scalar multiple of the Dijkgraaf–Witten invariant of $\mathbb{Z}/m\mathbb{Z}$.

We denote by $\mathcal{P}^{4\mathrm{f}}$ the following composite homomorphism used in [14]:

$$\Pi_2(X_{12}) \xrightarrow{\mathcal{S}_{\Pi_2}} \Pi_{2,\rho}(X_{12}) \xrightarrow{(i_{12})_*} \Pi_{2,\rho}(X) \xrightarrow{p^{4\mathrm{f}}} \Pi^{4\mathrm{f}}_{2,\rho}(X). \tag{26}$$

Besides, the invariant of 3–fold branched covering spaces conforms to a factorization similar to (25), as follow. Recall that any 3–manifold M can be presented by a 3–fold labeled diagram D_ϕ (see Section 2.2). Therefore, for a finite 4–fold symmetric quandle X, we may regard any X_ρ–coloring C of D_ϕ as a $p_X^{-1}(\mathcal{R}_3)$–coloring, where $\mathcal{R}_3 = \{(12),(23),(31)\}$. To study the homotopy invariant, we now introduce a quotient group of $\Pi_{2,\rho}(p_X^{-1}(\mathcal{R}_3))$ modulo X_ρ–colorings of trefoils shown in Figure 8. Denote the quotient group by $\Pi^{3\mathrm{f}}_{2,\rho}(X)$. The inclusion $i_{\mathcal{R}_3}\colon p_X^{-1}(\mathcal{R}_3) \hookrightarrow X$ induces $(i_{\mathcal{R}_3})_*\colon \Pi^{3\mathrm{f}}_{2,\rho}(X) \to \Pi^{4\mathrm{f}}_{2,\rho}(X)$. Hence, the map (3) factors through $\Pi^{3\mathrm{f}}_{2,\rho}(X)$ as follows:

$$\begin{aligned}\Xi^{4\mathrm{f}}_X(D_\phi;\bullet)\colon \mathrm{Col}_{X,\rho}(D_\phi) &\xrightarrow{\Xi_{p_X^{-1}(\mathcal{R}_3)}(D;\bullet)} \Pi_{2,\rho}(p_X^{-1}(\mathcal{R}_3)) \\ &\xrightarrow{\mathrm{proj}} \Pi^{3\mathrm{f}}_{2,\rho}(X) \xrightarrow{(i_{\mathcal{R}_3})_*} \Pi^{4\mathrm{f}}_{2,\rho}(X).\end{aligned}$$

Consequently, for the study of a 4–fold symmetric homotopy invariant of 3–manifolds, it is important to study $\Pi^{3\mathrm{f}}_{2,\rho}(X)$.

As the simplest case, we let $G = (\mathbb{Z}/2\mathbb{Z})^m$. Define a map $\pi_{\mathcal{R}_3}\colon \widetilde{G}_c \longrightarrow p_{\widetilde{G}_c}^{-1}(\mathcal{R}_3)$ by

$$\pi_{\mathcal{R}_3}\big(g,(1,2)\big) = \big(g,(1,2)\big), \pi_{\mathcal{R}_3}\big(g,(2,3)\big) = \big(g,(2,3)\big), \pi_{\mathcal{R}_3}\big(g,(1,3)\big) = \big(g,(1,3)\big),$$
$$\pi_{\mathcal{R}_3}\big(g,(3,4)\big) = \big(g,(1,2)\big), \pi_{\mathcal{R}_3}\big(g,(1,4)\big) = \big(g,(2,3)\big), \pi_{\mathcal{R}_3}\big(g,(4,2)\big) = \big(g,(1,3)\big).$$

We can see that $\pi_{\mathcal{R}_3}$ is a symmetric quandle homomorphism. By definition we have that $\pi_{\mathcal{R}_3} \circ i_{\mathcal{R}_3}$ is the identity on $p_{\widetilde{G}_c}^{-1}(\mathcal{R}_3)$. Then the induced map

$$(\pi_{\mathcal{R}_3} \circ i_{\mathcal{R}_3})_*\colon \Pi_{2,\rho}(p_{\widetilde{G}_c}^{-1}(\mathcal{R}_3)) \longrightarrow \Pi_{2,\rho}(p_{\widetilde{G}_c}^{-1}(\mathcal{R}_3))$$

implies that $\Pi_{2,\rho}(p_{\widetilde{G}_c}^{-1}(\mathcal{R}_3))$ is a direct summand of $\Pi_{2,\rho}(\widetilde{G}_c)$. Further, we see that $\Pi^{3\mathrm{f}}_{2,\rho}(X)$ is a direct summand of $\Pi^{4\mathrm{f}}_{2,\rho}(X)$ as well. In conclusion, to search the 4–fold symmetric quandle homotopy invariant of $G = (\mathbb{Z}/2\mathbb{Z})^m$, it suffices to determine $\Pi^{3\mathrm{f}}_{2,\rho}(X)$.

6.3 4–fold symmetric quandle homotopy invariant in the case $G = \mathbb{Z}/2\mathbb{Z}$

As a simplest case, we give an estimate for the 4–fold quandle homotopy invariants in two cases of $(G, c) = (\mathbb{Z}/2\mathbb{Z}, 0)$ and $(\mathbb{Z}/2\mathbb{Z}, 1)$. To see this, it suffices to calculate $\Pi^{3\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$ by the previous argument.

Proposition 6.4 *If* $(G, c) = (\mathbb{Z}/2\mathbb{Z}, 1)$, *then* $\Pi^{3\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$ *is a quotient of* $\mathbb{Z}/4\mathbb{Z}$.

Proof Note that the subquandle $p^{-1}_{\widetilde{G}_c}(\mathcal{R}_3)$ is isomorphic to the quandle which the author used in [21, Section 4.4]. According to [21, Proposition 4.9], $\Pi^{3\mathrm{f}}_{2,\rho}(p^{-1}_{\widetilde{G}_c}(\mathcal{R}_3))$ is either $\mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ or a quotient of $\mathbb{Z}/96\mathbb{Z}$, and the part $\mathbb{Z}/24\mathbb{Z}$ or $\mathbb{Z}/96\mathbb{Z}$ is generated by a $\widetilde{G}_c$–coloring of the trefoil shown in Figure 10 ; hence, so is $\Pi^{3\mathrm{f}}_{2,\rho}(p^{-1}_{\widetilde{G}_c}(\mathcal{R}_3))$ by the epimorphism (23). Under modulo $\widetilde{G}_c$–colored trefoils, $\Pi^{3\mathrm{f}}_{2,\rho}(p^{-1}_{\widetilde{G}_c}(\mathcal{R}_3))$ is a quotient of $\mathbb{Z}/4\mathbb{Z}$. $\square$

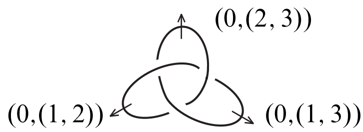
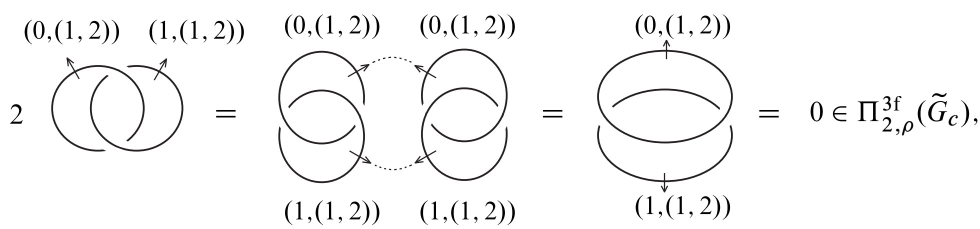


Figure 10: The $\widetilde{\mathbb{Z}/2\mathbb{Z}}_1$–coloring of the trefoil

For another quandle, we obtain:

Proposition 6.5 *If* $(G, c) = (\mathbb{Z}/2\mathbb{Z}, 0)$, *then* $\Pi^{3\mathrm{f}}_{2,\rho}(\widetilde{G}_c) \cong \mathbb{Z}/2\mathbb{Z}$ *or* 0.

Proof Note that the subquandle $p^{-1}_{\widetilde{G}_c}(\mathcal{R}_3)$ is isomorphic to $\mathcal{S}$. Then the author showed [21, Proposition 4.12] that $\Pi_2(\mathcal{S})$ is generated by such two $\widetilde{G}_c$–colorings of the trefoil and of Hopf link shown in Figure 11. Hence, $\Pi^{3\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$ is generated by such a $\widetilde{G}_c$–colorings of Hopf link. To complete the proof, we claim that the $\widetilde{G}_c$–colorings are annihilated by 2. Indeed,



where we use concordance relations along the dashed lines in the second equality. $\square$

Remark In [14, Section 5.3], Hatakenaka and the author show that $\Pi^{3\mathrm{f}}_{2,\rho}(\widetilde{G}_c) \cong \mathbb{Z}/2\mathbb{Z}$ and that the 4–fold quandle homotopy invariant of $\widetilde{G}_c$ coincides with Dijkgraaf–Witten invariant of $G = \mathbb{Z}/2\mathbb{Z}$.

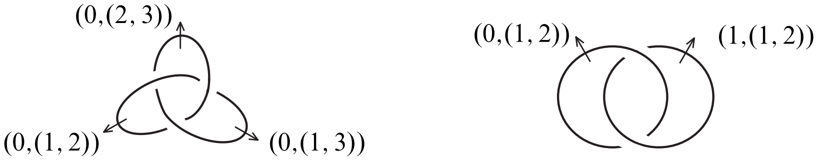


Figure 11: Two $\widetilde{\mathbb{Z}/2\mathbb{Z}}_0$–colorings of the trefoil and Hopf link

Remark 2 Let $(G,c)=(\{e\},e)$. Note $\widetilde{G}_e \cong \mathcal{S}$. Then we can show that $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_e)\cong 0$, similar to Proposition 6.5. Indeed, since $\Pi_2(\widetilde{G}_e)$ is generated by two colorings of the trefoil and Hopf link such as Figure 8, $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_e)$ vanishes by definition. Hence, it goes without saying that the 4–fold symmetric quandle homotopy invariant is trivial.

7 4–Fold symmetric quandle cocycle invariant

However, it is difficult to directly calculate the 4–fold symmetric homotopy invariants valued in $\Pi^{4\mathrm{f}}_{2,\rho}(X)$, since so is the computation of $\Pi^{4\mathrm{f}}_{2,\rho}(X)$. For a reduction of the invariants to a computable invariant, we introduce 4–fold symmetric quandle 2–cocycles, modifying symmetric quandle 2–cocycles introduced by Kamada and Oshiro. Under the influence of their work, we define a 4–fold symmetric quandle cocycle invariant of 3–manifolds. This is a slight generalization of the state sum invariant considered in Hatakenaka [13, Section 3.3]. Further, we show that the 4–fold symmetric cocycle invariant is derived from the 4–fold symmetric homotopy invariant (Proposition 7.3).

7.1 Review of symmetric quandle cocycles and weights

In this section, we review the symmetric quandle cocycle introduced by Kamada and Oshiro [17; 18]. For a symmetric quandle (X,ρ), an (X,ρ)*–set* is a set Λ equipped with a map $*\colon \Lambda\times X \longrightarrow \Lambda$ satisfying $(\lambda * x)*x' = (\lambda * x')*(x*x')$ and $(\lambda * x)*\rho(x)=\lambda$ for any $\lambda\in\Lambda$ and $x,x'\in X$. For example, when $\Lambda = X$ with the quandle operation, X is an (X,ρ)–set itself. For an abelian group A and an (X,ρ)–set Λ, a map $\theta\colon \Lambda\times X\times X\longrightarrow A$ is called a *symmetric quandle* 2*–cocycle*, if it satisfies the following three conditions:

(C1) For any $(\lambda,x,y,z)\in\Lambda\times X^3$,

$$\theta(\lambda,y,z)^{-1}\cdot\theta(\lambda*x,y,z)\cdot\theta(\lambda,x,z) = \theta(\lambda*y,x*y,z)\cdot\theta(\lambda,x,y)\cdot\theta(\lambda*z,x*z,y*z)^{-1}.$$

(C2) For any $(\lambda,x)\in\Lambda\times X$, $\theta(\lambda,x,x)=1_A$.

(C3) For any $(\lambda, x, y) \in \Lambda \times X^2$,

$$\theta(\lambda, x, y) = \theta(\lambda * x, \rho(x), y)^{-1}, \quad \theta(\lambda, x, y) = \theta(\lambda * y, x * y, \rho(y))^{-1}.$$

Let us review a symmetric quandle cocycle invariant of unoriented links introduced by Kamada and Oshiro as follows. Let D be an unoriented diagram. An *X_Λ–coloring of D* is defined to be an X_ρ–coloring of D with an assignment of elements of Λ to each complementary regions of D such that, for each regions separated by the arc with a color $x \in X$ as shown in Figure 12, $\lambda * x = \lambda'$ holds, where λ and $\lambda' \in \Lambda$. Fix $\lambda_0 \in \Lambda$. An X_Λ–coloring of D is said to be *at* λ_0, if the unbounded region containing the infinity point is assigned by λ_0. Denote by $\mathrm{Col}_{X_\Lambda}(D)_{\lambda_0}$ the set of all X_Λ–colorings of D at λ_0.

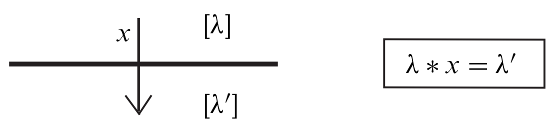


Figure 12: Coloring condition for regions

Given an X–coloring of D, we obtain the associated X_Λ–coloring of D at λ_0 whose assignments of each complementary regions are automatically determined by the rule in Figure 12. Therefore we obtain a bijection between $\mathrm{Col}_{X,\rho}(D)$ and $\mathrm{Col}_{X_\Lambda}(D)_{\lambda_0}$ (cf Kamada and Oshiro [18, Section 6]). Also, when X is a 4–fold symmetric quandle, for a labeled diagram D_ϕ, we denote by $\mathrm{Col}_{X_\Lambda}(D_\phi)_{\lambda_0}$ the set of all X_Λ–colorings of D at λ_0 whose restricted X_ρ–coloring is contained in $\mathrm{Col}_{X,\rho}(D_\phi)$. Similarly, we can obtain a bijection $\mathrm{Col}_{X,\rho}(D_\phi) \simeq \mathrm{Col}_{X_\Lambda}(D_\phi)_{\lambda_0}$.

For a symmetric quandle 2–cocycle θ, we will provide X_Λ–colorings of D at λ_0 with a grading by A. Let C be an X_Λ–coloring of D at λ_0. For a crossing v of C, we choose one of the four complementary regions of D around v. If the region is assigned by $\lambda \in \Lambda$, then the *weight* of v is defined to be $\theta(\lambda, x, y)^\epsilon \in A$, where $x, y \in X$ and the sign $\epsilon \in \{+1, -1\}$ are determined by the orientations as shown in Figure 13.

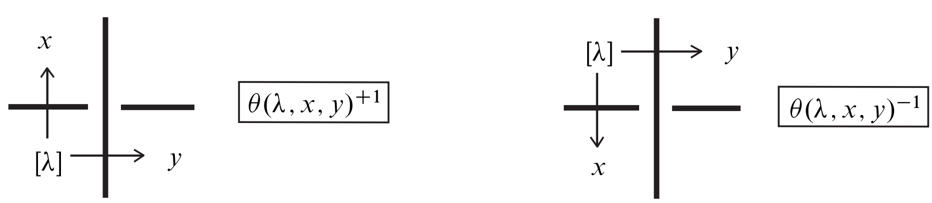


Figure 13: Weight of a crossing v

It is known [18, Lemma 6.2] that the weight of any crossing does not depend on the choice of four complementary regions and these orientations. Now we give $\Phi_\theta(D;C)_{\lambda_0} \in A$ by the sum of the weights of all crossing of D. Then the sum can be considered as a map

$$\Phi_\theta(D;\bullet)_{\lambda_0}\colon \mathrm{Col}_{X_\Lambda}(D)_{\lambda_0} \longrightarrow A. \tag{27}$$

It is shown [18, Theorem 6.3] that, given an X_Λ–coloring C_1 (resp. C_2) of an unoriented diagram D_1 (resp. D_2), if C_1 and C_2 are related by some finite sequences of Reidemeister moves, then $\Phi_\theta(D_1;C_1)_{\lambda_0} = \Phi_\theta(D_2;C_2)_{\lambda_0} \in A$. Then the *symmetric* 2*–cocycle invariant* of an unoriented link L is defined by

$$\Phi_\theta(L)_{\lambda_0} := \sum_{C \in \mathrm{Col}_\Lambda(D)_{\lambda_0}} \Phi_\theta(D;C)_{\lambda_0} \in \mathbb{Z}[A].$$

However, it is not so easy to find a nontrivial symmetric quandle 2–cocycle in general.

Remark 3 It is known [18, Theorem 6.7] that this invariant $\Phi_\theta(L)_{\lambda_0}$ coincides with the original (shadow) cocycle invariant $\Phi_\theta^{\mathrm{ori}}(L)_{\lambda_0}$ introduced by [3; 4].

7.2 Definition: 4–fold symmetric quandle cocycle invariant

In analogy, for a 4–fold symmetric quandle, we will construct an invariant of 3–manifolds. For this, we now introduce 4–fold symmetric 2–cocycles:

Definition 7.1 Let (X, p_X, ρ) be a 4–fold symmetric quandle, and Λ an (X,ρ)–set. A symmetric quandle 2–cocycle $\theta\colon \Lambda \times X \times X \longrightarrow A$ is 4*–fold*, if it satisfies the following two conditions:

(C4) For any $\lambda \in \Lambda$, $x_{ij} \in X_{ij}$ and $y_{jk} \in X_{jk}$, the cocycle θ satisfies

$$\theta(\lambda, x_{ij}, y_{jk}) \cdot \theta(\lambda, y_{jk}, x_{ij} * y_{jk}) \cdot \theta(\lambda, x_{ij} * y_{jk}, x_{ij}) = 1_A.$$

(C5) For any $\lambda \in \Lambda$, $z_{ij} \in X_{ij}$ and $w_{kl} \in X_{kl}$, the cocycle θ satisfies $\theta(\lambda, z_{ij}, w_{kl}) \cdot \theta(\lambda, w_{kl}, z_{ij}) = 1_A$.

Definition 7.2 Let X be a finite 4–fold symmetric quandle, Λ an (X,ρ)–set, and D_ϕ a labeled diagram. Fix $\lambda_0 \in \Lambda$. For a 4–fold symmetric quandle 2–cocycle θ, the 4*–fold symmetric quandle cocycle invariant* of D_ϕ is defined by

$$\Phi_\theta(D_\phi)_{\lambda_0} = \sum_{C \in \mathrm{Col}_{X_\Lambda}(D_\phi)_{\lambda_0}} \Phi_\theta(D;C)_{\lambda_0} \in \mathbb{Z}[A].$$

Proposition 7.3 *Let X be a finite 4–fold symmetric quandle, and Λ an (X,ρ)–set. We fix a 4–fold symmetric quandle 2–cocycle $\theta \in \mathrm{Map}(\Lambda \times X \times X, A)$. Then there exists a homomorphism $\mathcal{H}_\theta\colon \Pi_{2,\rho}^{4\mathrm{f}}(X) \to A$ satisfying that, for any labeled diagram D_ϕ,*

$$\mathcal{H}_\theta(\Xi_X^{4\mathrm{f}}(D_\phi)) = \Phi_\theta(D_\phi)_{\lambda_0} \in \mathbb{Z}[A]. \tag{28}$$

In particular, $\Phi_\theta(D_\phi)_{\lambda_0}$ is a topological invariant of the 3–manifold M presented by D_ϕ.

Proof We will construct a homomorphism (29) below. For an X_Λ–coloring C_1 (resp. C_2) of an unoriented diagram D_1 (resp. D_2), if C_1 and C_2 are related by a symmetric concordance relation as in Figure 7, then $\Phi_\theta(D_1;C_1)_{\lambda_0} = \Phi_\theta(D_2;C_2)_{\lambda_0} \in A$. Hence, running over all X_Λ–colorings of all unoriented diagrams D, the map (27) induces

$$\overline{\mathcal{H}}_\theta\colon \Pi_{2,\rho}(X) \longrightarrow A. \tag{29}$$

From the definitions of the multiplication of $\Pi_{2,\rho}(X)$ and of the weights, this map $\overline{\mathcal{H}}_\theta$ is multiplicative. Furtherer, notice that the left hand sides in (C4) and (C5) are the weights of X_Λ–colored trefoils and Hopf links in Figure 8, respectively. Therefore the homomorphism $\overline{\mathcal{H}}_\theta$ induces $\mathcal{H}_\theta\colon \Pi_{2,\rho}^{4\mathrm{f}}(X) \to A$ as required. To summarize this argument, we put a commutative diagram:

$$\begin{tikzcd}
\mathrm{Col}_{X_\Lambda}(D_\phi)_{\lambda_0} = \mathrm{Col}_{X,\rho}(D_\phi) \arrow[r, "\Xi_X(D_\phi;\bullet)"] \arrow[dr, "\Phi_\theta(D,\bullet)"'] & \Pi_{2,\rho}(X) \arrow[r, "\mathrm{proj}"] \arrow[d, "\overline{\mathcal{H}}_\theta"'] & \Pi_{2,\rho}^{4\mathrm{f}}(X) \arrow[dl, "\mathcal{H}_\theta"] \\
 & A &
\end{tikzcd}$$

where we identify $\mathrm{Col}_{X_\Lambda}(D_\phi)_{\lambda_0}$ with $\mathrm{Col}_{X,\rho}(D_\phi)$ mentioned above. Hence, for any $C \in \mathrm{Col}_{X_\Lambda}(D_\phi)_{\lambda_0}$, we have $\mathcal{H}_\theta\big(\Xi_X^{4\mathrm{f}}(D_\phi;C)\big) = \Phi_\theta(D_\phi;C)_{\lambda_0} \in A$, which implies (28) as desired. □

We give some remarks about Proposition 7.3.

Remark 4 (Cohomologous 2–cocycles) We consider a map $\delta_1\colon \mathrm{Map}(\Lambda \times X, A) \longrightarrow \mathrm{Map}(\Lambda \times X \times X, A)$ defined by

$$\delta_1(f)(\lambda,x,y) := f(\lambda,y)\cdot f(\lambda * x, y)^{-1} \cdot f(\lambda,x)^{-1} \cdot f(\lambda * y, x * y), \tag{30}$$

for $f \in \mathrm{Map}(\Lambda \times X, A)$. δ_1 is called the *coboundary map*. It is known [18, Theorem 6.3] that if $f \in \mathrm{Map}(\Lambda \times X, A)$ satisfies $f(\lambda,x) = f(\lambda * x, \rho(x))$ for any $(\lambda,x) \in \Lambda \times X$, then $\phi = \delta_1(f)$ is a symmetric 2–cocycle and the resulting map (29) is the zero map.

Moreover, we can verify that such $\phi = \delta_1(f)$ is 4–fold by direct calculation. Hence, for the detection of a nontrivial 4–fold cocycle invariant, we have to find a cocycle in $\mathrm{Map}(\Lambda \times X \times X, A)$ modulo the image of δ_1.

Remark 5 We comment on the choice of the coefficient group A. For a finite cored group (G,c), we showed that $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ is a finite abelian group whose elements are annihilated by $2^{12} \cdot 3^4 \cdot |G|^{12} \cdot |[G,G]|^4$ (Theorem 6.2). Therefore if $A \otimes_{\mathbb{Z}} (\mathbb{Z}/(6 \cdot |G|)\mathbb{Z}) = 0$ (eg, $A = \mathbb{Q}$), then the 4–fold symmetric quandle cocycle invariant is trivial by Proposition 7.3. Here the invariant is said to be *trivial*, if $\Phi_\theta(D_\phi)$ is contained in $\mathbb{Z}[1_A] \subset \mathbb{Z}[A]$. Therefore, we shall assume that $A \otimes_{\mathbb{Z}} (\mathbb{Z}/(6 \cdot |G|)\mathbb{Z}) \neq 0$.

Remark Given a group 3–cocycle ψ of G in a certain condition, Hatakenaka regarded G^4 as a $(\widetilde{G}_e, \rho)$–set and constructed a 4–fold symmetric quandle cocycle. Further, she reconstructed Dijkgraaf–Witten invariant with respect to ψ as a 4–fold symmetric quandle cocycle invariant (see Hatakenaka [13, Theorem 4.2]). Hence, such Dijkgraaf–Witten invariant is derived from $\Xi_X^{4\mathrm{f}}(M)$. However, in general, it is shown that so is all the Dijkgraaf–Witten invariant by her and the author [14].

Besides, for an application, some 4–fold cocycle invariants are used to estimate whether a 3–manifold can be presented by a double branched covering or not.

Proposition 7.4 *Let D_ϕ be a labeled diagram which presents a 3–manifold M. Let X be a finite 4–fold symmetric quandle. Let $\theta \in \mathrm{Map}(\Lambda \times X \times X, A)$ be a 4–fold symmetric quandle 2–cocycle. If the cocycle invariant $\Phi_\theta(M) \notin \mathbb{Z}[1_A]$ and the induced cocycle $(i_{12})^*(\theta) \in \mathrm{Map}(\Lambda \times X_{12} \times X_{12}, A)$ is the zero map, then M is not any double branched covering space of S^3. Here $i_{12}\colon X_{12} \hookrightarrow X$ is the inclusion.*

Proof Assume D_ϕ is 2–fold. Then we may regard that any $\widetilde{G}_c$–colorings of D_ϕ are contained in X_{12}. Hence, each X–colorings can be coupled together with the induced cocycle $(i_{12})^*(\theta)$. Then the cocycle invariant $\Phi_\theta(M)$ lies in $\mathbb{Z}[1_A]$, which implies a contradiction. □

Known approaches of such estimates are homological arguments (see, eg, Fox [12]). For example, Sakuma [24] discussed whether surface bundles over S^1 are double branched coverings of S^3 by considering their Heegaard splittings. On the other hand, our method is combinatorial and elementary for the estimate. However, unfortunately, we find no such 4–fold symmetric quandle cocycles yet.

Problem 7.5 Find a nontrivial 4–fold symmetric quandle cocycle of some (G,c) satisfying the condition in Proposition 7.4. Using such cocycles, find some 3–manifolds except surface bundles over S^1 that are not any double branched coverings of S^3.

8 4–Fold symmetric quandle cocycle invariants with trivial coefficients

This section discusses 4–fold symmetric quandle cocycle invariants with trivial $(\widetilde{G}_c, \rho)$-sets when $c = e$. In Section 8.1, we show that any symmetric 2–cocycle over $\mathbb{Z}/2\mathbb{Z}$ is 4–fold. In Section 8.2, we review the coloring polynomial introduced by Eisermann [7]. In Section 8.3, we apply the polynomial over $\mathbb{Z}/2\mathbb{Z}$ to labeled diagrams. In conclusion, we can calculate 4–fold symmetric quandle cocycle invariants of 3–manifolds of $\widetilde{G}_e$ without knowing presentations of the cocycles (Remark 6).

8.1 Symmetric quandle 2–cocycles with a trivial (X, ρ)–set

In this section, we assume that an (X, ρ)–set Λ is composed of a single point λ_0 as a trivial action from X. We omit writing the letter $\Lambda = \{\lambda_0\}$, and regard a 2–cocycle $\theta \in \mathrm{Map}(\Lambda \times X^2, A)$ as a map from X^2. Then, the 2–cocycle condition (C1) can be reformulated as

$$\theta(x, z) \cdot \theta(x * z, y * z) = \theta(x * y, z) \cdot \theta(x, y) \in A, \tag{31}$$

for any $x, y, z \in X$. Remark that, if $\rho = \mathrm{id}_X$, (C3) means $\theta(x, y)^2 = 1_A$ for any $x, y \in X$. Let us prepare a proposition:

Proposition 8.1 *Let X be a 4–fold symmetric quandle. If $a^2 = 1_A$ for $a \in A$, then any symmetric 2–cocycle $\theta \in \mathrm{Map}(X^2, A)$ is 4–fold. In particular, if $\rho = \mathrm{id}_X$, then any symmetric 2–cocycle is 4–fold.*

Proof We first show that θ satisfies the condition (C4). Applying $x = z = x_{ij} \in X_{ij},\ y = y_{jk} \in X_{jk}$ to (31), by the axiom (C2) we obtain

$$\theta(x_{ij}, y_{jk} * x_{ij}) = \theta(x_{ij}, y_{jk}) \cdot \theta(x_{ij} * y_{jk}, x_{ij}).$$

Further, using the axiom (C3), the left hand side becomes

$$\begin{aligned}\theta(x_{ij} * (y_{jk} * x_{ij}), \rho(y_{jk} * x_{ij}))^{-1} &= \theta((y_{jk} * x_{ij}) * \rho(x_{ij}), x_{ij} * y_{jk})^{-1} \\ &= \theta(y_{jk}, x_{ij} * y_{jk})^{-1},\end{aligned}$$

where the second equality is obtained from (F3). The equalities imply (C4).

Next, for the axiom (C5), by applying $x = x_{ij} \in X_{ij}$, $y = y_{kl} \in X_{kl}$ to (31), we have $\theta(x_{ij}, y_{kl}) = \theta(x_{ij} * z, y_{kl} * z)$ for any $z \in X$. Hence, for any $\Upsilon \in \mathrm{Inn}(X)$ we obtain

$$\theta(x_{ij}, y_{kl}) = \theta(x_{ij} \cdot \Upsilon, y_{kl} \cdot \Upsilon). \tag{32}$$

Put $\kappa_{jk} \in X_{jk}$, and define $\Upsilon_{xy} = (\bullet * \kappa_{jk}) * \big(y_{kl} * (x_{ij} * \kappa_{jk})\big) \in \mathrm{Inn}(X)$. Then by elementary calculation we have $x_{ij} \cdot \Upsilon_{xy} = y_{kl}$ and $y_{kl} \cdot \Upsilon_{xy} = x_{ij}$. Therefore by applying $\Upsilon = \Upsilon_{xy}$ to (32), (C5) follows from (C3), ie,

$$\theta(x_{ij}, y_{kl}) = \theta(x_{ij} \cdot \Upsilon_{xy}, y_{kl} \cdot \Upsilon_{xy}) = \theta(y_{kl}, x_{ij}) = \theta(y_{kl}, x_{ij})^{-1}. \qquad \square$$

Remark For a nontrivial $(\widetilde{G}_c, \rho)$–set, some symmetric quandle cocycles of $\widetilde{G}_c$ are not 4–fold. For example, we let $G = (\mathbb{Z}/2\mathbb{Z})^m$ and $A = \mathbb{Z}/3\mathbb{Z}$. Put the map $\pi_{\mathcal{R}_3}$ in Section 6.2. Then the composite

$$\widetilde{G}_c \xrightarrow{\pi_{\mathcal{R}_3}} p_{\widetilde{G}_c}^{-1}(\mathcal{R}_3) \xrightarrow{p_{\widetilde{G}_c}} \mathcal{R}_3$$

is a symmetric quandle homomorphism. Noting that $\mathcal{R}_3$ is a dihedral quandle of order 3, recall a 3–cocycle $\theta_{\mathcal{R}_3}$ of $\mathcal{R}_3$ found in Mochizuki's paper [19]. Then by the presentation of $\theta_{\mathcal{R}_3}$, the cocycle $\theta_{\mathcal{R}_3}$ can be regarded a symmetric quandle 2–cocycle with an $(\mathcal{R}_3, \mathrm{id}_{\mathcal{R}_3})$–set. Then the induced cocycle $(p_{\widetilde{G}_c} \circ \pi_{\mathcal{R}_3})^*(\theta_{\mathcal{R}_3})$ is a symmetric quandle 2–cocycle of $\widetilde{G}_c$. However, we see that the 2–cocycle invariants of trefoils are nonzero, which means that $(p_{\widetilde{G}_c} \circ \pi_{\mathcal{R}_3})^*(\theta_{\mathcal{R}_3})$ is not 4–fold.

8.2 Preliminaries: the coloring polynomial of knots

We review the coloring polynomial introduced by Eisermann [7]. Further, we modify his construction to apply to symmetric quandle 2–cocycles.

Let X be a finite connected quandle. We assume that X is of type 2 and that $\mathfrak{inn}_X\colon X \longrightarrow \mathrm{Inn}(X)$ given in (5) is injective. Put $x_0 \in X$. Let D be a knot diagram on $\mathbb{R}^2$ of an oriented knot K. Fix $m_K \in \pi_1(S^3 \setminus K)$ obtained from a meridian of K. Denote by $\mathrm{Col}_X^{m_K, x_0}(D)$ the set of X–colorings of D which sends the arc associated with m_K to $x_0 \in X$. Then it is known [7, Lemma 3.14] that there is a natural bijection

$$\Gamma_\bullet\colon \mathrm{Col}_X^{m_K, x_0}(D) \longrightarrow \mathrm{Hom}_{\mathrm{grp}}^{m_K, x_0}(\pi_1(S^3 \setminus K), \mathrm{Inn}(X)),$$

where $\mathrm{Hom}_{\mathrm{grp}}^{m_K, x_0}(\pi_1(S^3 \setminus K), \mathrm{Inn}(X))$ stands for the set of the homomorphisms sending m_K to $\mathfrak{inn}_X(x_0) \in \mathrm{Inn}(X)$. Eisermann introduced the following invariant of knots:

$$\mathcal{P}_{\mathrm{Inn}(X)}^{x_0}(K) := \sum_{C \in \mathrm{Col}_X^{m_K, x_0}(D)} \Gamma_C(l_K) \in \mathbb{Z}[\mathrm{Inn}(X)], \tag{33}$$

where $l_K \in \pi_1(S^3 \setminus K)$ is derived from the longitude of K. $\mathcal{P}_{\mathrm{Inn}(X)}^{x_0}(K)$ is called a *coloring polynomial* of K. Note that l_K lies in the commutator subgroup $[\pi_1(S^3 \setminus K), \pi_1(S^3 \setminus K)]$ and commutes with m_K. Hence, $\Gamma_C(l_K) \in Z(x_0) \cap [\mathrm{Inn}(X), \mathrm{Inn}(X)]$, where $Z(x_0)$ is the centralization subgroup of $\mathfrak{inn}_X(x_0)$. Let H be a quotient group of $Z(x_0) \cap [\mathrm{Inn}(X), \mathrm{Inn}(X)]$, and let $\pi_H \colon Z(x_0) \cap [\mathrm{Inn}(X), \mathrm{Inn}(X)] \to H$ be the projection. Then we put

$$\mathcal{P}^{x_0}_{\mathrm{Inn}(X),H}(K) := \sum_{C \in \mathrm{Col}_X^{m_K, x_0}(D)} \pi_H\big(\Gamma_C(l_K)\big) \in \mathbb{Z}[H]. \tag{34}$$

Remark that $Z(x_0) \cap [\mathrm{Inn}(X), \mathrm{Inn}(X)]$ is not always abelian. Then we consider

$$H_{\mathrm{ab}} := \big(Z(x_0) \cap [\mathrm{Inn}(X), \mathrm{Inn}(X)]\big)_{\mathrm{ab}}, \quad H_{2\mathbb{Z}} := H_{\mathrm{ab}}/2H_{\mathrm{ab}}. \tag{35}$$

Eisermann showed [7, Theorem 3.24] that this invariant $\mathcal{P}^{x_0}_{\mathrm{Inn}(X),H_{\mathrm{ab}}}(K)$ is equivalent to some 2–cocycle invariant. We will modify the theorem for symmetric quandles of type 2[2].

Proposition 8.2 *(Symmetric quandle version of [7, Theorem 3.24]) Let (X, id_X) be a finite connected symmetric quandle of type 2. Assume that $\mathfrak{inn}_X \colon X \longrightarrow \mathrm{Inn}(X)$ is injective. Then X admits a symmetric quandle 2–cocycle $\theta_{2\mathbb{Z}} \in \mathrm{Map}(X^2, H_{2\mathbb{Z}})$ such that $\Phi_{\theta_{2\mathbb{Z}}}(D; C) = \pi_{H_{2\mathbb{Z}}}(\Gamma_C(m_K)) \in H_{2\mathbb{Z}}$ for any $C \in \mathrm{Col}_X^{m_K, x_0}(D)$. In particular, $\Phi_{\theta_{2\mathbb{Z}}}(K) = |X| \cdot \mathcal{P}^{x_0}_{\mathrm{Inn}(X),H_{2\mathbb{Z}}}(K) \in \mathbb{Z}[H_{2\mathbb{Z}}]$.*

Before proving Proposition 8.2, we discuss a construction of the required 2–cocycle $\theta_{2\mathbb{Z}}$ following Eisermann [7, Lemma 3.16]. First, we set

$$Q_X := \{(a, g) \in X \times [\mathrm{Inn}(X), \mathrm{Inn}(X)] \mid a = x_0 \cdot g\}. \tag{36}$$

We see that the set Q_X is a symmetric quandle of type 2 with the following operations:

$$(a, g) * (b, h) := \big(a * b,\ g \cdot \mathfrak{inn}_X(a)^{-1} \cdot \mathfrak{inn}_X(b)\big), \quad \rho(a, g) := (a, g),$$

and that the natural projection $Q_X \to X$ is a symmetric quandle epimorphism (see [7, Lemma 3.24]). We can check that the fiber is $Z(x_0) \cap [\mathrm{Inn}(X), \mathrm{Inn}(X)]$. Note that its normal subgroup $\mathrm{Ker}(\pi_{H_{2\mathbb{Z}}})$ freely acts on each fiber. Then we can verify that the left quotient $\mathrm{Ker}(\pi_{H_{2\mathbb{Z}}}) \backslash Q_X$ is also a symmetric quandle, and has a symmetric epimorphism $p_{Q_{2\mathbb{Z}}} \colon \mathrm{Ker}(\pi_{H_{2\mathbb{Z}}}) \backslash Q_X \to X$ by definition. Remark the following property of $p_{Q_{2\mathbb{Z}}}$:

(37) • $p_{Q_{2\mathbb{Z}}}(\widetilde{x}) = p_{Q_{2\mathbb{Z}}}(\widetilde{y})$ implies $\widetilde{a} * \widetilde{x} = \widetilde{a} * \widetilde{y}$ for any $\widetilde{a}, \widetilde{x}, \widetilde{y} \in \mathrm{Ker}(\pi_{H_{2\mathbb{Z}}}) \backslash Q_X$.

[2] If X is not of type 2, then the result similar to Proposition 8.2 is not always true. For example, let $X = \widetilde{G}_c$ with $(G, c) = (\mathbb{Z}/2\mathbb{Z}, 1)$. The polynomial $\mathcal{P}^{x_0}_{\mathrm{Inn}(X),H_{2\mathbb{Z}}}(K)$ of the trefoil knot K is nontrivial, while we can verify that $\widetilde{G}_c$ has no nontrivial symmetric quandle cocycle by the help of the computer.

Put an arbitrary section $s_X\colon X \to \mathrm{Ker}(\pi_{H_{2\mathbb{Z}}})\backslash Q_X$. Then we define a map $\theta_{2\mathbb{Z}}\colon X^2 \to H_{2\mathbb{Z}}$ by the following relation:

$$s_X(a) * s_X(b) = \theta_{2\mathbb{Z}}(a,b)\cdot s_X(a*b) \in H_{2\mathbb{Z}}, \tag{38}$$

for $a, b \in X$. Then it is known [7, Lemma 3.1] that the property (37) enables the resulting $\theta_{2\mathbb{Z}}$ to satisfy the conditions (C1) and (C2). Further, it is known (see [7, Theorem 3.19]) that $\theta_{2\mathbb{Z}}$ does not depend on the choice of the section s_X up to cohomologous in the sense of Remark 4.

Proof First, in order to show that $\theta_{2\mathbb{Z}}$ is a symmetric quandle 2–cocycle, it suffices to show that $\theta_{2\mathbb{Z}}$ satisfies (C3). By applying $\bullet * s_X(b)$ to the equality (38), we have

$$s_X(a) = \theta_{2\mathbb{Z}}(a,b)\cdot\big(s_X(a*b)*s_X(b)\big) = \theta_{2\mathbb{Z}}(a,b)\cdot\theta_{2\mathbb{Z}}(a*b,b)^{-1}\cdot s_X(a),$$

where we use (38) again for the last equality. This implies (C3).

Next, we put the symmetric quandle cocycle invariant $\Phi_{\theta_{2\mathbb{Z}}}(K)$. Recall that the invariant coincides with the original cocycle invariant $\Phi^{\mathrm{ori}}_{\theta_{2\mathbb{Z}}}(K)$ by Remark 3. Eisermann showed [7, Theorem 3.25] that $\Phi^{\mathrm{ori}}_{\theta_{2\mathbb{Z}}}(K) = |X|\cdot\mathcal{P}^{x_0}_{\mathrm{Inn}(X),H_{2\mathbb{Z}}}(K) \in \mathbb{Z}[H_{2\mathbb{Z}}]$. Hence, $\Phi_{\theta_{2\mathbb{Z}}}(K) = \Phi^{\mathrm{ori}}_{\theta_{2\mathbb{Z}}}(K) = |X|\cdot\mathcal{P}^{x_0}_{\mathrm{Inn}(X),H_{2\mathbb{Z}}}(K) \in \mathbb{Z}[H_{2\mathbb{Z}}]$, which completes the proof. □

Remark 6 The point of Proposition 8.2 is as follows. In general, it is difficult to find a presentation of a 2–cocycle $\theta \in \mathrm{Map}(X^2, \mathbb{Z}/2\mathbb{Z})$. However, the invariant of $\mathcal{P}^{x_0}_{\mathrm{Inn}(X),H_{2\mathbb{Z}}}(K)$ is computable without knowing the presentation, and, further, admits every 2–cocycle invariant valued in $\mathbb{Z}/2\mathbb{Z}$. Therefore, if we determined the group $Z(x_0)\cap[\mathrm{Inn}(X),\mathrm{Inn}(X)]$, then we could calculate the universal invariant $\mathcal{P}^{x_0}_{\mathrm{Inn}(X),H_{2\mathbb{Z}}}(K)$.

8.3 Coloring polynomials of 4–fold symmetric quandles of type 2

We return to the theme of 4–fold symmetric cocycle invariants. Let G be a finite group. Recall that the quandle $\widetilde{G}_e$ is connected and of type 2 by Lemma 3.7 and Corollary 3.6. Following the previous section, we give an invariant of 3–manifolds which derives out of every symmetric quandle 2–cocycle invariants of $\widetilde{G}_e$. Denote $(e,(i,j)) \in \widetilde{G}_e$ by e_{ij} for simplicity.

Recall that any 3–manifold M is a 3–fold simple covering of S^3 branched over a knot K (see Section 2.2). Then, we may assume that the labeled diagram D_ϕ is 3–fold shown in Figure 4, and that the subdiagram $D_{\mathcal{R}_3}$ is a knot diagram of K. Let m_K and $l_K \in \pi_1(S^3 \setminus K)$ be a meridian and a longitude of K, respectively. Then we define

$$\mathcal{P}^{e_{12}}_{\widetilde{G}_e, H_{2\mathbb{Z}}}(D_\phi) = \sum_{C \in \mathrm{Col}^{e_{12}, e_{23}, e_{34}}_{\widetilde{G}_e, \rho}(D_\phi)} \pi_{H_{2\mathbb{Z}}}(\Gamma_C(l_K)) \in \mathbb{Z}[H_{2\mathbb{Z}}], \tag{39}$$

where $H_{2\mathbb{Z}}$ is given in (35).

Proposition 8.3 *Let $\widetilde{G}_e$ and $H_{2\mathbb{Z}}$ be as above. Let $\theta_{2\mathbb{Z}}$ be the resulting symmetric 2–cocycle in Proposition 8.2. Let a 3–fold labeled diagram D_ϕ present a 3–manifold M. Then, $\theta_{2\mathbb{Z}}$ is 4–fold, and the 4–fold symmetric cocycle invariant $\Phi_\theta(M)$ is equal to $|G|^3 \cdot \mathcal{P}^{e_{12}}_{\widetilde{G}_e, H_{2\mathbb{Z}}}(D_\phi) \in \mathbb{Z}[H_{2\mathbb{Z}}]$. In particular, the polynomial (39) is an invariant of M.*

Proof By Proposition 8.1, $\theta_{2\mathbb{Z}}$ is 4–fold. Put the map $\mathcal{H}_{\theta_{2\mathbb{Z}}}\colon \Pi^{4\mathrm{f}}_{2,\rho}(X) \to H_{2\mathbb{Z}}$ in Proposition 7.3. By Lemma 4.4, we have

$$\begin{aligned}\Phi_{\theta_{2\mathbb{Z}}}(M) = \mathcal{H}_{\theta_{2\mathbb{Z}}}(\Xi^{4\mathrm{f}}_{\widetilde{G}_e}(D_\phi)) &= |G|^3 \sum_{C \in \mathrm{Col}^{e_{12}, e_{23}, e_{34}}_{\widetilde{G}_e, \rho}(D_\phi)} \mathcal{H}_{\theta_{2\mathbb{Z}}}(\Xi^{4\mathrm{f}}_X(D_\phi; C)) \\ &= |G|^3 \sum_{C \in \mathrm{Col}^{e_{12}, e_{23}, e_{34}}_{\widetilde{G}_e, \rho}(D_\phi)} \Phi_{\theta_{2\mathbb{Z}}}(D_\phi; C).\end{aligned}$$

Further, by Proposition 8.2 we conclude that this equals

$$|G|^3 \sum_{C \in \mathrm{Col}^{e_{12}, e_{23}, e_{34}}_{\widetilde{G}_e, \rho}(D_\phi)} \pi_{H_{2\mathbb{Z}}}(\Gamma_C(l_K)) = |G|^3 \cdot \mathcal{P}^{e_{12}}_{\widetilde{G}_e, H_{2\mathbb{Z}}}(D_\phi). \qquad \square$$

Hence, it is important for the calculation of the invariant $\mathcal{P}^{e_{12}}_{\widetilde{G}_e, H_{2\mathbb{Z}}}(D_\phi)$ to compute $Z(e_{12}) \cap [\mathrm{Inn}(\widetilde{G}_e), \mathrm{Inn}(\widetilde{G}_e)]$ and the container $H_{2\mathbb{Z}}$ (see also Remark 6). For this, we recall here some notation $I_{G,e}$, $K_{G,e}$, $Z_{G,e}$ used in Section 5:

$$I_{G,e} = \{(x, y, z, w; \sigma) \in G^4 \rtimes \mathfrak{S}_4 \mid xyzw \in [G, G]\},$$

$$K_{G,e} = \{(x, x, z, w; \sigma) \in I_{G,e} \mid \sigma \in \{e,\ (12),\ (34),\ (12)(34)\}\}, \tag{40}$$

$$Z_{G,e} = \{(z, z, z, z; e) \in G^4 \rtimes \mathfrak{S}_4 \mid z \in Z(G),\ z^4 \in [G, G]\}. \tag{41}$$

We will show:

Proposition 8.4 *Let G be a finite group. Then there exists a group isomorphism*

$$Z(e_{12}) \cap [\mathrm{Inn}(\widetilde{G}_e), \mathrm{Inn}(\widetilde{G}_e)] \cong \{(x, x, z, w; \sigma) \in G^4 \rtimes \mathbb{Z}/2\mathbb{Z} \mid x^2 zw \in [G, G]\}/Z_{G,e},$$

where we identify a subgroup $\{e,\ (12)(34)\} \subset \mathfrak{S}_4$ with $\mathbb{Z}/2\mathbb{Z}$.

Proof To begin with, recall $\mathrm{Inn}(\widetilde{G}_e) \cong I_{G,e}/Z_{G,e}$ by Theorem 5.4 and the presentation of $K_{G,e}$ in (17). Since $K_{G,e}$ is the centralizer group of $(e,e,e,e;(12)) \in I_{G,e}$ and $Z_{G,e}$ is the center of $I_{G,e}$, we can easily verify $Z(e_{12}) = K_{G,e}/Z_{G,e}$. Further, we note $[(I_{G,e}/Z_{G,e}),(I_{G,e}/Z_{G,e})] = [I_{G,e}, I_{G,e}]/Z_{G,e}$. We thus have a group isomorphism

$$Z(e_{12}) \cap [\mathrm{Inn}(\widetilde{G}_e), \mathrm{Inn}(\widetilde{G}_e)] \cong (K_{G,e} \cap [I_{G,e}, I_{G,e}])/Z_{G,e}. \tag{42}$$

Next, we explicitly present the commutator subgroup $[\mathrm{Inn}(\widetilde{G}_e), \mathrm{Inn}(\widetilde{G}_e)]$ as follows. By Lemma 8.5 below, the abelianization of $\mathrm{Inn}(\widetilde{G}_e)$ is $\mathbb{Z}/2\mathbb{Z}$. Put $E := \{e\}$. Recall that $\mathrm{Inn}(\widetilde{E}_e) \cong \mathfrak{S}_4$ and that the abelianization of $\mathfrak{S}_4$ is given by the signature $\mathfrak{S}_4 \to \mathbb{Z}/2\mathbb{Z}$. Hence, putting the natural map $G \to E$, the induced composite $\mathrm{Inn}(\widetilde{G}_e) \to \mathrm{Inn}(\widetilde{E}_e) \cong \mathfrak{S}_4 \to \mathbb{Z}/2\mathbb{Z}$ gives rise to the abelianization of $\mathrm{Inn}(\widetilde{G}_e)$. Therefore,

$$[\mathrm{Inn}(\widetilde{G}_e), \mathrm{Inn}(\widetilde{G}_e)] \cong \{(x,y,z,w;\sigma) \in G^4 \rtimes A_4 \mid xyzw \in [G,G]\}/Z_{G,e},$$

where A_4 is the alternation group of order 12. Then, the presentation (40) gives rise to

$$K_{G,e} \cap [I_{G,e}, I_{G,e}] = \{\, (x,x,z,w;\sigma) \in G^4 \rtimes A_4 \mid x^2zw \in [G,G],\ \sigma = e \text{ or } (12)(34)\,\}.$$

Combing this formula with (42), we conclude the required formula. $\square$

Lemma 8.5 *For a connected quandle X of type m, the abelianization of $\mathrm{Inn}(X)$ is a quotient of $\mathbb{Z}/m\mathbb{Z}$.*

Proof Recall *the associated group* of X defined by the group presentation $\mathrm{As}(X) := \langle x \in X \mid x \cdot y = y \cdot (x * y)\rangle$. Then $\mathrm{As}(X)$ has a canonical right action on X. Hence we have an epimorphism $p_{\mathrm{AI}}\colon \mathrm{As}(X) \to \mathrm{Inn}(X)$.

Let us consider an epimorphism $\varepsilon_X\colon \mathrm{As}(X) \to \mathbb{Z}$ given by the length of words. Since X is connected, ε_X gives rise to the abelianization of $\mathrm{As}(X)$ by definition, that is, $\varepsilon_X\colon \mathrm{As}(X)/[\mathrm{As}(X), \mathrm{As}(X)] \cong \mathbb{Z}$. Let us fix $x_0 \in X$, and put a section s_{ε_X} of ε_X which sends 1 to x_0. Since X is of type m, the composite $p_{\mathrm{AI}} \circ s_{\varepsilon_X}$ induces an epimorphism $\mathbb{Z}/m\mathbb{Z} \to \mathrm{Inn}(X)/[\mathrm{Inn}(X), \mathrm{Inn}(X)]$, which completes the proof. $\square$

In order to compute the abelianization of $Z(e_{12}) \cap [\mathrm{Inn}(\widetilde{G}_e), \mathrm{Inn}(\widetilde{G}_e)]$, the exact sequence (43) below is useful. For this, we consider the central group extension

$$0 \longrightarrow Z_{G,e} \xrightarrow{\ i\ } K_{G,e} \cap [I_{G,e}, I_{G,e}] \longrightarrow (K_{G,e} \cap [I_{G,e}, I_{G,e}])/Z_{G,e} \longrightarrow 0.$$

Using the Lyndon–Hochschild–Serre spectral sequence of the group extension, we have an exact sequence

$$\begin{aligned} Z_{G,e} \xrightarrow{i_*} H_1(K_{G,e} \cap [I_{G,e}, I_{G,e}]; \mathbb{Z}) \qquad\qquad & \\ \to H_1(Z(e_{12}) \cap [\mathrm{Inn}(\widetilde{G}_e), \mathrm{Inn}(\widetilde{G}_e)]; \mathbb{Z}) \to 0. & \end{aligned} \tag{43}$$

Here we identify $H_1(K;\mathbb{Z})$ with the abelianization of K. For the last term, the key is to determine the induced map i_*.

As examples, we compute H_{ab} and $H_{2\mathbb{Z}}$ in the cases where G is perfect, cyclic or quaternion group. However, the computations are easily done by using the sequence (43) and Proposition 8.4. We will roughly explain the computations.

Example 8.6 Let G be a finite perfect group, that is, $G=[G,G]$. We thus see $K_{G,e}\cap[I_{G,e},I_{G,e}]\cong G\times(G^2\rtimes\mathbb{Z}/2\mathbb{Z})$ by Proposition 8.4. Then the abelianization of $K_{G,e}\cap[I_{G,e},I_{G,e}]$ is $\mathbb{Z}/2\mathbb{Z}$. In particular, $H_{2\mathbb{Z}}\cong\mathbb{Z}/2\mathbb{Z}$.

As a special case, let us consider the group composed of a single point $\{e\}$. By Remark 2 we see that the 4–fold cocycle invariant of $\{e\}$ is trivial for any 3–manifolds. Put the terminal map $G\to\{e\}$. The induced map on $H_{2\mathbb{Z}}$ is then an isomorphism on $\mathbb{Z}/2\mathbb{Z}$. In conclusion, for a finite perfect group G, the polynomial $\mathcal{P}^{\varrho_{12}}_{\tilde{G}_e,H_{2\mathbb{Z}}}(D_\phi)$ turns out to be a trivial invariant.

Example 8.7 Let $G=\mathbb{Z}/n\mathbb{Z}$. Proposition 8.4 indicates that the abelianization

$$K_{G,e}\cap[I_{G,e},I_{G,e}]=\{(x,x,z,z^{-1}x^{-2};\sigma)\in G^4\rtimes\mathbb{Z}/2\mathbb{Z}\}\longrightarrow G\oplus G/2G\oplus\mathbb{Z}/2\mathbb{Z},$$

is given by $(x,x,z,z^{-1}x^{-2};\sigma)\mapsto(x,[z],\sigma)$. The group $H_{2\mathbb{Z}}$ can be computed by the induced map i_* in (43). Without the detailed proof, we state only the conclusion:

$$H_{2\mathbb{Z}}\cong\begin{cases}\mathbb{Z}/2\mathbb{Z} & n \text{ odd},\\ (\mathbb{Z}/2\mathbb{Z})^2 & n=2\cdot m \text{ or } 4\cdot m, \text{ where } m \text{ is odd},\\ (\mathbb{Z}/2\mathbb{Z})^3 & n=2^k\cdot m, \text{ where } m \text{ is odd and } k>2.\end{cases}$$

Example 8.8 Let G be the quaternion group Q_8. Regarding Q_8 as a subset of the quaternion field $\mathbb{H}$, Q_8 is composed of $\{\pm1,\ \pm i,\ \pm j,\ \pm k\}$. Notice that, for any $x\in Q_8$, $x^2=\pm1$ and that the center of Q_8 is $[Q_8,Q_8]=\{\pm1\}$. It follows from Proposition 8.4 that a homomorphism

$$\begin{aligned}&K_{G,e}\cap[I_{G,e},I_{G,e}]\\&\qquad=\{(x,x,z,\pm z;\sigma)\in G^4\rtimes\mathbb{Z}/2\}\to(Q_8)_{\mathrm{ab}}\oplus(Q_8)_{\mathrm{ab}}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2,\\&\qquad\qquad(x,x,z,\pm z;\sigma)\longmapsto([x],[z],\mathrm{sgn}(\pm),\sigma),\end{aligned}$$

is the abelianization of $K_{G,e}\cap[I_{G,e},I_{G,e}]$. Note that $Z_{G,e}=\{(g,g,g,g;e)\in G^4\rtimes\mathbb{Z}/2\mathbb{Z}\mid g=\pm1\}$, and that i_* in the sequence (43) is injective. Therefore, noting $(Q_8)_{\mathrm{ab}}\cong(\mathbb{Z}/2\mathbb{Z})^2$, we have reached the conclusion of $H_{2\mathbb{Z}}\cong(\mathbb{Z}/2\mathbb{Z})^5$.

Although we can calculate the invariant $\mathcal{P}^{e_{12}}_{\widetilde{G}_e, H_{2\mathbb{Z}}}(D_\phi)$ of 3–manifolds without knowing the presentations of 4–fold symmetric quandle cocycles, unfortunately, we have not been able to find examples of a nontrivial invariant. On the other hand, Hatakenaka and the author [14] will show that our 4–fold symmetric quandle homotopy invariant of $\widetilde{G}_e$ is at least as strong as Dijkgraaf–Witten invariants [5] of G.

Problem 8.9 Find an example of a nontrivial 4–fold symmetric quandle cocycle invariant which is stronger than the Dijkgraaf–Witten invariant of G. Develop an algorithm to construct 4–fold symmetric quandle cocycles of $\widetilde{G}_c$, similar to those of Alexander quandles in Mochizuki [19].

9 Appendix: (Symmetric) quandle homotopy invariant of links

In this section, we discuss symmetric quandle homotopy invariants of unoriented links. Fenn, Rourke and Sanderson [9; 10; 11] introduced the rack space of a quandle, and studied an invariant of framed links valued in its second homotopy group. The author [21] calculated the homotopy groups of some quandles and studied invariants of oriented links. We first review (resp. symmetric) quandle homotopy invariants of (resp. un)oriented links. Our goal is to show that the symmetric quandle homotopy invariants are derived from the quandle homotopy invariants without good involution (Proposition 9.1).

To begin with, we review the quandle homotopy invariants. Let X be a finite quandle, and let $D_\circ$ be an oriented link diagram. Then the *quandle homotopy invariant* of $D_\circ$ is

$$\Xi^o_X(D_\circ) = \sum_{C \in \mathrm{Col}_X(D_\circ)} \Xi^o_X(D_\circ; C) \in \mathbb{Z}[\Pi_2(X)],$$

where $\Pi_2(X)$ is the group defined in Section 6.1 and the map $\Xi^o_X(D_\circ; \bullet)\colon \mathrm{Col}_X(D_\circ) \to \Pi_2(X)$ sends an X–coloring to the canonical class. Remark that, although the quandle homotopy invariants considered in the author [21] are defined by a topological method, it is shown [21, Theorem A.2] that they entirely coincide with our invariants $\Xi^o_X(D_\circ)$.

Next, we construct the symmetric quandle homotopy invariants in a similar fashion. Let (X, ρ) be a finite symmetric quandle, and let D be an unoriented link diagram. Then the *symmetric quandle homotopy invariant* of D is defined by the expression

$$\Xi_X(D) = \sum_{C \in \mathrm{Col}_{X,\rho}(D)} \Xi_{X,\rho}(D; C) \in \mathbb{Z}[\Pi_{2,\rho}(X)],$$

where $\Pi_{2,\rho}(X)$ is the group defined in Section 3.1, and $\Xi_{X,\rho}(D;\bullet)\colon \mathrm{Col}_{X,\rho}(D) \to \Pi_{2,\rho}(X)$ is a map given by (2). This is invariant under Reidemeister moves.

Proposition 9.1 *Let X be a finite quandle, and let D be an unoriented diagram. The symmetric quandle homotopy invariant of D is derived from the quandle homotopy invariant. More precisely, for any good involution ρ of X and any orientation of D, the homomorphism $\mathcal{S}_\Pi\colon \Pi_2(X) \longrightarrow \Pi_{2,\rho}(X)$ in* (23) *gives rise to*

$$\Xi_X(D) = \mathcal{S}_\Pi(\Xi^o_X(D_o)) \in \mathbb{Z}[\Pi_{2,\rho}(X)], \tag{44}$$

where D_o means the diagram D with the orientation.

Proof Recall the bijection $\mathfrak{P}_X$ given in (1) and the construction of $\mathcal{S}_\Pi$. Then, we have the commutative diagram

$$\begin{tikzcd} \mathrm{Col}_X(D_o) \arrow[r, "\mathfrak{P}_X"] \arrow[d, "\Xi^o_X(D_o;\bullet)"'] & \mathrm{Col}_{X,\rho}(D) \arrow[d, "\Xi_X(D;\bullet)"] \\ \Pi_2(X) \arrow[r, "\mathcal{S}_\Pi"] & \Pi_{2,\rho}(X) \end{tikzcd}$$

which immediately gives (44). □

We will give a summary of (symmetric) quandle homotopy invariants as follows. Kamada and Oshiro [18, Section 6] introduced the symmetric quandle cocycle invariants of 1–dimensional links. Similar to Proposition 7.3, the symmetric quandle cocycle invariants are derived from symmetric quandle homotopy invariants, and, hence, are from the quandle homotopy invariants without good involution by Proposition 9.1. In conclusion, the symmetric quandle structure has no expansion in the quandle homotopy invariant of 1–dimensional links.

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