The cactus tree of a metric space

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We extend the cactus theorem of Dinitz, Karzanov, Lomonosov to metric spaces. In particular we show that if X is a separable continuum which is not separated by $n-1$ points then the set of all n–tuples of points separating X can be encoded by an $\mathbb{R}$–tree.

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1 Introduction

Finite cuts have been studied in several different contexts. Whyburn [26] in 1928 showed that the set of cut points of a Peano continuum has the structure of a "dendrite". This "dendritic" decomposition of continua has been extended and used to prove several results in continua theory. We recall here that a continuum is a compact, connected Hausdorff space and a Peano continuum is a locally connected metric continuum. If X is a continuum, we say that a point c is a cut point of X if $X-\{c\}$ is not connected. Whyburn's work was extended by others and the more abstract framework of pretrees proposed by Ward [25] has proved adequate for studying cuts; see Bowditch [6], Adeleke and Neumann [2] and our paper [18].

Cuts of continua became relevant for group theory after Bestvina and Mess [3] showed that the boundary of a hyperbolic group is locally connected if and only if it has no cut points. Bowditch [5] shows how to pass from the action of a hyperbolic group G on its boundary ∂G to an action on an $\mathbb{R}$–tree, assuming that ∂G has a cut point. A crucial ingredient of this is a representation of all cut points of the boundary of the group (a continuum) by a tree. The general version of this is done by the first author [21; 22]. Further work from Swarup [20] and Levitt [14] implies that ∂G has no cut points. Bowditch [4] pursues these ideas further and shows that JSJ–decompositions of 1–ended hyperbolic groups over 2–ended groups are reflected on the boundary. To do this, he constructed a tree encoding all cut pairs of a locally connected continuum. In [18] we extended this to cut pairs of general continua and we applied it to JSJ–decompositions of CAT(0) groups [19].

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We note that the observation that leads to the construction of a tree encoding cut pairs is that cut pairs that “cross” each other lie in “circles”. These “circles” are encoded by vertices of the tree. It is worth noting that in the group theoretic setting, the stabilizers of these “circles” are the hanging-orbifold vertices of the JSJ–decomposition of the group. So in [18] the tree encoding the cut pairs of a continuum is termed the JSJ–decomposition of the continuum. One could remark that the classical JSJ–decomposition deals with separating annuli and tori “crossing” each other and that the context of separating cut pairs which cross each other is quite similar. In the manifold setting, all crossing annuli and tori are contained in a Seifert fibered piece (so they have a quite simple structure) while in the continua setting all crossing pairs are contained in a “circle”.

Guralnik [11] studied general finite cuts of locally connected continua using techniques similar to what we use here. He showed in particular that one can “encode” minimal cut triples by trees in the case of “cut-rigid” spaces admitting a cusp-uniform action by an infinite group.

Finite cuts have also been studied by graph theorists, who have obtained results similar to the continua theory results. It is easy to see that if Γ is a finite graph, then the cut vertices of Γ can be represented by a tree. Tutte [23, Chapter 11; 24, Chapter IV] proved that in fact one can represent by a tree the set of pairs of vertices that separate Γ (assuming that Γ has no cut vertex). In this case there might be cut pairs of vertices that “cross” each other, but Tutte shows that such pairs are arranged in circuits, and this is what allows him to encode cut pairs by a tree.

Networks are represented by finite graphs, and the cardinality of edge cuts give a good measure of the robustness of the network. So edge cuts have been studied extensively in network theory. An edge cut with the minimum possible number of edges is called a *mincut*. A classic theorem of Dinitz, Karzanov and Lomonosov [8] shows that mincuts have a rich combinatorial structure; they can be represented by a *cactus*, a tree-like structure. A simpler proof of this “cactus theorem” was given recently by Fleiner and Frank [9] (see also Kammer and Täubig [12, Section 7.5] and Naor and Vazirani [16] for applications). We now give a brief summary of the cactus theorem as it is relevant to this paper. For more details we refer to [8; 9]. A *cactus* is a graph in which any two simple cycles have at most one vertex in common. We note that if K is a mincut of a graph Γ then $\Gamma - K$ has exactly two components. We say that two mincuts K, L of a graph Γ cross if L intersects both components of $\Gamma - K$. It is quite easy to see that if no two mincuts of a graph Γ cross each other, then the set of mincuts of Γ is encoded by a tree. The central observation of [8] is that crossing mincuts have a circular structure. This allows them to show that the set of all mincuts of the graph can be represented by a cactus. More precisely there is a map from Γ to C where C is a cactus such that the inverse image of any mincut of C is a mincut of Γ and every mincut of Γ arises in this way. To put it differently, even though the cardinality of mincuts is (in general) greater than 2, crossing mincuts have the same structure as crossing separating pairs. One could think of the cactus theorem of Dinitz, Karzanov and Lomonosov as an analogue of the manifold (or group) JSJ–theory for graphs.

In this paper we generalize the cactus theory from graphs to continua. Guralnik [11] asked whether it is possible to encode all cut triples of a continuum by a tree. We show here more generally that minimal separators of any cardinality are represented by a tree. In a forthcoming paper [17] we apply the results of this paper to study convergence actions on continua.

Before describing our results, we explain the differences from the graph theoretic setting: Finite cuts of continua may separate in more than two pieces. In the graph theoretic setting, mincuts that cross have empty intersection. This is not necessarily the case in metric spaces. Still crossing cuts have a circular structure given by "wheels" (see below for a definition). Finally there are issues related to the nonlocal connectedness of continua; the tree we obtain is not finite as it is for graphs and it is not even necessarily a simplicial tree. In fact it takes some work to show that it is an $\mathbb{R}$–tree. We observe finally that the cactus theorem for continua is indeed a generalization of the classical cactus theorem for graphs: If Γ is a graph, one can thicken the vertices of Γ to discs and also thicken the edges so that they have exactly one cut point to obtain a continuum X. The minimal separators of the continuum X correspond then to the mincuts of Γ. So the cactus that we obtain for X is the same as the cactus encoding the mincuts of Γ.

We now give an outline of our results. A continuum X is called *n–thick* if it cannot be separated by $n-1$ points and there is some $A \subset X$ with $|A| = n$ such that $X - A$ is not connected. We say then that A is a *minimal separator* of X. If A, B are minimal separators, then we say that A *separates* B if B intersects at least two components of $X - A$. It turns out that if A separates B then B separates A.

Definition Let X be a continuum and $A \subset X$ finite. If there are nonsingleton continua $Y_1, \dots, Y_r$ with

- $\bigcup_i Y_i = X$
- $\bigcup_{i \neq j} Y_i \cap Y_j = A$,

then we say that A *decomposes* X (into $Y_1, \dots, Y_r$). Note that $\partial Y_i = Y_i \cap A$ for all i.

Definition Let X be an m–thick continuum. A finite set $W \subset X$ is called a *wheel* if W decomposes X into continua $M_0, \dots, M_{n-1}$ with $n > 3$ satisfying the following (see Figure 1):

- There is a (possibly empty) $I = \bigcap M_i$ called the *center* of the wheel with $|I| < m$ and $M_i \cap M_j = I$ for all $i - j \neq \pm 1 \mod n$.
- For each i, $|\partial M_i| = m$.

More generally if W is an infinite subset of X and every finite subset of W is contained in some finite wheel contained in W, then we say that W is *a wheel.*

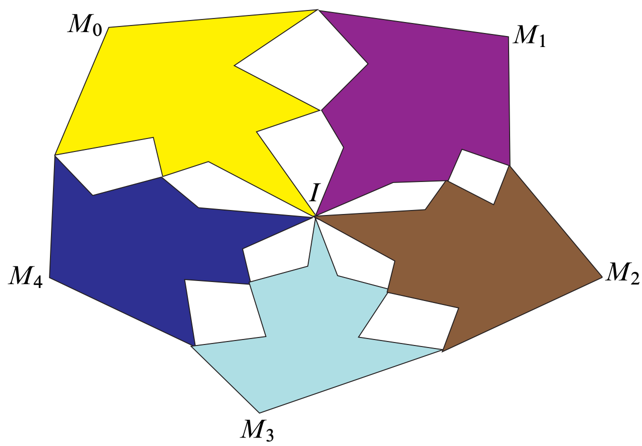

Figure 1

In order to show that all minimal separators of X are encoded by a tree, we use the notion of a pretree. A pretree is a set with a betweenness relation. The basic example of a pretree is the set of vertices of a tree where betweenness is defined in the obvious way. It turns out that given a pretree (satisfying certain conditions) one can construct a canonical enveloping tree. See Section 5 for a formal definition of pretrees and more details on the correspondence between pretrees and trees.

We show that every minimal separator of X either is contained in a maximal wheel or it does not cross any other minimal separator. If a minimal separator K does not cross any other minimal separator we say that it is *isolated*.

We define a pretree $\mathcal{R}$ with elements the maximal wheels of X and the isolated minimal separators of X. We define a betweenness relation in $\mathcal{R}$:

Let $x, y, z \in \mathcal{R}$ be distinct. For y a minimal separator, we say that y is between x, z if there are continua A, B such that

$$x \subset A, \quad z \subset B, \quad A \cup B = X, \quad A \cap B = y.$$

If y is a maximal wheel we say that y is between x, z if for some minimal separator $w \in y$, w is between x, z. We can now state the main theorem of this paper:

Theorem 6.5 *Let X be a separable, m–thick continuum, where $m > 1$. Then the pretree $\mathcal{R}$ embeds into an $\mathbb{R}$–tree T.*

If X is locally connected, we show that in fact one obtains a tree rather than an $\mathbb{R}$–tree:

Theorem 6.6 *Let X be a locally connected, separable, m–thick continuum, where $m > 1$. Then the $\mathbb{R}$–tree corresponding to the minimal separators of X is simplicial.*

The authors wish to thank the referee for correcting our mistakes.

2 Preliminaries

Definition A compact connected Hausdorff space is called a *continuum*.

Definition Let X be a topological space. We say that a set C *separates* the nonempty sets $A, B \subset X$ if there are disjoint open sets U, V of $X - C$, such that $A \subset U$, $B \subset V$ and $U \cup V = X - C$. We say C *separates* the points $a, b \in X$ if C separates $\{a\}$ and $\{b\}$. We say that C *separates* $D \subset X$ if C separates two points of D. If $C = \{c\}$, then we call c a *cut point*. If $C = \{c, d\}$ where $c \neq d$ and *neither* c nor d is cut point, then we call $\{c, d\}$ a (unordered) *cut pair*.

The proof of the following Lemma is an elementary exercise in Topology and will be left to the reader.

Lemma 2.1 *Let A be a connected subset of the space X and B closed in X. If $A \cap \operatorname{Int} B \neq \varnothing$, then either $A \subset B$ or $A \cap \partial B$ separates the subspace A.*

Lemma 2.2 [18] *Let X be a continuum and $C \subset X$ be minimal with the property that $X - C$ is not connected. Then C is closed in X, and the set C separates $A \subset X - C$ from $B \subset X - C$ if and only if there exist continua Y, Z such that $A \subset Y$, $B \subset Z$, $Y \cup Z = X$ and $Y \cap Z = C = \partial Y = \partial Z$.*

We now introduce the Freudenthal compactification, which is a generalization of the ends compactifications (see Aarts and Nishiura [1] for a general reference). This will allow us to throw away any finite set that offends us, and proceed as if the world were a much more friendly place.

Definition A regular Hausdorff space is called *rim-compact* if there is a base $\mathcal{B}$ for the topology with ∂U compact for each $U \in \mathcal{B}$.

2.1 Definition and properties of the Freudenthal compactification [10]

Let X be a rim-compact space. Let $\mathcal{B}$ be the collection of all open sets with compact boundary. Consider the collection of all nets $(U_i) \subset \mathcal{B}$ satisfying:

- $\bar{U}_j \subset U_i$ for all $j > i$.
- $\bigcap U_i = \varnothing$.
- For any $V \in \mathcal{B}$,
  - either $U_i \subset V$ for all $i \gg 0$ (in which case we say V *contains* the end e defined by (U_i))
  - or $U_i \cap V = \varnothing$ for all $i \gg 0$.

We define two such nets (U_i) and (V_j) to be *equivalent* if $U_i \cap V_j \neq \varnothing$ for all i, j. The set of equivalence classes of such nets is called the *set of ends* of X, denoted $\mathrm{End}(X)$. If (U_i) is such a net whose equivalence class is $e \in \mathrm{End}(X)$, then for each i we say that U_i is a *representative* of e.

Consider $FX = X \cup \mathrm{End}(X)$. For $U \in \mathcal{B}$ we define FU to be the union of U with all ends contained in U. The collection $F\mathcal{B} = \{FU : U \in \mathcal{B}\}$ is a base for a topology on FX satisfying:

(1) FX is a Hausdorff compactification of X.

(2) If A, B are disjoint closed subsets of X with compact boundaries, then in FX $\bar{A} \cap \bar{B} = \varnothing$ [15].

(3) ∂FU in FX is equal to ∂U in X for all $U \in \mathcal{B}$.

The topological space FX is called the Freudenthal compactification of X.

Definition A subset A of X is *zero-dimensionally embedded* in X if there is a base $\mathcal{B}$ for the topology of X such that $\partial U \cap A = \varnothing$ for each $U \in \mathcal{B}$.

Notice that if A is zero-dimensionally embedded in X, then $\dim A = 0$. The converse is not true [1], but it is easily seen that any finite subset of a Hausdorff space X is zero-dimensionally embedded in X.

Theorem 2.3 [1] *A space X is rim-compact if and only if X has a Hausdorff compactification Y with $Y - X$ zero-dimensionally embedded in Y.*

Notice that by (3) the Freudenthal compactification, FX, of a rim-compact space X has the property that $FX - X$ is zero-dimensionally embedded in FX.

Theorem 2.4 [1, VI, 3.7] *If X is a rim-compact space and Y is a Hausdorff compactification of X with $Y - X$ zero-dimensionally embedded in Y, then there is a surjective map ρ: $FX \to Y$ which is the identity on X.*

We now condense out the result we will need here.

Corollary 2.5 *If Y is a compact Hausdorff space and A is a finite collection of nonisolated points of Y, then $Y - A$ is rim-compact and there is a unique surjective map ρ: $F(Y-A) \to Y$ from the Freudenthal compactification of $Y-A$ onto Y which is the identity on $Y-A$. Thus there is a natural bijection between the quasicomponents of $Y - A$ and the quasicomponents of $F(Y - A)$.*

Proof This follows from property (2) of the Freudenthal compactification. □

Now recall a basic result from topology:

Lemma 2.6 [13, Section 47 II, Theorem 2] *If X is a compact Hausdorff space, then the components of X are exactly the quasicomponents of X.*

Combining the previous two results we see that for any continuum X and finite subset $A \subset X$, *the quasicomponents of $X - A$ are naturally equivalent to the components of* $F(X - A)$. This together with the following Lemma is what we mean by: This will allow us to throw away any finite set that offends us, and proceed as if the world were a much more friendly place.

Lemma 2.7 *If X is a connected rim-compact space and B is a compact subset of X, then the inclusion i: $(X - B) \hookrightarrow (FX - B)$ induces a bijection on quasicomponents.*

Proof Let $\alpha \subset X$ be a quasicomponent of $X - B$. Clearly α will be contained in a single quasicomponent of $FX - B$. Let β be some other quasicomponent of $X - B$ and find open sets U, V of $X - B$ with

- $\alpha \subset U$
- $\beta \subset V$
- $U \cup V = X - B$
- $U \cap V = \varnothing$.

In $F(X)$, consider the closures $\bar{U}$ and $\bar{V}$. Since X is dense in FX, it follows that $FX - B = \bar{U} \cup \bar{V} - B$.

Suppose that $e \in [\bar{U} \cap \bar{V}] - B$. It follows that $e \in FX - X = \mathrm{End}(X)$. Since B is compact, we can choose a representative W of e such that $W \cap B = \varnothing$. Working in X, since ∂U is compact, we may also assume that (in X) either $\bar{U} \cap \overline{W} = \varnothing$ or $\overline{W} \subset U$ (in X). Similarly we may assume that (in X) $\bar{V} \cap \overline{W} = \varnothing$ or $\overline{W} \subset V$. Since $U \cap V = \varnothing$, we may assume that $\bar{U} \cap \overline{W} = \varnothing$ in X which implies by (2) that in FX, $\bar{U} \cap \overline{W} = \varnothing$, however $e \in \overline{W} \cap \bar{U}$ which is a contradiction.

Thus in FX, $\bar{U} \cap \bar{V} \subset B$, and so α and β are in different quasicomponents of $FX - B$. Thus the induced function on quasicomponents is 1–to–1.

Now consider a quasicomponent γ of $FX - B$. We must show that $\gamma \cap X \neq \varnothing$. Since X is connected, so is FX and so $\bar{\gamma} \cap B \neq \varnothing$. However $FX - X$ is zero dimensionally embedded in FX, so if $\gamma \subset FX - X$, then γ is a singleton, contradicting $\bar{\gamma} \cap B \neq \varnothing$. Thus the induced function on quasicomponents is onto. □

The following is now easy.

Lemma 2.8 *If X is a continuum and C is a finite subset which is minimal with respect to the property that $X - C$ is not connected, then for every quasicomponent α of $X - C$, $C \subset \bar{\alpha}$, the closure of α.*

Proof By induction on $|C|$. When C is a cut point, by [13, Section 47 III, Theorem 2], C meets the closure of every component of $X - C$. Since every component is contained in a quasicomponent, then C is contained in the closure of each quasicomponent of $X - C$.

When $|C| > 1$, let $c \in C$ and consider $\hat{X} = F(X - [C - \{c\}])$. Notice that c is a cut point of the continuum $\hat{X}$, and so c is contained in the closure of each quasicomponent of $\hat{X} - \{c\}$; However by Corollary 2.5 and Lemma 2.7, it follows that c is contained in the closure of each quasicomponent of $X - C$. Thus every point of C is contained in the closure of each quasicomponent of $X - C$ as required. □

3 Decompositions of continua

Definition Let X be a continuum and $A \subset X$ finite. If there are nonsingleton continua $Y_1, \dots, Y_r$ with

- $\bigcup_i Y_i = X$
- $\bigcup_{i \neq j} Y_i \cap Y_j = A$.

Then we say that A *decomposes* X (into $Y_1, \dots, Y_r$). Note $\partial Y_i = Y_i \cap A$ for all i.

Definition Let X be a continuum and A a closed subset of X. We say that A is an *irreducible separator* if for any $B \subsetneq A$ the map of quasicomponents induced by the inclusion $(X - A) \to (X - B)$ is not one to one.

Lemma 3.1 *Suppose that A decomposes the continuum X into continua $Y_1, \dots, Y_r$. If B is a finite separating subset of X, then either $B \cap A$ separates X or for some i, B separates Y_i.*

Proof Since B is a closed separating set, $X - B = U \cup V$ where U and V are disjoint nonempty open sets.

Assume that for any i, $Y_i - B$ is contained in one of U or V. Let $J_U = \{1 \le i \le r : Y_i - B \subset U\}$ and $J_V = \{1 \le i \le r : Y_i - B \subset V\}$. Since no nondegenerate continuum is finite, J_U and J_V form a partition of $\{1, 2 \dots, r\}$. Let

$$C = \bigcup_{p \in J_U, q \in J_V} [Y_p \cap Y_q].$$

Clearly $C \subset B \cap A$ (otherwise $U \cap V \neq \varnothing$). Also notice that $\bigcup_{i \in J_U} Y_i$ and $\bigcup_{i \in J_V} Y_i$ are closed sets whose union is X and whose intersection is C. Thus C separates X. □

Theorem 3.2 *Let X be a continuum. Let $\mathcal{A}$ be a finite collection of finite irreducible separators of X. Then $\bigcup \mathcal{A}$ decomposes X. That is:*

There are continua $Y_1, \dots, Y_k$ with

- $X = Y_1 \cup \dots \cup Y_k$
- $\bigcup_{i \neq j} Y_i \cap Y_j = \bigcup \mathcal{A}$.

Proof We proceed by induction on $|\bigcup \mathcal{A}|$. When $|\bigcup \mathcal{A}| = 1$, the result follows from Lemma 2.2.

Case I $\bigcup \mathcal{A}$ consists entirely of cut points. Let $a \in \bigcup \mathcal{A}$. Clearly $[\bigcup \mathcal{A}] - \{a\}$ also satisfies our hypothesis. Thus $[\bigcup \mathcal{A}] - \{a\}$ decomposes X into continua $Z_1, \dots, Z_r$. By Lemma 3.1, a is a cut point of Z_i for some i. Thus $Z_i = S \cup T$ where S and T are continua and $S \cap T = \{a\}$. It follows that $\bigcup \mathcal{A}$ decomposes X into $Z_1, \dots, Z_{i-1}, S, T, Z_{i+1} \dots, Z_r$ as required.

Case II There is $a \in \bigcup \mathcal{A}$ where a is not a cut point of X. Thus $X - \{a\}$ is connected and we consider the Freudenthal compactification $\widehat{X} = F(X - \{a\})$.

Claim *$\widehat{X}$ and $\bigcup \mathcal{A} - \{a\}$ satisfy the hypothesis of our theorem. That is, $C - \{a\}$ is an irreducible separator of $\widehat{X}$ for each $C \in \mathcal{A}$.*

Let $A \in \mathcal{A}$. We first consider the case where $a \notin A$. Let $B \subsetneq A$. We will use Lemma 2.7 to work in the Freudenthal compactifications. By hypothesis, there is a quasicomponent (also component see Lemma 2.6) γ of $F(X - B)$ which is the image of more than one (quasi)component by the map $\iota\colon F(X - A) \to F(X - B)$ of Theorem 2.4. If $a \notin \gamma$ the claim is trivial, so we may assume that $a \in \gamma$.

Let α be the component of a in $F(X - A)$. Since α is a component of a compact space, it is a continuum. If a is not a cut point of α then we are done. Thus we may assume that a is a cut point of α. Let β be a component of $F(X - A)$ such that $\iota(\alpha) \cap \iota(\beta) \neq \varnothing$ (β exists by hypothesis). Let $p \in \alpha$ with $\iota(p) \in \iota(\beta)$. It follows that $\iota(p) \in A - B$. Let $\hat{\alpha}$ be the (quasi)component of p in $F(X - [A \cup \{a\}])$, and $\hat{\beta}$ the (quasi)component of $F(X - [A \cup \{a\}])$ corresponding to β. Under the map $\pi : F(X - [A \cup \{a\}]) \to F(X - [B \cup \{a\}])$, $\pi(p) \in \pi(\hat{\alpha}) \cap \pi(\hat{\beta})$ as required.

Now consider the case where $a \in A$. Let $\hat{A} = A - \{a\}$, and $\hat{B} \subsetneq \hat{A}$. Let $B = \hat{B} \cup \{a\}$. The inclusion $X - A \to X - B$ is not one to one on quasicomponents, and this implies that the inclusion $\hat{X} - \hat{A} \to \hat{X} - \hat{B}$ is also not one to one by Lemma 2.7. The Claim is proven.

Thus by induction on $|\bigcup \mathcal{A}|$, we are done. $\square$

Corollary 3.3 *Let X be a continuum, F a finite subset of X, and $\mathcal{A}$ a finite collection of finite irreducible separators of X. Then there is a connected graph Γ, connected subgraphs $\Gamma_1, \dots, \Gamma_k$, and a decomposition $Y_1, \dots, Y_k$ of X over $\bigcup \mathcal{A}$ with the following properties:*

- *There is an inclusion $\bigcup \mathcal{A} \cup F \to \Gamma$ such that $Y_i \cap Y_j = \Gamma_i \cap \Gamma_j$ for all $1 \le i, j \le k$.*
- *For any $a, b \in \bigcup \mathcal{A} \cup F$, and $B \subset \bigcup \mathcal{A}$, B separates a from b in X if and only it separates a from b in Γ.*

Proof Redo the proof of the theorem ensuring that a and b are separated by $B \subset \mathcal{A}$, then $\{a, b\} \not\subset Y_i$ for any i. Now construct graphs Γ_i as follows: Let $n = |\bigcup \mathcal{A} \cup F|$ and Γ_i be the 1–skeleton of the barycentric subdivision of the n–dimensional simplex. Embed $[\bigcup \mathcal{A} \cup F] \cap Y_i$ into the vertex set of the n–dimensional simplex which is subset of Γ_i. Now declare $\Gamma_i \cap \Gamma_j = Y_i \cap Y_j$ for all $i \neq j$. This identification defines the graph Γ with the desired properties. $\square$

Definition Let $A \subset X$ with $X - A$ disconnected and $|A| < \infty$ minimal. Then we say X is $|A|$*–thick* and A is a *minimal separator*.

Lemma 3.4 *Let A, B be disjoint minimal separators of the continuum X where A decomposes X into M_0, M_1. If A separates B, then B separates A. Also there are disjoint B_0, B_1, A_0, A_1 with $B_0 \cup B_1 = B$, $A_0 \cup A_1 = A$ and $|B_i| = |A_i| = n/2$ for $i = 0, 1$ and there are subcontinua $P^i_j \subset M_j$ for $i, j = 0, 1$ with $\bigcup_{i,j=0,1} P^i_j = X$, $P^i_j \cap P^{1-i}_j = B_j$, $P^i_j \cap P^i_{1-j} = A_i$, and $P^i_j \cap P^{1-i}_{1-j} = \varnothing$.* (See Figure 2.)

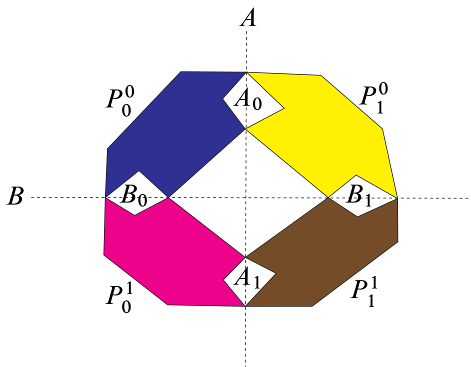

Figure 2

Proof Let $n = |A| = |B|$. By hypothesis $M_0 \cap M_1 = A = \partial M_i$, for $i = 0, 1$, and $M_0 \cup M_1 = X$. Let $B_j = M_j \cap B$ for $j = 0, 1$. By hypothesis, $\varnothing \neq B_j \subset \operatorname{Int} M_j$ for $j = 0, 1$. Also there exist subcontinua N_0, N_1 with $N_0 \cap N_1 = B = \partial N_i$, for $i = 0, 1$ and $N_1 \cup N_2 = X$. Let $A_i = N_i \cap A$ for $i = 0, 1$. By Lemma 2.1, A_i separates N_i for $i = 0, 1$ (in particular $A_i \neq \varnothing$). Applying Lemma 2.2 again we see that for $i, j = 0, 1$ there are subcontinua P^i_j such that $N_i = P^i_0 \cup P^i_1$, and $P^i_0 \cap P^i_1 = A_i$ for $j = 0, 1$. Using minimality and applying Lemma 2.1, we may assume that $P^i_j \subset M_j$ for $j = 0, 1$. Observe that $\partial P^i_j \subset A_i \cup B_j$ for $i, j = 0, 1$. Since $|\partial P^i_j| \geq n$ for $i, j = 0, 1$, it follows that $|A_i| = |B_j| = n/2$ for $i, j = 0, 1$ and $\partial P^i_j = A_i \cup B_j$. The result follows. $\square$

Theorem 3.5 *Let A, B be minimal separators of the continuum X where A decomposes X into M_0, M_1. If A separates B, then B separates A and there are disjoint subsets $A_0, A_1 \subset A$, $B_0, B_1 \subset B$ and subcontinua $P^i_j \subset M_j$ for $i, j = 0, 1$ such that*

- $A \subset A_0 \cup A_1 \cup B$
- $B \subset B_0 \cup B_1 \cup A$
- $B_i \cap A = \varnothing = A_i \cup B$ *for* $i = 0, 1$
- $|A_i| = |B_j|$ *for* $i, j = 0, 1$
- $\bigcup_{i,j=0,1} P^i_j = X$

- $P_j^i \cap P_j^{1-i} = B_j \cup [A \cap B]$
- $P_j^i \cap P_{1-j}^i = A_i \cup [A \cap B]$
- $P_j^i \cap P_{1-j}^{1-i} = A \cap B$.

This decomposition is unique up to relabeling, that is for each quasicomponent α of $X-(A \cup B)$, $\partial P_j^i \subset \overline{\alpha}$ for some i, j. (See Figure 3.)

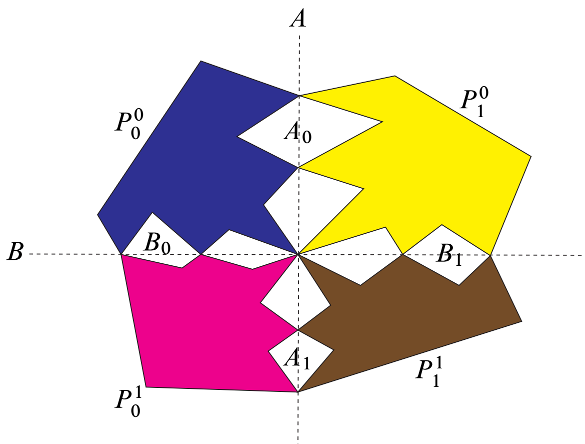

Figure 3

Proof Since A separates B, then $B \not\subset A$, and $A \cap B$ doesn't separate X. Consider the continuum $\widehat{X}$, the Freudenthal compactification of $X-[A \cap B]$. By Corollary 2.5, the inclusion $X-[A \cap B] \hookrightarrow X$ induces a canonical projection $\pi\colon \widehat{X} \to X$ with

$$\pi\big(\widehat{X}-(X-[A \cap B])\big) \subset A \cap B$$

By Lemma 2.7, a subset D of $X-[A \cap B]$ separates $X-[A \cap B]$ if and only if D separates $\widehat{X}$.

The sets $A-B$ and $B-A$ are minimal separators of $\widehat{X}$ and $A-B$ separates $B-A$. By Lemma 3.4 we have equal size partitions A_0, A_1 and B_0, B_1 of $A-B$ and $B-A$ respectively, where $A-B$ decomposes $\widehat{X}$ into $\widehat{M}_0 \cup \widehat{M}_1$ where $\pi(\widehat{M}_j) = M_j$. We also have continua $\widehat{P}_j^i \subset \widehat{M}_j = \pi^{-1}(M_j)$ for $i, j = 0, 1$ with $\bigcup_{i,j=0,1} \widehat{P}_j^i = \widehat{X}$, $\widehat{P}_j^i \cap \widehat{P}_j^{1-i} = B_j$, $\widehat{P}_j^i \cap \widehat{P}_{1-j}^i = A_i$ and $\widehat{P}_j^i \cap \widehat{P}_{1-j}^{1-i} = \varnothing$.

For $i, j = 0, 1$ let $P_j^i = \pi(\widehat{P}_j^i)$. Notice that $\partial P_j^i \subset A_i \cup B_j \cup [A \cap B]$, and since $|A_i \cup B_j \cup [A \cap B]| = |A|$, it follows by minimality that $\partial P_j^i = A_i \cup B_j \cup [A \cap B]$.

To prove uniqueness, it suffices to show that for any (quasi)component α of the Freudenthal compactification $F(X-[A\cup B])$, for some i,j, $\partial P_j^i\subset\iota(\alpha)$ where $\iota\colon F(X-[A\cup B])\to X$ is the map of Corollary 2.5. Clearly $\iota(\alpha)\subset P_j^i$ for some i,j, say $i,j=0$. If $\partial P_0^0\not\subset\alpha$, then by Lemma 2.7 there is a open set U of X such that $\alpha\cap(X-[A\cup B])\subset U$ with $\partial U\subset A\cup B$ and $\partial U\subsetneq\partial P_0^0$ which violates minimality. □

4 Wheels

Definition Let X be an m–thick continuum. A finite set $W\subset X$ is called a *wheel* if W decomposes X into continua $M_0,\dots,M_{n-1}$ with $n>3$ satisfying the following:

- There is a (possibly empty) $I=\bigcap M_i$ called the *center* of the wheel with $|I|<m$ and $M_i\cap M_j=I$ for all $i-j\neq\pm1 \mod n$.
- For each i, $|\partial M_i|=m$.

Notice the following:

- ∂M_i is a minimal separator for all i.
- $[M_i\cap M_{i+1}]\cup[M_j\cap M_{j+1}]$ (where the addition is mod n) is a minimal separator for all $i\neq j$.
- $m=|I|+2k$ where $|(M_i\cap M_{i+1})-I|=k>0$ for $0\le i\le n-1$.

The collection $M_0,\dots,M_{n-1}$ is called the *wheel decomposition* of X by W. This decomposition is unique by Lemma 2.8. (See Figure 1.)

Definition For wheels W and V, we say that W is a *subwheel* of V if $W\subset V$.

- Every continuum of the wheel decomposition of V will be contained in one of the continua in the wheel decomposition of W.
- The center of W is the same as the center of V.

If W is an infinite subset of X and every finite subset of W is contained in some finite wheel contained in W, then we say that W is *a wheel.*

Definition Let X be a n–thick continuum. A nondegenerate nonempty set $A\subset X$ is called *inseparable* if no pair of points in A can be separated by a minimal separator.

Every inseparable set is contained in a maximal inseparable set. A maximal inseparable subset is closed (its complement is the union of open subsets).

Example 4.1 A maximal inseparable set need not be connected.

For example, take X a finite thin simplicial chamber complex of dimension n. (Recall that this means that every $n-1$ simplex is a face of exactly two chambers (n–simplices), and that we can get from one point to any other point by a sequence of chambers (n–simplices), where adjacent chambers share a $n-1$ face.)

Now cut X along all open simplices of dimension at least one. We obtain a cell complex $\widehat{X}$ where n–cells intersect only in vertices. Since X was a chamber complex, it could only be separated by removing a subset of dimension at least $n-1$. It follows that:

- $\widehat{X}$ is $n+1$ thick.
- Minimal separators coincide with vertex sets of n–cells of $\widehat{X}$.
- There are no wheels.
- The maximal inseparable subsets of $\widehat{X}$ are:
  - Each closed n–cell of $\widehat{X}$.
  - The set of vertices of $\widehat{X}$.

For example if X is the icosahedral subdivision of the 2–sphere, then the maximal inseparable subsets of $\widehat{X}$ are the triangles and the set of all vertices (which is not connected).

Theorem 4.2 *Let X be an n–thick continuum. Let W be a wheel of X and let K be a minimal separator which separates two points a, b of W. Then $W \cup K$ is a wheel.*

Proof We assume of course that $K \not\subset W$. We may assume that W is finite. Let I be the center of W and $M_0, \dots, M_{k-1}$ be the wheel decomposition of W.

We show first that $I \subset K$. If I is not contained in K, then we may assume that K separates some $a \in I$ from some b in, say, $M_1 \cap M_2$. We argue now that this is impossible. To fix some notation let's say that

$$|K| = |I| + 2m$$

(by Lemma 3.4, $|K| - |I|$ is even) and that

$$|K \cap I| = |I| - s$$

for some $s > 0$.

We distinguish 2 cases:

Case 1 There is an $i \neq 1$ and some $b' \in M_i \cap M_{i+1}$ such that K separates a from b'.

It follows that K intersects all M_1, M_2, M_i, M_{i+1}. At least 3 of these are distinct, let's say that M_1, M_2, M_i are distinct.

We set

$$n_1 = |K \cap M_1 \cap M_2 - I|, \qquad n_2 = |K \cap M_2 \cap M_i - I|, \qquad n_3 = |K \cap M_i \cap M_1 - I|$$
$$k_1 = |K \cap M_1 - (M_k \cup M_2)|, \; k_2 = |K \cap M_2 - (M_1 \cup M_i)|, \; k_3 = |K \cap M_i - (M_2 \cup M_1)|$$

(note that it is possible that $M_{i+1} = M_1$ if $k = i = 3$).

We remark that

$$(*) \qquad \sum_{j=1}^{3} (n_j + k_j) + |I| - s \le |I| + 2m.$$

Since $K \cap M_1$ separates $a, b \in M_1$, there are two continua M_1^1, M_1^2 such that $a \in M_1^1$, $b \in M_1^2$ and

$$M_1^1 \cup M_1^2 = M_1, \quad M_1^1 \cap M_1^2 \subset K.$$

We have now

$$|\partial M_1^1| + |\partial M_1^2| \le 2(n_1 + n_3 + k_1 + |I| - s) + s + (m - n_1) + (m - n_3).$$

On the other hand,

$$|\partial M_1^1| + |\partial M_1^2| \ge 2(|I| + 2m).$$

We obtain the inequality

$$(1) \qquad n_1 + n_3 + 2k_1 - s \ge 2m.$$

Similarly using that $K \cap M_2$ separates $a, b \in M_2$ and $K \cap M_i$ separates $a, b' \in M_i$, we see that

$$(2) \qquad n_1 + n_2 + 2k_2 - s \ge 2m,$$
$$(3) \qquad n_2 + n_3 + 2k_3 - s \ge 2m.$$

Adding (1), (2), (3) we have

$$\sum_{j=1}^{3} (n_j + k_j) - \frac{3s}{2} \ge 3m$$

which contradicts $(*)$.

Case 2 The hypothesis of Case 1 does not hold.

Since K separates also a from $b \in M_1 \cap M_2$, we have that $K \cap M_1$ and $K \cap M_2$ separate a from b in M_1, respectively in M_2. We set

$$n_1 = |K \cap M_1 \cap M_2 - I|, \quad k_1 = |(K \cap M_1) - M_2|, \quad k_2 = |(K \cap M_2) - M_1|.$$

We have the inequality

$$|I| - s + n_1 + k_1 + k_2 \leq 2m + |I|. \tag{†}$$

Since $K \cap M_1$ separates $a, b \in M_1$, there are two continua M_1^1, M_1^2 such that

$$M_1^1 \cup M_1^2 = M_1, \quad M_1^1 \cap M_1^2 \subset K$$

and $b \in M_1^1, \quad a \in M_1^2$. We have now

$$|\partial M_1^1| \leq m + k_1 + |I| - s.$$

On the other hand,

$$|\partial M_1^1| \geq |I| + 2m$$

We obtain the inequality

$$k_1 - s \geq m. \tag{4}$$

Arguing similarly using M_2 we obtain the inequality

$$k_2 - s \geq m. \tag{5}$$

Adding (4), (5) we get

$$k_1 + k_2 - 2s \geq 2m$$

which contradicts (†) since $s > 0$ and $n_1 < m$.

We have shown therefore in both cases that $I \subset K$. Replacing X with $F(X - I)$ and applying Lemma 2.7, we may assume that $I = \varnothing$. Thus by hypothesis, $|M_i \cap M_j| = 2m$ if $j - i = \pm 1 \mod r$ and is empty otherwise.

We show now that $W \cup K$ is a wheel. Since $K \not\subset W$, then $K \cap W$ doesn't separate and by Lemma 3.1, for some i, K separates M_i. Renumbering, we may assume that K separates M_1.

If there is a quasicomponent of α of $X - K$ which is contained in M_1, then by Lemma 2.8, $K \subset \overline{\alpha} \subset \overline{M}_1 = M_1$. However, $X - M_1$ is connected by Lemma 2.8 applied to the points of $\partial M_i - M_1$ for $i \neq 1$. This and Lemma 2.8 contradict the fact that K separates points of W.

Thus every quasicomponent of $X - K$ hits $\mathrm{Int}(M_2 \cup \cdots \cup M_0)$, and by Lemma 2.8 every quasicomponent of $X - K$ hits $\mathrm{Int}\, M_1$ as well. We know that K decompose X

into continua N_0, and N_1 and it follows that $(\operatorname{Int} N_i)\cap M_1\neq\varnothing$ by Lemma 2.1. Thus K separates the minimal separator $\partial M_1\subset W$.

Now we apply Theorem 3.5 to K and ∂M_1. Since ∂M_1 decomposes X into the continua M_1 and $M_2\cup\cdots\cup M_0$, we find that K decomposes M_1 into continua P and Q, with $P\cap M_0\neq\varnothing$ and $Q\cap M_2\neq\varnothing$.

Notice that $X-M_1=[M_2\cup\cdots\cup M_0]-\partial M_1$, so this space has a Freudenthal compactification. Now consider $Y=F(X-M_1)$ and $H=K-M_1$.

Since no quasicomponent of $X-K$ is contained in M_1, it follows from Lemma 2.7 that H separates Y. Let $J=W-M_1$. Clearly J decomposes Y into $\widehat{M}_2,\dots,\widehat{M}_0$ where $\pi(\widehat{M}_i)=M_i$, $i=2,\dots,0$ for the canonical map $\pi\colon Y\to X-\operatorname{Int} M_1$. By Lemma 3.1, either $H\cap J$ separates Y or for some $i\neq 1$, H separates $\widehat{M}_i$.

Case I $H\cap J$ separates Y.

By Lemma 2.8, $H\cap J$ doesn't separate $\widehat{M}_i$ for any $i>1$, and it follows that $H\cap J$ must separate $\widehat{M}_i$ from $\widehat{M}_{i+1}$ for some $i>1$. Thus by Lemma 2.8 $\widehat{M}_i\cap\widehat{M}_{i+1}=M_i\cap M_{i+1}\subset H\cap J$. Thus $|H|\geq m=|M_i\cap M_{i+1}|$. Since $H=K-M_1$ has order at least m, then by Theorem 3.5 applied to ∂M_1 and K, $K\cap\partial M_1=\varnothing$. Thus Theorem 3.5 implies that $|K\cap\operatorname{Int} M_1|=m=|K-M_1|$, and so $H\cap J=H$. Since $K\cap\partial M_1=\varnothing$, by Lemma 2.8 $\operatorname{Int} M_k$ is contained in a single quasicomponent of $X-K$, and so $M_k\subset P$. Similarly $M_2\subset Q$. It follows now that $Q\cap M_k$, and $P\cap M_2$ are empty, and so $P, Q, M_2,\dots,M_k$ is a wheel decomposition of X, so $\{K_1,K_2,\dots,K_r,K\}$ is a wheel.

Case II H separates $\widehat{M}_i$ for some $i\neq 0$ and $H\cap J$ doesn't separate Y.

Since $\partial\widehat{M}_i\not\subset H$, this implies that $H\cap\operatorname{Int}\widehat{M}_i\neq\varnothing$. However $\operatorname{Int}\widehat{M}_i=\operatorname{Int} M_i$, therefore $\operatorname{Int} M_i\cap K\neq\varnothing$. Let K decompose X into continua A, B.

Case II(a) For some j, M_j is contained in one of A, B.

We may assume $M_j\subset B$. Reordering if need be, we have $A\subset M_1\cup M_2\cup\cdots\cup M_i$ where $K\cap\operatorname{Int} M_1\neq\varnothing$ and $K\cap\operatorname{Int} M_i\neq\varnothing$. Consider $A_1=A\cap M_1$ and $A_i=A\cap M_i$. Notice $\partial A_1\subset(M_1\cap M_2)\sqcup(M_1\cap K-M_2)$ and $\partial A_i\subset(M_{i-1}\cap M_i)\sqcup(M_i\cap K-M_{i-1})$. By Theorem 3.5 applied to K and ∂M_1, A_1 is a continuum with $|\partial A_1|=2m$ and similarly A_i is a continuum with $|\partial A_i|=2m$. It now follows $|M_i\cap K-M_{i-1}|=m=|M_1\cap K-M_2|$. This implies that $K\subset[M_1-M_2]\cup[M_i-M_{i-1}]$. (In particular $K\cap M_1\cap M_2=\varnothing=K\cap M_{i-1}\cap M_i$). Also since $\partial A_1=2m$, $M_1\cap M_2\subset\partial A_1$ which implies that $M_1\cap M_2\subset\operatorname{Int} A$, and similarly $M_{i-1}\cap M_i\subset\operatorname{Int} A$.

Now let $B_1=B\cap M_1$ and $B_i=B\cap M_i$. Applying Theorem 3.5 to ∂M_1 and K, we get that B_1 is a continuum with $|\partial B_1|=2m$ and similarly B_i is a continuum with $|\partial B_i|=2m$. Since $M_1\cap M_2\subset \operatorname{Int} A$, we know $\partial B_1\subset[M_0\cap M_1]\cup[M_1\cap K-M_2]$. But $|M_0\cap M_1|=m=|M_1\cap K-M_2|$, hence $M_0\cap M_1\cap[M_1\cap K-M_2]=\varnothing$. Therefore $M_0\cap M_1\cap K=\varnothing$, and similarly $M_i\cap M_{i+1}\cap K=\varnothing$, so we have $K\cap W=\varnothing$.

Notice that $A_1\cap M_k$, $B_1\cap M_2$, $A_i\cap M_{i+1}$ and $B_i\cap M_{i-1}$ are all empty. So starting with the wheel decomposition $M_0, M_1, \dots, M_{k-1}$ of X and replacing M_1 with A_1 and B_1 and replacing M_i with A_i B_i, we have a wheel decomposition for $W\cup K$.

Case II(b) For each j, we have $\operatorname{Int} M_j\cap A\neq\varnothing\neq\operatorname{Int} M_j\cap B$.

Notice that K separates ∂M_j for all j. Applying Theorem 3.5 to ∂M_j and K, we get that $A_j=M_j\cap A$ and $B_j=M_j\cap B$ are continua with $|\partial A_j|=2m=|\partial B_j|$. Notice that $A_j\cap B_j\subset K$ for each J.

$$\begin{aligned}4m=|\partial A_j|+|\partial B_j|&=|\partial M_j-K|+2|K\cap M_j|\\&=2m-|K\cap\partial M_j|+2|K\cap M_j|\\&=2m+|K\cap M_j|+|K\cap\operatorname{Int} M_j|\\&=2m+2|K\cap\operatorname{Int} M_j|+|K\cap\partial M_j|.\end{aligned}$$

Define $k_j=|K\cap\operatorname{Int} M_j|$ and $n_j=|K\cap\partial M_j|$, so $2m=2k_j+n_j$ and

$$2\sum_{j=0}^{k-1}k_j+\sum_{j=0}^{k-1}n_j=2km.$$

Notice that

$$2m=|K|=\sum_{j=0}^{k-1}k_j+\frac{1}{2}\sum_{j=0}^{k-1}n_j.$$

It follows that $k=2$ contradicting the fact that $k\geq 3$. □

5 Pretrees and trees

Pretrees are tree-like structures used to construct trees or $\mathbb{R}$–trees (see [6]). Not all pretrees however embed in $\mathbb{R}$–trees. In this section we give necessary and sufficient conditions for a pretree to embed to an $\mathbb{R}$–tree. We remark that Chiswell has shown [7] that any countable pretree embeds to an $\mathbb{R}$–tree.

Definition Let $\mathcal{P}$ be a set and let $R\subset\mathcal{P}\times\mathcal{P}\times\mathcal{P}$. We say then that R is a *betweenness* relation. If $(x,y,z)\in R$, then we write xyz and we say that y is between x,z. $\mathcal{P}$ equipped with this betweenness relation is called a *pretree* if the following hold:

(1) There is no y such that xyx for any $x \in \mathcal{P}$.

(2) $xzy \Leftrightarrow yzx$.

(3) For all x, y, z if y is between x, z then z is not between x, y.

(4) If xzy and $z \neq w$ then either xzw or yzw.

The obvious example of a pretree is a tree (simplicial or $\mathbb{R}$–tree). Note that any subset of a pretree is a pretree.

Example 5.1 Not all pretrees are subsets of $\mathbb{R}$–trees. Indeed any linearly ordered set $(P, <)$ can be seen as a pretree: we define betweenness by: xyz if $x < y < z$ or $z < y < x$. Consider now the first uncountable ordinal $\aleph_1$. Clearly $\aleph_1$ cannot be embedded in an order preserving way to an $\mathbb{R}$–tree.

Definition Let $\mathcal{P}$ be a pretree and let $x, y \in \mathcal{P}$. We define the *open interval* (x, y) to be the set of all $z \in \mathcal{P}$ between x, y. Similarly we define the *closed interval* $[x, y]$ and the half open intervals $[x, y)$, $(x, y]$.

Definition A point x of a pretree $\mathcal{P}$ is called a terminal point if $x \notin (a, b)$ for all $a, b \in \mathcal{P}$.

Definition A subset I of a pretree $\mathcal{P}$ is called *linearly ordered*, if for each distinct triple of points in I, one of them is contained in the open interval between the other two.

It is shown in [6] that every linearly ordered subset comes with a linear order (and its opposite) defined using the betweenness relation. Every interval is a linearly ordered set, but not every linearly ordered set is an interval. Notice that by Zorn's Lemma, every linearly ordered set is contained in a maximal linearly ordered set.

Remark 1 It is useful to note that if $\mathcal{P}$ is a pretree, one can recover the betweenness relation on $\mathcal{P}$ from the set of maximal linearly ordered subsets of $\mathcal{P}$. To be precise, xyz holds in $\mathcal{P}$ if and only if x, y, z lie in a maximal linearly ordered subset of $\mathcal{P}$ and xyz holds in this subset. So one way to define betweenness in a pretree $\mathcal{P}$ is by specifying all maximal linearly ordered subsets of $\mathcal{P}$.

Definition If every maximal linearly ordered subset of a pretree $\mathcal{P}$ is order isomorphic to an interval of $\mathbb{R}$, then we say that $\mathcal{P}$ is an $\mathbb{R}$ tree. Notice that this doesn't define a topology on $\mathcal{P}$.

Definition For I a linearly ordered subset of a pretree $\mathcal{P}$, we say that I is *complete* if I has the supremum property, that is if every nonempty subset of I with an upper bound in I has a supremum in I. If every maximal linearly ordered subset of $\mathcal{P}$ is complete, then we say $\mathcal{P}$ is *complete*.

Definition For $\mathcal{P}$ a pretree and $A \subset \mathcal{P}$, the convex hull of A is defined by

$$\mathrm{Hull}(A) = \bigcup_{a,b \in A} [a,b].$$

We say that a set $A \subset \mathcal{P}$ is *convex* if $A = \mathrm{Hull}(A)$. Maximal linearly ordered sets are of course convex.

Definition A subset A of a pretree X is *predense* in X if for every distinct $a, b \in X$, $[a,b] \cap A \neq \varnothing$.

Let $\mathcal{P}$ be a pretree; A maximal linearly ordered $I \subset \mathcal{P}$ is called *preseparable* if I has a countable predense subset. A pretree is *preseparable* if every maximal linearly ordered subset in it is preseparable.

Definition Let $\mathcal{P}$ be a pretree. We say that $\mathcal{P}$ is a *median* pretree if for any three points $x, y, z \in \mathcal{P}$ the intersection $[x,y] \cap [y,z] \cap [z,x]$ is nonempty. Note that if this intersection is nonempty then it consists of a single point called the *median* of x, y, z.

Example 5.2 Consider the $\mathbb{R}$–tree T given by the union of x, y–axes in the plane. The subset $\mathcal{P}$ of T consisting of the intervals $(-\infty, -1] \cup (0, \infty)$ of the x–axis together with the interval $(0, \infty)$ of the y–axis is not median. It becomes median if we add 0.

Clearly if a pretree $\mathcal{P}$ embeds in an $\mathbb{R}$–tree, then $\mathcal{P}$ is preseparable. So being preseparable is a necessary condition for an embedding to an $\mathbb{R}$–tree. However this condition is not sufficient.

Example 5.3 Consider the upper half plane $P = \{(x,y) : y \geq 0\}$. We define a betweenness relation on P by specifying all maximal linearly ordered subsets of P. The x–axis is one of them. Now if we denote by (a,b) intervals on the x–axis and by $(a,b)_x$, the interval $\{(x,t) : a < t < b\}$ parallel to the y–axis the rest of the maximal linearly ordered subsets of P are of the following two types:

$$(-\infty, x] \cup (0,\infty)_x, \quad (0,\infty)_x \cup (x, \infty)$$

Clearly this defines a pretree structure on P. With this structure, P is a preseparable pretree which is not median. It is not very hard to see that P does not embed in an

$\mathbb{R}$–tree. The reason is that if one tries to complete P to a median pretree, one is forced to introduce a "gap" at every point on the x–axis. So the resulting median pretree is not preseparable anymore.

Definition Let $(L,<)$ be a linearly ordered set. If $A, B \subset L$, we write $A < B$ if $a < b$ for all $a \in A, b \in B$.

(A,B) is a *Dedekind cut* of L if A, B are nonempty, $L = A \cup B$ and $A < B$.

Definition Let $\mathcal{P}$ be a pretree and let $(L,<)$ be a maximal linearly ordered subset of $\mathcal{P}$. We say that L has a *gap* at $x \in L$ if one of the following two holds:

(i) (A,B) is a Dedekind cut of L, $x = \sup A$ lies in A and there is a linearly ordered subset C of $\mathcal{P}$ such that $A \cup C$ and $B \cup C$ are maximal linearly ordered subsets of $\mathcal{P}$.

(ii) (A,B) is a Dedekind cut of L, $x = \inf B$ lies in B and there is a linearly ordered subset C of $\mathcal{P}$ such that $A \cup C$ and $B \cup C$ are maximal linearly ordered subsets of $\mathcal{P}$.

If every maximal linearly ordered subset of $\mathcal{P}$ has at most countably many gaps, then we say that $\mathcal{P}$ has *few gaps*.

We can now state the main result of this section:

Theorem 5.4 *Let $\mathcal{P}$ be a pretree. Then there is an embedding of $\mathcal{P}$ into an $\mathbb{R}$–tree if and only if $\mathcal{P}$ is preseparable and has few gaps.*

We recall the following:

Theorem 5.5 [18] *Let $\mathcal{P}$ be a complete preseparable median pretree, then there is a canonical embedding of $\mathcal{P}$ into an $\mathbb{R}$–tree T, with $\mathrm{Hull}(\mathcal{P}) = T$.*

So in order to prove Theorem 5.4, it suffices to show that $\mathcal{P}$ can be embedded to a median complete preseparable pretree. Bowditch [6] has described a way to embed any pretree to a complete median pretree using flows. For the sake of completeness and also because we need to preserve preseparability, we give here a brief account of a procedure that embeds a pretree to a median complete pretree. Instead of flows, we will use Dedekind cuts.

Lemma 5.6 *Let $(L,<)$ be a linearly ordered set. Let D be the set of Dedekind cuts of L. We extend the order of L to $\overline{L} = L \cup D$ as follows: For $a \in L, x = (A,B) \in D$ we define $a < x$ if $a \in A$. For $x_1 = (A_1,B_1)$, $x_2 = (A_2,B_2)$, we define $x_1 < x_2$ if $A_1 \subset A_2$. With this ordering, $\overline{L}$ is a complete linearly ordered set, and L is predense in $\overline{L}$.*

Proof It is easy to see that $\overline{L}$ is a linearly ordered set. We show that $\overline{L}$ is complete. Let S be a bounded subset of $\overline{L}$. Consider $S_1 = S \cap L$ and $S_2 = \bigcup\{A : (A,B) \in D \cap S\}$. If $S_1 \subset S_2$ then $(S_2, \overline{L} - S_2) = \sup S$. Otherwise there are two cases. If $S_1 \cap L$ has a supremum m in L then $m = \sup S_1$. If not consider

$$A = \{a \in L : a < b \text{ for some } b \in S_1\}.$$

Then $(A, L - A) = \sup S$. By construction L is predense in $\overline{L}$. □

This lemma gives a way to complete any linearly ordered set. However, this completion is not very economical. For example, if one starts with $\mathbb{R}$, which is already complete, one still adds infinitely many points to get $\overline{\mathbb{R}}$. Clearly one can do better than that. With the notation of Lemma 5.6 we have:

Lemma 5.7 *Let $(L,<)$ be a linearly ordered set. Let $L' \subset \overline{L}$ such that L' contains all Dedekind cuts (A,B) of L for which $\sup A$ does not lie in L. Then L' is complete.*

Proof The proof of Lemma 5.6 applies in this case too. □

We can turn a pretree complete and median using Dedekind cuts as we did in order to complete linearly ordered sets.

Definition Let $\mathcal{P}$ be a pretree. A set of nonempty linearly ordered subsets of $\mathcal{P}$, $D = \{A_i : i \in I\}$ is called a *Dedekind cut* of $\mathcal{P}$ if the following hold:

(i) For any $i \neq j$, $A_i \cup A_j$ is a maximal linearly ordered subset of $\mathcal{P}$ and $A_i < A_j$ in one of the two linear orderings on $A_i \cup A_j$.

(ii) D is a maximal collection of nonempty linearly ordered sets which satisfies (i).

If D is a Dedekind cut of $\mathcal{P}$ and $A \in D$ then we say that $B \subset \mathcal{P}$ is *D–equivalent* to A if $(D - \{A\}) \cup \{B\}$ is a Dedekind cut. We then write $A \sim B$. It is easy to see that this defines indeed an equivalence relation and we denote the equivalence class of A by $[A]$. We say that two Dedekind cuts D, D' are *equivalent* if every $A \in D$ is equivalent to some $A' \in D'$. We denote by $[D]$ the equivalence class of D. We say that a Dedekind cut is *essential* if either $|D| > 2$ or $D = \{A,B\}$ with $A < B$ and $\sup A \notin \mathcal{P}$.

If $\mathcal{P}$ is a pretree we define the median completion $C(\mathcal{P})$ of $\mathcal{P}$ by

$$C(\mathcal{P}) = \mathcal{P} \cup \{[D] : D \text{ is an essential Dedekind cut of } \mathcal{P}\}.$$

We now turn $C(\mathcal{P})$ into a pretree. To define betweenness it is enough to specify all maximal linearly ordered subsets of $C(\mathcal{P})$. If L is a maximal linearly ordered subset of $\mathcal{P}$, consider the set D_L of all essential Dedekind cuts of $\mathcal{P}$ that contain some Dedekind cut of L (to be precise Dedekind cuts of L were defined as ordered pairs but here we just see them as 2 element sets). The maximal linearly ordered sets of $C(\mathcal{P})$ are of the form

$$\widetilde{L} = L \cup \{[D] : D \in D_L\},$$

where L is maximal linearly ordered subset of $\mathcal{P}$. Note that if (A, B) is a Dedekind cut of L contained in D, then for any $D' \in [D]$ if D' contains a Dedekind cut of L, D' contains in fact (A, B). So we can associate to each $[D] \in \widetilde{L}$ a unique Dedekind cut of L and see $\widetilde{L}$ as a subset of $\overline{L}$. By Lemma 5.7 $\widetilde{L}$ is complete. We show now that $C(\mathcal{P})$ is a pretree. We will need a lemma:

Lemma 5.8 (1) *If $x \in \mathcal{P}$, L a maximal linearly ordered subset of $\mathcal{P}$ and (A_1, A_2) a Dedekind cut of L. Then either $\{x\} \cup A_1$ or $\{x\} \cup A_2$ is a linearly ordered subset of $\mathcal{P}$.*

(2) *Let L_1, L_2 be maximal linearly ordered subsets of $\mathcal{P}$ and $(A_1, A_2), (B_1, B_2)$ be Dedekind cuts of L_1, L_2 respectively. Then for some i, j $A_i \cup B_j$ is a linearly ordered subset of $\mathcal{P}$.*

Proof Left to the reader. □

Axioms (1), (2) of the definition clearly hold for $C(\mathcal{P})$.

Lemma 5.9 *If xyz holds in $C(\mathcal{P})$, then xzy does not hold.*

Proof We argue by contradiction, that is we assume that xyz and xzy both hold. We distinguish 2 cases.

Case 1 x is a Dedekind cut.

Case 2 x is an element of $\mathcal{P}$.

Because of the symmetry between y, z, we split Case 1 to three further cases:

Case 1a y, z are Dedekind cuts.

Case 1b z is an element of $\mathcal{P}$ and y is a Dedekind cut.

Case 1c Both y, z are elements of $\mathcal{P}$.

We now treat Case 1a.

Since xyz holds, there is a maximal linearly ordered subset L of $\mathcal{P}$ and x, y, z correspond respectively to Dedekind cuts $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ of L. Without loss of generality, we may assume that $A_1 \subset A_2 \subset A_3$ where all inclusions are proper. Since xzy holds, then there is a maximal linearly ordered subset L' of $\mathcal{P}$ so that x, y, z correspond respectively to Dedekind cuts $(A'_1, B'_1), (A'_2, B'_2), (A'_3, B'_3)$ of L' and $A'_1 \subset A'_3 \subset A'_2$ (proper inclusions). If $A_3 \cup A'_3$ is a maximal linearly ordered subset of $\mathcal{P}$, then $A_1 \cup A'_1$ is a maximal linearly ordered of $\mathcal{P}$ and it is a proper subset of $A_3 \cup A'_3$, which is impossible. Therefore $A_3 \sim A'_3$. Then $A_3 \cup B'_3$ is a maximal linearly ordered subset of $\mathcal{P}$. But $A_2 \cup B'_2 \subset A_3 \cup B'_3$ (proper inclusion) which leads to a contradiction.

Case 1b There is a maximal linearly ordered subset L of $\mathcal{P}$ and x, y correspond respectively to Dedekind cuts $(A_1, B_1), (A_2, B_2)$ of L, while $z \in L$. Without loss of generality, we may assume that $A_1 \subset A_2$ and $z \notin A_2$. Similarly there is a maximal linearly ordered subset L' of $\mathcal{P}$ so that x, y correspond respectively to Dedekind cuts $(A'_1, B'_1), (A'_2, B'_2)$ of L', $A'_1 \subset A'_2$ (proper inclusions) and $z \in A'_2$. As in the previous case, we have $A_2 \sim A'_2$ so (A'_2, B_2) is a Dedekind cut of a maximal linearly ordered subset of $\mathcal{P}$. This is a contradiction since $z \in A'_2 \cap B_2$.

Case 1c As before, we may assume that x corresponds to Dedekind cuts (A_1, B_1), (A'_1, B'_1) of linearly ordered subsets L, L' and $y, z \in L \cap L'$ with $y, z \notin A_1$, $y, z \notin A'_1$. Clearly $L'' = A_1 \cup B'_1$ is a linearly ordered subset of $\mathcal{P}$ and $A_1 < y < z$, $A_1 < z < y$ both hold in L''. This is a contradiction.

We now treat Case 2. As in Case 1, we split it into 3 cases.

Case 2a Assume that y, z correspond to Dedekind cuts. Since xyz holds, there is a maximal linearly ordered subset L of $\mathcal{P}$, y, z correspond respectively to Dedekind cuts $(A_1, B_1), (A_2, B_2)$ of L and $x \in A_1 \subset A_2$. Since xzy holds, there is a maximal linearly ordered subset L' of $\mathcal{P}$ so that y, z correspond respectively to Dedekind cuts $(A'_1, B'_1), (A'_2, B'_2)$ of L' and $x \in A'_3 \subset A'_2$ (proper inclusion). We remark that (A_2, A'_2) is not a Dedekind cut since $x \in A_2 \cap A'_2$. It follows that (A_2, B'_2) is a Dedekind cut. But $A_1 \subset A_2$, $B'_1 \subset B'_2$ (proper inclusions) so $A_1 \cup B'_1$ is a proper subset of $A_2 \cup B'_2$ which is impossible since $A_1 \cup B'_1$ is a maximal linearly ordered subset of $\mathcal{P}$.

Case 2b Assume that z corresponds to a Dedekind cut and $y \in \mathcal{P}$. Since xyz holds, there is a maximal linearly ordered subset L of $\mathcal{P}$, z corresponds to a Dedekind cut (A, B) of L and $x, y \in A, x < y$. Since xzy holds there is a maximal linearly ordered subset L' of $\mathcal{P}$, z corresponds to a Dedekind cut (A', B') of L' and $x \in A'$,

$y \in B'$. Now we remark that (A, A') cannot be a Dedekind cut since $x \in A \cap A'$. It follows that (A, B') is a Dedekind cut. But this is also impossible since $y \in A \cap B'$.

Case 2c All $x, y, z \in \mathcal{P}$. Clearly then we cannot have xyz and xzy since $\mathcal{P}$ is a pretree. □

Lemma 5.10 *If xzy and $z \neq w$ in $C(\mathcal{P})$, then either xzw or yzw.*

Proof We distinguish two cases:

Case 1 $w \in cP$.

Case 2 $w \notin \mathcal{P}$.

We first treat Case 1. Since xzy, there is a maximal linearly ordered subset L of $\mathcal{P}$ so that $x, z, y \in L$. If z is a Dedekind cut, consider (A, B) to be the Dedekind cut of L corresponding to z. Otherwise let $A = \{t \in L, t \leq z\}$ and $B = L - A$. Without loss of generality, we may assume that if $x \in \mathcal{P}$ $x \in A$, if x is represented by (A_1, B_1), $A_1 \subset A$, and if $y \in \mathcal{P}$, $y \in B$ while if y is represented by (A_2, B_2), then $A \subset A_2$. By Lemma 5.8 either $\{w\} \cup A$ or $\{w\} \cup B$ is a linearly ordered subset of L. In the first case xzw and in the second case yzw.

Case 2 Since xzy there is a maximal linearly ordered subset L of $\mathcal{P}$ so that $x, z, y \in L$. We define (A, B) as in Case 1. If x, y are Dedekind cuts, we represent them as in Case 1 by Dedekind cuts (A_1, B_1), (A_2, B_2) of L. Let (A', B') be a Dedekind cut of a maximal linearly ordered set representing w. By Lemma 5.8, one of $A \cup A'$, $A \cup B'$, $B \cup A'$, $B \cup B'$ is a linearly ordered subset of $\mathcal{P}$. In the first case xzw in the second xzw in the third yzw and in the fourth yzw. □

The previous lemmas show that $C(\mathcal{P})$ is a pretree. As we noted earlier $C(\mathcal{P})$ is complete.

Lemma 5.11 *$C(\mathcal{P})$ is median.*

Proof Let $x, y, z \in C(\mathcal{P})$. Consider $w = \sup([x, y] \cup [x, z])$. Then $w \in [y, z]$ so w is the median of x, y, z. □

Lemma 5.12 *$C(\mathcal{P})$ is preseparable, and $\mathcal{P}$ is predense in $C(\mathcal{P})$.*

Proof Let L be a maximal linearly ordered subset of $C(\mathcal{P})$. Then L is obtained from a maximal linearly ordered subset L' of $\mathcal{P}$, and by Lemma 5.6 L' is predense in L. It follows that $\mathcal{P}$ is predense in $C(\mathcal{P})$. Since $C(\mathcal{P})$ has few gaps, the set S of points of L that are not limit points of L' is countable. Since $\mathcal{P}$ is preseparable, there is a countable dense set of L', Q. Then $Q \cup S$ is a countable dense set of L. So $C(\mathcal{P})$ is preseparable. □

6 The cactus of a continuum

Let X be an n–thick continuum. We explain how to associate a pretree to all minimal separators of the continuum. We also show that when X is separable, this pretree can be embedded into an $\mathbb{R}$–tree.

If a minimal separator K_1 separates two points a, b of a minimal separator K_2, then we say that K_1 *crosses* K_2. From Theorem 3.5 we have that K_2 crosses K_1 too.

We have seen in the previous section that every minimal separator of X either is contained in a maximal wheel or it does not cross any other minimal separator. If a minimal separator K does not cross any other minimal separator, we say that it is *isolated*.

We define now a pretree $\mathcal{R}$ with elements all the maximal wheels of X and all the isolated minimal separators of X. We define a betweenness relation in $\mathcal{R}$:

Let $x, y, z \in \mathcal{R}$. If y is a minimal separator, we say that y is between x, z if there are continua A, B such that

$$x \subset A, \quad z \subset B, \quad A \cup B = X, \quad A \cap B = y.$$

If y is a maximal wheel, we say that y is between x, z if for some minimal separator $w \in y$, w is between x, z.

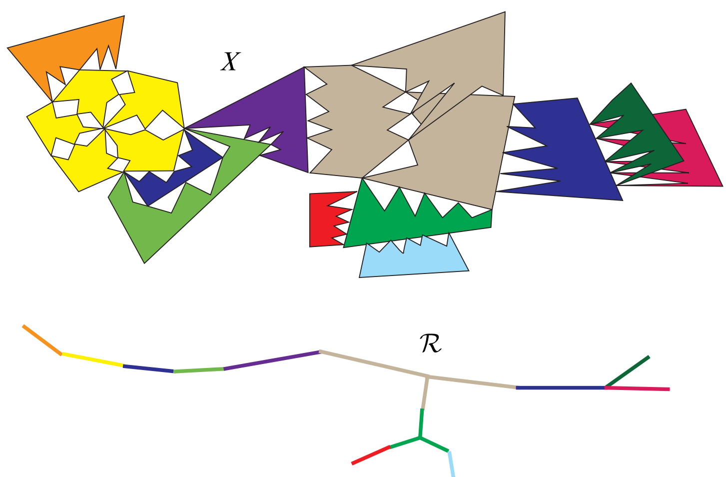

Figure 4

In Figure 4, there are two wheels, a yellow one and a gray one. In the pretree $\mathcal{R}$, the isolated cuts are the vertices where the colors change. The yellow wheel is the center of the yellow edge and the gray wheel is the center of the gray tripod.

Lemma 6.1 *$\mathcal{R}$ with the betweenness relation defined above, is a pretree.*

Proof We show that $\mathcal{R}$ satisfies the 4 axioms of the pretree definition.

(1) It is clear that xyx does not hold.

(2) xyz holds if and only if zyx holds.

(3) Assume that a minimal separator y is between x, z. Then there are continua A_1, B_1 such that

$$x \subset A_1, \quad z \subset B_1, \quad A_1 \cup B_1 = X, \quad A_1 \cap B_1 = y.$$

Assume that z is also a minimal separator and xzy holds. Then there are continua A_2, B_2 such that

$$x \subset A_2, \quad y \subset B_2, \quad A_2 \cup B_2 = X, \quad A_2 \cap B_2 = z.$$

We have

$$A_1 = (A_1 \cap A_2) \cup (A_1 \cap B_2), \quad (A_1 \cap A_2) \cap (A_1 \cap B_2) \subset y \cap z.$$

Note that y is not contained in $A_1 \cap A_2$. We set

$$B_3 = B_1 \cup (A_1 \cap B_2), \quad A_3 = A_1 \cup A_2$$

Then $X = A_3 \cup B_3$ and

$$A_3 \cap B_3 = (B_1 \cap A_1 \cap A_2) \cup (A_1 \cap B_2) \cap A_1 \cap A_2 \subset y \cap z.$$

We note also that $A_3 = A_1 \cap A_2$ contains x, while B_3 does not contain x. It follows that $y \cap z$ separates two points of X, a contradiction since X is n–thick. This shows that axiom 3 holds if x, z are minimal separators. However the case where one of y, z or both y, z are wheels reduces easily to the case where both are minimal separators. This shows that axiom 3 holds.

(4) Assume that xyz holds and let $w \neq y$. We will show that either xyw or zyw holds. Indeed let A, B be continua such that

$$x \subset A, \quad z \subset B, \quad A \cup B = X$$

and $A \cap B = y$ if y is a minimal separator or $A \cap B = y_1 \in y$ if y is a wheel. Then $w \subset A$ or $w \subset B$ since $w \neq y$. If $w \subset A$ then zyw holds and if $w \subset B$, then xyw holds. This proves axiom 4. □

Let X be an n–thick separable continuum. We showed above how to associate a pretree $\mathcal{R}$ to X which "encodes' all minimal separators of X. We now use the construction of Section 5 (see also [6]) and we complete $\mathcal{R}$ to a median pretree $\mathcal{P}$.

Lemma 6.2 *The set of maximal wheels of $\mathcal{R}$ is countable.*

Proof Let Q be a countable dense set of X. If W is a maximal wheel, we consider a wheel decomposition

$$X = M_0 \cup M_1 \cup M_2 \cup M_3$$

corresponding to a subwheel of W. We pick $q_i \in Q$ such that $q_i \in \mathrm{Int}\, M_i$ $(i = 0, 1, 2, 3)$. We map W to $\{q_0, q_1, q_2, q_3\}$. This gives a map from the set of maximal wheels to Q^4.

Suppose this map were not injective, then there would be two distinct maximal wheels W and $\widehat{W}$ having subwheels decompositions $X = M_0 \cup M_1 \cup M_2 \cup M_3$ and $X = \widehat{M}_0 \cup \widehat{M}_1 \cup \widehat{M}_2 \cup \widehat{M}_3$, respectively with $q_i \in \mathrm{Int}\, M_i \cap \mathrm{Int}\, \widehat{M}_i$ for some $q_i \in Q$ for $i = 0, 1, 2, 3$. Since W and $\widehat{W}$ don't cross, it must be the case that for some $i = 0, 1, 2, 3$, $W \subset \widehat{M}_i$, and at most one of $M_0, \dots, M_3$ is not contained in $\widehat{M}_i$. This contradicts the fact that $q_i \in \mathrm{Int}\, M_i \cap \mathrm{Int}\, \widehat{M}_i$ for $i = 0, 1, 2, 3$.

Thus this map from the set of maximal wheels to Q^4 is injective so the set of maximal wheels is countable. □

Lemma 6.3 *$\mathcal{R}$ is preseparable.*

Proof Let I be a maximal linearly ordered subset of $\mathcal{R}$. We define a dense subset S of I. Since the set of maximal wheels of $\mathcal{R}$ is countable, we take S to include all maximal wheels that lie in I. Let Q be a dense set of X. For any q_1, q_2 in Q, if q_1, q_2 are separated by some minimal separator in I, we pick a minimal separator K in I which separates q_1, q_2 and we add this to S. Clearly S is a countable dense subset of I. □

Lemma 6.4 *The set of added medians $\mathcal{P} - \mathcal{R}$ is countable.*

Proof Any added median m corresponds to a triple of points $x_0, x_1, x_2 \in \mathcal{R}$, none of which is between the other two. If some of x_0, x_1, x_2 are not minimal separators, then we substitute it by a minimal separator lying in it. Now there are continua A_i, B_i, $i = 0, 1, 2$ such that $A_i \cap B_i = x_i$, $X = A_i \cup B_i$ and $x_{i+1}, x_{i+2} \in A_i$, $i \in \mathbb{Z}_3$.

Since X is separable, it has a countable dense subset Q. We pick $q_i \in B_i, i = 1, 2, 3$ and we map m to the set $\{q_1, q_2, q_3\}$. It is clear that this map from the set of medians to $Q \times Q \times Q$ is injective so $\mathcal{P} - \mathcal{R}$ is countable. □

Theorem 6.5 *Let X be a separable, m–thick continuum, where $m > 1$. Then the pretree $\mathcal{R}$ embeds into an $\mathbb{R}$–tree T.*

Proof From the previous lemma it follows that $\mathcal{R}$ has few gaps. Therefore by Theorem 5.4 the completion $\mathcal{P}$ is preseparable and embeds canonically in an $\mathbb{R}$–tree T. □

The $\mathbb{R}$–tree T is called the *cactus tree* of the continuum X. Since this construction is canonical, the homeomorphism group of X acts naturally on the cactus tree of X.

Definition We say that a pretree $\mathcal{P}$ is *discrete* if for any $x, y \in \mathcal{P}$ there are finitely many $z \in \mathcal{P}$ such that xzy.

Theorem 6.6 *Let X be a locally connected, separable, m–thick continuum, where $m > 1$. Then the $\mathbb{R}$–tree corresponding to the minimal separators of X is simplicial.*

Proof So X is an m–thick metric continuum. We show that either X is not locally connected or $m = 1$. Suppose not, then there exists a strictly monotone sequence of branch points $A_i \subset T$ such that $A_i \to A \in T$.

We first consider the case where A is not terminal in T. By construction of T, all branch points of T are elements of $\mathcal{P}$. Since $\mathcal{P}$ is complete and $(A_n) \subset \mathcal{P}$, $A \in \mathcal{P}$. Since A is not terminal in T, it is not terminal in $\mathcal{P}$, so there is $B \in \mathcal{P}$ such that $A \in (A_n, B)$ for all A_n. Since the elements of $\mathcal{P} - \mathcal{R}$ are Dedekind cuts, we may assume that $B \in \mathcal{R}$. Since $\mathcal{R}$ is predense in $\mathcal{P}$, we can replace (A_n) with a monotone sequence in $\mathcal{R}$ (which we will also call (A_n)) which also has the property that $A_n \to A$ and $A \in (A_n, B)$ for all n. Notice that A_n need no longer be a branch point of T. Notice that the set $\{A_n, A, B\}$ is a linearly ordered subset of $\mathcal{P}$.

Let $n > 1$.

Consider the case where A_n is a minimal separating set. Since $A_n \in (A_{n-1}, A_{n+1})$, A_n decomposes X into continua Y_n, Z_n with $A_{n-1} \subset Y_n$ and $A_{n+1} \subset Z_n$. It follows that $A_i \subset Y_n$ for all $i < n$, and $B, A_j \subset Z_n$ for all $j > n$.

Next consider the case where A_n is a wheel. Since $A_n \subset (A_{n-1}, A_{n+1})$, there is a minimal separator $\widehat{A}_n \subset A_n$ with $\widehat{A}_n$ decomposing X into continua Y_n, Z_n with $A_{n-1} \subset Y_n$ and $A_{n+1} \subset Z_n$. It follows that $A_i \subset Y_n$ for all $i < n$, and $B, A_j \subset Z_n$ for all $j > n$. In this case, we replace A_n with the minimal separator $\widehat{A}_n$.

Thus for each n, we have minimal separators A_n with $|A_n| = m$ and A_n decomposes X into Y_n, Z_n with $A_i \subset Y_n$ for all $i < n$, and $B, A_j \subset Z_n$ for all $j > n$. By Lemma 3.1, A_{n+1} decomposes Z_n into continua W_n, Z_{n+1}. where $\partial W_n = A_n \cup A_{n+1}$.

For each n, let $A_n = \{a_n^1, \dots, a_n^m\}$. Passing to a subsequence, we may assume that for each $i = 1, \dots, m$ $a_n^i \to a^i$. Choose open sets $U^i \ni a^i$, $i = 1, \dots, m$ such that for $a^j \neq a^i$, $U^j \cap U^i = \varnothing$.

Case 1 For some choice of $U^i\,(i=1,\dots,m)$, for all $N\in\mathbb{N}$ there is $n>N$ with $W_n\not\subset\cup U^i$. Passing to a subsequence, we may assume that for each n there is $x_n\in W_n-\bigcup U^i$. Passing to a subsequence, we may assume that $x_n\to x\notin\bigcup U^i$. Making U^i smaller if need be, we can choose open $U\ni x$ with $U\cap U^i=\varnothing$ for all i. For $n\gg 0$, $x_n\in U$ and $a_n^i\notin U$, and so W_i and W_j are in different quasicomponents of U. It follows that there is not a basis of connected sets for x, and so X is not locally connected.

Case 2 For every choice of $U^i\,(i=1,\dots,m)$, for $n\gg 0$, then $W_n\subset\cup U^i$. Since W_n is a continuum, it follows that $a^i=a^j$ for all i,j. Notice now that $\partial[\bigcup Y_n]=\{a^1\}$, so a^1 is a cut point separating A_1 from B. Thus $m=1$.

Now consider the case where A is terminal in T. It follows that $A\in\mathcal{R}$. Replacing A with a minimal cut subset of itself (when A is a maximal wheel), we have that A decomposes X into continua B,C where $\operatorname{Int}B\cap\bigcup[\mathcal{R}-\{A\}]=\varnothing$. The proof proceeds as above with this new B playing the same roll as the old B. $\square$

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