# $\mathcal{Z}$–Structures on product groups

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A $\mathcal{Z}$–*structure* on a group $G$, defined by M Bestvina, is a pair $(\hat{X}, Z)$ of spaces such that $\hat{X}$ is a compact ER, $Z$ is a $\mathcal{Z}$–set in $\hat{X}$, $G$ acts properly and cocompactly on $X = \hat{X} \backslash Z$ and the collection of translates of any compact set in $X$ forms a null sequence in $\hat{X}$. It is natural to ask whether a given group admits a $\mathcal{Z}$–structure. In this paper, we show that if two groups each admit a $\mathcal{Z}$–structure, then so do their free and direct products.

# 1 Introduction

## 1.1 Preliminaries

Introduced by M Bestvina in [1], a $\mathcal{Z}$–structure on a group is an extension of the notion of a boundary on a CAT(0) or hyperbolic group to more general groups. Specifically, a $\mathcal{Z}$–structure mimics not only the concept of compactifying the space on which the group $G$ acts in a particularly nice way but also the fact that boundaries on CAT(0) and hyperbolic groups satisfy a "null condition," essentially meaning that compact sets get "small" as they are pushed toward the boundary by elements of $G$. Here we will review the definitions of $\mathcal{Z}$– and $\mathcal{E}\mathcal{Z}$–structures along with several other preliminary definitions and results which will be used later.

**Note** Bestvina's original definition of a $\mathcal{Z}$–structure implies that $G$ is torsion-free. The definition was generalized by A Dranishnikov in [4] to include groups with torsion. We will use the more general definition in this paper.

**Convention** In this paper we assume that all spaces are locally compact separable metric spaces.

**Definition 1.1** A subspace $A$ of a space $X$ is a *retract* of $X$ if there exists a map $r \colon X \to A$ extending $\mathrm{id}_A \colon A \to A$. A subspace $A$ of a space $X$ is a *strong deformation retract* of $X$ if there exists a homotopy $H \colon X \times [0, 1] \to X$ satisfying $H_0 \equiv \mathrm{id}_X$, $H_1(X) \subseteq A$ and $H_t(A) \equiv \mathrm{id}_A$ for all $t \in [0, 1]$.

**Definition 1.2** A separable metric space $X$ is an *absolute retract* (or AR) if, whenever $X$ is embedded as a closed subset of another separable metric space $Y$, its image is a retract of $Y$. A space $X$ is an *absolute neighborhood retract* (or ANR) if, whenever $X$ is embedded as a closed subset of another separable metric space $Y$, some neighborhood of $X$ in $Y$ retracts onto $X$.

**Definition 1.3** A space $X$ is a *Euclidean retract* (or ER) if it can be embedded in some Euclidean space as its retract. $X$ is a *Euclidean neighborhood retract* (or ENR) if it can be embedded in some Euclidean space $\mathbb{R}^n$ in such a way that a neighborhood of $X$ in $\mathbb{R}^n$ retracts onto $X$.

We recite here a well-known fact concerning a relationship between ANRs, ARs, and ERs, and two useful properties of ARs:

**Fact 1.4** *If $X$ is a finite dimensional space, then the following are equivalent:*

  (i)  *$X$ is an AR.*

 (ii)  *$X$ is a contractible ANR.*

(iii)  *$X$ is an ER.*

(iv)  *$X$ is contractible and locally contractible.*

**Fact 1.5** *If $X$ is an AR and $A$ is a closed subspace of a separable metric space $Y$, then every map $f\colon A \to X$ extends to $Y$.*

**Fact 1.6** *Every retract of an AR is an AR.*

**Fact 1.7** (Hu [8, Theorem III.7.10]) *Let $X$ be an AR. Then a closed subspace $A$ of $X$ is an AR if and only if $A$ is a strong deformation retract of $X$.*

**Proof of Fact 1.7** The sufficiency follows from Fact 1.6. To prove the necessity, consider the closed subspace $Q := (X \times \{0\}) \cup (A \times I) \cup (X \times \{1\})$ of $X \times I$. Since $A$ is an AR and is closed in $X$, there is a retraction $r\colon X \supseteq A$. Define a map $F\colon Q \to X$ by taking

$$F(x,t) := \begin{cases} x & \text{if } x \in X,\ t = 0, \\ x & \text{if } x \in A,\ t \in I, \\ r(x) & \text{if } x \in X,\ t = 1. \end{cases}$$

Since $X$ is an AR and $Q$ is closed in $X \times I$, then $F$ extends to $H\colon X \times I \to X$, which is a strong deformation retraction of $X$ to $A$. □

**Definition 1.8** A closed subset $Z$ of an ANR $\widehat{X}$ is a $\mathcal{Z}$–*set* in $\widehat{X}$ if there exists a homotopy $H\colon \widehat{X} \times [0,1] \to \widehat{X}$ such that $H_0 \equiv \mathrm{id}_{\widehat{X}}$ and $H_t(\widehat{X}) \cap Z = \varnothing$ for all $t > 0$. In this situation, we will call $H$ a $\mathcal{Z}$–*set homotopy*.

**Definition 1.9** An ANR $\widehat{X}$ is a $\mathcal{Z}$–*compactification* of $X$ if $X \subseteq \widehat{X}$, $\widehat{X}$ is compact, and $\partial X := \widehat{X} \backslash X$ is a $\mathcal{Z}$–set in $\widehat{X}$. In such a situation, we call the set $\partial X$ a *boundary* for the space $X$.

**Remark** It is easy to see that if $\widehat{X}$ is a $\mathcal{Z}$–compactification of $X$, then the inclusion map $X \hookrightarrow \widehat{X}$ is a homotopy equivalence. In fact, the inclusion $U \backslash \partial X \hookrightarrow U$ is a homotopy equivalence for every open subset $U$ of $\widehat{X}$.

**Lemma 1.10** *If $\widehat{X}$ is an AR which is a $\mathcal{Z}$–compactification of $X$, then there is a homotopy $\widehat{F}\colon \widehat{X} \times [0,1] \to \widehat{X}$ and a base point $x_0 \in X$ such that $\widehat{F}_0 \equiv \mathrm{id}_{\widehat{X}}$, $\widehat{F}_t(\widehat{X}) \cap \partial X = \varnothing$ if $t > 0$, $\widehat{F}_1(\widehat{X}) = \{x_0\}$ and $\widehat{F}(x_0, t) = x_0$ for all $t \in [0,1]$.*

**Proof** The proof is analogous to that of [Fact 1.7]:

Since $\widehat{X}$ is a $\mathcal{Z}$–compactification of $X$, there is a homotopy $F\colon \widehat{X} \times [0,1] \to \widehat{X}$ satisfying $F_0 \equiv \mathrm{id}_{\widehat{X}}$ and $F_t(\widehat{X}) \cap \partial X = \varnothing$ whenever $t > 0$. Moreover, since $\widehat{X}$ is contractible, then we may assume that $F$ is a contraction to some base point $x_0 \in X$.

Now $F(\{x_0\} \times [0,1]) \cap \partial X = \varnothing$, so we may choose an open neighborhood $U$ of $F(\{x_0\} \times [0,1])$ in $X$.

Define $Q := ((X \backslash U) \times [0,1]) \cup (\{x_0\} \times [0,1]) \cup (X \times \{1\})$, and $H\colon Q \to X$ by

$$H(x,t) := \begin{cases} F(x,t) & \text{if } (x,t) \in (X \backslash U) \times [0,1], \\ x_0 & \text{if } x = x_0,\ t \in [0,1], \\ x_0 & \text{if } x \in X,\ t = 1. \end{cases}$$

Since $X$ is an AR and $Q$ is closed in $X \times [0,1]$, then $H$ extends to $\widehat{H}\colon X \times [0,1] \to X$. Now the map $\widehat{F}\colon \widehat{X} \times [0,1] \to \widehat{X}$ defined by

$$\widehat{F}(x,t) := \begin{cases} F(x,t) & \text{if } x \in V = \partial X,\ t \in [0,1], \\ \widehat{H}(x,t) & \text{if } x \in X,\ t \in [0,1] \end{cases}$$

has the desired properties. $\square$

**Lemma 1.11** *Suppose $\widehat{X}$ is an AR which is a $\mathcal{Z}$–compactification of $X$, $\{C_i\}_{i=1}^{\infty}$ is an exhaustion of $X$ by compact sets satisfying $\overline{C_i} \subseteq \mathrm{int}(C_{i+1})$ for all $i \in \mathbb{N}$, and $\{t_i\}_{i=1}^{\infty} \subseteq (0,1)$ satisfies $t_i > t_{i+1}$ for all $i \in \mathbb{N}$. Then there is a $\mathcal{Z}$–set homotopy*

$F\colon \widehat{X} \times [0,1] \to \widehat{X}$ which is a strong deformation retraction of $\widehat{X}$ to a base point $x_0 \in X$ and having the additional property that

$$F(x,t) = x \text{ whenever } (x,t) \in \bigcup_{i=1}^{\infty} (C_i \times [0,t_i])$$

**Proof**  Let $\bar{F}\colon \widehat{X} \times [0,1] \to \widehat{X}$ be a $\mathcal{Z}$–set homotopy which is a strong deformation retraction of $\widehat{X}$ to $x_0 \in X$, as in [Lemma 1.10](#).

Let $A := (\widehat{X} \times \{0,1\}) \cup (\partial X \times [0,1]) \cup \left( \bigcup_{i=1}^{\infty} (C_i \times [0,t_i]) \right)$ and define $f\colon A \to [0,1]$ by

$$f(x,t) = \begin{cases} 0 & \text{if } (x,t) \in (\widehat{X} \times \{0\}) \cup \left( \bigcup_{i=1}^{\infty} (C_i \times [0,t_i]) \right), \\ t & \text{if } (x,t) \in (\widehat{X} \times \{1\}) \cup (\partial X \times [0,1]). \end{cases}$$

Now, since $A$ is a closed subset of $\widehat{X} \times [0,1]$, the map $f$ extends to $f\colon \widehat{X} \times [0,1] \to [0,1]$.

Then $F\colon \widehat{X} \times [0,1] \to \widehat{X}$ defined by $F(x,t) := \bar{F}(x, f(x,t))$ for all $(x,t) \in \widehat{X} \times [0,1]$ has the required attributes.  $\square$

**Definition 1.12**  The action of a group $G$ on a space $X$ is *proper* if every point $x \in X$ has a neighborhood $U$ satisfying $g(U) \cap U = \varnothing$ for all but finitely many $g \in G$.

**Definition 1.13**  The action of $G$ on $X$ is *cocompact* if there is a compactum $K$ in $X$ so that $\bigcup_{g \in G} gK = X$.

**Definition 1.14**  Suppose $G$ is a group acting properly and cocompactly on $X$, and $\widehat{X}$ is a $\mathcal{Z}$–compactification of $X$. We say that $\widehat{X}$ satisfies the *null condition with respect to the action of $G$ on $X$* if the following condition holds:

For any compactum $C$ in $X$ and any open cover $\mathcal{U}$ of $\widehat{X}$, there is a finite subset $\Gamma$ of $G$ so that if $g \in G \backslash \Gamma$, then $gC$ is contained in a single element of $\mathcal{U}$.

**Definition 1.15**  (Bestvina [1])  Let $G$ be a group. A $\mathcal{Z}$–*structure* on $G$ is a pair $(\widehat{X}, Z)$ of spaces such that:

(1)  $\widehat{X}$ is a compact ER.

(2)  $\widehat{X}$ is a $\mathcal{Z}$–compactification of $X := \widehat{X} \backslash Z$.

(3)  $G$ acts properly and cocompactly on $X := \widehat{X} \backslash Z$.

(4)  $\widehat{X}$ satisfies the null condition with respect to the action of $G$ on $X$.

**Remark**  Note that if $G$ admits a $Z$–structure $(\hat{X}, \partial X)$, then $G$ acts on the contractible finite-dimensional ANR $X$.

In [1], Bestvina discusses the possibility of requiring that the $G$–action on $X$ extend to an action on $\hat{X}$. This variation on the notion of $Z$–structure was formalized by Farrell and LaFont in the following way:

**Definition 1.16**  (Farrell–LaFont [6])  The pair $(\hat{X}, Z)$ is an $\mathcal{E}Z$–*structure* on the group $G$ if $(\hat{X}, Z)$ is a $Z$–structure on $G$, and the action of $G$ on $X := \hat{X} \backslash Z$ extends to an action on $\hat{X}$.

In general, the conditions from Definition 1.15 which are most difficult to verify when showing the existence of a $Z$–structure are (1) and (2). When proving the theorems in this paper, we found the following to be helpful:

**Definition 1.17**  For an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of a space $Z$, we say that a homotopy $H: Z \times [0, 1] \to Z$ is a $\mathcal{U}$–*homotopy* if for each $z \in Z$, there is an $\alpha \in A$ such that $H(\{z\} \times [0, 1]) \subseteq U_\alpha$.

Similarly, if $(Z, d)$ is a metric space, then we say that $H: Z \times [0, 1] \to Z$ is an $\epsilon$–*homotopy* if for each $z \in Z$, $\mathrm{diam}_d(H(\{z\} \times [0, 1])) < \epsilon$.

**Definition 1.18**  A space $W$ $\mathcal{U}$–*dominates* (respectively, $\epsilon$–*dominates*) the space $Z$ if there exist maps $\phi: W \to Z$ and $\psi: Z \to W$ such that the composition $\phi \circ \psi: Z \to Z$ is $\mathcal{U}$–homotopic (respectively, $\epsilon$–homotopic) to $\mathrm{id}_Z$.

**Theorem 1.19**  (Hanner [7])  *Each of the following conditions is a sufficient condition for a space $X$ to be an ANR:*

  (a)  *For each covering $\mathcal{U}$ of $X$ there is an ANR which $\mathcal{U}$–dominates $X$.*
  (b)  *For some metric on $X$ there exists for each $\epsilon > 0$ an ANR which $\epsilon$–dominates $X$.*

**Definition 1.20**  A map $f: X \to Y$ between metric spaces $(X, d)$ and $(Y, d')$ is an $\epsilon$–*mapping* if $\mathrm{diam}_d f^{-1}(\{y\}) < \epsilon$ for every $y \in Y$.

**Theorem 1.21**  (See Engelking [5, page 107].)  *If $X$ is a compact metric space and for every $\epsilon > 0$ there exists an $\epsilon$–mapping $f: X \to Y$ of $X$ to a compact space $Y$ such that $\dim Y \leq n$, then $\dim X \leq n$.*

**Corollary 1.22**  *If $X$ is a metric space with $\dim X \leq n$ and $\hat{X}$ is a metric space which is a $Z$–compactification of $X$, then $\dim \hat{X} \leq n$.*

**Proof**  Let $\epsilon > 0$ and let $H := \widehat{X} \times [0, 1]$ be a $\mathcal{Z}$–set homotopy. We may choose $t \in (0, 1]$ such that the corestriction $H_t \colon \widehat{X} \to H_t(\widehat{X})$ is an $\epsilon$–mapping. Since $H_t(\widehat{X}) \subseteq X$ and $\dim X \leq n$, then $\dim H_t(\widehat{X}) \leq n$. Moreover, $H_t(\widehat{X})$ is compact, so Theorem 1.21 applies.                                                                                          □

## 1.2  Examples

While the task (posed by Bestvina) of classifying all groups which admit $\mathcal{Z}$–structures remains open, there are various classes of groups which are known to admit $\mathcal{Z}$–structures:

**Example 1.23**  A geodesic space $X$ is CAT(0) if geodesic triangles in $X$ are "no fatter than" those in the Euclidean plane. (See Bridson and Haefliger [3, Chapter II.1] for more background.)

If $X$ is a CAT(0) space, the *visual boundary* of $X$, denoted $\partial X$, is the set of geodesic rays emanating from a chosen base point $x_0$. This boundary on $X$ is well-defined and independent of the base point. The *cone topology* on $\overline{X} := X \cup \partial X$ has as a basis all open balls $B(x, r) \subseteq X$ and all sets of the form $U(c, r, \epsilon)$, where, given a geodesic ray $c$ based at $x_0$ and $r, \epsilon > 0$,

$$U(c, r, \epsilon)$$
$$:= \{x \in X \mid d(x, x_0) > r, \; d(p_r(x), c(r)) < \epsilon\} \cup \{x \in \partial X \mid d(p_r(x), c(r)) < \epsilon\},$$

where $p_r$ is the natural projection map to $\overline{B}(x_0, r)$. These neighborhoods $U(c, r, \epsilon)$ of boundary points contain those points in $\overline{X}$ which are sufficiently far from $x_0$ (ie sufficiently close to $\partial X$) and which emanate from $x_0$ at the appropriate "angle."

A metric space $(X, d)$ is *proper* if every closed metric ball in $X$ is compact.

A group $G$ is CAT(0) if $G$ acts properly and cocompactly by isometries on a proper CAT(0) space.

**Fact 1.24**  *If $G$ is a CAT(0) group acting properly and cocompactly by isometries on the proper CAT(0) space $X$, then $(\overline{X}, \partial X)$ is a $\mathcal{Z}$–structure on $G$.*

It is easy to see that $\overline{X}$ is a $\mathcal{Z}$–compactification of $X$; $\overline{X}$ can be pulled off of $\partial X$ via a homotopy which runs all the geodesic rays in reverse.

The following statement follows easily from the CAT(0)–inequality and the fact that $G$ acts isometrically on $X$:

Given a compact $C \subseteq X$, $r > 0$ and $\epsilon > 0$, there is a number $R > 0$ such that if $gC \cap B(x_0, R) = \varnothing$, then there is some $c \in \partial X$ such that $gC \subseteq U(c, r, \epsilon)$.

This fact and the properness of the action of $G$ on $X$ together imply that $\overline{X}$ satisfies the null condition with respect to the action of $G$ on $X$.

In addition, the action of $G$ on $X$ extends naturally to $\partial X$, giving:

**Fact 1.25** *If $G$ is a CAT$(0)$ group, then $G$ admits an $\mathcal{EZ}$–structure.*

**Example 1.26** (See [3] for a more thorough treatment.) A geodesic metric space $(X, d)$ is $\delta$–*hyperbolic* (where $\delta \geq 0$) if for any triangle with geodesic sides in $X$, each side of the triangle is contained in the $\delta$–neighborhood of the union of the other two sides.

A group $G$ is *hyperbolic* if its Cayley graph is $\delta$–hyperbolic for some $\delta \geq 0$.

**Theorem 1.27** (Bestvina–Mess [2]) *If $G$ is a torsion-free hyperbolic group, then $G$ admits a $\mathcal{Z}$–structure.*

The proof in [2] takes as $X$ an appropriately chosen Rips complex of $G$ and as $\partial X$ the Gromov boundary of $X$.

**Example 1.28** Systolic groups are groups that act simplicially and cocompactly on simplicial complexes which satisfy a combinatorial version of nonpositive curvature. In [9], D Osajda and P Przytycki show that every systolic group admits an $\mathcal{EZ}$–structure.

**Example 1.29** Bestvina constructs in [1] multiple $\mathcal{Z}$–structures on the Baumslag–Solitar group $BS(1, 2) = \langle x, t \mid t^{-1}xt = x^2 \rangle$, one of which is clearly not an $\mathcal{EZ}$–structure. It is known that this group is not CAT$(0)$, hyperbolic or systolic.

## 1.3 Statement of main results

In this paper, we prove that, if groups $G$ and $H$ each admit $\mathcal{Z}$–structures, then so do their free and direct products:

**Theorem 2.9** *If both $G$ and $H$ admit $\mathcal{Z}$–structures, then so does $G * H$.*

The proof of Theorem 2.9 involves the construction of a tree-like space $W$ on which $G * H$ acts properly and cocompactly, and the fabrication of a metric $d$ in such a way

that the metric completion $\overline{W}$ of $W$, with $\partial W := \overline{W} \setminus W$, satisfies the axioms of a $\mathcal{Z}$–structure. The space $W$ is constructed by gluing copies of $X$ and $Y$ in an equivariant manner, where $(\widehat{X}, \partial X)$ and $(\widehat{Y}, \partial Y)$ are $\mathcal{Z}$–structures on $G$ and $H$, respectively.

The ability to extend the action of $G * H$ on $W$ to $\overline{W}$ is a consequence of the assumption that the actions of $G$ and $H$ extend to $\widehat{X}$ and $\widehat{Y}$, which allows us to obtain:

**Theorem 2.10** If $G$ and $H$ each admit $\mathcal{EZ}$–structures, then so does $G * H$.

The other main results found in this paper pertain to direct products of groups which admit $\mathcal{Z}$–structures.

**Theorem 3.21** If both $G$ and $H$ admit $\mathcal{Z}$–structures, then so does $G \times H$.

The proof of Theorem 3.21 is motivated by its analog in the CAT(0) setting (see [3, Example II.8.11(6)]):

If $X$ and $Y$ are CAT(0) spaces, then so is $X \times Y$ under the Euclidean product metric. If $\partial X$ and $\partial Y$ denote the visual boundaries of $X$ and $Y$ (see Example 1.23 for definitions), let $\partial X * \partial Y$ represent the spherical join of $\partial X$ and $\partial Y$, ie $\partial X * \partial Y = \partial X \times \partial Y \times [0, \pi/2]/\sim$, where $(c_1, c_2, \theta) \sim (c_1', c_2', \theta)$ if and only if ($\theta = 0$ and $c_1 = c_1'$) or ($\theta = \pi/2$ and $c_2 = c_2'$).

Then $\partial(X \times Y)$ is naturally homeomorphic to $\partial X * \partial Y$:

For each $\theta \in [0, \pi/2]$, $c_1 \in \partial X$, and $c_2 \in \partial Y$, denote by $(\cos \theta)c_1 + (\sin \theta)c_2$ the point of $\partial(X \times Y)$ represented by the ray $t \mapsto (c_1(t \cos \theta), c_2(t \sin \theta))$. Then every point of $\partial(X \times Y)$ can be represented by a ray of this form. Of course, the rays $(\cos \theta)c_1 + (\sin \theta)c_2$ and $(\cos \theta)c_1' + (\sin \theta)c_2'$ are equal when $\theta = 0$ and $c_1 = c_1'$ (regardless of whether or not we have $c_2 = c_2'$), and when $\theta = \pi/2$ and $c_2 = c_2'$. This is consistent with the equivalence relation defining $\partial X * \partial Y$.

Intuitively, for points in $\overline{X \times Y}$ to be "close" to a given boundary point $(c_1, c_2, \theta) \in \partial X * \partial Y$, it is not sufficient to have $X$–coordinate near $c_1$ and $Y$–coordinate near $c_2$; they must also have "angle" near $\theta$.

Now, given CAT(0) groups $G$ and $H$ which act properly and cocompactly on $X$ and $Y$, respectively, the pair $(\overline{X \times Y}, \partial X * \partial Y)$ is a $\mathcal{Z}$–structure on $G \times H$.

The fact that the null condition with respect to the action of $G \times H$ on $X \times Y$ is satisfied by $\overline{X \times Y}$ is an implication of the CAT(0)–inequality, the general idea being

that the span of "angles" achieved by a compactum shrinks as it is translated outside of a large metric ball.

To prove the theorem for general direct products, we define a notion of "slope" which cooperates with the given $\mathcal{Z}$–set homotopies and certain carefully chosen metrics on the factors $X$ and $Y$. In the CAT(0) case, we can take as slope function $(x, y) \mapsto d_Y(y, y_0)/d_X(x, x_0)$ thanks to the properties of the CAT(0) metrics; in the general case, we construct functions $p\colon X \to [0, \infty)$ and $q\colon Y \to [0, \infty)$ to have similar properties and use these to define slope.

To compactify $X \times Y$, then, we glue to it the join $\partial X * \partial Y$ and topologize with neighborhoods of boundary points analogous to those used in the CAT(0) setting.

By extending the action of $G \times H$ on $X \times Y$ to the $\mathcal{Z}$–compactification $\widehat{X \times Y}$ described above, we obtain:

**Theorem 3.22**  *If $G$ and $H$ each admit $\mathcal{EZ}$–structures, then so does $G \times H$.*

# 2   $\mathcal{Z}$– and $\mathcal{EZ}$–structures on free products of groups

Suppose $(\hat{X}, \partial X)$ and $(\hat{Y}, \partial Y)$ are $\mathcal{Z}$–structures on $G$ and $H$, respectively.

Let $\rho$ and $\tau$ be metrics on $\hat{X}$ and $\hat{Y}$ satisfying $\mathrm{diam}_\rho \hat{X} = \mathrm{diam}_\tau \hat{Y} = 1$.

**Notation**   (i)   Denote by $1_G$ and $1_H$ the identity elements from $G$ and $H$, respectively, and by $\mathbf{1}$ the identity element in $G * H$.

(ii)   Whenever we refer to a word $\mathbf{1} \neq w \in G * H$, it is always assumed that $w$ is reduced, ie that consecutive letters of $w$ come from alternating factors, with no letter being an identity element from either group. With this in mind, we define, for $w \neq \mathbf{1}$:

- $|w| :=$ the length of $w$
- $w(k) :=$ the $k$–th letter of $w$, counting from left to right
- $w|_k :=$ the leftmost length $k$ subword of $w$

(iii)   We will use the convention that $|\mathbf{1}| = 0$, that $\mathbf{1}(|\mathbf{1}|) = \mathbf{1}(0) = \mathbf{1}$, and that $\mathbf{1} \in G \cap H$.

**Definition 2.1**   (The space $W$)   Let $X_0$ and $Y_0$ be "base" copies of $X$ and $Y$, respectively. Define

$$W := \left( \left( \bigcup_{\substack{w \in G*H \\ w(|w|) \in H}} wX_0 \right) \cup \left( \bigcup_{\substack{w \in G*H \\ w(|w|) \in G}} wY_0 \right) \right) \Big/ \sim .$$

To define the equivalence relation $\sim$, first note that if $w(|w|) \in H$, then $wX_0$ contains all the points of the form $wgx_0$ for $g \in G$, including the point $wx_0$. Similarly, if $w(|w|) \in G$, then $wY_0$ contains all the points of the form $why_0$ for $h \in H$, including the point $wy_0$.

In other words, if $w(|w|) \in H$, then $wx_0 \in wX_0$; otherwise $wx_0 \in w|_{|w|-1}X_0$. Likewise, if $w(|w|) \in G$, then $wy_0 \in wY_0$; otherwise $wy_0 \in w|_{|w|-1}Y_0$.

Therefore, we define $\sim$ by

$$wx_0 \sim wy_0 \quad \text{for all } w \in G * H.$$

The result of this gluing is that, if $w(|w|) \in H$, then $wX_0$ is glued to $w|_{|w|-1}Y_0$ by identifying the points $wx_0 \in wX_0$ and $wy_0 \in w|_{|w|-1}Y_0$. Analogously, if $w(|w|) \in G$, then $wY_0$ is glued to $w|_{|w|-1}X_0$ by identifying the points $wy_0 \in wY_0$ and $wx_0 \in w|_{|w|-1}X_0$. (See Figure 1 below.)
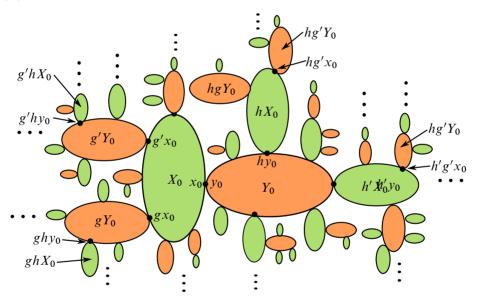


Figure 1: The gluing schematic for $W$

**Warning** Although it is included with the intention of providing some intuition about the construction of $W$, Figure 1 has the potential to be a bit misleading, due to its two-dimensionality. We warn the reader that the points $wx_0$ and $wy_0$ are *not* boundary points of translates of $X_0$ and $Y_0$, despite their appearance in the graphic.

**Remark** (i) The above construction of $W$ is similar to that found in the proof of [3, Theorem II.11.16]; said theorem produces a complete CAT(0) space on which

$\Gamma_0 *_\Gamma \Gamma_1$ acts properly (and cocompactly) by isometries when each of $\Gamma_0, \Gamma_1$, and $\Gamma$ acts properly (and cocompactly) by isometries on a CAT(0) space. Our construction allows more general spaces but yields essentially the same underlying space under the hypotheses of the cited theorem in the case where $\Gamma$ is trivial.

(ii)  The action of $G * H$ on $W$ is as follows:

Note that each point of $wX_0$ where $w(|w|) \in H$ has the form $wx$ for some $x \in X_0$. Thus if $x \in X_0$, we define $w \cdot x := wx \in wX_0$.

We define $w \cdot y$ for $y \in Y_0$ and $w(|w|) \in G$ similarly.

If $x \in X_0$ and $w(|w|) \in G$, then $wX_0 = w|_{|w|-1}w(|w|)X_0 = w|_{|w|-1}X_0$, and $w \cdot x := wx = w|_{|w|-1} \cdot w(|w|)x \in w|_{|w|-1}X_0$. Similarly, if $y \in Y_0$ and $w(|w|) \in H$, then $wY_0 = w|_{|w|-1}Y_0$, and $w \cdot y := wy \in w|_{|w|-1}Y_0$.

Now for a general point $z \in W$, there is some $x \in X_0$ or $y \in Y_0$ and some $w' \in G * H$ such that $z = w' \cdot x$ or $z = w' \cdot y$. In the first case, we define $w \cdot z := (ww') \cdot x$; otherwise we set $w \cdot z := (ww') \cdot y$.

(iii)  For the rest of this chapter, it is to be understood that the use of the symbol $wX_0$ implies that $w(|w|) \in H$, and the use of the symbol $wY_0$ implies that $w(|w|) \in G$.

**Definition 2.2**  (Metric $d$ on $W$)  Define $r \colon G \cup H \to \mathbb{N}$ by

$$r(g) = n \iff gx_0 \in B_\rho\left(\partial X, \frac{1}{2^{n-1}}\right) \setminus B_\rho\left(\partial X, \frac{1}{2^n}\right)$$

$$r(h) = n \iff hy_0 \in B_\tau\left(\partial Y, \frac{1}{2^{n-1}}\right) \setminus B_\tau\left(\partial Y, \frac{1}{2^n}\right)$$

and $r^* \colon G * H \to (0, 1]$ by

$$\mathbf{1} \in G * H \longmapsto 1$$

$$g \in G \longmapsto \frac{1}{2^{r(g)}}$$

$$h \in H \longmapsto \frac{1}{2^{r(h)}}$$

$$w \in G * H \longmapsto \prod_{k=1}^{|w|} \frac{1}{2^{r(w(k))}} = \prod_{k=1}^{|w|} r^*(w(k)).$$

We use the function $r^*$ to define a metric $d$ on $W$:

The restriction of $d$ to $wX_0$ (respectively $wY_0$) is declared to be a rescaling of $\rho$ (respectively $\tau$) so that $\mathrm{diam}_d\, wX_0 = \mathrm{diam}_d\, wY_0 = r^*(w)$.

For points $x, x' \in W$ which do not lie in a single translate of $X_0$ or $Y_0$, we say that a finite sequence $\{w_i x_0\}_{i=1}^{k}$ *connects* $x$ and $x'$ if each of the pairs $(x, w_1 x_0)$, $(w_k x_0, x')$, and $(w_i x_0, w_{i+1} x_0)$ for $i = 1, \ldots, k-1$, lives in a single translate of $X_0$ or $Y_0$.

To define $d(x, x')$ when $x$ and $x'$ do not lie in a single translate of $X_0$ or $Y_0$, let $\{w_i x_0\}_{i=1}^{k}$ be the shortest sequence which connects $x$ and $x'$, and set

$$d(x, x') := d(x, w_1 x_0) + \sum_{i=1}^{k-1} d(w_i x_0, w_{i+1} x_0). + d(w_k x_0, x').$$

It is an easy exercise to check that $d$ is indeed a metric on $W$. The proof that $d$ satisfies the triangle inequality resembles its counterpart for a tree, using in addition the triangle inequality on the components $w X_0$ and $w Y_0$.

Now $(W, d)$ is a metric space, and we denote by $\overline{W}$ the metric completion of $(W, d)$ and set $\partial W := \overline{W} \backslash W$.

Let us discuss briefly the convention to be used from this point forward when referring to points of $\partial W$. We may view $\overline{W}$ as the set of equivalence classes of Cauchy sequences in $W$, where $\sim$ is generated by $\{x_i\}_{i=1}^{\infty} \sim \{x_i'\}_{i=1}^{\infty}$ if $d(x_i, x_i') \to 0$ as $i \to \infty$. It is not difficult to see that, if a Cauchy sequence $\vec{x} = \{x_i\}_{i=1}^{\infty} \subseteq W$ does not converge in $W$, then $\vec{x} \sim \vec{x}'$ where $\vec{x}' = \{x_i'\}_{i=1}^{\infty}$ falls under one of three possible categories:

  (i)   There exists $w \in G * H$ such that $x_i' \in w X_0$ for all $i \in \mathbb{N}$.

  (ii)  There exists $w \in G * H$ such that $x_i' \in w Y_0$ for all $i \in \mathbb{N}$.

  (iii) There exists a sequence $\{w_i\}_{i=1}^{\infty} \subseteq G * H$ with $|w_i| = i$, $w_i|_{i-1} = w_{i-1}$, and $x_i' = w_i x_0$ for all $i \in \mathbb{N}$.

If $\vec{x}'$ falls under category (i), then $x_i, x_i' \to \overline{x}$ for some $\overline{x} \in \partial w X_0 = w \partial X_0$; if $\vec{x}'$ falls under category (ii), then $x_i, x_i' \to \overline{y}$ for some $\overline{y} \in \partial w Y_0 = w \partial Y_0$. Otherwise, $\vec{x}'$ corresponds to a unique element of $\mathcal{A}$, where

$$\mathcal{A} = \{\overline{\alpha} \mid \overline{\alpha} = \{\alpha_i\}_{i=1}^{\infty}, \ \alpha_i = w_i x_0, \ w_i \in G * H, \ |w_i| = i, \ w_{i+1}|_i = w_i \ \forall i \in \mathbb{N}\}.$$

Hence, we will refer to points $\alpha \in \partial W$ as having three possible types, each of which corresponds to one of the above-named categories for Cauchy sequences in $W$ which do not converge in $W$:

  (i)   $\alpha \in w \partial X_0$

  (ii)  $\alpha \in w \partial Y_0$

  (iii) $\alpha \in \mathcal{A}$

**Proposition 2.3** $\overline{W}$ is compact.

**Proof** Since $\overline{W}$ is complete, it suffices to show that $\overline{W}$ is totally bounded, ie that for any $\epsilon > 0$, there is a finite cover of $\overline{W}$ by $\epsilon$–balls.

Let $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that $1/2^{k-1} < \epsilon/4$.

Then if a reduced word $w \in G * H$ satisfies $|w| \geq k$, we have

$$\text{diam}_d\left(\left(\bigcup_{\substack{w'|_{|w|}=w \\ w'(|w'|)\in H}} w'\widehat{X}_0\right) \cup \left(\bigcup_{\substack{w'|_{|w|}=w \\ w'(|w'|)\in G}} w'\widehat{Y}_0\right)\right) \leq \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}} < \frac{\epsilon}{4}.$$

Thus if $x \in v\widehat{X}_0$, where $v|_{|w|} = w$ and $|w| \geq k$, then

$$d(x, wx_0) \leq \text{diam}_d\left(\left(\bigcup_{\substack{w'|_{|w|}=w \\ w'(|w'|)\in H}} w'\widehat{X}_0\right) \cup \left(\bigcup_{\substack{w'|_{|w|}=w \\ w'(|w'|)\in G}} w'\widehat{Y}_0\right)\right) + \text{diam}_d\, w\widehat{X}_0$$

$$\leq \frac{1}{2^{k-1}} + \frac{1}{2^k} < 2 \cdot \frac{1}{2^{k-1}} < \frac{\epsilon}{2}.$$

Similarly, if $y \in v\widehat{Y}_0$, where $v|_{|w|} = w$ and $|w| \geq k$, then $d(y, wx_0) < \epsilon/2$.

Therefore, if $|w| \geq k$, then

$$\left(\left(\bigcup_{\substack{w'|_{|w|}=w \\ w'(|w'|)\in H}} w'\widehat{X}_0\right) \cup \left(\bigcup_{\substack{w'|_{|w|}=w \\ w'(|w'|)\in G}} w'\widehat{Y}_0\right)\right) \subseteq B_d\left(wx_0, \frac{\epsilon}{2}\right).$$

For any $j \in \mathbb{N}$, denote by $\overline{W_j}$ the union of all translates of $\widehat{X}_0$ and $\widehat{Y}_0$ by elements of $G * H$ having length no more than $j$, ie

$$\overline{W_j} := \left(\left(\bigcup_{\substack{|w'|\leq j \\ w'(|w'|)\in H}} w'\widehat{X}_0\right) \cup \left(\bigcup_{\substack{|w'|\leq j \\ w'(|w'|)\in G}} w'\widehat{Y}_0\right)\right).$$

Now suppose we have a finite cover $\mathcal{U}$ of $\overline{W_k}$ (where $k$ satisfies $1/2^{k-1} < \epsilon/4$, as earlier) by $(\epsilon/2)$–balls, and let $\mathcal{U}'$ be the finite cover of $\overline{W_k}$ by $\epsilon$–balls obtained by increasing the radius of each element of $\mathcal{U}$ to $\epsilon$.

We claim that $\mathcal{U}'$ covers all of $\overline{W}$:

First, consider a word $w \in G * H$ having $|w| = k$. Since $wx_0 \in w\widehat{X}_0$ or $wx_0 \in w\widehat{Y}_0$, and $|w| = k$, there is some $y \in \overline{W_k}$ such that $wx_0 \in B_d(y, \epsilon/2) \in \mathcal{U}$. Then $B_d(y, \epsilon) \in \mathcal{U}'$,

and by earlier comments, we have

$$\left(\left(\bigcup_{\substack{w'|_{|w|}=w \\ w'(|w'|)\in H}} w'\widehat{X}_0\right)\cup\left(\bigcup_{\substack{w'|_{|w|}=w \\ w'(|w'|)\in G}} w'\widehat{Y}_0\right)\right)\subseteq B_d\left(wx_0,\frac{\epsilon}{2}\right)\subseteq B_d(y,\epsilon).$$

Therefore

$$\left(\left(\bigcup_{w'\in G*H} w'\widehat{X}_0\right)\cup\left(\bigcup_{w'\in G*H} w'\widehat{Y}_0\right)\right)$$
$$=\left(\left(\bigcup_{\substack{w'|_{|w|}=w \\ w'(|w'|)\in H}} w'\widehat{X}_0\right)\cup\left(\bigcup_{\substack{w'|_{|w|}=w \\ w'(|w'|)\in G}} w'\widehat{Y}_0\right)\right)\cup\overline{W_k}$$

is covered by $\mathcal{U}'$.

Moreover, any $\overline{\alpha}\in\mathcal{A}$ also satisfies $d(\overline{\alpha},\alpha_k)=d(\overline{\alpha},w_kx_0)\le\epsilon/4$ by similar calculations. Since $w_kx_0\in w_k\widehat{X}_0$ or $w_kx_0\in w_k\widehat{Y}_0$, and $|w_k|=k$, then, like above, there is some $y\in\overline{W_k}$ such that $B_d(y,\epsilon)\in\mathcal{U}'$ and

$$d(\overline{\alpha},y)\le d(\overline{\alpha},\alpha_k)+d(\alpha_k,y)=d(\overline{\alpha},w_kx_0)+d(w_kx_0,y)<\frac{\epsilon}{4}+\frac{\epsilon}{2}<\epsilon.$$

Therefore $\mathcal{U}'$ covers $\overline{W}$.

We finish the proof of the proposition by constructing a finite cover $\mathcal{U}$ of $\overline{W_k}$ by $(\epsilon/2)$–balls:

Begin with a finite cover $\mathcal{U}_0$ of $\widehat{X}_0\cup\widehat{Y}_0$ by $(\epsilon/2)$–balls; add in finitely many $(\epsilon/2)$–balls centered at points of $\partial X_0$ and $\partial Y_0$ to cover $\partial X_0$ and $\partial Y_0$. Let $\frac{1}{2}>\delta_0>0$ be such that if $d(x,\partial X_0)<3\delta_0$, then $x$ lies in some element of $\mathcal{U}_0$ based at a point of $\partial X_0$, and if $d(y,\partial Y_0)<3\delta_0$, then $y$ lies in an element of $\mathcal{U}_0$ based at a point of $\partial Y_0$. Choose $N>0$ such that $1/2^N<\delta_0\le1/2^{N-1}$.

Now if $g\in G$ satisfies $d(gx_0,\partial X_0)<\delta_0$, then $r(g)\ge N$, so that $\mathrm{diam}_d\,gY_0\le1/2^N$. Similarly, if $d(hy_0,\partial Y_0)<\delta_0$, then $\mathrm{diam}_d\,hX_0\le1/2^N$.

Let $A_1=\{g\in G\mid d(gx_0,\partial X_0)\ge\delta_0\}\cup\{h\in H\mid d(hy_0,\partial Y_0)\ge\delta_0\}$.

Then $A_1$ is finite, and if $g\in G\backslash A_1$, then $d(gx_0,\partial X_0)<\delta_0$, and for any $x\in w\widehat{X}_0$ or $x\in w\widehat{Y}_0$ where $w|_1=g$, we have

$$d(x,gx_0)\le\sum_{n=N}^{\infty}\frac{1}{2^n}=\frac{1}{2^{N-1}}<2\delta_0$$

so that $\qquad d(x,\partial X_0)\le d(x,gx_0)+d(gx_0,\partial X_0)<2\delta_0+\delta_0=3\delta_0$

which implies that $x$ lies in some element of $\mathcal{U}_0$.

Therefore, $\mathcal{U}_0$ is a finite cover of

$$\widehat{X}_0 \cup \widehat{Y}_0 \cup \left( \bigcup_{\substack{w \in G * H \\ w|_1 \notin A_1}} w\widehat{X}_0 \right) \cup \left( \bigcup_{\substack{w \in G * H \\ w|_1 \notin A_1}} w\widehat{Y}_0 \right)$$

by $(\epsilon/2)$–balls.

Now let $\mathcal{U}_1$ be a finite cover of $\left( \bigcup_{g \in A_1} g\widehat{Y}_0 \right) \cup \left( \bigcup_{h \in A_1} h\widehat{X}_0 \right)$ by $(\epsilon/2)$–balls. Use a similar argument to the above to obtain, for each $g \in G \backslash A_1$ a finite subset $A_2^g \subseteq H$ such that if $h \in H \backslash A_2^g$, then $\left( \bigcup_{w|_2 = gh} w\widehat{X}_0 \right) \cup \left( \bigcup_{w|_2 = gh} w\widehat{Y}_0 \right)$ is contained in an element of $\mathcal{U}_1$.

Continue in this manner, letting $\mathcal{U}'_m = \bigcup_{i=0}^m \mathcal{U}_i$ for each $m = 0, \dots, k$.

Then $\mathcal{U}'_m$ covers $\overline{W_m}$ by finitely many $(\epsilon/2)$–balls for each $m = 0, \dots, k$, so that $\mathcal{U} := \mathcal{U}'_k$ is a finite cover of $\overline{W_k}$ by $(\epsilon/2)$–balls, as desired. $\qquad \square$

To see that $\overline{W}$ is an ANR, we will construct for each $\epsilon > 0$ an ANR $Z_\epsilon \subseteq \overline{W}$ which $\epsilon$–dominates $\overline{W}$, and apply Theorem 1.19.

Given $\epsilon > 0$, define

$$Z_\epsilon := \widehat{X}_0 \cup \widehat{Y}_0 \cup \left( \bigcup_{\substack{|w| \leq k \\ w(1) \in A_1 \cap H \\ w(i) \in A_i^{w|_{i-1}}}} w\widehat{X}_0 \right) \cup \left( \bigcup_{\substack{|w| \leq k \\ w(1) \in A_1 \cap G \\ w(i) \in A_i^{w|_{i-1}}}} w\widehat{Y}_0 \right),$$

where $k$, $A_1$, and $A_i^{w|_{i-1}}$ are defined as in the proof of Proposition 2.3. Then $Z_\epsilon$ is a finite connected union of translates of $\widehat{X}_0$ and $\widehat{Y}_0$ with the property that

if $w\widehat{X}_0 \nsubseteq Z_\epsilon$, then $\text{diam}_d\left( \left( \bigcup_{w'|_{|w|} = w} w'\widehat{X}_0 \right) \cup \left( \bigcup_{w'|_{|w|} = w} w'\widehat{Y}_0 \right) \right) < \epsilon$,

if $w\widehat{Y}_0 \nsubseteq Z_\epsilon$, then $\text{diam}_d\left( \left( \bigcup_{w'|_{|w|} = w} w'\widehat{X}_0 \right) \cup \left( \bigcup_{w'|_{|w|} = w} w'\widehat{Y}_0 \right) \right) < \epsilon$.

Let $M_\epsilon$ denote the finite set of words in $G * H$ corresponding to the translates of $\widehat{X}_0$ and $\widehat{Y}_0$ in $Z_\epsilon$, and define a function $m: G * H \to \mathbb{N} \cup \{0\}$ by

$$m(w) := \max \{k \mid w|_k \in M_\epsilon\}.$$

Note that $m(w) = |w|$ if and only if $w\widehat{X}_0 \subseteq Z_\epsilon$ (or $w\widehat{Y}_0 \subseteq Z_\epsilon$).

Define maps $\phi: Z_\epsilon \to \overline{W}$ and $\psi: \overline{W} \to Z_\epsilon$ to be inclusion and "projection" maps, respectively. By "projection," we mean that $\psi|_{Z_\epsilon} \equiv \text{id}_{Z_\epsilon}$, and if, for example, $x \in w\widehat{X}_0$, where $w \notin M_\epsilon$, then $\psi(x) = w|_{m(w)+1} x_0 \in Z_\epsilon$.

**Lemma 2.4** *For any fixed $\epsilon > 0$, let $Z_\epsilon$, $\phi$, and $\psi$ be defined as above. Then there is a homotopy $K \colon \overline{W} \times [0,1] \to \overline{W}$ having the following properties:*

(i) $K$ *is a $(2\epsilon)$–homotopy with $K_0 \equiv \mathrm{id}_{\overline{W}}$ and $K_1 \equiv \phi \circ \psi$.*

(ii) $K_t(\overline{W} \setminus Z_\epsilon) \cap \partial W = \varnothing$ *for all $t > 0$.*

Note that, by [Lemma 1.10](), we may choose homotopies $F \colon \widehat{X} \times [0,1] \to \widehat{X}$ and $J \colon \widehat{Y} \times [0,1] \to \widehat{Y}$ and basepoints $x_0 \in X$, $y_0 \in Y$ such that

$$F_0 \equiv \mathrm{id}_{\widehat{X}}, \qquad\qquad\qquad J_0 \equiv \mathrm{id}_{\widehat{Y}},$$
$$F_t(\widehat{X}) \cap \partial X = J_t(\widehat{Y}) \cap \partial Y = \varnothing \text{ for all } t \in (0,1],$$
$$F_1(\widehat{X}) = \{x_0\}, \qquad\qquad\qquad J_1(\widehat{Y}) = \{y_0\},$$
$$F(x_0, t) = x_0 \text{ for all } t \in [0,1], \quad J(y_0, t) = y_0 \text{ for all } t \in [0,1].$$

Observe also that, by [Lemma 1.11](), we may assume that $F$ and $J$ satisfy (in addition to being $\mathcal{Z}$–set homotopies which are strong deformation retractions)

$$F_t|_{\widehat{X} \setminus B_\rho(\partial X, 1/2^k)} \equiv \mathrm{id}_{\widehat{X} \setminus B_\rho(\partial X, 1/2^k)} \text{ if } t \in [0, 1/2^k]$$

and

$$J_t|_{\widehat{Y} \setminus B_\tau(\partial Y, 1/2^k)} \equiv \mathrm{id}_{\widehat{Y} \setminus B_\tau(\partial Y, 1/2^k)} \text{ if } t \in [0, 1/2^k].$$

These homotopies are used to construct $K$ and also to prove [Proposition 2.8](). We refer the reader to the end of the chapter for the proof of [Lemma 2.4]().

**Proposition 2.5** $\overline{W}$ *is an ANR.*

**Proof** By [Theorem 1.19](), it suffices to show that for every $\epsilon > 0$, there is an ANR which $(2\epsilon)$–dominates $\overline{W}$.

Fix $\epsilon > 0$, and let $Z_\epsilon$ be defined as above.

As a subspace of $\overline{W}$, it is clear that $Z_\epsilon$ is metrizable. That $Z_\epsilon$ is an ANR follows from the fact that translates of $\widehat{X}_0$ and $\widehat{Y}_0$ are glued together along at most one point, and the inductive application of the following theorem:

**Theorem 2.6** (See [8, Section VI.1].) *If $A$, $B$, and $C$ are ANR's, with $A \subseteq B$, and if the adjunction space $Z$ of the map $g \colon A \to C$ is metrizable, then $Z$ is an ANR.*

In this situation, we take as $B$ a finite connected union of translates of $\widehat{X}_0$ and $\widehat{Y}_0$, as $C$ another translate of $\widehat{X}_0$ or $\widehat{Y}_0$ which is to be connected to $B$, and as $A$ the single point in $B$ at which $C$ is to be attached. The map $g \colon A \to C$ is the obvious one, and the adjunction space $Z$ is the disjoint union of $B$ and $C$ modulo the equivalence

relation which identifies the single point in $A$ to its image under $g$. It is clear that the spaces $A$, $B$, and $C$ are ANR's and that $Z$ is metrizable, so the theorem applies.

Lemma 2.4 implies that $Z_\epsilon$ ($2\epsilon$)–dominates $\overline{W}$. Therefore, Theorem 1.19 applies, and $\overline{W}$ is an ANR. □

**Corollary 2.7** $\overline{W}$ *is an ER.*

**Proof** By Proposition 2.5 and Fact 1.4, it suffices to show that $\overline{W}$ is finite dimensional and contractible.

Fix $\epsilon > 0$. Lemma 2.4 shows that $\overline{W}$ is $\epsilon$–dominated by a compact metric space $Z_{\epsilon/2}$ whose dimension is bounded above by the maximum of the dimensions of $\widehat{X}$ and $\widehat{Y}$. We claim that the map $K_1\colon \overline{W} \to Z_{\epsilon/2}$, where $K\colon \overline{W} \times [0, 1] \to \overline{W}$ is the ($\epsilon/2$)–homotopy given by Lemma 2.4, is an $\epsilon$–mapping:

For each $z \in Z_{\epsilon/2}$, either $K_1^{-1}(\{z\}) = \{z\}$ (in which case, it is certainly true that $\mathrm{diam}_d\, K_1^{-1}(\{z\}) = 0 < \epsilon$), or $z = wx_0$ for some $w \in G * H$ satisfying $m(w) = |w| - 1$. In this second case, we have $K_1^{-1}(\{z\}) = \mathcal{B}_w$, where

$$\mathcal{B}_w := \left( \bigcup_{w'|_{|w|} = w} w'\widehat{X}_0 \right) \cup \left( \bigcup_{w'|_{|w|} = w} w'\widehat{Y}_0 \right) \cup \{\{w_i x_0\}_{i=1}^\infty \in \mathcal{A} \mid (w_i)|_{|w|} = w\}$$

consists of all the branches coming off of (and including) $w\widehat{X}_0$ (or $w\widehat{Y}_0$). Then $\mathrm{diam}_d\, K_1^{-1}(\{z\}) = \mathrm{diam}_d\, \mathcal{B}_w \leq \epsilon/2 < \epsilon$ by definition of $Z_{\epsilon/2}$.

Hence we have, for each $\epsilon > 0$, an $\epsilon$–mapping of $\overline{W}$ to a compact metric space $Z_{\epsilon/2}$ with $\dim Z_{\epsilon/2} \leq \max \dim \widehat{X}, \dim \widehat{Y}$. Therefore Theorem 1.21 applies, and $\overline{W}$ is finite-dimensional.

Moreover, $\overline{W}$ is contractible, since it is homotopy equivalent to the contractible $Z_\epsilon$.

Therefore, $\overline{W}$ is an ER. □

**Proposition 2.8** $\partial W$ *is a* $\mathcal{Z}$–*set in* $\overline{W}$.

**Proof** We must construct a homotopy $P\colon \overline{W} \times [0, 1] \to \overline{W}$ with the property that $P_0 \equiv \mathrm{id}_{\overline{W}}$ and $P_t(\overline{W}) \subseteq W$ for all $t > 0$.

Recall that, given any $\epsilon > 0$ and a space $Z_\epsilon \subseteq \overline{W}$ with the property that branches outside of $Z_\epsilon$ have diameter smaller than $\epsilon$, Lemma 2.4 gives a ($2\epsilon$)–homotopy $K\colon \overline{W} \times [0, 1] \to \overline{W}$ which satisfies $K_t(\overline{W} \setminus \partial W) = \varnothing$ for any $t > 0$. This homotopy, of course, depends on both $\epsilon$ and the choice of the space $Z_\epsilon$.

To build the $\mathcal{Z}$–set homotopy $P$, we first fix $\epsilon = 1$ and $Z_\epsilon = \widehat{X}_0 \cup \widehat{Y}_0$. Then we let $K$ be the homotopy given by Lemma 2.4 with these choices in place. Now we have $K_0 \equiv \mathrm{id}_{\overline{W}}$ and $K_1(\overline{W}) \subseteq Z_\epsilon = \widehat{X}_0 \cup \widehat{Y}_0$.

Observe that for each $x \in \overline{W}$, either $x \in \widehat{X}_0 \cup \widehat{Y}_0$ or there exists a unique $g \in G$ (or $h \in H$) such that $x \in \mathcal{B}_g$ (or $x \in \mathcal{B}_h$), where

$$
\mathcal{B}_g := \left( \bigcup_{w'|_1=g} w'\widehat{X}_0 \right) \cup \left( \bigcup_{w'|_1=g} w'\widehat{Y}_0 \right) \cup \{\{w_i x_0\}_{i=1}^\infty \in \mathcal{A} \mid (w_i)|_1 = g\}
$$

and, similarly,

$$
\mathcal{B}_h := \left( \bigcup_{w'|_1=h} w'\widehat{X}_0 \right) \cup \left( \bigcup_{w'|_1=h} w'\widehat{Y}_0 \right) \cup \{\{w_i x_0\}_{i=1}^\infty \in \mathcal{A} \mid (w_i)|_1 = h\}.
$$

Now we define

$$
P(x,t) := \begin{cases}
F(x,t) & \text{for any } t \in [0,1] \text{ if } x \in \widehat{X}_0, \\
J(x,t) & \text{for any } t \in [0,1] \text{ if } x \in \widehat{Y}_0, \\
K(x, 2^{r(h)} \cdot t) & \text{if } x \in \mathcal{B}_h \text{ and } t \in [0, 2^{r(h)}], \\
K(x, 2^{r(g)} \cdot t) & \text{if } x \in \mathcal{B}_g \text{ and } t \in [0, 2^{r(g)}], \\
F(x,t) & \text{if } x \in \mathcal{B}_h \text{ and } t \in [2^{r(h)}, 1], \\
J(x,t) & \text{if } x \in \mathcal{B}_g \text{ and } t \in [2^{r(g)}, 1],
\end{cases}
$$

where $r\colon G \cup H \to [0,1]$ is as defined at the beginning of the chapter.

That $P$ is continuous follows from the pasting lemma for continuous functions, and the properties of $F$, $J$, and $K$ imply that $P$ has the desired attributes. $\qquad\square$

**Theorem 2.9**  *If both $G$ and $H$ admit $\mathcal{Z}$–structures, then so does $G * H$.*

**Proof**  First, it is clear that $G * H$ acts cocompactly on $W$, since if the translates of $C$ and $D$ under the actions of $G$ and $H$ cover $X$ and $Y$, respectively, then the translates of $C \cup D$ under the action of $G * H$ cover $W$. Moreover, given a translate $Z$ of $X_0$ or $Y_0$, each element of $G * H$ either fixes $Z$ (in which case, the action is proper) or moves $Z$ completely off itself, so that the action of $G * H$ on $W$ is also proper. Therefore $(\overline{W}, \partial W)$ satisfies condition (3) of Definition 1.15.

Proposition 2.3 and Proposition 2.8, and Corollary 2.7 prove that conditions (1) and (2) are satisfied by the pair $(\overline{W}, \partial W)$.

It remains only to show that $\overline{W}$ satisfies the null condition with respect to the action of $G * H$ on $W$. This follows directly from the facts each of the original actions have this

property and that for any $\epsilon > 0$, there are only finitely many translates of $\widehat{X}_0$ and $\widehat{Y}_0$ with $d$–diameter more than $\epsilon$. Hence condition (4) of Definition 1.15 is also satisfied.

Therefore $(\overline{W}, \partial W)$ is a $\mathcal{Z}$–structure on $G * H$. □

**Theorem 2.10** *If $G$ and $H$ each admit $\mathcal{E}\mathcal{Z}$–structures, then so does $G * H$.*

**Proof** We show that $(\overline{W}, \partial W)$, as defined in the proof of Theorem 2.9, satisfies the axioms for an $\mathcal{E}\mathcal{Z}$–structure. By Theorem 2.9, it remains only to show that the action of $G * H$ on $W$ extends to an action on $\overline{W}$.

Recall that a point $\alpha \in \partial W$ has one of three types: (i) $\alpha \in w\partial X_0$, (ii) $\alpha \in w\partial Y_0$ or (iii) $\alpha \in \mathcal{A}$, where

$$\mathcal{A} = \big\{ \overline{\alpha} \mid \overline{\alpha} = \{\alpha_i\}_{i=1}^\infty , \ \alpha_i = w_i x_0 , \ w_i \in G * H , \ |w_i| = i , \ w_{i+1}|_i = w_i \ \forall i \in \mathbb{N} \big\}.$$

Under the assumption that the actions of $G$ and $H$ extend to actions on $\widehat{X}_0$ and $\widehat{Y}_0$, the action of $G * H$ on $W$ extends to points of $\partial W$ having type (i) and (ii) in the obvious way.

For a point $\alpha = \{\alpha_i\}_{i=1}^\infty \in \mathcal{A}$, let $w \cdot \alpha := \{w \cdot \alpha_i\}_{i=1}^\infty \in \mathcal{A}$, and the theorem is proved. □

We conclude the chapter with the proof of Lemma 2.4:

**Proof of Lemma 2.4** For a given word $w$, $j(w) := |w| - m(w)$ indicates in some sense how "far" $w\widehat{X}_0$ (or $w\widehat{Y}_0$) is projected by $\psi$.

Recall the function $r \colon G \cup H \to \mathbb{N}$ defined earlier in the chapter by

$$r(g) = n \Longleftrightarrow gx_0 \in B_\rho(\partial X, 1/2^{n-1}) \backslash B_\rho(\partial X, 1/2^n),$$
$$r(h) = n \Longleftrightarrow hy_0 \in B_\tau(\partial Y, 1/2^{n-1}) \backslash B_\tau(\partial Y, 1/2^n).$$

Also recall that $F \colon \widehat{X} \times [0, 1] \to \widehat{X}$ and $J \colon \widehat{Y} \times [0, 1] \to \widehat{X}$ satisfy

$$F_t|_{\widehat{X} \backslash B_\rho(\partial X, 1/2^k)} \equiv \mathrm{id}_{\widehat{X} \backslash B_\rho(\partial X, 1/2^k)} \text{ if } t \in [0, 1/2^k],$$
$$J_t|_{\widehat{Y} \backslash B_\tau(\partial Y, 1/2^k)} \equiv \mathrm{id}_{\widehat{Y} \backslash B_\tau(\partial Y, 1/2^k)} \text{ if } t \in [0, 1/2^k].$$

This implies that each point in $X$ (resp. $Y$) remains fixed under the homotopy $F$ (resp. $J$) on a predetermined interval around $t = 0$. In particular, for any $g \in G$, we have $F(\{gx_0\} \times [0, 1/2^{r(g)}]) = \{gx_0\}$, and similarly for $h \in H$. We use this fact to define a homotopy $K \colon \overline{W} \times [0, 1] \to \overline{W}$ from $\mathrm{id}_{\overline{W}}$ to $\phi \circ \psi$ by concatenating translates of $F$ and $J$ in such a way that two translated homotopies agree when they intersect at a gluing point and the entire "branch" of $\overline{W}$ coming off of any given gluing point is

pulled in by $K$ during the time that the gluing point remains fixed. This systematic concatenation of the $\mathcal{Z}$–set homotopies allows the definition of $K$ to be extended to points of $\mathcal{A}$. Here we give an inductive definition for $K$, and, in hopes of simplifying the ideas used, we give a figure below illustrating an example of its execution on a specific branch of $\overline{W}$.

We first define $K^0\colon Z_\epsilon \times [0,1] \to \overline{W}$ by $K^0(z,t) := z$ for all $(z,t) \in Z_\epsilon \times [0,1]$.

To define $K$ on the rest of $\overline{W}\setminus\mathcal{A}$, first note that $z \in \overline{W}\setminus(\mathcal{A}\cup Z_\epsilon) \implies z \in w\widehat{X}_0$ or $z \in w\widehat{Y}_0$, where $j(w) \in \mathbb{N}$.

To each $w \notin M_\epsilon$ with $j(w) \geq 2$, we associate a number

$$t(w) = \prod_{i=1}^{j(w)-1} \frac{1}{2^{r(w(|w|-(i-1)))}} \in (0,1).$$

(The entire branch coming off the gluing point $wx_0$ will be pulled in by $K$ to $wx_0$ on the interval $[0, t(w)]$.)

Define

$$Q_n := \left( \bigcup_{j(w)=n} w\widehat{X}_0 \right) \cup \left( \bigcup_{j(w)=n} w\widehat{Y}_0 \right) \quad \text{and} \quad Q^n := Z_\epsilon \cup \left( \bigcup_{i=1}^n Q_i \right)$$

for each $n \in \mathbb{N}$.

We will use induction on $n$ to define a homotopy $K^n\colon Q^n \times [0,1] \to \overline{W}$ and set $K := \bigcup_{n=0}^\infty K^n\colon \overline{W}\setminus\mathcal{A}$. Then we will extend $K$ to $\mathcal{A}$ by taking appropriate limits.

First let $K_1\colon Q_1 \times [0,1] \to \overline{W}$ be defined by

$$K_1(z,t) := \begin{cases} wF(z,t) & \text{if } z \in w\widehat{X}_0 \text{ with } j(w) = 1, \\ wJ(z,t) & \text{if } z \in w\widehat{Y}_0 \text{ with } j(w) = 1, \end{cases}$$

and set $K^1 := K^0 \cup K_1\colon Q^1 \times [0,1] \to \overline{W}$.

We show that $K^0$ and $K_1$ agree on the intersection $Z_\epsilon \cap Q_1$ and conclude that $K^1$ is continuous:

Note that $Z_\epsilon \cap Q_1 = \{wx_0 \mid j(w) = 1\}$. For any such $wx_0 \in Z_\epsilon \cap Q_1$, either $w(|w|) \in G$ or $w(|w|) \in H$; assume without loss of generality that $w(|w|) \in G$. Then $wx_0 = wy_0 \in Z_\epsilon \cap w\widehat{Y}_0$, and $K_1(wx_0, t) = K_1(wy_0, t) = wJ(wy_0, t) = wy_0$ for all $t \in [0,1]$ since $J$ is a strong deformation retraction. Thus $K_1(wx_0, t) = K^0(wx_0, t)$ for all $t \in [0,1]$.

Next we define $K_2 \colon Q_2 \times [0,1] \to \overline{W}$ by

$$K_2(z,t) := \begin{cases} wF(z,t/t(w)) & \text{if } z \in w\widehat{X}_0 \text{ with } j(w) = 2 \text{ and } t \in [0, t(w)], \\ K^1(wy_0, t) & \text{if } z \in w\widehat{X}_0 \text{ with } j(w) = 2 \text{ and } t \in [t(w), 1], \\ wJ(z,t) & \text{if } z \in w\widehat{Y}_0 \text{ with } j(w) = 2, \\ K^1(wx_0, t) & \text{if } z \in w\widehat{Y}_0 \text{ with } j(w) = 2 \text{ and } t \in [t(w), 1], \end{cases}$$

and set $K^2 := K^1 \cup K_2 \colon Q^2 \times [0,1] \to \overline{W}$.

We again show that $K^1$ and $K_2$ agree on the intersection $Q^1 \cap Q_2 = \{wx_0 \mid j(w) = 2\}$ to conclude that $K^2$ is continuous:

Given $wx_0$ with $j(w) = 2$, assume without loss of generality that $w(|w|) = g \in G$. Then $wx_0 \in w|_{|w|-1}\widehat{X}_0 \cap w\widehat{Y}_0 \subseteq Q^1 \cap Q_2$.

Then $K^1(wx_0, t) = K_1(wx_0, t) = w|_{|w|-1}F(wx_0, t)$ for all $t \in [0,1]$. Because $F(gx_0, t) = gx_0$ for all $t \leq 1/2^{r(g)}$, we have $K^1(wx_0, t) = w|_{|w|-1}F(wx_0, t) = wx_0$ for all $t \leq 1/2^{r(g)} = t(w)$. On the other hand, $K_2(wx_0, t) = wJ(wx_0, 1/t(w)) = wx_0$ for all $t \leq t(w)$. Moreover, $K^1(wx_0, t) = K_1(wx_0, t) = K_2(wx_0, t)$ for all $t \geq t(w)$, by definition. Therefore $K^1$ and $K_2$ agree on $\{wx_0\} \times [0,1]$ for every $wx_0 \in Q^1 \cap Q_2$, so $K^2$ is continuous.

Lastly, we observe that if $j(w) = 3$, then $K^2(wx_0, t) = wx_0$ for all $t \leq t(w)$:

Suppose $j(w) = 3$; then $wx_0 \in w|_{|w|-1}\widehat{X}_0$ (if $w(|w|) \in G$) or $wx_0 \in w|_{|w|-1}\widehat{Y}_0$ (if $w(|w|) \in H$). Assume, without loss of generality, that $w(|w|) = g \in G$. Then $K^2(wx_0, t) = K_2(wx_0, t) = w|_{|w|-1}F(wx_0, t/t(w|_{|w|-1}))$ for all $t \leq t(w|_{|w|-1})$. Since $F(gx_0, t) = gx_0$ for all $t \leq 1/2^{r(g)}$, then $w|_{|w|-1}F(wx_0, t/t(w|_{|w|-1})) = wx_0$ whenever $t/t(w|_{|w|-1}) \leq 1/2^{r(g)}$, which holds when $t \leq t(w|_{|w|-1}) \cdot (1/2^{r(g)}) = t(w)$.

Continue inductively: Suppose $K^{n-1} \colon Q^{n-1} \times [0,1] \to \overline{W}$ is continuous and satisfies, for each $w \in G * H$ with $j(w) \leq n$,

$$K^{n-1}(wx_0, t) = wx_0 \quad \text{for all } t \leq t(w).$$

Define $K_n \colon Q_n \times [0,1] \to \overline{W}$ by

$$K_n(z,t) := \begin{cases} wF(z,t/t(w)) & \text{if } z \in w\widehat{X}_0 \text{ with } j(w) = n \text{ and } t \in [0, t(w)], \\ K^{n-1}(wy_0, t) & \text{if } z \in w\widehat{X}_0 \text{ with } j(w) = n \text{ and } t \in [t(w), 1], \\ wJ(z,t) & \text{if } z \in w\widehat{Y}_0 \text{ with } j(w) = n, \\ K^{n-1}(wx_0, t) & \text{if } z \in w\widehat{Y}_0 \text{ with } j(w) = n \text{ and } t \in [t(w), 1]. \end{cases}$$

An identical argument to the above, showing that $K^2$ is continuous, shows that $K^n := K^{n-1} \cup K_n \colon Q^n \times [0,1] \to \overline{W}$ is continuous. Moreover, if $j(w) = n+1$, then, assuming

$w(|w|) = g \in G$ and setting $w' := w|_{|w|-1}$, we have $wx_0 \in w'\widehat{X}_0$, and $K^n(wx_0, t) = K_n(wx_0, t) = w'F(wx_0, t/t(w'))$ for any $t \leq t(w')$. But $F(gx_0, t) = gx_0$ for all $t \leq 1/2^{r(g)}$, so that $w'F(wx_0, t/t(w')) = wx_0$ whenever $t/t(w') \leq 1/2^{r(g)}$, which occurs for any $t \leq t(w') \cdot (1/2^{r(g)}) = t(w)$.

**Example** (See Figure 2.) Suppose $j(w) = 4$, and $w = w'ghg'$, where $w'x_0 \in Z_\epsilon$, $r(g') = 1$, $r(h) = 3$, and $r(g) = 2$. Then we have

$$t(w) = \frac{1}{2^{r(g')}} \cdot \frac{1}{2^{r(h)}} \cdot \frac{1}{2^{r(g)}} = \frac{1}{2} \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{1}{64}$$

$$t(w'gh) = \frac{1}{2^{r(h)}} \cdot \frac{1}{2^{r(g)}} = \frac{1}{8} \cdot \frac{1}{4} = \frac{1}{32}$$

$$t(w'g) = \frac{1}{2^{r(g)}} = \frac{1}{4}.$$



Figure 2: The homotopy $K$ on a single branch of $\overline{W}$

We conclude the construction of $K$ by extending to $\mathcal{A}$:

Suppose $z = \{w_i x_0\}_{i=1}^\infty \in \mathcal{A}$. Then $w_i x_0 \to z$ as $i \to \infty$, and we set, for each $t \in [0, 1]$,

$$K(z, t) := \lim_{i \to \infty} K(w_i x_0, t).$$

Continuity of $K: \overline{W} \times [0, 1] \to \overline{W}$ is implied by the induction argument above and the following simple facts:

- $K(w\widehat{X}_0 \times [0, t(w)]) \subseteq w\widehat{X}_0$ for any $w \in G * H$.
- $K(w\widehat{Y}_0 \times [0, t(w)]) \subseteq w\widehat{Y}_0$ for any $w \in G * H$.
- $K(w\widehat{X}_0 \times [t(w|_{|w|-(i-1)}), t(w|_{|w|-i})])$.

$$\subseteq \begin{cases} w|_{|w|-i}\widehat{Y}_0 & \text{if } 1 \le i \le j(w) - 2 \text{ is odd,} \\ w|_{|w|-i}\widehat{X}_0 & \text{if } 1 \le i \le j(w) - 2 \text{ is even.} \end{cases}$$

- $K(w\widehat{Y}_0 \times [t(w|_{|w|-(i-1)}), t(w|_{|w|-i})])$

$$\subseteq \begin{cases} w|_{|w|-i}\widehat{X}_0 & \text{if } 1 \le i \le j(w) - 2 \text{ is odd,} \\ w|_{|w|-i}\widehat{Y}_0 & \text{if } 1 \le i \le j(w) - 2 \text{ is even.} \end{cases}$$

- If $w$ satisfies $|w| > m(w)$ (ie $j(w) > 0$), and

$$\mathcal{B}_w := \left( \bigcup_{w'|_{|w|}=w} w'\widehat{X}_0 \right) \cup \left( \bigcup_{w'|_{|w|}=w} w'\widehat{Y}_0 \right) \cup \{\{w_i x_0\}_{i=1}^\infty \in \mathcal{A} \mid (w_i)|_{|w|} = w\},$$

consists of all the branches coming off of (and including) $w\widehat{X}_0$ (or $w\widehat{Y}_0$), then

$$K(\mathcal{B}_w \times [0, t(w)]) \subseteq \mathcal{B}_w$$

and $\qquad\qquad K(\mathcal{B}_w \times [0, 1]) \subseteq \mathcal{B}_{w|_{m(w)+1}}.$

The properties of $Z_\epsilon$ imply that $\text{diam}_d\, \mathcal{B}_w \le \epsilon < 2\epsilon$ whenever $j(w) > 0$, so that $\text{diam}_d\,(K\,(\{z\} \times [0, 1])) < 2\epsilon$ for any $z \in \overline{W}$.

Hence, $K$ is a $(2\epsilon)$–homotopy between $\text{id}_{\overline{W}}$ and $\phi \circ \psi$.

Moreover, $K_t(\overline{W} \setminus Z_\epsilon) \cap \partial W = \varnothing$ for all $t > 0$ since $F$ and $J$ are $\mathcal{Z}$–set homotopies, and due to the limit definition of $K$ at points of $\mathcal{A}$. $\qquad\square$

# 3 $\quad\mathcal{Z}$– and $\mathcal{EZ}$–structures on direct products of groups

**Fact 3.1** The product $\widehat{X} \times \widehat{Y}$ of $\mathcal{Z}$–compactifications $\widehat{X}$ and $\widehat{Y}$ of $X$ and $Y$, respectively, is a $\mathcal{Z}$–compactification of $X \times Y$.

**Proof** It is a standard fact that a product of ANR's is an ANR. Thus $\widehat{X} \times \widehat{Y}$ is an ANR.

Suppose $F\colon \widehat{X} \times [0, 1] \to \widehat{X}$ and $G\colon \widehat{Y} \times [0, 1] \to \widehat{Y}$ are $\mathcal{Z}$–set homotopies. Define $H\colon \widehat{X} \times \widehat{Y} \times [0, 1] \to \widehat{X} \times \widehat{Y}$ by $H(\hat{x}, \hat{y}, t) := (F(\hat{x}, t), G(\hat{y}, t))$ for all $(\hat{x}, \hat{y}, t) \in \widehat{X} \times \widehat{Y} \times [0, 1]$. Since $F_0 \equiv \text{id}_{\widehat{X}}$ and $G_0 \equiv \text{id}_{\widehat{Y}}$, we have $H_0 \equiv \text{id}_{\widehat{X} \times \widehat{Y}}$. Moreover, $F_t(\widehat{X}) \subseteq X$ and $G_t(\widehat{Y}) \subseteq Y$ for any $t > 0$ implies $H_t(\widehat{X} \times \widehat{Y}) \subseteq X \times Y$ whenever $t > 0$. Hence $H_t(\widehat{X} \times \widehat{Y}) \cap ((\partial X \times Y) \cup (X \times \partial Y)) = \varnothing$ for any $t > 0$.

Therefore, $(\partial X \times Y) \cup (X \times \partial Y))$ is a $\mathcal{Z}$–set in $\widehat{X} \times \widehat{Y}$. $\qquad\square$

Unfortunately, the analogous result for $\mathcal{Z}$–structures does not hold: Suppose $(\hat{X}, \partial X)$ and $(\hat{Y}, \partial Y)$ are $\mathcal{Z}$–structures on $G$ and $H$, respectively. Although, by Fact 3.1, $\hat{X} \times \hat{Y}$ is a $\mathcal{Z}$–compactification of $X \times Y$, the space $\hat{X} \times \hat{Y}$ does not, in general, satisfy the null condition with respect to the action of $G \times H$ on $X \times Y$.
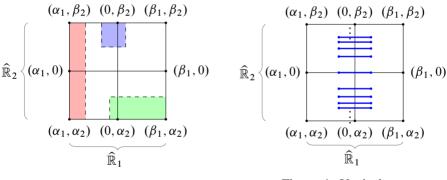
**Example 3.2** Let $\hat{\mathbb{R}}$ denote the $\mathcal{Z}$–compactification of the real line $\mathbb{R}$ by two points, and consider the $\mathcal{Z}$–compactification $\hat{\mathbb{R}} \times \hat{\mathbb{R}}$ of $\mathbb{R}^2$, the Euclidean plane.

Observe that $\hat{\mathbb{R}}$ is a $\mathcal{Z}$–structure on $\mathbb{Z}$, but, with the product topology, $\hat{\mathbb{R}} \times \hat{\mathbb{R}}$ is not a $\mathcal{Z}$–structure on $\mathbb{Z} \times \mathbb{Z}$:
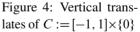
For $i = 1, 2$, let $\mathbb{R}_i = \mathbb{R} = (-\infty, \infty)$, and $\hat{\mathbb{R}}_i := \{\alpha_i\} \cup (-\infty, \infty) \cup \{\beta_i\}$. The set $\mathcal{B}_i = \{(a, b) \mid a, b \in \mathbb{R}_i\} \cup \{\{\alpha_i\} \cup (-\infty, a) \mid a \in \mathbb{R}_i\} \cup \{(b, \infty) \cup \{\beta_i\} \mid b \in \mathbb{R}_i\}$ is a basis for the topology on $\hat{\mathbb{R}}_i$.

Now $\widehat{\mathbb{R}^2} := \hat{\mathbb{R}}_1 \times \hat{\mathbb{R}}_2$, with the product topology. Note in Figure 3 some typical neighborhoods of boundary points in $\widehat{\mathbb{R}^2}$.

Now consider the compact subset $C := [-1, 1] \times \{0\}$ of $\mathbb{R}_1 \times \mathbb{R}_2$. Then for any $n \in \mathbb{Z}$, $(0, n) \cdot C = [-1, 1] \times \{n\}$. (See Figure 4.)



Figure 3: Neighborhoods of boundary points in $\widehat{\mathbb{R}^2}$

Figure 4: Vertical translates of $C := [-1, 1] \times \{0\}$

Let $\mathcal{U} := \{U_0, U_1, U_2, U_3\}$, where

$$U_0 := (-\tfrac{3}{4}, \tfrac{3}{4}) \times ((-\tfrac{1}{2}, \infty) \times \{\beta_2\}), \quad U_1 := (-\tfrac{3}{4}, \tfrac{3}{4}) \times (\{\alpha_2\} \cup (-\infty, \tfrac{1}{2}))$$
$$U_2 := (\{\alpha_1\} \cup (-\infty, -\tfrac{1}{2})) \times \hat{\mathbb{R}}_2, \quad U_3 := ((\tfrac{1}{2}, \infty) \cup \{\beta_1\}) \times \hat{\mathbb{R}}_2.$$

Then, as displayed in Figure 5, $\mathcal{U}$ is an open cover of $\widehat{\mathbb{R}^2}$, but $(0, n) \cdot C$ is not contained in any $U_i$ for any $n \in \mathbb{Z}$.

Therefore $\widehat{\mathbb{R}^2}$ does not satisfy the null condition with respect to the action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{R} \times \mathbb{R}$.
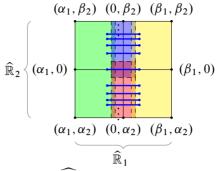
Figure 5: An open cover of $\widehat{\mathbb{R}^2}$ whose elements contain no vertical translate of $C$

**Example 3.3** Now consider the $\mathcal{Z}$–compactification $\widehat{\mathbb{R}^{2\prime}}$ of $\mathbb{R}^2$ obtained instead by adjoining to $\mathbb{R}^2$ a circle $Z := [0, 2\pi]/\sim$, where $0 \sim 2\pi$ and having as basis for its topology

$$\mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_\partial$$

where $\mathcal{B}_0$ contains the standard open sets of $\mathbb{R}^2$, and $\mathcal{B}_\partial$ contains all sets of the form $B(z, R, \epsilon)$, where, for each $z \in Z, R > 0, \epsilon > 0$,

$$B(z, R, \epsilon) := \big\{ (r, \theta) \in \mathbb{R}^2 \mid r > R, \; |\theta - z| < \epsilon \big\} \cup \big\{ z' \in Z \mid |z - z'| < \epsilon \big\}$$

Note in Figure 6 some examples of typical neighborhoods of boundary points in $\widehat{\mathbb{R}^{2\prime}}$.
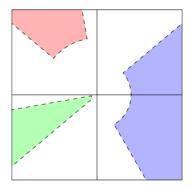


Figure 6: Neighborhoods of boundary points in $\widehat{\mathbb{R}^{2\prime}}$

Now the variation in angles achieved by the translates of a given compactum in $\mathbb{R}^2$ shrinks as the compactum is pushed by the elements of $\mathbb{Z} \times \mathbb{Z}$ outside of metric balls of larger and larger radius. Figure 7 illustrates, for example, that all but finitely many translates $(0, n) \cdot C$ ($n \geq 0$) of the compactum $C = [-1, 1] \times \{0\}$ fall into $B(\pi/2, R, \epsilon)$, no matter how small $\epsilon$ is chosen.
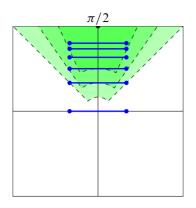
Figure 7: Translates of $C$ eventually fit into small neighborhoods of $\pi/2$

Therefore $\widehat{\mathbb{R}^{2\prime}}$ satisfies the null condition with respect to the action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{R}^2$.

Perhaps the following depiction of this example is more appropriate in this paper, as its essence is analogous to the techniques used to prove Theorem 3.21:

It is not difficult to see that the set $Z$ as described above is homeomorphic to $\{\alpha_1, \beta_1\} * \{\alpha_2, \beta_2\}$, where $*$ indicates a join. In other words,

$$Z \approx \{\alpha_1, \beta_1\} \times \{\alpha_2, \beta_2\} \times [0, \infty]/\sim \, ,$$

where $\sim$ is the equivalence relation given by $(\alpha_1, \alpha_2, 0) \sim (\alpha_1, \beta_2, 0)$, $(\beta_1, \alpha_2, 0) \sim (\beta_1, \beta_2, 0)$, $(\alpha_1, \alpha_2, \infty) \sim (\beta_1, \alpha_2, \infty)$, and $(\alpha_1, \beta_2, \infty) \sim (\beta_1, \beta_2, \infty)$.

To each point of $\mathbb{R}^2$ we may assign a "slope" (say $(x, y) \longmapsto |y/x|$) and, in this example, points in $Z$ can be pulled in to $\mathbb{R}^2$ via a homotopy which keeps the slope coordinate constant.

This is essentially the kind of structure we impose on $X \times Y$ when $(\widehat{X}, \partial X)$ and $(\widehat{Y}, \partial Y)$ are $\mathcal{Z}$–structures on $G$ and $H$, respectively, to prove Theorem 3.21.

Suppose from this point forward that $(\widehat{X}, \partial X)$ and $(\widehat{Y}, \partial Y)$ are $\mathcal{Z}$–structures on $G$ and $H$, respectively. We will denote by $\bar{x}$ (respectively, $\bar{y}$) a point in $\partial X$ (respectively, $\partial Y$), and by $\hat{x}$ (respectively, $\hat{y}$) a general point of $\widehat{X}$ (respectively, $\widehat{Y}$).

Since $\partial X$ and $\partial Y$ are $\mathcal{Z}$–sets in $\widehat{X}$ and $\widehat{Y}$, respectively, there exist homotopies $\alpha\colon \widehat{X} \times [0, 1] \to \widehat{X}$ and $\beta\colon \widehat{Y} \times [0, 1] \to \widehat{Y}$ such that $\alpha_0 \equiv \mathrm{id}_{\widehat{X}}$, $\beta_0 \equiv \mathrm{id}_{\widehat{Y}}$, $\alpha_t(\widehat{X}) \subseteq X$ for all $t \in (0, 1)$, and $\beta_t(\widehat{Y}) \subseteq Y$ for all $t \in (0, 1)$. By Lemma 1.10, we may assume in addition that $\alpha$ and $\beta$ are strong deformation retractions to base points $x_0 \in X$ and $y_0 \in Y$, so that $\alpha_1(\widehat{X}) = \{x_0\}$, and $\beta_1(\widehat{Y}) = \{y_0\}$.

**Definition 3.4** A metric $d\colon X \times X \to [0, \infty)$ is *proper* if every closed metric ball in $X$ is compact. A map $f\colon X \to Y$ is *proper* if for every compact $C \subseteq Y$, the preimage $f^{-1}(C)$ is compact in $X$.

**Lemma 3.5** *There exists a proper metric $d$ on $X$.*

**Proof** First, note that, since $\hat{X}$ is metrizable (it is an ER), we may choose a metric $\hat{d}$ on $\hat{X}$. Then $\hat{D}\colon (\hat{X} \times [0, \infty)) \times (\hat{X} \times [0, \infty)) \to [0, \infty)$ defined by

$$\hat{D}((x_1, t_1), (x_2, t_2)) := \sqrt{(\hat{d}(x_1, x_2))^2 + |t_1 - t_2|^2}$$

is a proper metric on $\hat{X} \times [0, \infty)$.

Let $f\colon \hat{X} \to [0, 1]$ be a continuous function satisfying $f(x_0) = 0$, $f(\partial X) = \{1\}$, and $f(x) \in (0, 1)$ if $x \in X \setminus \{x_0\}$.

Let $h\colon [0, \infty) \to [0, 1)$ be a homeomorphism, and consider the graph

$$G := \{(x, f(h(x))) \mid x \in X\} \subseteq \hat{X} \times [0, \infty)$$

of $f \circ h$. Since $\hat{X} \times [0, \infty)$ is a proper metric space and $g\colon X \to \hat{X} \times [0, \infty)$ with $g(x) = (x, f(h(x)))$ is a proper embedding of $X$ in $\hat{X} \times [0, \infty)$, then $X$ inherits a proper metric $d$ from $\hat{X} \times [0, \infty)$. $\qquad \square$

From now on, we will assume that $(X, \rho)$ and $(Y, \tau)$ are proper metric spaces, and that $\bar{\rho}$ and $\bar{\tau}$ are metrics on $\hat{X}$ and $\hat{Y}$, respectively.

**Lemma 3.6** *There exists a proper map $p\colon X \to [0, \infty)$ having the following properties:*

(i) *The variation of $p$ over translates of a given compactum in $X$ is bounded, ie*

(†) $\qquad R_p(C) := \sup \{\max \{p(x) - p(x') \mid x, x' \in gC\} \mid g \in G\} < \infty$

*for any compactum $C$ in $X$.*

(ii) *For some sequence $1 = t_0 > t_1 > t_2 > \cdots > 0$, we have*

(††) $\qquad p(\alpha(\partial X \times [t_i, t_{i-1}))) \subseteq (i - 1, i + 1]$.

**Proof** Let $t_0 := 1$. Let $C_1$ be a connected compact subset of $X$ containing $x_0$ with the property that the translates of $C_1$ cover $X$, ie $\bigcup_{g \in G} gC_1 = X$. Since $C_1$ is compact, there exists $r_1 > 0$ such that $B_\rho(x_0, r_1) \supseteq C_1$.

Let $t_1 \in (0, 1)$ be such that $\alpha(\partial X \times [0, t_1)) \cap \overline{B_\rho(x_0, r_1)} = \varnothing$, and choose $r_2'$ so that $B_\rho(x_0, r_2') \supseteq \alpha(\partial X \times [t_1, 1])$.

Choose $r_2$ such that

$$B_\rho(x_0, r_2) \supseteq \overline{B_\rho(x_0, r_2')} \cup \left(\bigcup\{gC_1 \mid gC_1 \cap B_\rho(x_0, r_1) \neq \varnothing\}\right)$$

and $t_2 \in (0, 1)$ such that $\alpha(\partial X \times [0, t_2)) \cap \overline{B_\rho(x_0, r_2)} = \varnothing$.

Continue inductively.

For each i, let $r_i' > 0$ satisfy $B_\rho(x_0, r_i') \supseteq \alpha(\partial X \times [t_{i-1}, 1])$. Then choose $r_i > 0$ so that

$$(\star) \qquad B_\rho(x_0, r_i) \supseteq \overline{B_\rho(x_0, r_i')} \cup \left(\cup \{gC_1 \mid gC_1 \cap B_\rho(x_0, r_{i-1}) \neq \varnothing\}\right)$$

and $t_i \in (0, 1)$ such that

$$(\star\star) \qquad\qquad\qquad \alpha(\partial X \times [0, t_i)) \cap \overline{B_\rho(x_0, r_i)} = \varnothing.$$

We have $0 < r_1 < r_2 < \cdots$ with $r_i \to \infty$ as $i \to \infty$, so that $X = \bigcup_{i=1}^\infty B_\rho(x_0, r_i)$.

Moreover, we have $1 = t_0 > t_1 > t_2 > \cdots > 0$ with $t_i \to 0$ as $i \to \infty$.

Define $p \colon X \to [0, \infty)$ to be a piecewise rescaling of the map $\rho(\cdot, x_0)$ measuring distance to the point $x_0$ in such a way that $p(x_0) = 0$, and $p(\overline{B_\rho(x_0, r_i)} - B_\rho(x_0, r_{i-1})) = [i - 1, i]$. Since $(X, \rho)$ is a proper metric space, it is clear that $p$ is a proper map.

**Claim 3.7** *The map* $p \colon X \to [0, \infty)$ *satisfies* (†).

**Proof** First we note that (†) holds for $C_1$:

For each $g \in G$, let $i_g := \min\{k \in \mathbb{N} \mid gC_1 \cap B_\rho(x_0, r_k) \neq \varnothing\}$. Then, by definition, $p(gC_1) \subseteq [i_g - 1, i_g + 1]$. Thus

$$\max\{p(x) - p(x') \mid x, x' \in gC_1\} \leq 2 \quad \text{for every } g \in G,$$

so $R_p(C_1) \leq 2$.

Now consider any compactum $C$ in $X$. We may assume, without loss of generality, that $x_0 \in C$. Since $C$ is compact, it is contained in a metric ball in $X$, so there is a minimal finite collection $\{g_1, g_2, \ldots, g_{k_C}\} \subseteq G$ so that $\{g_1 C_1, g_2 C_1, \ldots, g_{k_C} C_1\}$ covers $C$ and $\bigcup_{i=1}^{k_C} g_i C_1$ is connected. Then, in fact, given any $g \in G$, $\bigcup_{i=1}^{k_C} g g_i C_1$ is connected and contains $gC$. Therefore, by connectedness and a simple inductive argument,

$$\max\{p(x) - p(x') \mid x, x' \in gC\} \leq 2k_C \quad \text{for all } g \in G.$$

Hence $R_p(C) \leq 2k_C < \infty$ for any compactum $C$ in $X$.

By Claim 3.7, we have constructed a proper map $p\colon X \to [0,\infty)$ satisfying (†) for any compactum $C$ in $X$.

Moreover, ($\star$) and ($\star\star$) guarantee that (††) is satisfied by the constructed $p$. □

Certainly we may define, using the same methods, a proper map $q\colon Y \to [0,\infty)$ satisfying conditions analogous to (†) and (††).

**Lemma 3.8** *There are reparametrizations $\widehat{\alpha}$ and $\widehat{\beta}$ of the homotopies $\alpha$ and $\beta$ so that $p(\widehat{\alpha}(\overline{x},t)) \in [1/t - 1, 1/t + 2]$ and $q(\widehat{\beta}(\overline{y},t)) \in [1/t - 1, 1/t + 2]$ for all $t \in (0,1]$, $\overline{x} \in \partial X$, $\overline{y} \in \partial Y$.*

**Proof** Note that, using the notation from Lemma 3.6, we have, for any $t \in (0,1)$, some $i \in \mathbb{N}$ so that $t_i \le t < t_{i-1}$. Condition (††) gives, then, that $p(\alpha(\partial X \times \{t\})) \subseteq (i-1, i+1]$.

Let $\xi\colon [0,1] \to [0,1]$ be the piecewise linear homeomorphism satisfying $\xi(0) = 0$, $\xi(1) = 1$, and $\xi(1/i) = t_{i-1}$ for all $i \in \mathbb{N}$, and define $\widehat{\alpha}\colon \widehat{X} \times [0,1] \to \widehat{X}$ by $\widehat{\alpha}(\widehat{x},t) := \alpha(\widehat{x}, \xi(t))$ for all $\widehat{x} \in \widehat{X}$ and all $t \in [0,1]$.

Now we have arranged that, given $t \in [0,1]$ and $i \in \mathbb{N}$ such that $t \in [1/i, 1/(i-1))$, $\xi(t) \in [t_i, t_{i-1})$, so

$$p(\widehat{\alpha}(\overline{x},t)) = p(\alpha(\overline{x},\xi(t))) \in (i-1, i+1] \subseteq (1/t - 1, 1/t + 2]$$

for any $\overline{x} \in \partial X$.

Moreover, $p(\widehat{\alpha}(\overline{x},1)) = p(\alpha(\overline{x},1)) = p(x_0) = 0 \in [0,3]$ for any $\overline{x} \in \partial X$, so $\widehat{\alpha}$ satisfies the requirement at $t = 1$.

Define $\widehat{\beta}$ similarly, and the result holds. □

**Definition 3.9** We define $\widehat{X \times Y}$ as follows:

The *join* $\partial X * \partial Y$ of the boundaries $\partial X$ and $\partial Y$ is

$$\partial X * \partial Y = \partial X \times \partial Y \times [0,\infty]/{\sim} ,$$

where $\sim$ is the equivalence relation generated by $\langle \overline{x}, \overline{y}, \mu \rangle \sim \langle \overline{x}', \overline{y}', \mu' \rangle$ if and only if ($\mu = \mu' = 0$ and $\overline{x} = \overline{x}'$) or ($\mu = \mu' = \infty$ and $\overline{y} = \overline{y}'$).

We will denote by $\langle \overline{x}, 0 \rangle$ the equivalence class under $\sim$ containing $\langle \overline{x}, \overline{y}, 0 \rangle$ for all $\overline{y} \in \partial Y$, and by $\langle \overline{y}, \infty \rangle$ the equivalence class containing $\langle \overline{x}, \overline{y}, \infty \rangle$ for all $\overline{x} \in \partial X$.

Now we define a slope function $\mu\colon X \times Y \to [0,\infty]$ by

$$\mu(x,y) = \begin{cases} q(y)/p(x) & \text{if } p(x) \neq 0, \\ \infty & \text{if } p(x) = 0. \end{cases}$$

As a set, $\widehat{X \times Y} := (X \times Y) \cup (\partial X * \partial Y)$.

The topology on $\widehat{X \times Y}$ is generated by the basis $\mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_\partial$, where

$$\mathcal{B}_0 := \{U \times V \mid U \text{ is open in } X, V \text{ is open in } Y\},$$
$$\mathcal{B}_\partial := \{U(z,\epsilon) \mid z \in \partial X * \partial Y, \epsilon > 0\}.$$

The neighborhoods $U(z,\epsilon)$ of boundary points are defined by:

Given $\overline{x} \in \partial X$ and $\epsilon > 0$,

$$U(\langle \overline{x}, 0 \rangle, \epsilon) := \{(x,y) \mid \overline{\rho}(x,\overline{x}), \mu(x,y) < \epsilon\}$$
$$\cup \left\{\langle \overline{x}', \overline{y}', \mu' \rangle \mid \overline{\rho}(\overline{x},\overline{x}'), \mu' < \epsilon\right\} \cup \left\{\langle \overline{x}', 0 \rangle \mid \overline{\rho}(\overline{x},\overline{x}') < \epsilon\right\}.$$

For $\overline{y} \in \partial Y$ and $\epsilon > 0$,

$$U(\langle \overline{y}, \infty \rangle, \epsilon) := \{(x,y) \mid \overline{\tau}(y,\overline{y}), 1/\mu(x,y) < \epsilon\}$$
$$\cup \left\{\langle \overline{x}', \overline{y}', \mu' \rangle \mid \overline{\tau}(\overline{y},\overline{y}'), 1/\mu' < \epsilon\right\} \cup \left\{\langle \overline{y}', \infty \rangle \mid \overline{\tau}(\overline{y},\overline{y}') < \epsilon\right\}.$$

For $\langle \overline{x}, \overline{y}, \mu \rangle \in \partial X \times \partial Y \times (0,\infty)$ and $\epsilon < \mu$,

$$U(\langle \overline{x}, \overline{y}, \mu \rangle, \epsilon) := \{(x,y) \mid \overline{\rho}(x,\overline{x}), \overline{\tau}(y,\overline{y}), |\mu(x,y) - \mu| < \epsilon\}$$
$$\cup \left\{\langle \overline{x}', \overline{y}', \mu' \rangle \mid \overline{\rho}(\overline{x},\overline{x}'), \overline{\tau}(\overline{y},\overline{y}'), |\mu - \mu'| < \epsilon\right\}.$$

**Note**  Recall that $\overline{\rho}$ and $\overline{\tau}$ are metrics on the compactifications $\widehat{X}$ and $\widehat{Y}$, respectively.

**Proposition 3.10**  $\widehat{X \times Y}$ *is a compactification of* $X \times Y$.

**Proof**  We first observe that the topology inherited by $X \times Y$ as a subspace of $\widehat{X \times Y}$ is the same as the original topology on $X \times Y$.

It remains to show that $\widehat{X \times Y}$ is compact.

Let $\mathcal{U}$ be an open cover of $\widehat{X \times Y}$ by basic open sets. Since $\partial X * \partial Y$ is compact, we may choose a finite subset $\mathcal{U}_\partial = \{U_i\}_{i=1}^k$ of $\mathcal{U}$ which covers $\partial X * \partial Y$.

**Claim 3.11**  *There exists* $1 > \delta > 0$ *such that for every* $z \in \partial X * \partial Y$ *there is some* $i \in \{1,\ldots,k\}$ *such that* $U(z,\delta) \subseteq U_i$.

**Proof** For each $i = 1, \ldots, k$, define $\eta_i \colon \partial X * \partial Y \to [0, \infty)$ by

$$\eta_i(z) := \begin{cases} 0 & \text{if } z \notin U_i, \\ \sup\{\eta > 0 \mid U(z, \eta) \subseteq U_i\} & \text{if } z \in U_i. \end{cases}$$

Then each $\eta_i$ is a continuous function, and for each $z \in \partial X * \partial Y$, there is some $i \in \{1, \ldots, k\}$ so that $\eta_i(z) > 0$.

Now $\eta := \max\{\eta_i \mid i = 1, \ldots, k\}$ is a continuous and strictly positive function on the compact set $\partial X * \partial Y$, so

$$\delta := \min\{\delta', 1/2\},$$

where
$$\delta' := \min\{\eta(z) \mid z \in \partial X * \partial Y\}$$

is a positive number which satisfies the desired condition. □

We will show that there is a compactum $C \subseteq X \times Y$ such that if $(x, y) \notin C$, then $(x, y) \in U(z, \delta)$ for some $z \in \partial X * \partial Y$.

**Claim 3.12** *Given a compactum $J \subseteq X$, there is a compactum $P_J \subseteq Y$ such that if $(x, y) \in J \times (Y \setminus P_J)$ then $(x, y) \in U(\langle \bar{y}, \infty \rangle, \delta)$ for some $\bar{y} \in \partial Y$.*

**Proof** Let $M_J := \max\{p(x) \mid x \in J\}$, and choose $P_J$ sufficiently large so that if $y \notin P_J$, then $\bar{\tau}(y, \partial Y) < \delta$ and $q(y) > M_J \cdot (1/\delta)$.

Then if $(x, y) \in J \times (Y \setminus P_J)$, there is some $\bar{y} \in \partial Y$ such that $\bar{\tau}(y, \bar{y}) < \delta$, and $\mu(x, y) = q(y)/p(x) > 1/\delta$, so $(x, y) \in U(\langle \bar{y}, \infty \rangle, \delta)$. □

Similarly, we have:

**Claim 3.13** *Given a compactum $K \subseteq Y$, there is a compactum $Q_K \subseteq X$ such that if $(x, y) \in (X \setminus Q_K) \times K$, then $(x, y) \in U(\langle \bar{x}, 0 \rangle, \delta)$ for some $\bar{x} \in \partial X$.*

We define
$$C := (C_X \times P_{C_X}) \cup (Q_{C_Y} \times C_Y),$$

where
$$C_X := \hat{X} \setminus B_{\bar{\rho}}(\partial X, \delta) \quad \text{and} \quad C_Y := \hat{Y} \setminus B_{\bar{\tau}}(\partial Y, \delta).$$

Now suppose $(x, y) \in (X \times Y) \setminus C$. If $x \in C_X$ or $y \in C_Y$, then Claim 3.12 or 3.13 gives the result. Otherwise we have $x \notin C_X$ and $y \notin C_Y$, which implies that there are $\bar{x} \in \partial X$ and $\bar{y} \in \partial Y$ such that $\bar{\rho}(x, \bar{x}), \bar{\tau}(y, \bar{y}) < \delta$. Therefore $(x, y) \in U(\langle \bar{x}, \bar{y}, \mu(x, y) \rangle, \delta)$, and the proposition is proved. □

Let us clarify here a future abuse of notation: by $\partial X \subseteq \partial X * \partial Y$ (respectively $\partial Y \subseteq \partial X * \partial Y$), we mean the homeomorphic copy $\partial X \times \partial Y \times \{0\}/\sim$ (respectively $\partial X \times \partial Y \times \{\infty\}/\sim$) of $\partial X$ (respectively $\partial Y$) in $\partial X * \partial Y$.

**Proposition 3.14** $\widehat{X \times Y}$ satisfies the null condition with respect to the action of $G \times H$ on $X \times Y$.

**Proof** Consider a compactum $C \times D$ in $X \times Y$, where $C$ is compact in $X$ and $D$ is compact in $Y$. Let $\mathcal{U}$ be an open cover of $\widehat{X \times Y}$ by basic open sets. We may assume, without loss of generality, that $\mathcal{U}$ is finite, since $\widehat{X \times Y}$ is compact.

Let $\mathcal{U}_\partial = \{U_i\}_{i=1}^k$ denote the finite subset of $\mathcal{U}$ which covers $\partial X * \partial Y$.

Choose $1 > \delta > 0$ as in [Claim 3.11](#).

**Notation** Let $\text{diam}_\mu(A) := \sup \{\mu(x, y) - \mu(x', y') \mid (x, y), (x', y') \in A\}$ for any $A \subseteq X \times Y$.

We also denote by $W$ the set of points $\langle \bar{x}, \bar{y}, \mu \rangle$ in $\partial X * \partial Y$ with $0 < \mu < \infty$.

**Claim 3.15** Suppose $(gC \times hD) \cap U(w, \delta/2) \neq \varnothing$ for some $w \in W$ and $\text{diam}_{\bar{\rho}} gC$, $\text{diam}_{\bar{\tau}} hD$, $\text{diam}_\mu(gC \times hD) < \delta/2$. Then there is some $i \in \{1, \ldots, k\}$ so that $gC \times hD \subseteq U_i$.

**Proof** This claim follows easily from simple calculations using the triangle inequality and the definition of $U(w, \epsilon)$. $\qquad\square$

Next, define

$$R_p := \sup \left\{ \max \left\{ p(x) - p(x') \mid x, x' \in gC \right\} \mid g \in G \right\}$$
$$R_q := \sup \left\{ \max \left\{ q(y) - q(y') \mid y, y' \in hD \right\} \mid h \in H \right\}$$

Note that by [Lemma 3.6](#), both $R_p$ and $R_q$ are finite.

**Claim 3.16** Given a compactum $J \subseteq X$, there is a compactum $P_J \subseteq Y$ so that if $(gC \times hD) \cap (J \times P_J) = \varnothing$, but $gC \cap J \neq \varnothing$, then there is some $\bar{y} \in \partial Y$ such that $gC \times hD \subseteq U(\langle \bar{y}, \infty \rangle, \delta)$.

**Proof** Since the action of $G$ on $X$ is proper, then $|\{g \in G \mid gC \cap J \neq \varnothing\}| < \infty$, so $M_J := \max \{p(x) \mid x \in gC, \ gC \cap J \neq \varnothing\} < \infty$ since $C$ is compact.

Choose $P_J$ sufficiently large so that if $hD \nsubseteq P_J$, then

(1)
$$q(y) > M_J \cdot \frac{1}{\delta} \quad \forall y \in hD$$

(2)
$$\overline{\tau}(hD, \partial Y) < \frac{\delta}{2}$$

(3)
$$\mathrm{diam}_{\overline{\tau}} hD < \frac{\delta}{2}.$$

Note that (1) can be achieved by the properness of the function $q$, (2) by cocompactness of the action of $H$ on $Y$, and (3) by the fact that $\widehat{Y}$ satisfies the null condition with respect to the action of $H$ on $Y$.

Now if $(gC \times hD) \cap (J \times P_J) = \varnothing$ and $gC \cap J \neq \varnothing$, then $hD \nsubseteq P_J$, so by (2) and (3), we have $hD \subseteq B_{\overline{\tau}}(\overline{y}, \delta)$ for some $\overline{y} \in \partial Y$.

Therefore for any $(x, y) \in gC \times hD$, we have

$$\overline{\tau}(y, \overline{y}) < \delta$$

$$\mu(x, y) = \frac{q(y)}{p(x)} > \frac{M_J \cdot (1/\delta)}{M_J} = \frac{1}{\delta}.$$

Hence, $gC \times hD \subseteq U(\langle \overline{y}, \infty \rangle, \delta)$, and the claim is proved. $\qquad \square$

Clearly we may use analogous techniques to obtain:

**Claim 3.17** *Given a compactum $K \subseteq Y$, there is a compactum $Q_K \subseteq X$ so that if $(gC \times hD) \cap (K \times Q_K) = \varnothing$, but $hD \cap K \neq \varnothing$, then there is some $\overline{x} \in \partial X$ such that $gC \times hD \subseteq U(\langle \overline{x}, 0 \rangle, \delta)$.*

Now choose a compact subset $J \subseteq X$ containing $x_0$ such that if $gC \nsubseteq J$, then

$$\mathrm{diam}_{\overline{\rho}} gC < \frac{\delta}{4}$$

$$p(x) > \frac{4}{\delta}\left(R_q + \frac{1}{\delta} \cdot R_p\right) \quad \forall x \in gC$$

$$\overline{\rho}(gC, \partial X) < \frac{\delta}{4}$$

and a compact subset $K \subseteq Y$ containing $y_0$ such that if $hD \nsubseteq K$, then

$$\mathrm{diam}_{\overline{\tau}} hD < \frac{\delta}{4}$$

$$\overline{\tau}(hD, \partial Y) < \frac{\delta}{4}.$$

Let $P_J \subseteq Y$ and $Q_K \subseteq X$ be as in Claims 3.16 and 3.17, respectively.

**Claim 3.18** *If* $(gC \times hD) \cap [(J \times P_J) \cup (Q_K \times K)] = \varnothing$, *then* $gC \times hD$ *is contained in a single element of* $\mathcal{U}$.

**Proof** By Claims 3.16, 3.17, and the choice of $\delta$, if $gC \cap J \neq \varnothing$ or $hD \cap K \neq \varnothing$, then we are done.

Assume that $gC \cap J = hD \cap K = \varnothing$. Then $\operatorname{diam}_{\overline{\rho}} gC, \operatorname{diam}_{\overline{\tau}} hD < \delta/4$, and there exist $(\overline{x}, \overline{y}) \in \partial X \times \partial Y$ and $(\hat{x}, \hat{y}) \in gC \times hD$ such that $\overline{\rho}(\hat{x}, \overline{x}), \overline{\tau}(\hat{y}, \overline{y}) < \delta/4$.

**Case 1** There exists $(x', y') \in gC \times hD$ such that $\delta \le \mu(x', y') \le 1/\delta$.

Then $(\overline{x}, \overline{y}, \mu(x', y')) \in W$, and since

$$\overline{\rho}(x', \overline{x}) \le \overline{\rho}(x', \hat{x}) + \overline{\rho}(\hat{x}, \overline{x}) < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}$$

$$\overline{\tau}(y', \overline{y}) \le \overline{\tau}(y', \hat{y}) + \overline{\tau}(\hat{y}, \overline{y}) < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}$$

we have $(x', y') \in (gC \times hD) \cap U((\overline{x}, \overline{y}, \mu(x', y')), \delta/2)$.

Moreover, for any $(x, y) \in gC \times hD$, we have

$$
\begin{aligned}
|\mu(x, y) - \mu(x', y')| &= \left| \mu(x, y) - \mu(x, y') + \mu(x, y') - \mu(x', y') \right| \\
&= \left| \frac{q(y)}{p(x)} - \frac{q(y')}{p(x)} + \frac{q(y')}{p(x)} - \frac{q(y')}{p(x')} \right| \\
&\le \frac{1}{p(x)} \cdot |q(y) - q(y')| + \frac{q(y')}{p(x')} \cdot \frac{|p(x') - p(x)|}{p(x)} \\
&= \frac{1}{p(x)} \cdot |q(y) - q(y')| + \mu(x', y') \cdot \frac{|p(x') - p(x)|}{p(x)} \\
&< \frac{1}{(4/\delta) \cdot (R_q + (1/\delta) \cdot R_p)} \cdot R_q + \frac{1}{\delta} \cdot \frac{R_p}{(4/\delta) \cdot (R_q + (1/\delta) \cdot R_p)} \\
&= \frac{\delta}{4}.
\end{aligned}
$$

Hence $\operatorname{diam}_\mu gC \times hD < \delta/2$, so the conditions of Claim 3.15 are satisfied, and $gC \times hD$ is contained in a single element of $\mathcal{U}$.

**Case 2** There is no $(x', y') \in gC \times hD$ with $\delta \le \mu(x', y) \le 1/\delta$.

Then we have $\mu(x, y) < \delta$ for all $(x, y) \in gC \times hD$, or $\mu(x, y) > 1/\delta$ for all $(x, y) \in gC \times hD$.

In the case where $\mu(x, y) < \delta$ for all $(x, y) \in gC \times hD$, we have

$$\bar{\rho}(x, \bar{x}) \leq \bar{\rho}(x, \hat{x}) + \bar{\rho}(\hat{x}, \bar{x}) < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}$$

for all $(x, y) \in gC \times hD$, so that, in fact, $gC \times hD \subseteq U(\bar{x}, \delta)$.

A similar argument shows that if $\mu(x, y) > 1/\delta$ for all $(x, y) \in gC \times hD$, then $gC \times hD \subseteq U((\bar{y}, \delta))$.

This proves the claim. □

Finally, let

$$\Gamma := \left\{ (g, h) \in G \times H \mid (gC \times hD) \cap \big((J \times P_J) \cup (Q_K \times K)\big) \neq \varnothing \right\}.$$

Then $\Gamma$ is finite by cocompactness of the actions of $G$ and $H$ on $X$ and $Y$, respectively, and Claim 3.18 shows that if $(g, h) \notin \Gamma$, then $gC \times hD$ is contained in a single element of the original cover $\mathcal{U}$.

So $\widehat{X \times Y}$ satisfies the null condition with respect to the action of $G \times H$ on $X \times Y$. □

To prove that $\widehat{X \times Y}$ is an ANR, we will construct a homotopy $\gamma \colon \widehat{X \times Y} \times [0, 1] \to \widehat{X \times Y}$ which pulls $\widehat{X \times Y}$ off of $\partial X * \partial Y$ into the ANR $X \times Y$. In analogy with the CAT(0) case, we describe the homotopy by first constructing a "ray" from the base point $(x_0, y_0)$ to each point of $\widehat{X \times Y}$. The homotopy $\gamma$ then pulls points inward along these rays. The subtle point of the argument, and the key to obtaining continuity, is the parametrization of the rays in such a way that the slope function $\mu$ is respected near $\partial X * \partial Y$. After $\gamma$ is constructed, we apply Theorem 1.19 to conclude that $\widehat{X \times Y}$ is an ANR. The existence of $\gamma$ will also imply that $\partial X * \partial Y$ is a $\mathcal{Z}$–set in $\widehat{X \times Y}$:

Define $\alpha' \colon \hat{X} \times [0, \infty) \to X$ and $\beta' \colon \hat{Y} \times [0, \infty) \to Y$ by

$$\alpha'(\hat{x}, t) := \hat{\alpha}(\hat{x}, \delta(t)) \quad \text{and} \quad \beta'(\hat{y}, t) := \hat{\beta}(\hat{y}, \delta(t))$$

for all $\hat{x} \in \hat{X}, \hat{y} \in \hat{Y}, t \in [0, \infty)$, where $\delta \colon [0, \infty) \to (0, 1]$ is given by $\delta(t) := 1/(1+t)$ and $\hat{\beta}$ and $\hat{\alpha}$ are as defined in Lemma 3.8.

Now a simple calculation shows that for any $t \in [0, \infty), \bar{x} \in \partial X, \bar{y} \in \partial Y$, we have $p(\alpha'(\bar{x}, t)), q(\beta'(\bar{y}, t)) \in (t-1, t+3)$. This will allow us to construct rays in $X \times Y$ which respect the slope function $\mu$ by controlling the speeds at which $\alpha'$ and $\beta'$ are traced.

Let $\gamma'\colon \widehat{X \times Y} \times [0, \infty) \to X \times Y$ be given by

$$\gamma'((x, y), t) := \left( \alpha'\left( x, \frac{t}{\sqrt{(\mu(x, y))^2 + 1}} \right), \beta'\left( y, \frac{\mu(x, y) \cdot t}{\sqrt{(\mu(x, y))^2 + 1}} \right) \right)$$
$$\text{if } (x, y) \in X \times Y, \ t \geq 0,$$

$$\gamma'(\langle \overline{x}, \overline{y}, \mu \rangle, t) := \left( \alpha'\left( \overline{x}, \frac{t}{\sqrt{\mu^2 + 1}} \right), \beta'\left( \overline{y}, \frac{\mu \cdot t}{\sqrt{\mu^2 + 1}} \right) \right)$$
$$\text{if } \langle \overline{x}, \overline{y}, \mu \rangle \in \partial X * \partial Y, \ 0 < \mu < \infty, \ t \geq 0,$$

$$\gamma'((\overline{x}, 0), t) := (\alpha'(\overline{x}, t), y_0) \text{ if } \overline{x} \in \partial X, \ t \geq 0,$$

$$\gamma'((\overline{y}, \infty), t) := (x_0, \beta'(\overline{y}, t)) \text{ if } \overline{y} \in \partial Y, \ t \geq 0$$

The map $\gamma'$ applied to a boundary point $z$ returns a ray in $X \times Y$ which converges (in $\widehat{X \times Y}$) to $z$:

If $z = \langle \overline{x}, 0 \rangle$, then $\gamma'(z) = (\alpha'(\overline{x}, t), y_0)$ for all $t \geq 0$. Since $\alpha'(\overline{x}, t) \to \overline{x}$ in $\widehat{X}$ and $\mu(\gamma'(z)) = \mu(\alpha'(\overline{x}, t), y_0) = q(y_0)/p(\alpha'(\overline{x}, t)) = 0$ for sufficiently large $t$, then $\gamma'(z)$ gets arbitrarily close to $\langle \overline{x}, 0 \rangle$ in $\widehat{X \times Y}$.

A similar argument holds when $z = \langle \overline{y}, \infty \rangle$.

Finally, if $z = \langle \overline{x}, \overline{y}, \mu \rangle$, where $0 < \mu < \infty$, then for any $t \geq 0$, we have

$$\mu\left( \gamma'\left( \left\langle \overline{x}, \overline{y}, \mu \right\rangle, t \right) \right) = \frac{q\left( \beta'\left( \overline{y}, \frac{\mu \cdot t}{\sqrt{\mu^2 + 1}} \right) \right)}{p\left( \alpha'\left( \overline{x}, \frac{t}{\sqrt{\mu^2 + 1}} \right) \right)} \in \left( \frac{\frac{\mu \cdot t}{\sqrt{\mu^2 + 1}} - 2}{\frac{t}{\sqrt{\mu^2 + 1}} + 3}, \frac{\frac{\mu \cdot t}{\sqrt{\mu^2 + 1}} + 3}{\frac{t}{\sqrt{\mu^2 + 1}} - 2} \right)$$
$$= \left( \frac{\mu \cdot t - 2\sqrt{\mu^2 + 1}}{t + 3\sqrt{\mu^2 + 1}}, \frac{\mu \cdot t + 3\sqrt{\mu^2 + 1}}{t - 2\sqrt{\mu^2 + 1}} \right).$$

Therefore $\mu\left( \gamma' \langle \overline{x}, \overline{y}, \mu \rangle, t \right) \to \mu$ as $t \to \infty$, so that $\gamma'\left( \langle \overline{x}, \overline{y}, \mu \rangle, t \right) \to \langle \overline{x}, \overline{y}, \mu \rangle$ in $\widehat{X \times Y}$ as $t \to \infty$.

This allows us to define $\gamma$, which begins at $\mathrm{id}_{\widehat{X \times Y}}$ and then runs $\gamma'$ in reverse, and get a continuous map in doing so:

Let $\gamma\colon \widehat{X \times Y} \times [0, 1] \to \widehat{X \times Y}$ be defined by

$$\gamma(z, t) := \begin{cases} z & \text{if } t = 0, \\ \gamma'\left( z, \delta^{-1}(t) \right) & \text{if } t \in (0, 1]. \end{cases}$$

**Proposition 3.19** $\widehat{X \times Y}$ *is an ER.*

**Proof** Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of $\widehat{X \times Y}$. Choose $\epsilon > 0$ such that for each $z \in \widehat{X \times Y}$ there is an $\alpha \in A$ so that $\gamma\left(\{z\} \times [0, \epsilon]\right) \subseteq U_\alpha$. Note that such an $\epsilon$ exists since $\widehat{X \times Y}$ is compact.

Also note that $X \times Y$ is an ANR, being a product of ANR's. We will show that $X \times Y$ is a $\mathcal{U}$–dominating space for $\widehat{X \times Y}$.

Consider the map $\psi := \gamma_\epsilon \colon \widehat{X \times Y} \to X \times Y$, along with $\phi \colon X \times Y \hookrightarrow \widehat{X \times Y}$, where $\phi$ is the inclusion map. Then $\phi \circ \psi \colon \widehat{X \times Y} \to \widehat{X \times Y}$ is $\mathcal{U}$–homotopic to $\mathrm{id}_{\widehat{X \times Y}}$ via the homotopy $\gamma|_{\widehat{X \times Y} \times [0, \epsilon]}$. Thus $X \times Y$ is a $\mathcal{U}$–dominating space for $\widehat{X \times Y}$. Hence $\widehat{X \times Y}$ is an ANR by Theorem 1.19. Since $\widehat{X \times Y}$ is also contractible, then $\widehat{X \times Y}$ is an ER (Recall Fact 1.4). $\qquad \square$

**Proposition 3.20** $\partial X * \partial Y$ is a $\mathcal{Z}$–set in $\widehat{X \times Y}$.

**Proof** By construction, we have $\gamma_0 \equiv \mathrm{id}_{\widehat{X \times Y}}$ and $\gamma_t(\widehat{X \times Y}) \cap \partial X * \partial Y = \varnothing$ whenever $t > 0$. Therefore $\partial X * \partial Y$ is a $\mathcal{Z}$–set in $\widehat{X \times Y}$. $\qquad \square$

**Theorem 3.21** Let $G$ and $H$ be groups which admit $\mathcal{Z}$–structures $(\widehat{X}, \partial X)$ and $(\widehat{Y}, \partial Y)$, respectively. Then $(\widehat{X \times Y}, \partial X * \partial Y)$ is a $\mathcal{Z}$–structure on $G \times H$.

**Proof** Propositions 3.19, 3.20 and 3.14 show that conditions (1), (2) and (4) in Definition 1.15 are satisfied by $\widehat{X \times Y}$. Moreover, $G \times H$ acts properly and cocompactly on $X \times Y$, since each of $G$ and $H$ acts accordingly on each of $X$ and $Y$, so condition (3) is also satisfied.

Therefore $(\widehat{X \times Y}, \partial X * \partial Y)$ is a $\mathcal{Z}$–structure on $G \times H$. $\qquad \square$

**Theorem 3.22** If $G$ and $H$ each admit $\mathcal{EZ}$–structures, then so does $G \times H$.

**Proof** By Theorem 3.21, it suffices to show that the action of $G \times H$ on $X \times Y$ extends to an action on $\widehat{X \times Y}$.

By hypothesis, the actions of $G$ and $H$ on $X$ and $Y$ extend to actions on $\widehat{X}$ and $\widehat{Y}$, respectively, ie we have maps $\pi \colon G \times \widehat{X} \to \widehat{X}$ and $\overline{\rho} \colon H \times \widehat{Y} \to \widehat{Y}$ which satisfy the axioms of a group action.

Define the action of $G \times H$ on $\widehat{X \times Y}$ via the map $\overline{\tau} \colon (G \times H) \times \widehat{X \times Y} \to \widehat{X \times Y}$, where

$$\overline{\tau}((g, h), (x, y)) := (\pi(g, x), \overline{\rho}(h, y))$$
$$\overline{\tau}((g, h), \langle \overline{x}, \overline{y}, \mu \rangle) := (\pi(g, \overline{x}), \overline{\rho}(h, \overline{y}), \mu)$$
$$\overline{\tau}((g, h), \langle \overline{x}, 0 \rangle) := (\pi(g, \overline{x}), 0)$$
$$\overline{\tau}((g, h), \langle \overline{y}, \infty \rangle) := (\overline{\rho}(h, \overline{y}), \infty).$$
$\qquad \square$

# 4 Applications and open questions

It is known that groups within certain classes admit $\mathcal{Z}$–structures, such as CAT(0), hyperbolic, and systolic groups. However, it is more often than not the case that the direct product of two hyperbolic groups is not hyperbolic and that the direct product of two systolic groups is not systolic. In addition, it is not clear how to handle the product (direct or free) of two groups when they come from distinct classes. Theorem 2.9 and Theorem 3.21 imply the following:

**Corollary 4.1** Let $\mathcal{F}$ denote the family of groups consisting of all CAT(0), hyperbolic, and systolic groups. If $G, H \in \mathcal{F}$, then $G * H$ and $G \times H$ both admit $\mathcal{EZ}$–structures.

We end the paper with some open questions related to this work:

(1) Does a modification of the construction in the proof of Theorem 2.9 give an analogous result pertaining to free products with amalgamation over finite subgroups?

(2) Does a variation of Theorem 2.9 hold for HNN extensions over finite subgroups?

(3) If $G$, $H$, and $K$ all admit $\mathcal{Z}$–structures, does $G *_K H$ admit a $\mathcal{Z}$–structure? What about $G *_K$, again under the hypothesis that $G$ and $K$ admit $\mathcal{Z}$–structures?

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