Sutured Floer homology, sutured TQFT and noncommutative QFT

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We define a "sutured topological quantum field theory", motivated by the study of sutured Floer homology of product 3–manifolds, and contact elements. We study a rich algebraic structure of suture elements in sutured TQFT, showing that it corresponds to contact elements in sutured Floer homology. We use this approach to make computations of contact elements in sutured Floer homology over $\mathbb{Z}$ of sutured manifolds $(D^2 \times S^1, F \times S^1)$ where F is finite. This generalises previous results of the author over $\mathbb{Z}_2$ coefficients. Our approach elaborates upon the quantum field theoretic aspects of sutured Floer homology, building a noncommutative Fock space, together with a bilinear form deriving from a certain combinatorial partial order; we show that the sutured TQFT of discs is isomorphic to this Fock space.

1 Introduction

1.1 Chord diagrams and signs

This paper, like its prequel [17], is about fun with chord diagrams. A chord diagram Γ is a finite collection of nonintersecting properly embedded arcs in a 2–dimensional disc D^2, up to homotopy relative to endpoints. Fixing $2n$ points on ∂D^2, the number of chord diagrams of n arcs (or chords) connecting them is C_n, the n–th Catalan number. In [17] we considered the $\mathbb{Z}_2$ vector space $SFH_{\mathrm{comb}}(T, n)$ generated by chord diagrams of n chords, subject to a relation called the *bypass relation*. The bypass relation says that whenever 3 chord diagrams coincide, except within a disc on which the chords appear as in Figure 1, they sum to zero.

In [17], we showed that this vector space describes precisely contact elements in the sutured Floer homology (SFH) of $(D^2 \times S^1, F \times S^1)$, where $|F| = 2n$, with $\mathbb{Z}_2$ coefficients. However SFH can be defined over $\mathbb{Z}$; our vector space SFH_{comb} should generalise to $\mathbb{Z}$ coefficients.

Over $\mathbb{Z}$, contact elements have a sign ambiguity; see Honda, Kazez and Matić [10]. So a chord diagram should represent an element in an abelian group V, *up to sign*, as in
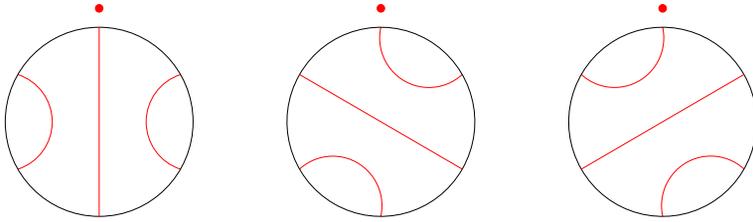
Figure 1: A bypass triple

the *lax vectors* of Conway [4]. The sum of two lax vectors $\pm v, \pm w$ is not well-defined: choosing representatives v, w, we could have $(\pm v) + (\pm w)$ equal to $\pm(v + w)$ or $\pm(v - w)$. The bypass relation then says that three lax vectors $\pm u, \pm v, \pm w$ somehow sum to zero; $\pm(u + v)$ or $\pm(u - v)$ equals $\pm w$.

This paper will resolve these ambiguities, and along the way unearth further structure.

1.2   Stackability resolves signs

The key to this resolution is a bilinear form $\langle \cdot \,|\, \cdot \rangle$ on our abelian group V. Over $\mathbb{Z}_2$ this was called the *stackability map* $m(\cdot, \cdot)$ in [17]. Place two chord diagrams as the lids of a cylinder with vertical arcs running along its sides; round corners and arcs in a specific way to obtain a sphere with curves on it; chord diagrams are *stackable* if we obtain a single connected curve on the sphere. The map

$$\langle \cdot \,|\, \cdot \rangle \colon V \otimes V \longrightarrow \mathbb{Z}$$

takes stackable chord diagrams to ± 1, and nonstackable chord diagrams to 0; it reduces to $m(\cdot, \cdot)$ mod 2. The key idea we shall use to obtain coherent signs for chord diagrams is that while there is ambiguity in ± 1, there is no ambiguity in ± 0. The bypass relation tells us that a chord diagram should be given by $\pm(u + v)$ or $\pm(u - v)$. However if we know, say, that $\langle u|w \rangle = \langle v|w \rangle = 1$ for some w, then $\langle \pm(u - v)|w \rangle = 0$ while $\langle \pm(u + v)|w \rangle = \pm 2$; we can then usefully distinguish between the two.

1.3   A noncommutative Fock space and partial order

In quantum field theory a Fock space $\mathcal{F}$ is an algebraic object whose elements can describe states of several particles. For instance, $x \in \mathcal{F}$ can denote the presence of one particle, $y \in \mathcal{F}$ a different particle, and xy the presence of both. A Fock space has creation and annihilation operators, adjoint with respect to an inner product. In a commutative or bosonic (resp. anticommutative or fermionic) Fock space, $xy = yx$ (resp. $xy = -yx$). We shall consider a *noncommutative* $\mathcal{F}$, however, in which neither

$xy = yx$ nor $xy = -yx$ holds; roughly, $\mathcal{F}$ contains noncommutative polynomials in x, y.

There are natural creation and annihilation operators on $\mathcal{F}$, which insert or delete a specified symbol in a word in a specified place – many more creation and annihilation operators than in the commutative or anticommutative cases; they obey the relations of a *simplicial set*.

In [17] we defined a partial order on words in $\{x, y\}$ (in [17] we used $\{-, +\}$). Say that $w_1 \leq w_2$ if w_2 can be obtained from w_1 by moving some (possibly none) of the x's to the right (equivalently, by moving some of the y's to the left). Thus $xxyy \leq yxyx$ but $xyyx \not\leq yxxy$. Now define a bilinear form $\langle \cdot \,|\, \cdot \rangle \colon \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathbb{Z}$ as a boolean version of $\leq$, as follows. For two words w_1, w_2 in $\{x, y\}$, let $\langle w_1 | w_2 \rangle = 1$ if $w_1 \leq w_2$ and 0 otherwise; then extend to $\mathcal{F}$ linearly. We imagine $\langle \cdot \,|\, \cdot \rangle$ as a "noncommutative inner product" and $\mathcal{F}$ as a "Fock space of two noncommuting particles". Creation and annihilation operators are adjoint with respect to $\langle \cdot \,|\, \cdot \rangle$ (in a noncommutative way).

We shall prove the following. (A precise version is Theorem 3.4; a more detailed statement is Theorem 5.1.)

Theorem 1.1 *The Fock space and bilinear form defined from $\leq$ are isomorphic to the abelian group V generated by chord diagrams, and bilinear form defined by stacking:*

$$(\mathcal{F}, \langle \cdot \,|\, \cdot \rangle) \cong (V, \langle \cdot \,|\, \cdot \rangle).$$

Thus chord diagrams give lax elements of $\mathcal{F}$; three bypass-related chord diagrams give lax elements of $\mathcal{F}$ summing to zero; and stackability is described by the boolean $\leq$. This was shown in [17] mod 2; we show it holds over $\mathbb{Z}$.

An element of $\mathcal{F}$ corresponding to a chord diagram Γ is a linear combination of words in $\{x, y\}$ (up to sign); among these there are well-defined first and last words w_-, w_+. In [17] we showed (over $\mathbb{Z}_2$) that $w_- \leq w_+$; in fact the correspondence $\Gamma \mapsto (w_-, w_+)$ gives a bijection between chord diagrams and pairs of words comparable under $\leq$. We will show the same properties also hold over $\mathbb{Z}$; further, every chord diagram gives a linear combination of words with all coefficients ± 1.

It is not difficult to show that $\langle \cdot \,|\, \cdot \rangle$ on $\mathcal{F}$ is nondegenerate. Thus there is a *duality operator* H such that for all u, v, $\langle u | v \rangle = \langle v | Hu \rangle$. Note that for commutative $\langle \cdot \,|\, \cdot \rangle$, ie $\langle u | v \rangle = \langle v | u \rangle$, we have $H = 1$; and in the anticommutative case, $H = -1$. We will show that H is *periodic*, in the sense that some power of H is the identity; this is, in a sense, a generalisation of commutativity and anticommutativity. We prove this statement by showing that H is equivalent to the operation of *rotating* a chord diagram.

The key feature distinguishing our Fock space from the usual sort is the form $\langle \cdot \mid \cdot \rangle$, based on $\leq$. Partial orders appear prominently in the theory of *causal sets* (see eg Bombelli et al [3] and Sorkin [19]), one approach to quantum gravity. But we have not seen anything like $\leq$ in the theoretical physics literature.

1.4 Sutured Topological Quantum Field Theory

We have been somewhat vague about V. We now explain what V is: a TQFT-like object.

The idea is to build an algebraic structure purely by reference to surfaces (like discs), arcs on them (like chord diagrams), and various topological operations. To a surface Σ with some markings F on the boundary, we associate an abelian group $V(\Sigma, F)$. When arcs Γ are drawn appropriately on the surface (*sutures*) we associate an element (up to sign) $c(\Gamma)$ in $V(\Sigma, F)$, called a *suture element*. These associations are natural with respect to topological operations such as gluing and stacking. This defines what we call *sutured topological quantum field theory*.

The V above is the sutured TQFT of discs $V(D^2) = \bigoplus_F V(D^2, F)$ (direct sum over all possible F). We impose enough axioms (explaining their rationale as we introduce them) that $V(D^2)$ is unique.

It is interesting to note that although the bypass relation appears in sutured TQFT, we need not impose it as an axiom. Instead we can define $\langle \cdot \mid \cdot \rangle$ from stackability (which is just a certain gluing of surfaces); the bypass relation is equivalent to a nondegeneracy condition on $\langle \cdot \mid \cdot \rangle$ (Proposition 3.7).

The point of this paper is that, in describing sutured TQFT axiomatically, we find surprisingly familiar structures from (nontopological!) quantum field theory, namely all the structure of the Fock space $\mathcal{F}$. The sutured TQFT of discs is the QFT of two noncommuting particles, where the two usages of "QFT" are quite distinct.

We remark that sutured TQFT, as we define it, is very similar in some respects to a *planar algebra*; see Jones [13]. To a surface with boundary is associated an algebraic object; to curves on the surface, dividing it into positive and negative regions, are associated distinguished elements of the algebraic object; surfaces may be glued together, giving maps of algebraic objects which are natural with respect to the distinguished elements; surfaces with many boundary components give operators. However there are several distinctions: sutured TQFT is not restricted to planar surfaces; sutured TQFT has no canonical form of multiplication (except on discs, where it does not coincide with planar algebra multiplication); the element associated to a set of curves in sutured TQFT has a sign ambiguity; and in the author's limited knowledge, nothing like a bypass relation

or stackability has been studied in the context of planar algebras. Clearly the two subjects have wildly disparate motivations: planar algebras from subfactors and von Neumann algebras; sutured TQFT from contact geometry and sutured Floer homology. We wonder how deep these similarities run.

1.5 SFH gives a sutured TQFT

Although V is defined from sutured TQFT axioms, it is motivated by sutured Floer homology. As discussed by Honda, Kazez and Matić in [10] and at length by the author in [17], the SFH of product manifolds $(\Sigma \times S^1, F \times S^1)$, and their contact elements, have properties similar to a $(1+1)$–dimensional TQFT. Sutured TQFT is this TQFT, abstracted from SFH, and now independent of it.

We will show that $SFH(\Sigma \times S^1, F \times S^1)$ forms a sutured TQFT; in particular, for any sutured TQFT, $V(D^2, F) \cong SFH(D^2 \times S^1, F \times S^1)$. In particular, we obtain an explicit description of contact elements, with $\mathbb{Z}$ coefficients, in $SFH(D^2 \times S^1, F \times S^1)$, extending the results over $\mathbb{Z}_2$ in [17]. Sutured TQFT on more complicated surfaces is not unique, but we also consider an additional axiom, satisfied by $SFH(\Sigma \times S^1, F \times S^1)$, which ensures nontriviality at higher genus.

This paper thus shows that three structures are equivalent: $SFH(D^2 \times S^1, F \times S^1)$, $V(D^2)$ and $\mathcal{F}$. Moreover, it shows that all the structure of SFH of product manifolds can be described without considering Heegaard decompositions, holomorphic curves or contact structures. We obtain proofs about contact elements which are both "holomorphic curve free" and "contact geometry free". In a subsequent paper we shall use sutured TQFT to give a proof that the contact element of a torsion contact structure is zero; see Ghiggini et al [6] and Massot [16].

1.6 Structure of this paper

Several considerations determine the structure of this paper. To prove our main result that $V(D^2) \cong \mathcal{F}$, it is easiest first to define the formal algebraic structure of $\mathcal{F}$, and then show SFH is isomorphic to it. Moreover, we may eschew sutured Floer homology until the last minute; it is not necessary to the discussion of sutured TQFT or noncommutative QFT, although we do use it to prove that a sutured TQFT exists! In a sense, our results are more general than SFH: they are about suture elements in sutured TQFT, of which contact elements in SFH of product manifolds form an example.

Thus, this paper begins by establishing the formal algebraic structure of $\mathcal{F}$ (Section 2). Much of the detail in this section (especially the bisimplicial structure in Section 2.3, differentials, commutation relations, normal form in Section 2.7 and Temperley–Lieb

representation in Section 2.8) can be skipped on a first reading: the important details are the Fock space $\mathcal{F}$, the creation and annihilation operations, the bilinear form $\langle \cdot | \cdot \rangle$ and the duality operator H.

Then in Section 3 we axiomatically introduce sutured TQFT, and deduce properties of it. Some of this is also quite technical (eg variations on axioms) and can be skipped on a first reading. In Section 4 we show that $SFH(\Sigma \times S^1, F \times S^1)$ forms a sutured TQFT, hence that a sutured TQFT exists. In Section 5 we demonstrate $V(D^2) \cong \mathcal{F}$, along with isomorphisms of detailed structures.

2   Algebraic noncommutative QFT

2.1   Fock space

Let $S = \{x, y\}$ and let $\mathcal{M}$ be the free monoid on S, ie the set of all finite words (including the empty word, which is the identity 1) on $\{x, y\}$, under the operation of concatenation. Let $\mathcal{F}$ be the monoid ring of $\mathcal{M}$ over $\mathbb{Z}$. That is, $\mathcal{F}$ consists of finite $\mathbb{Z}$–linear combinations of finite words on $\{x, y\}$; multiplication is concatenation, now extended linearly. Alternatively, $\mathcal{F}$ is the polynomial ring generated over $\mathbb{Z}$ by two noncommuting indeterminates x, y. The empty word, denoted 1, is a multiplicative identity; it is also called the *vacuum*.

Clearly $\mathcal{M}$ and $\mathcal{F}$ have several gradings: degree n_x in x, degree n_y in y, and linear combinations of these. Multiplication adds these gradings, making $\mathcal{F}$ into a bigraded ring. Let $\mathcal{M}_n$ denote the subset of $\mathcal{M}$ consisting of words of length n, ie with total degree n, and $\mathcal{F}_n \subset \mathcal{F}$ the additive subgroup generated by $\mathcal{M}_n$, ie linear combinations of length-n words. As graded abelian groups, $\mathcal{F} = \bigoplus_{n \geq 0} \mathcal{F}_n$ and $\mathcal{F}_n = (\mathbb{Z}x \oplus \mathbb{Z}y)^{\otimes n}$. Denote by $\mathcal{M}_{n_x, n_y}$ the subset of $\mathcal{M}$ consisting of words of degree n_x, n_y in x, y respectively; and by $\mathcal{F}_{n_x, n_y} \subset \mathcal{F}$ the additive subgroup generated by $\mathcal{M}_{n_x, n_y}$.

As an alternative notation, thinking of x as having degree -1 and y having degree 1, let $\mathcal{M}_n^e$ denote the subset of $\mathcal{M}$ consisting of words of length n and degree e (ie such that $n_y - n_x = e$), and $\mathcal{F}_n^e$ the additive subgroup of $\mathcal{F}$ generated by $\mathcal{M}_n^e$. So n, e, n_x, n_y are related by

$$n = n_x + n_y, \quad e = -n_x + n_y, \quad n_x = \frac{n - e}{2}, \quad n_y = \frac{n + e}{2}.$$

Obviously, $\mathcal{M}_{n_x,n_y} = \mathcal{M}_{n_x+n_y}^{n_y-n_x}$, $\mathcal{M}_n^e = \mathcal{M}_{(n-e)/2,n+e/2}$, $\mathcal{F}_{n_x,n_y} = \mathcal{F}_{n_x+n_y}^{n_y-n_x}$ and $\mathcal{F}_n^e = \mathcal{F}_{n-e/2,n+e/2}$. Further,

$$\mathcal{M} = \bigsqcup_n \mathcal{M}_n = \bigsqcup_{n,e} \mathcal{M}_n^e = \bigsqcup_{n_x,n_y} \mathcal{M}_{n_x,n_y}, \quad \mathcal{F} = \bigoplus_n \mathcal{F}_n = \bigoplus_{n,e} \mathcal{F}_n^e = \bigoplus_{n_x,n_y} \mathcal{F}_{n_x,n_y},$$

where $0 \le n \in \mathbb{Z}$, and $e \in \mathbb{Z}$ satisfies $|e| \le n$, and $e \equiv n$ mod 2. As abelian groups

$$\mathcal{F}_n \cong \mathbb{Z}^{2^n} \quad \text{and} \quad \mathcal{F}_n^e \cong \mathbb{Z}^{\binom{n}{n_x}} = \mathbb{Z}^{\binom{n}{n_y}} = \mathbb{Z}^{\binom{n}{(n-e)/2}} = \mathbb{Z}^{\binom{n}{(n+e)/2}}.$$

2.2 Creation and annihilation operators

Operations on words give operations on $\mathcal{F}$. Precisely, any function $\mathcal{M} \longrightarrow \mathcal{M} \cup \{0\}$ extends to a linear operator on $\mathcal{F}$. Thus we define the following operations; s denotes a letter in $\{x, y\}$.
  (i) (a) *Initial annihilation $a_{s,0}w$*: if w begins with an s, delete it; else return 0.
      (b) *Internal annihilation $a_{s,i}w$*, for $1 \le i \le n_s w$: delete the i–th s in w.
      (c) *Final annihilation $a_{s,n_s w+1}w$*: if w ends with an s, delete it; else return 0.

 (ii) (a) *Initial creation $a_{s,0}^*w$*: prepend an s to the beginning of w.
      (b) *Internal creation $a_{s,i}^*w$*, for $1 \le i \le n_s w$: replace the i–th s in w with ss.
      (c) *Final creation a_{s,n_s+1}^*w*: append an s to the end of w.

The names initial, internal, and final should be clear; initial and final annihilation (resp. creation) are collectively called *terminal* annihilation (resp. creation). We then have

$$a_{x,i}: \mathcal{F}_n^e \longrightarrow \mathcal{F}_{n-1}^{e+1}, \quad a_{y,i}: \mathcal{F}_n^e \longrightarrow \mathcal{F}_{n-1}^{e-1},$$
$$a_{x,i}^*: \mathcal{F}_n^e \longrightarrow \mathcal{F}_{n+1}^{e-1}, \quad a_{y,i}^*: \mathcal{F}_n^e \longrightarrow \mathcal{F}_{n+1}^{e+1}.$$

Here we take $\mathcal{F}_{-1}^e = \{0\}$: on the vacuum 1 all annihilation operators give 0.

These operators satisfy various relations which are easily checked, as follows
  (i) *Between x and y; interspecies.*
      (a) In almost every case, x–annihilation/creation and y–creation/annihilation commute. That is, for $0 \le i \le n_x + 1$ and $0 \le j \le n_y + 1$, except for $(i, j) = (0, 0)$ or $(n_x + 1, n_y + 1)$,

$$a_{x,i} \circ a_{y,j} = a_{y,j} \circ a_{x,i}, \quad a_{x,i}^* \circ a_{y,j} = a_{y,j} \circ a_{x,i}^*,$$
$$a_{x,i} \circ a_{y,j}^* = a_{y,j}^* \circ a_{x,i}, \quad a_{x,i}^* \circ a_{y,j}^* = a_{y,j}^* \circ a_{x,j}^*.$$

      (b) Initial x–annihilation/creation and initial y–annihilation/creation never commute.
      (c) Final x–annihilation/creation and final y–annihilation/creation never commute.

(ii) *Among x or among y; intraspecies.* Set s to be x or y.

(a) Annihilations commute after a shift. For $0 \leq i < j \leq n_s + 1$ (not $i = j$),

$$a_{s,i} \circ a_{s,j} = a_{s,j-1} \circ a_{s,i}.$$

(b) Annihilations and creations usually commute after a shift, and are sometimes inverses. For $0 \leq i, j \leq n_s + 1$,

$$a_{s,i} \circ a_{s,j}^* = \begin{cases} a_{s,j-1}^* \circ a_{s,i} & i < j, \\ 1 & i = j, \; j+1, \\ a_{s,j}^* \circ a_{s,i-1} & i > j+1. \end{cases}$$

(c) Creations commute, after a shift. For $0 \leq i \leq j \leq n_s + 1$,

$$a_{s,i}^* \circ a_{s,j}^* = a_{s,j+1}^* \circ a_{s,i}^*.$$

2.3 Bisimplicial structure

As noted in [17], the intraspecies relations are those of a *simplicial set*. We have *two* simplicial structures on $\mathcal{F}$, one for x and one for y. Let $s \in \{x, y\}$. Words with s–degree n_s have $n_s + 2$ annihilation operators (n_s internal and 2 terminal), regarded as face maps, and $n_s + 2$ creation operators, regarded as degeneracy maps; thus words of s–degree n_s can be regarded as $(n_s + 1)$–dimensional simplices.

As a bisimplicial object in the category of abelian groups, $\mathcal{F}$ has contravariant functors

$$\mathbb{F}_x, \mathbb{F}_y \colon \Delta \longrightarrow \mathrm{Ab}.$$

Here Δ is the simplicial category; its objects are the (set-theoretic) nonnegative integers $\mathbf{n} = \{0, 1, \ldots, n-1\}$ and its morphisms are order-preserving functions $\mathbf{m} \to \mathbf{n}$. Ab is the category of abelian groups. These functors have "image $\mathcal{F}$", $\mathcal{F} = \bigoplus_n \mathbb{F}_x(\mathbf{n}) = \bigoplus_n \mathbb{F}_y(\mathbf{n})$, as we now describe.

In fact the monoid $\mathcal{M}$ also has a bisimplicial structure, arising from functors

$$\mathbb{F}_x, \mathbb{F}_y \colon \Delta \longrightarrow \mathrm{Set},$$

with $\bigsqcup_n \mathbb{F}_x(\mathbf{n}) = \bigsqcup_n \mathbb{F}_y(\mathbf{n}) = \mathcal{M} \cup \{0\}$. (We adjoin 0 since terminal annihilations may return 0.) The functors to Set for $\mathcal{M}$ extend linearly to the functors to Ab, as the elements of $\mathcal{M}$ form a basis for $\mathcal{F}$.

The functors $\mathbb{F}_s$ can be described by regarding a word w on $\{x, y\}$ as a function. It will be useful to describe words by functions in several different ways, and we now pause to describe these functions.

Definition 2.1 Let $w \in \mathcal{M}_{n_x,n_y}$, $n = n_x + n_y$ and number the letters in w left to right from x_1 to x_{n_x} and y_1 to y_{n_y}. Define functions $f_w^x, f_w^y, g_w^x, g_w^y, h_w^x, h_w^y$ as follows.

(i) $f_w^x : \{1, 2, \ldots, n_x\} \longrightarrow \{0, 1, \ldots, n_y\}$, $f_w^x(i)$ is the number of y's (strictly) to the left of x_i in w.

(ii) $f_w^y : \{1, 2, \ldots, n_y\} \longrightarrow \{0, 1, \ldots, n_x\}$, $f_w^y(i)$ is the number of x's (strictly) to the left of y_i in w.

(iii) $g_w^x : \{1, 2, \ldots, n\} \longrightarrow \{0, 1, \ldots, n_x\}$, $g_w^x(i)$ is the number of x's in the first i letters of w.

(iv) $g_w^y : \{1, 2, \ldots, n\} \longrightarrow \{0, 1, \ldots, n_y\}$, $g_w^y(i)$ is the number of y's in the first i letters of w.

(v) $h_w^x : \{1, 2, \ldots, n_x\} \longrightarrow \{1, 2, \ldots, n\}$, $h_w^x(i)$ is the position of x_i in w.

(vi) $h_w^y : \{1, 2, \ldots, n_y\} \longrightarrow \{1, 2, \ldots, n\}$, $h_w^y(i)$ is the position of y_i in w.

Clearly the f_w^s are increasing. The g_w^s are slowly increasing, $g_w^s(i+1) - g_w^s(i) \in \{0, 1\}$, and $g_w^s(1) \in \{0, 1\}$, $g_w^s(n_x + n_y) = n_s$. (Note $g_w^s(i)$ is like the baseball team's score after i innings of [17].) The h_w^s are strictly increasing. It's clear that any of these functions corresponds to a unique word w; moreover, there is a bijection between $\mathcal{M}_{n_x,n_y}$ and functions with these respective properties. So one of these functions determines all the others: for instance, $h_w^s(i) = f_w^s(i) + i$; the images of h_w^x and h_w^y form a partition of $\{1, \ldots, n\}$; and $h_w^s(i) = \min\{(g_w^s)^{-1}(i)\}$; also $g_w^x(i) + g_w^y(i) = i$.

For the moment we only need the f_w^s. As the f_w^s are increasing (order-preserving), they can be regarded as morphisms in Δ; after a shift, we may regard $f_w^x \in \mathrm{Mor}_\Delta(\mathbf{n_x}, \mathbf{n_y} + \mathbf{1})$; so $\mathcal{M}_{n_x,n_y}$ and $\mathrm{Mor}_\Delta(\mathbf{n_x}, \mathbf{n_y} + \mathbf{1})$ are bijective.

The idea of $\mathbb{F}_x$ is to take each object $\mathbf{n}$ in Δ to the set of all words in $\mathcal{M}$ with degree n in x, which is $\bigsqcup_{n_y} \mathcal{M}_{n_x,n_y} \cong \bigsqcup_{n_y} \mathrm{Mor}_\Delta(\mathbf{n}, \mathbf{n_y} + \mathbf{1})$; we can denote this $\mathrm{Mor}_\Delta(\mathbf{n}, \cdot)$. Then $\mathbb{F}_x$ takes the morphism $g : \mathbf{m} \to \mathbf{n}$ of Δ to the function

$$\mathrm{Mor}_\Delta(\mathbf{n}, \cdot) \xrightarrow{\mathbb{F}_x g} \mathrm{Mor}_\Delta(\mathbf{m}, \cdot)$$

that is given by precomposition with g. Note $\mathbb{F}_x(g)$ takes each $\mathrm{Mor}_\Delta(\mathbf{n}, \mathbf{n_y} + \mathbf{1})$ to $\mathrm{Mor}_\Delta(\mathbf{m}, \mathbf{n_y} + \mathbf{1})$, hence preserves n_y (which makes sense as this is the structure of adding and deleting x's).

However this structure does not allow for terminal creation and annihilation operators. The idea is to modify the above description by appending and prepending an x to the beginning and end of each word. Define the subset $\mathrm{Mor}_\Delta^T(\mathbf{m}, \mathbf{n})$ of $\mathrm{Mor}_\Delta(\mathbf{m}, \mathbf{n})$ to be those order-preserving maps $\mathbf{m} \to \mathbf{n}$ which take $0 \mapsto 0$ and

$m - 1 \mapsto n - 1$; call them *terminal-preserving morphisms*. There is a natural bijection $\mathrm{Mor}_\Delta(\mathbf{m}, \mathbf{n}) \cong \mathrm{Mor}_\Delta^T(\mathbf{m} + 2, \mathbf{n})$ given by shifting a map by one and setting its values on 0 and $m + 1$. The composition of a morphism in $\mathrm{Mor}_\Delta^T(\mathbf{m}, \mathbf{n})$ and a morphism in $\mathrm{Mor}_\Delta^T(\mathbf{n}, \mathbf{k})$ is a morphism in $\mathrm{Mor}_\Delta^T(\mathbf{m}, \mathbf{k})$: terminal-preserving morphisms are closed under composition.

Words of degree n_x, n_y in x, y are in bijective correspondence with words of degree $n_x + 2, n_y$ in x, y which begin and end with x. Similarly, increasing functions $\mathbf{n_x} \to \mathbf{n_y} + \mathbf{1}$ are in bijective correspondence with $\mathrm{Mor}_\Delta^T(\mathbf{n_x} + \mathbf{2}, \mathbf{n_y} + \mathbf{1})$. We define $\mathbb{F}_x$ to take $\mathbf{n} \in \mathrm{Ob}\,\Delta$ to the set of all words of degree $n - 2$ in x, along with 0. That is,

$$\mathbb{F}_x(\mathbf{n}) = \{0\} \cup \{\text{words of } x\text{–degree } n - 2 \text{ in } \mathcal{M}\}$$

$$\cong \{0\} \cup \left\{ \begin{array}{l} \text{words of } x\text{–degree } n \text{ in } \mathcal{M} \\ \text{which begin and end with } x \end{array} \right\}$$

$$\cong \{0\} \cup \bigsqcup_{n_y} \mathrm{Mor}_\Delta(\mathbf{n} - \mathbf{2}, \mathbf{n_y} + \mathbf{1})$$

$$\cong \{0\} \cup \bigsqcup_{n_y} \mathrm{Mor}_\Delta^T(\mathbf{n}, \mathbf{n_y} + \mathbf{1}) \cong \{0\} \cup \mathrm{Mor}_\Delta^T(\mathbf{n}, \cdot)$$

For $n = 0, 1$ then $\mathbb{F}_x(\mathbf{n}) = \{0\}$. We define $\mathbb{F}_x$ to take a morphism $g \in \mathrm{Mor}_\Delta(\mathbf{m}, \mathbf{n})$ to the function

$$\{0\} \cup \mathrm{Mor}_\Delta^T(\mathbf{n}, \cdot) \xrightarrow{\mathbb{F}_x g} \{0\} \cup \mathrm{Mor}_\Delta^T(\mathbf{m}, \cdot)$$

which takes $0 \mapsto 0$ and which precomposes functions by g, if such precomposition gives a terminal-preserving morphism; else gives 0. (If g is terminal-preserving then such a composition is certainly terminal-preserving; if g is not terminal-preserving then the composition may or may not be terminal-preserving.) This $\mathbb{F}_x g$ preserves y–degree unless it maps to 0.

This gives the contravariant functor $\mathbb{F}_x \colon \Delta \longrightarrow \mathrm{Set}$ with image $\mathcal{M}$, which extends to $\mathbb{F}_x \colon \Delta \longrightarrow \mathrm{Ab}$ with image $\mathcal{F}$.

For each $n_x \geq 0$ and $0 \leq i \leq n_x + 1$, define $a_{x,i}$ to be the unique morphism in $\mathrm{Mor}_\Delta(\mathbf{n_x} + \mathbf{1}, \mathbf{n_x} + \mathbf{2})$ which has image $(\mathbf{n_x} + \mathbf{2}) \backslash \{i\}$, in other words, which takes $(0, 1, \ldots, i-1, i, \ldots, n_x) \mapsto (0, 1, \ldots, i-1, i+1, \ldots, n_x+1)$. Such an $a_{x,i}$, under $\mathbb{F}_x$, gives a map $\mathbb{F}_x(\mathbf{n_x} + \mathbf{2}) \to \mathbb{F}_x(\mathbf{n_x} + \mathbf{1})$, ie

$$\{0\} \cup \{\text{words of } x\text{–degree } n_x \text{ in } \mathcal{M}\} \longrightarrow \{0\} \cup \{\text{words of } x\text{–degree } n_x - 1 \text{ in } \mathcal{M}\},$$

which is seen to be $a_{x,i}$ as originally defined. Note that the terminal annihilation operators $a_{x,0}, a_{x,n_x+1}$ are not terminal-preserving (rather "terminal-annihilating")

but all other $a_{x,i}$ are terminal-preserving; hence $a_{x,0}, a_{x,n_x+1}$ sometimes return zero, but other $a_{x,i}$ do not.

Similarly, for each $n_x \geq 0$ and $0 \leq i \leq n_x+1$, let $a^*_{x,i}$ be the unique surjective morphism in $\mathrm{Mor}_\Delta(\mathbf{n_x + 3}, \mathbf{n_x + 2})$ taking the value i twice, ie $(0, 1, \ldots, i, i+1, \ldots, n_x+2) \mapsto (0, 1, \ldots, i, i, \ldots, n_x+1)$. Then $\mathbb{F}_x(a^*_{x,i})\colon \mathbb{F}_x(\mathbf{n_x + 2}) \to \mathbb{F}_x(\mathbf{n_x + 3})$ is a map

$$\{0\} \cup \{\text{words of } x\text{–degree } n_x \text{ in } \mathcal{M}\} \longrightarrow \{0\} \cup \{\text{words of } x\text{–degree } n_x + 1 \text{ in } \mathcal{M}\}$$

which is $a^*_{x,i}$ as originally defined.

The functor $\mathbb{F}_y$ is defined similarly, using f^y_w. This gives two "dual" simplicial structures.

2.4 Partial order

Inequalities on all the functions f^s_w, g^s_w, h^s_w are equivalent, in the following sense.

Lemma 2.2 Let $w_0, w_1 \in \mathcal{M}_{n_x,n_y}$. The following inequalities are all equivalent:

$$f^x_{w_0} \leq f^x_{w_1}, \quad f^y_{w_0} \geq f^y_{w_1}, \quad g^x_{w_0} \geq g^x_{w_1}, \quad g^y_{w_0} \leq f^y_{w_1}, \quad h^x_{w_0} \leq h^x_{w_1}, \quad h^y_{w_0} \geq h^y_{w_1}.$$

Proof The inequalities on $f^s_{w_i}$ and $h^s_{w_i}$ are clearly equivalent, since $h^s_w(i) = f^s_w(i)+i$. The inequalities on $g^s_{w_i}$ and $h^s_{w_i}$ are equivalent since the $g^s_{w_i}$ are slowly increasing and $h^s_w(i) = \min\{(g^s_w)^{-1}(i)\}$. The inequalities on $h^x_{w_i}$ and $h^y_{w_i}$ are equivalent since their images are complementary. $\qquad\square$

If these inequalities hold, we say $w_0 \leq w_1$. It's clear this gives a partial order on each $\mathcal{M}_{n_x,n_y}$, which is a suborder of the lexicographic (total) ordering (x comes before y). Words with different x– or y–degrees are not comparable.

If h is order-preserving and $f \leq g$ then $h \circ f \leq h \circ g$. Because the creations $a^*_{s,i}$ are order-preserving ($\mathbf{n_s + 3} \to \mathbf{n_s + 2}$, under $\mathbb{F}_s$ mapping $\mathrm{Mor}^T_\Delta(\mathbf{n_s + 2}, \cdot) \to \mathrm{Mor}^T_\Delta(\mathbf{n_s + 3}, \cdot)$ by precomposition),

$$w_0 \leq w_1 \quad \text{implies} \quad a^*_{s,i} w_0 \leq a^*_{s,i} w_1$$

and in fact the converse is also true.

The same is not true for annihilation operators. Terminal annihilations may map to zero, so that no comparison is possible. Even when nonzero we may have $w_0 \not\leq w_1$ but $a_{s,i} w_0 \leq a_{s,i} w_1$; eg $yxxy \not\leq xyyx$ but, applying $a_{1,x}$ to both sides, $yxy \leq yyx$. It is however true that if $w_0 \leq w_1$ and $a_{s,i} w_0 \neq 0$, $a_{s,i} w_1 \neq 0$ then $a_{s,i} w_0 \leq a_{s,i} w_1$.

Later on (Section 3.13) we shall examine this partial order in more detail; we will need notions of *difference* between words, and *minimum* and *maximum* of two words.

Definition 2.3 For two words $w_0, w_1 \in \mathcal{M}_{n_x, n_y}$ (comparable or not), their *difference* is

$$d(w_0, w_1) = \sum_{i=1}^{n_x} h_{w_1}^x(i) - h_{w_0}^x(i).$$

As $h_w^x(i)$ is the position of the i–th x in w, $h_{w_1}^x(i) - h_{w_0}^x(i)$ gives the difference in position between the i–th x's in w_0 and w_1. If we regard x's as pawns and y's as empty squares on a $1 \times n$ chessboard, then $d(w_0, w_1)$ is the number of signed (rightwards = positive) pawn moves required to go from w_0 to w_1. If $w_0 \leq w_1$, then only rightwards moves are required and $d(w_0, w_1)$ is the number of such moves.

Lemma 2.4 *For any $w_0, w_1 \in \mathcal{M}_{n_x, n_y}$ (comparable or not), there are decompositions*

$$w_0 = w_0^0 w_0^1 \cdots w_0^{2k-1}, \quad w_1 = w_1^0 w_1^1 \cdots w_1^{2k-1},$$

where $w_0^i \leq w_1^i$ if i is even and $w_1^i \leq w_0^i$ if i is odd. The words $w_0^0, w_1^0, w_0^{2k-1}, w_1^{2k-1}$ might be empty, but the other w_j^i are not. (In particular, each w_0^i, w_1^i have equal x– and y–degrees.)

Proof Consider the baseball interpretation of [17]: when scores are level, split the words w_0, w_1; once on each interval of the game in which scores remain level. Any baseball game splits into subgames on which the leading team alternates, giving the desired inequality. $\qquad\square$

Having done this, let

$$w_- = w_0^0 w_1^1 w_0^2 w_1^3 \cdots w_1^{2k-1}, \quad w_+ = w_1^0 w_0^1 w_1^2 w_0^3 \cdots w_0^{2k-1}.$$

One can easily verify

$$f_{w_-}^x = \min(f_{w_0}^x, f_{w_1}^x), \quad g_{w_-}^y = \min(g_{w_0}^y, g_{w_1}^y), \quad h_{w_-}^x = \min(h_{w_0}^x, h_{w_1}^x),$$
$$f_{w_+}^x = \max(f_{w_0}^x, f_{w_1}^x), \quad g_{w_+}^y = \max(g_{w_0}^y, g_{w_1}^y), \quad h_{w_+}^x = \max(h_{w_0}^x, h_{w_1}^x),$$

etc. In particular, $w_- \leq w_+$ and, although there might be many ways to split w_0, w_1 (baseball can remain tied for several innings), the resulting w_-, w_+ are unique. Call w_- the *minimum* and w_+ the *maximum* of the pair (w_0, w_1). If $w_0 \leq w_1$ then $\min(w_0, w_1) = w_0$ and $\max(w_0, w_1) = w_1$.

2.5 Adjoints, bilinear form

The notation $a_{s,i}, a_{s,i}^*$ for annihilation and creation operators suggests that they should be adjoint. They are indeed adjoint with respect to $\leq$, but only in one direction each. We have

$$a_{y,i}^* w_0 \leq w_1 \quad \text{if and only if} \quad w_0 \leq a_{y,i} w_1$$

for $0 \leq i \leq n_y(w_1)$, the y–degree of w_1 (note w_1 has y–degree one larger than that of w_0). Also,

$$a_{x,i} w_0 \leq w_1 \quad \text{if and only if} \quad w_0 \leq a_{x,i}^* w_1$$

for $0 \leq i \leq n_x(w_0)$ (here w_0 has x–degree one larger than that of w_1). These inequalities hold even for terminal creations and annihilations: if a terminal annihilation gives 0, the inequality is false.

These "adjoint inequalities" are not true in the other direction. For instance, it is not true that $a_{y,i} w_0 \leq w_1$ if and only if $w_0 \leq a_{y,i}^* w_1$, since $xxy = a_{y,1} yxxy \leq xyx$ but $yxxy \not\leq a_{y,1}^* xyx = xyyx$.

As mentioned in the introduction, define a "booleanization" $\langle \cdot | \cdot \rangle \colon \mathcal{M} \times \mathcal{M} \longrightarrow \{0, 1\}$. For words $w_0, w_1 \in \mathcal{M}$, $\langle w_0 | w_1 \rangle = 1$ if and only if $w_0 \leq w_1$; otherwise $\langle w_0 | w_1 \rangle = 0$. We then extend linearly to $\langle \cdot | \cdot \rangle \colon \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathbb{Z}$. The following properties of $\langle \cdot | \cdot \rangle$ are easy to verify.

- It is bilinear over $\mathbb{Z}$ (by definition).

- It is not symmetric, indeed far from it: if $\langle w_0 | w_1 \rangle = \langle w_1 | w_0 \rangle = 1$ then $w_0 = w_1$.

- "Words have norm one": $\langle w | w \rangle = 1$.

- The decomposition $\mathcal{F} = \bigoplus_{n_x, n_y} \mathcal{F}_{n_x, n_y}$ is orthogonal with respect to $\langle \cdot | \cdot \rangle$: if w_0, w_1 have different x– or y–degree then $\langle w_0 | w_1 \rangle = 0$.

- Creation and annihilation operators $a_{s,i}, a_{s,i}^*$ are partially adjoint,

  $$\langle a_{x,i} w_0 | w_1 \rangle = \langle w_0 | a_{x,i}^* w_1 \rangle \quad \text{and} \quad \langle w_0 | a_{y,i} w_1 \rangle = \langle a_{y,i}^* w_0 | w_1 \rangle,$$

  for $0 \leq i \leq n_x(w_0)$, $0 \leq i \leq n_y(w_1)$ respectively.

- "Creation operators are isometries" (but annihilation operators are not):

  $$\langle w_0 | w_1 \rangle = \langle a_{s,i}^* w_0 | a_{s,i}^* w_1 \rangle.$$

- "Multiplicativity": if $(a, c) \in \mathcal{M}_{n_x, n_y}$ and $(b, d) \in \mathcal{M}_{n_x', n_y'}$, then $\langle ab | cd \rangle = \langle a | c \rangle \langle b | d \rangle$.

The bilinear form $\langle \cdot | \cdot \rangle$ is also nondegenerate. Suppose $\langle v | \cdot \rangle = 0$, for some $v \neq 0$, $v = \sum_i a_i w_i$, where w_i are distinct words and $0 \neq a_i \in \mathbb{Z}$. By orthogonality of each $\mathcal{F}_{n_x, n_y}$ we may assume all w_i have the same x– and y–degree. Let w_- be the lexicographically first among the w_i; then we have $\langle w_- | w_- \rangle = 1$ but for every other w_i, $\langle w_i | w_- \rangle = 0$. Thus $0 = \langle v | w_- \rangle = \langle a_- w_- | w_- \rangle = a_- \neq 0$, a contradiction. Similarly if $\langle \cdot | v \rangle = 0$ then by taking w_+, the lexicographically last among the w_i, we obtain $\langle w_+ | v \rangle \neq 0$, a contradiction. This gives nondegeneracy.

In fact, the bilinear form $\langle \ | \ \rangle$ is essentially unique, in the following sense.

Proposition 2.5 *Suppose* $B \colon \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathbb{Z}$ *is a bilinear form such that:*

(i) *Distinct* $\mathcal{F}_n^e$ *are orthogonal: if* $w_0 \in \mathcal{M}_{n_0}^{e_0}$, $w_1 \in \mathcal{M}_{n_1}^{e_1}$, $(n_0, e_0) \neq (n_1, e_1)$, *then* $B(w_0, w_1) = 0$.

(ii) *Creations and annihilations are partially adjoint:* $B(a_{x,i} w_0, w_1) = B(w_0, a_{x,i}^* w_1)$ *for* $0 \leq i \leq n_x(w_0)$ *and* $B(w_0, a_{y,i} w_1) = B(a_{y,i}^* w_0, w_1)$ *for* $0 \leq i \leq n_y(w_1)$.

(iii) $B(1, 1) = 1$.

Then $B = \langle \cdot \mid \cdot \rangle$.

Proof   First note that condition (ii) implies that creation operators are isometries:

$$B(a_{x,i}^* w_0, a_{x,i}^* w_1) = B(a_{x,i} a_{x,i}^* w_0, w_1) = B(w_0, w_1),$$
$$B(a_{y,i}^* w_0, a_{y,i}^* w_1) = B(w_0, a_{y,i} a_{y,i}^* w_1) = B(w_0, w_1).$$

Given any two words $w_0, w_1 \in \mathcal{M}_n^e$, we note that $B(w_0, w_1)$ can be simplified if w_0 has a repeated y or if w_1 has a repeated x, since then $w_0 = a_{y,i}^* w_0'$ or $w_1 = a_{x,i}^* w_1'$; so suppose there are no such repeated symbols. If w_0, w_1 begin with the same symbol then we may simplify, since then $w_0 = a_{y,0}^* w_0'$ or $w_1 = a_{x,0}^* w_1'$; so suppose they do not. If w_0 begins with y and w_1 begins with x, then let $w_1 = a_{x,0}^* w_1'$ so that $B(w_0, w_1) = B(w_0, a_{x,0}^* w_1') = B(a_{x,0} w_0, w_1') = 0$. Thus we may simplify to the case where w_0 begins with x and has no repeated y; and w_1 begins with y and has no repeated x; but w_0, w_1 lie in the same $\mathcal{M}_n^e$.

Thus w_0 has $e \leq 0$; and w_1 has $e \geq 0$; lying in the same $\mathcal{M}_n^e$, we must have $e = 0$, and in fact $w_0 = (xy)^m$, $w_1 = (yx)^m$. We now compute, for $m \geq 1$:

$$\begin{aligned}
B\big((xy)^m, (yx)^m\big) &= B\big((xy)^m y, (yx)^m y\big) \\
&= B\big(a_{y,m}^* (xy)^m, (yx)^m y\big) \\
&= B\big((xy)^m, a_{y,m}(yx)^m y\big) \\
&= B\big((xy)^m, (yx)^{m-1} xy\big) = B\big((xy)^{m-1}, (yx)^{m-1}\big).
\end{aligned}$$

(In the first line we apply a terminal creation. In the second line we use adjoint relations to reorder some symbols. In the third line we remove terminal creations.) Applying this repeatedly we have $B((xy)^m, (yx)^m) = B(1, 1) = 1$, and hence $B = \langle \cdot \mid \cdot \rangle$. $\square$

We remark that, since the "stackability map" of [17], mod 2, easily satisfies the hypotheses of this proposition, this gives another proof of the result, proved directly in [17], that the stackability map mod 2 is the boolean version of $\leq$.

2.6 Duality

By nondegeneracy, the bilinear form $\langle \cdot \,|\, \cdot \rangle$ gives a duality map on each $\mathcal{F}_{n_x, n_y}$, which is an isomorphism, at least over the rationals. In fact, as $\langle \cdot \,|\, \cdot \rangle$ is asymmetric, there are two such maps, which are isomorphisms over the rationals. (We will shortly see that these are also isomorphisms over $\mathbb{Z}$.)

$$\iota_- \colon \mathcal{F}_{n_x, n_y} \xrightarrow{\;\cong\;} (\mathcal{F}_{n_x, n_y})^*, \quad v \mapsto \langle v| \cdot \rangle$$

$$\iota_+ \colon \mathcal{F}_{n_x, n_y} \xrightarrow{\;\cong\;} (\mathcal{F}_{n_x, n_y})^*, \quad v \mapsto \langle \cdot |v \rangle$$

Composing these two maps in the two possible directions gives two inverse automorphisms of each $\mathcal{F}_n^e$:

$$H = \iota_+^{-1} \circ \iota_- \colon \mathcal{F}_n^e \longrightarrow \mathcal{F}_n^e,$$

$$H^{-1} = \iota_-^{-1} \circ \iota_+ \colon \mathcal{F}_n^e \longrightarrow \mathcal{F}_n^e.$$

It's clear from the definition that $\langle u|v \rangle = \langle v|Hu \rangle = \langle H^{-1}v|u \rangle$. Thus H is "unitary" with respect to $\langle \cdot \,|\, \cdot \rangle$, $H^* = H^{-1}$; H is an isometry, $\langle u|v \rangle = \langle Hu|Hv \rangle$.

We will show in Section 5.4 that H is periodic; we know no direct algebraic proof of this result.

Theorem 2.6 On $\mathcal{F}_{n_x, n_y}$, $H^{n+1} = (-1)^{n_x n_y}$. *In particular* $H^{2n+2} = 1$. *The period of H is* $2n + 2$, *if both* n_x, n_y *are odd; else the period is* $n + 1$.

There is another bilinear form on $\mathcal{F}$, which we denote by a dot $\cdot$, with respect to which the basis $\mathcal{M}$ of words is orthonormal, ie $w_0 \cdot w_1 = 1$ if $w_0 = w_1$ and $w_0 \cdot w_1 = 0$ otherwise. This $\cdot$ is clearly symmetric, $\mathbb{Z}$–bilinear, and nondegenerate. By nondegeneracy of both bilinear forms $\langle \cdot \,|\, \cdot \rangle$ and $\cdot$, it follows that there are operators $Q_+, Q_- \colon \mathcal{F}_n^e \longrightarrow \mathcal{F}_n^e$, over $\mathbb{Q}$, which are isomorphisms over $\mathbb{Q}$, intertwining the two forms, ie such that for all $u, v \in \mathcal{F}_n^e$,

$$u \cdot v = \langle u|Q_+v \rangle = \langle Q_-u|v \rangle.$$

But since $\cdot$ is symmetric, we have also

$$u \cdot v = v \cdot u = \langle v|Q_+u \rangle = \langle Q_-v|u \rangle = \langle H^{-1}Q_+u|v \rangle = \langle u|HQ_-v \rangle.$$

It follows that

$$H = Q_+ Q_-^{-1}.$$

Reading a word w from left to right, an x which is immediately followed by a y is called an *exceptional* x; E_w^x denotes the set of exceptional x's in w. For every subset $T \subseteq E_w^x$, let $\psi_T^x w$ denote the operation of taking each x in T, and its immediately following y, and replacing this xy with yx, ie "moving the x forwards one position".

For example, if $w = x_1 x_2 y_1 y_2 x_3 y_3$ (we use subscripts to distinguish letters), then $E_w^x = \{x_2, x_3\}$, and if $T = \{x_2\}$ then $\psi_T^x w = x_1 y_1 x_2 y_2 x_3 y_3$. Similarly E_w^y denotes the set of exceptional y's in w, ie those which are followed by an x; and for $T \subseteq E_w^y$, $\psi_T^y w$ denotes the operation of taking each y in T, and its following x, and replacing this yx with xy, "moving the y forwards one position". If our word is a one-dimensional chessboard, x's are pawns, and y's are empty squares, then ψ^x advances pawns to the right, and ψ^y advances pawns to the left.

We have the following explicit formulas for $Q_\pm^{\pm 1}$.

Proposition 2.7 For any word $w \in \mathcal{M}_n^e$,

$$Q_+ w = \sum_{T \subseteq E_w^y} (-1)^{|T|} \psi_T^y w, \quad Q_+^{-1} w = \sum_{w_i \leq w} w_i,$$

$$Q_- w = \sum_{T \subseteq E_w^x} (-1)^{|T|} \psi_T^x w, \quad Q_-^{-1} w = \sum_{w_i \geq w} w_i.$$

(So, for example, if $w = xyxy$ then $Q_- w = xyxy - xyyx - yxxy + yxyx$ and $Q_+ w = xyxy - xxyy$.)

Proof First, consider Q_+^{-1}. For any words $w_0, w_1 \in \mathcal{M}_n^e \subset \mathcal{F}_n^e$ we have

$$w_0 \cdot Q_+^{-1} w_1 = \langle w_0 | w_1 \rangle = \begin{cases} 1 & w_0 \leq w_1, \\ 0 & \text{otherwise.} \end{cases}$$

As words are all orthogonal with respect to $\cdot$, $Q_+^{-1} w_1$ contains w_0 with coefficient 1 whenever $w_0 \leq w_1$, otherwise with coefficient 0. Thus $Q_+^{-1} w$ is as claimed. The proof for Q_-^{-1} is similar.

Now consider Q_-; the case of Q_+ is similar. We will show that, for all $w, w_0 \in \mathcal{M}_n^e$,

$$\sum_{T \subseteq E_w^x} (-1)^{|T|} \langle \psi_T^x w \mid w_0 \rangle = \left\langle \sum_{T \subseteq E_w^x} (-1)^{|T|} \psi_T^x w \,\middle|\, w_0 \right\rangle = w \cdot w_0 = \begin{cases} 1 & w_0 = w, \\ 0 & \text{otherwise,} \end{cases}$$

which gives the desired expression for Q_-. Consider $f_w^x : \{1, \ldots, n_x\} \longrightarrow \{0, \ldots, n_y\}$, ie $f_w^x(i)$ is the number of y's to the left of the i–th x in w. Note that ψ_T^x has the following effect on f_w^x:

$$f_{\psi_T^x w}^x(i) = \begin{cases} f_w^x(i) & \text{if the } x \text{ numbered } i \text{ is not in } T, \\ f_w^x(i) + 1 & \text{if the } x \text{ numbered } i \text{ is in } T. \end{cases}$$

Thus, $\langle \psi_T^x w | w_0 \rangle = 1$ if and only if $f_w^x(i) \leq f_{w_0}^x(i)$ for all i numbering x's of w not in T, and $f_w^x(i) + 1 \leq f_{w_0}^x(i)$ for all i numbering x's of w in T. Let $\{1, \ldots, n_x\}$ be partitioned into three sets

$$
\begin{aligned}
S_0 &= \left\{ i : f_w^x(i) \leq f_{w_0}^x(i) - 1 \right\}, \\
S_1 &= \left\{ i : f_w^x(i) = f_{w_0}^x(i) \right\}, \\
S_2 &= \left\{ i : f_w^x(i) \geq f_{w_0}^x(i) + 1 \right\}.
\end{aligned}
$$

We then have $\langle \psi_T^x w | w_0 \rangle = 1$ if and only if $T \subseteq S_0$ and $S_2 = \varnothing$.

If S_2 is nonempty, then $\langle w | w_0 \rangle = 0$, and any $\langle \psi_T^x w | w_0 \rangle = 0$; also $w_0 \cdot w = 0$. Thus, we may assume $S_2 = \varnothing$. Then $\langle \psi_T^x w | w_0 \rangle = 1$ if and only if $T \subseteq S_0$. Letting $T_0 = S_0 \cap E_w^x$, we then have

$$
\sum_{T \subseteq E_w^x} (-1)^{|T|} \langle \psi_T^x w | w_0 \rangle = \sum_{T \subseteq T_0} (-1)^{|T|} \langle \psi_T^x w | w_0 \rangle = \sum_{T \subseteq T_0} (-1)^{|T|}.
$$

If T_0 is nonempty then this sum is 0; and also S_0 is nonempty, so $w_0 \neq w$. If T_0 is empty then this sum is 1 and $E_w^x \subseteq S_1$, which means that $w \leq w_0$ and the exceptional x's in w lie in the same positions as the corresponding x in w_0; this implies $w = w_0$. So the expression for Q_- is as claimed. $\qquad\square$

Corollary 2.8

$$
Hw = Q_+ Q_-^{-1} w = \sum_{w_i \geq w} \sum_{T \subseteq E_{w_i}^y} (-1)^{|T|} \psi_T^y w_i \qquad\qquad \square
$$

We shall discuss H in more detail in Sections 5.2–5.4. Note that all $Q_\pm^{\pm 1}$ and H have integer coefficients. It follows that all of the isomorphisms are over $\mathbb{Z}$, not just over $\mathbb{Q}$.

2.7 Differentials, commutation relations, normal form

In quantum field theory, creation and annihilation operators a^*, a usually satisfy $[a, a^*] = 1$ or $\{a, a^*\} = 1$; a^* creates a particle, while a annihilates each particle in turn and sums the results. Thus the compositions aa^* and a^*a count an identical set of situations, except that aa^* counts the original state once more. Thus (for bosons) $aa^* - a^*a = 1$ or (for fermions) $aa^* + a^*a = 1$.

Our noncommutative setting is more symmetric: creations create once, and annihilations annihilate once. But if we take a sum (or alternating sum) over i of annihilation

operators $a_{s,i}$, we obtain the usual relations. Define operators

$$a_s = \sum_{i=1}^{n_s} a_{s,i}, \quad d_s = \sum_{i=1}^{n_s} (-1)^i a_{s,i} \quad \text{on } \mathcal{F}_{n_x, n_y}.$$

In the first case we obtain $[a_s, a_{s,0}^*] = 1$; in the second case we have $\{d_s, a_{s,0}^*\} = 1$.

Note that a_s is simply partial differentiation by the symbol s, $a_s = \partial/\partial s$. It follows that a_x, a_y commute and obey the Leibniz rule.

On the other hand, d_s behaves like an exterior differential; $d_s^2 = 0$; $d_s(w_0 w_1) = (d_s w_0) w_1 + (-1)^k w_0 (d_s w_1)$ where k is the degree of w_0 in s; and $d_x d_y = d_y d_x$. We may regard d_x, d_y as boundary operators arising from the two simplicial structures on $\mathcal{F}$. (For this we could also take $d_s = \sum_{i=0}^{n_s+1} (-1)^i a_{s,i}$, the sum including terminal annihilations; this is also a differential, $d_s^2 = 0$ and $[d_x, d_y] = 0$. Our choice fits better with the physical analogy.)

The differentials d_x, d_y equip $\mathcal{F}$ with the structure of a double chain complex, and the relation $\{d_s, a_{s,0}^*\} = 1$ says that $a_{s,0}^*$ is a chain homotopy from 1 to 0; thus the homology of this complex is trivial. In fact, the relation $d_s a_{s,0}^* + a_{s,0}^* d_s = 1$ explicitly shows that any "closed" element z, $d_s z = 0$, is a "boundary" since $d_s a_{s,0}^* z + a_{s,0}^* d_s z = d_s(a_{s,0}^* z) = z$. (cf [17, Proposition 7.5], following Frabetti [5].)

The simplicial set relations show us how to commute creation and annihilation operators of the same species $s \in \{x, y\}$. It follows that any sequence of l s–creation and k s–annihilation operators can be written uniquely in the form

$$a_{s,i_1}^* a_{s,i_2}^* \cdots a_{s,i_k}^* a_{s,j_1} a_{s,j_2} \cdots a_{s,j_l},$$

where $i_1 > i_2 > \cdots > i_k$ (note strict inequality), and $j_1 \geq j_2 \geq \cdots \geq j_l$.

2.8 Temperley–Lieb representation

We now introduce more operators on $\mathcal{F}$: for $s \in \{x, y\}$ and $0 \leq i \leq n_s$,

$$T_{s,i} = a_{s,i} - a_{s,i+1}, \quad T_{s,i}^* = a_{s,i}^* - a_{s,i+1}^*.$$

These $T_{s,i}$ have various obvious properties. For instance, with $1 \leq i \leq n_s - 1$, $T_{s,i} w = 0$ if and only if w has the i–th and $(i+1)$–st s symbols adjacent. Further,

$$a_{s,i+1} T_{s,i}^* = a_{s,i+1}(a_{s,i}^* - a_{s,i+1}^*) = 1 - 1 = 0,$$

and so if we define

$$U_{s,i} = T_{s,i}^* a_{s,i+1} = (a_{s,i}^* - a_{s,i+1}^*) a_{s,i+1}$$

then $(U_{s,i}^*)^2 = 0$. For $1 \le i \le n_s - 1$, the effect of $U_{s,i}$ on a word w is to return 0, if the i–th and $(i+2)$–nd s symbols are in the same block. Otherwise it gives difference between the two words, one obtained by moving the $(i+1)$–st s back to be adjacent to the i–th s, the other obtained by moving the $(i+1)$–st s forward to the $(i+2)$–nd s. This makes combinatorially clear why $(U_{s,i})^2 = 0$, and that $U_{s,i}$ and $U_{s,j}$ commute when $|i - j| \ge 2$.

Consider $U_{s,i}, U_{s,j}$ when $|i - j| = 1$. We can compute (eg putting in normal form) that

$$U_{s,i} U_{s,i+1} U_{s,i} = -U_{s,i},$$
$$U_{s,i+1} U_{s,i} U_{s,i+1} = -U_{s,i+1}.$$

Thus the $U_{s,i}$ satisfy relations similar to the *Temperley–Lieb algebra*. This algebra is defined by the relations $U_i^2 = \delta U_i$, $U_i U_{i+1} U_i = U_i$, $U_{i+1} U_i U_{i+1} = U_{i+1}$, and $U_i U_j = U_j U_i$ for $|i - j| \ge 2$ [20; 1; 2]; we have $\delta = 0$ and some sign changes, a "twisted" representation.

2.9 A distinguished subset

We will now define some distinguished elements in $\mathcal{F}$. Define

$$\mathcal{C}^1 = \{a_{s,i}^*, a_{s,i}, T_{s,i}^*\} \cdot 1.$$

That is, $\mathcal{C}^1$ is the orbit of 1 under the action of the operators $a_{s,i}^*, a_{s,i}, T_{s,i}^*$, over all $s \in \{x, y\}$ and all $0 \le i \le n_s + 1$ in each $\mathcal{F}_n^e$. Similarly, define

$$\mathcal{C}^2 = \{a_{s,0}^*, a_{s,0}, H\} \cdot 1,$$
$$\mathcal{C}^3 = \{a_{s,n_s+1}^*, a_{s,n_s+1}, H\} \cdot 1.$$

In fact these are all the same (and one can easily find other sets of operators giving the same orbit).

Proposition 2.9 $\mathcal{C}^1 = \mathcal{C}^2 = \mathcal{C}^3$.

Denote this set $\mathcal{C}$. Let $\mathcal{C}_n = \mathcal{C} \cap \mathcal{F}_n$, $\mathcal{C}_{n_x,n_y} = \mathcal{C} \cap \mathcal{F}_{n_x,n_y}$, $\mathcal{C}_n^e = \mathcal{C} \cap \mathcal{F}_n^e$.

Theorem 2.10

(i) *The set $\mathcal{C}$ is not closed under addition or multiplication by $\mathbb{Z}$. However $\mathcal{C}$ is closed under negation, multiplication, creation and annihilation operators, and the operators $T_{s,i}, T_{s,i}^*, U_{s,i}$.*

(ii) (a) Each element $v \in \mathcal{C}$ lies in some $\mathcal{F}_n^e$ ($v \in \mathcal{C}_n^e$) and is of the form $v = \sum_i a_i w_i$ where $a_i = \pm 1$ and the w_i are distinct words in $\mathcal{M}_n^e$.

   (b) Among the words w_i there is a lexicographically first w_- and last w_+. Then for all i, $w_- \leq w_i \leq w_+$. If a word w_0 among the w_i is comparable to all these w_i with respect to $\leq$, then $w_0 \in \{w_-, w_+\}$. If $w_- \neq w_+$ then $\sum a_i = 0$.

   (c) For every pair $w_- \leq w_+$ in $\mathcal{M}_n^e$ there are precisely two $v \in \mathcal{C}_n^e$ with w_-, w_+ being the lexicographically first and last words occurring in v; one is the negative of the other.

   (d) The number of pairs $w_- \leq w_+$ in $\mathcal{M}_n^e$ is the Narayana number N_n^e, and $\sum_e N_n^e = C_n$, the n–th Catalan number. Thus $|\mathcal{C}_n| = 2C_n$ and $|\mathcal{C}_n^e| = 2N_n^e$.

(iii) The operators Q_+, Q_- do not preserve $\mathcal{C}$; but $Q_+\mathcal{C} = Q_-\mathcal{C}$, and these have the same cardinality as $\mathcal{C}$ in each grading: $|Q_\pm \mathcal{C}_n^e| = 2N_n^e$.

(iv) For any $v \in \mathcal{C}$, $\langle v|v \rangle = 1$ and $\langle v|Hv \rangle = 1$. For any $v_0, v_1 \in \mathcal{C}$, $\langle v_0|v_1 \rangle \in \{-1, 0, 1\}$.

(v) If the sum or difference of two distinct nonzero elements u, v of $\mathcal{C}$ is also a nonzero element of $\mathcal{C}$, then after switching signs of u, v and swapping u, v if necessary, the triple is $u, v, u - v$, and $\langle u|v \rangle = 1$, $\langle v|u \rangle = 0$. There exists an operator $A^*: \mathcal{F}_2^0 \longrightarrow \mathcal{F}_n^e$, which is a composition of initial creation operators and applications of H, such that $A(xy) = u$ and $A(yx) = v$.

(vi) If $u, v \in \mathcal{C}$ and $\langle u|v \rangle = 1$, then there exists a sequence $u = v_0, v_1, \ldots, v_m = v$ in $\mathcal{C}$ such that each $v_i - v_{i+1} \in \mathcal{C}$; and for each $i \leq j$, $\langle v_i|v_j \rangle = 1$.

The proof of this theorem will be given via isomorphism with sutured TQFT in Section 5.1. The Narayana numbers N_n^e here are as defined in [17].

There are some further tenuous physical analogies and speculations. Elements of $\mathcal{C}$ can be regarded as "pure states". The operator H can perhaps be considered as a Hamiltonian generating a time evolution, its periodicity some manifestation of a term e^{iHt}. The operators $Q_\pm$, which taken together define H, can be considered as supersymmetry; $Q_+\mathcal{C}$ can be regarded as super-partner-states. Is there some interpretation of sutured manifolds with corners having spacelike and timelike boundaries, and a Hamiltonian giving something like a partition function for contact manifolds?

3 Sutured TQFT

3.1 Sutured surfaces

For our purposes, a *sutured 3–manifold* (M, Γ) is a 3–manifold with boundary M, with $\Gamma \subset \partial M$ an embedded oriented 1–manifold, such that $\partial M \backslash \Gamma = R_+ \cup R_-$, where

$\partial R_\pm = \Gamma$ as oriented 1–manifolds, and $R_\pm$ is oriented as $\pm \partial M$. (Crossing Γ along ∂M goes from $R_\pm$ to $R_\mp$.)

A sutured 3–manifold may have corners on its boundary; there may be a curve C on ∂M along which two smooth surfaces meet. If so, sutures are required not to match along C, but to *interleave*; see Figure 2. The surface may be *smoothed*; in doing so, sutures are rounded and complementary regions $R_\pm$ joined as shown.
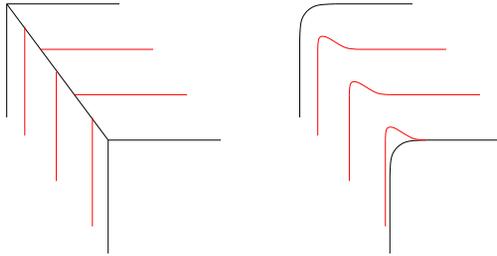
Figure 2: Edge smoothing and unsmoothing

In the following, we take sutures on surfaces, but without any associated 3–manifold. Define a *sutured surface* (Σ, Γ) to be a compact oriented surface Σ, possibly disconnected, with $\Gamma \subset \Sigma$ a properly embedded oriented 1-submanifold; Γ must have the property that $\Sigma \backslash \Gamma = R_+ \cup R_-$, where $R_\pm$ is oriented as $\pm \Sigma$, and $\overline{\partial R_\pm \backslash \partial \Sigma} = \Gamma$ as oriented 1–manifolds. Again (Σ, Γ) may have corners, with interleaving sutures, which may be smoothed as above. In this paper, we will only consider sutured surfaces with $\partial \Sigma \neq \varnothing$.

We define also a *sutured background* surface (Σ, F), which is a compact oriented surface Σ (possibly disconnected) with $\partial \Sigma \neq \varnothing$, together with a finite set of signed points $F \subset \partial \Sigma$, such that $\partial \Sigma \backslash F = C_+ \cup C_-$, where $C_\pm$ are oriented as $\pm \partial \Sigma$, and $\partial C_\pm = F$ as signed points. (Hence each boundary component C of Σ has a positive even number of points of F, which cut it alternately into arcs of C_+ and C_-.) Again a sutured background surface may have corners. A set of *sutures* Γ on a sutured background surface (Σ, F) is an oriented properly embedded 1-submanifold of Σ such that $\partial \Gamma = \partial \Sigma \cap \Gamma = F$ and such that (Σ, Γ) is a sutured surface, with $\partial R_\pm = \Gamma \cup C_\pm$.

Given a sutured background surface (Σ, F) with $\partial \Sigma \backslash F = C_+ \cup C_-$ as above, we define *gluings* of it. Consider two disjoint 1–manifolds $G_0, G_1 \subseteq \partial \Sigma$, and a homeomorphism $\tau \colon G_0 \xrightarrow{\cong} G_1$ which identifies marked points and positive/negative arcs, $G_0 \cap F \xrightarrow{\cong} G_1 \cap F$, $G_0 \cap C_\pm \xrightarrow{\cong} G_1 \cap C_\pm$. Then gluing (Σ, F) along τ gives a surface $\#_\tau(\Sigma, F)$. If there remain marked points on each boundary component then $\#_\tau(\Sigma, F)$ is also a sutured background surface and we call τ a *sutured gluing map*. If Γ is a set of sutures on (Σ, F) then a sutured gluing map gives a glued set of sutures $\#_\tau \Gamma$ on $\#_\tau(\Sigma, F)$.

A set of sutures Γ on (Σ, F) has a (relative) *Euler class* $e(\Gamma) = \chi(R_+) - \chi(R_-)$.

On the disc sutured background (D^2, F_{n+1}) with $|F_{n+1}| = 2(n+1)$, drawn in the plane and inheriting its orientation, we label the points of F_{n+1} with integers mod $2(n+1)$ as in [17]. We choose a basepoint, numbered 0; the arc of ∂D^2 immediately clockwise (resp. anticlockwise) of 0 must be signed positive (resp. negative). The points of F_{n+1} are numbered clockwise, modulo $2(n+1)$.

A set of sutures Γ on (D^2, F_{n+1}) without contractible components is just a chord diagram; it clearly has $|e(\Gamma)| \leq n$ and $e(\Gamma) \equiv n \bmod 2$. Given a chord diagram Γ on (D^2, F_{n+1}), from n and e we define $n_- = (n-e)/2$, $n_+ = (n+e)/2$, so n_-, n_+ are nonnegative integers, $n = n_- + n_+$ and $e = n_+ - n_-$. The point of F numbered $2n_+ + 1 \equiv -2n_- - 1 \bmod 2(n+1)$ is called the *root point*. Cutting ∂D^2 at base and root points gives two arcs. The arc containing $1, \ldots, 2n_+$ is called the *eastside*. The arc containing $-1, \ldots, -2n_-$ is called the *westside*.

3.2 TQFT axioms

As in any TQFT, we associate algebraic objects to our topological objects. Here the topological objects are sutured background surfaces, sets of sutures, and gluings. A *sutured TQFT* is a set of assignments as described by the following axioms.

Axiom 1 To each sutured background surface (Σ, F), assign an abelian group $V(\Sigma, F)$, depending only on the homeomorphism type of the pair (Σ, F).

Axiom 2 To a set of sutures Γ on (Σ, F), assign a subset of *suture elements* $c(\Gamma) \subset V(\Sigma, F)$, depending only on the isotopy class of Γ relative to boundary.

(We would have liked a single suture element, but this will turn out not to be possible, as we discuss in Section 3.4. Quantum states differing by a unit are physically indistinguishable.)

Axiom 3 For a sutured gluing map τ of a sutured background surface (Σ, F), assign a collection of linear maps $\Phi_\tau^i \colon V(\Sigma, F) \longrightarrow V(\#_\tau(\Sigma, F))$.

(We would have liked one canonical map, but this will turn out not to be possible.)

Axiom 4 For a finite disjoint union of sutured background surfaces $\bigsqcup_i (\Sigma_i, F_i)$,

$$V(\textstyle\bigsqcup_i (\Sigma_i, F_i)) = \bigotimes_i V(\Sigma_i, F_i).$$

Axiom 5 If Γ is a set of sutures on (Σ, F) and τ is a gluing of (Σ, F) then each Φ^i_τ takes suture elements to suture elements surjectively, $c(\Gamma) \to c(\#_\tau \Gamma)$.

These conditions are not standard TQFT axioms but are in a similar spirit.

It follows from the above that any $V(\Sigma, F)$ can be interpreted as a space of operators. Take some components of $\partial \Sigma$ and call them *incoming*; call the rest *outgoing*. Write $\partial \Sigma = (\partial \Sigma)_{\mathrm{in}} \cup (\partial \Sigma)_{\mathrm{out}}$ and $F = F_{\mathrm{in}} \cup F_{\mathrm{out}}$. Suppose we have a sutured background surface $(\Sigma_{\mathrm{in}}, F_{\mathrm{in}})$ which has boundary $(\partial \Sigma_{\mathrm{in}})$ identified with $(\partial \Sigma)_{\mathrm{in}}$ by a gluing τ; and τ identifies F_{in} and positive/negative boundary arcs on both surfaces. The gluing gives a sutured background surface $(\Sigma_{\mathrm{out}}, F_{\mathrm{out}})$, where $\Sigma_{\mathrm{out}} = \Sigma_{\mathrm{in}} \cup \Sigma$, and a (possibly not unique) map

$$\Phi_\tau \colon V(\Sigma_{\mathrm{in}}, F_{\mathrm{in}}) \otimes V(\Sigma, F) \longrightarrow V(\Sigma_{\mathrm{out}}, F_{\mathrm{out}}),$$

which is natural with respect to suture elements: if $\Gamma, \Gamma_{\mathrm{in}}$ are respectively sets of sutures on (Σ, F) and $(\Sigma_{\mathrm{in}}, F_{\mathrm{in}})$, then $\Gamma_{\mathrm{out}} = \Gamma_{\mathrm{in}} \cup \Gamma$ is a set of sutures on $(\Sigma_{\mathrm{out}}, F_{\mathrm{out}})$, and Φ_τ maps $c(\Gamma_{\mathrm{in}}) \otimes c(\Gamma) \to c(\Gamma_{\mathrm{out}})$.

Thus, a choice of sutures Γ on (Σ, F), together with a choice of representative $c \in c(\Gamma) \subset V(\Sigma, F)$ gives a specific $V(\Sigma_{\mathrm{in}}, F_{\mathrm{in}}) \longrightarrow V(\Sigma_{\mathrm{out}}, F_{\mathrm{out}})$ which we denote $\Phi_{\tau,c}$:

$$\Phi_{\tau,c} \colon V(\Sigma_{\mathrm{in}}, F_{\mathrm{in}}) \longrightarrow V(\Sigma_{\mathrm{out}}, F_{\mathrm{out}}), \quad x \mapsto \Phi_\tau(x \otimes c).$$

This is perhaps closer in spirit to a cobordism map. In this formulation, we view the situation as an inclusion of background surfaces $(\Sigma_{\mathrm{in}}, F_{\mathrm{in}}) \hookrightarrow (\Sigma_{\mathrm{out}}, F_{\mathrm{out}})$, together with a set of sutures Γ on $(\Sigma_{\mathrm{out}} \backslash \Sigma_{\mathrm{in}}, F_{\mathrm{in}} \cup F_{\mathrm{out}})$. Note that the inclusion must be strict in the sense that Σ_{in} lies in the interior of Σ_{out}. In this way we can regard our "TQFT" is a "$2 + 1 = 2$–dimensional TQFT".

In fact, we can reformulate gluing Axioms 3 and 5 in terms of inclusions, in line with [10].

Axiom 3$'$ To an inclusion $\iota(\Sigma_{\mathrm{in}}, F_{\mathrm{in}}) \hookrightarrow (\Sigma_{\mathrm{out}}, F_{\mathrm{out}})$ of sutured background surfaces, with Σ_{in} lying in the interior of Σ_{out}, together with Γ a set of sutures on $(\Sigma_{\mathrm{out}} \backslash \Sigma_{\mathrm{in}}, F_{\mathrm{in}} \cup F_{\mathrm{out}})$, assign a collection of linear maps $\Phi^i_{\iota, \Gamma} \colon V(\Sigma_{\mathrm{in}}, F_{\mathrm{in}}) \longrightarrow V(\Sigma_{\mathrm{out}}, F_{\mathrm{out}})$.

Axiom 5$'$ If Γ_{in} is a set of sutures on $(\Sigma_{\mathrm{in}}, F_{\mathrm{in}})$, let $\Gamma_{\mathrm{out}} = \Gamma_{\mathrm{in}} \cup \Gamma$ be the corresponding set of sutures on $(\Sigma_{\mathrm{out}}, F_{\mathrm{out}})$. Then each $\Phi^i_{\iota, \Gamma}$ maps takes suture elements to suture elements surjectively, $c(\Gamma_{\mathrm{in}}) \to c(\Gamma_{\mathrm{out}})$.

Lemma 3.1 *Axioms 3 and 5 are equivalent to Axioms 3$'$ and 5$'$.*

Proof Assuming Axioms 3 and 5, the above discussion shows how to regard the inclusion $\iota\colon (\Sigma_{\text{in}}, F_{\text{in}}) \hookrightarrow (\Sigma_{\text{out}}, F_{\text{out}})$ as a gluing τ; for each choice of suture element $c \in c(\Gamma)$, there are maps $V(\Sigma_{\text{in}}, F_{\text{in}}) \longrightarrow V(\Sigma_{\text{out}}, F_{\text{out}})$. We take all the maps so obtained over the possible choices for c.

In the other direction, assume Axioms $3'$ and $5'$. Given a gluing τ of (Σ, F), removing a neighbourhood of the boundary of (Σ, F), after gluing, gives an inclusion $\iota\colon (\Sigma, F) \hookrightarrow \#_\tau(\Sigma, F)$, and lying in its interior. Moreover, there is a natural set of sutures Γ on a neighbourhood of the boundary of (Σ, F), taking a product neighbourhood of the boundary. We obtain inclusion maps $\Phi^i_{\iota,\Gamma}\colon V(\Sigma, F) \longrightarrow V(\#_\tau(\Sigma, F))$, which we define to be Φ^i_τ. (See [10, Figure 13].) $\qquad\qquad\square$

We now impose the condition that "overtwisted contact elements are zero".

Axiom 6 If Γ contains a closed contractible loop then $c(\Gamma) = \{0\}$.

(Perhaps more generally we could set $c(\Gamma)$ to be some power of an indeterminate δ; this would be closer to analogous situations in knot theory. But the analogy from contact geometry suggests that one closed loop is equivalent to many.)

Consider (D^2, F_1), the disc with two marked points. There is only one chord diagram on (D^2, F_1). We call this Γ the *vacuum* $\Gamma_\varnothing$, and (D^2, F_1) the *vacuum background*. We will impose the standard QFT condition that the vacuum is nonzero, and normalise.

Axiom 7 $V(D^2, F_1) = \mathbb{Z}$ and $c(\Gamma_\varnothing) \subseteq \{-1, 1\}$.

In more generality, we could set $V(D^2, F_1)$ to be a ground ring R and then say that sutured TQFT is *over R coefficients*. In [17] we essentially considered sutured TQFT over $\mathbb{Z}_2$; here we focus on $\mathbb{Z}$ coefficients, though we will need to make reference to $\mathbb{Z}_2$ coefficients and previous work at times.

For any sutured background (Σ, F) with sutures Γ, there is an inclusion $(D^2, F_1) \hookrightarrow (\Sigma, F)$, and sutures Γ' on $(\Sigma \backslash D^2, F \cup F_1)$, taking $\Gamma_\varnothing \mapsto \Gamma_\varnothing \cup \Gamma' = \Gamma$. Axiom $3'$ gives a map $\mathbb{Z} \cong V(D^2, F_1) \longrightarrow V(\Sigma, F)$ taking $c(\Gamma_\varnothing) \subseteq \{-1, 1\} \rightarrow c(\Gamma)$ surjectively and we immediately have the following.

Lemma 3.2 For any sutures Γ on any (Σ, F), $c(\Gamma)$ is either a singleton or is of the form $\{c, -c\}$. $\qquad\qquad\square$

Over $\mathbb{Z}_2$ coefficients, these two possibilities are the same, and $c(\Gamma)$ is always a single element.

Any disc with a chord diagram may be included into a larger sutured disc in which the sutures simplify to the vacuum. The corresponding inclusion gives a map $V(D^2, F_n) \longrightarrow V(D^2, F_1) = \mathbb{Z}$ which takes $c(\Gamma) \to c(\Gamma_\varnothing) \subseteq \{-1, 1\}$ surjectively.

Lemma 3.3 *If Γ is a chord diagram then every $c \in c(\Gamma)$ is nonzero, primitive and nontorsion.* □

The "interesting" elements in each $V(\Sigma, F)$ are the suture elements. Largely for reasons of convenience, we will require our sutured TQFT to be "minimal" in the following sense.

Axiom 8 Every $V(\Sigma, F)$ is spanned by suture elements.

A "free presutured TQFT" is easily defined satisfying Axioms 1–8; it will not satisfy Axiom 9, but we shall consider it in Section 3.13. Take $V(\Sigma, F) = \bigoplus_\Gamma \mathbb{Z} c_\Gamma$, freely generated by elements c_Γ, one for each isotopy class of sutures Γ on (Σ, F) without contractible loops. We set $c(\Gamma) = \{c_\Gamma\}$ if Γ has no contractible loops, otherwise $c(\Gamma) = \{0\}$. Gluing maps are defined in the obvious way, $\Phi_\tau c_\Gamma = c_{\#_\tau \Gamma}$ or 0 accordingly as contractible components are seen or not. Clearly Axioms 1–8 are satisfied.

The final axiom, which essentially gives all the structure in the TQFT, involves the "stacking" bilinear form $\langle \cdot | \cdot \rangle \colon V(D^2, F_n) \otimes V(D^2, F_n) \longrightarrow \mathbb{Z}$. Consider the cylinder $S^1 \times [0, 1]$, with $2n$ parallel sutures $\{\cdot\} \times [0, 1]$; and then remove a small neighbourhood of a point on one of those sutures (a "leak"). Call this sutured surface (Σ, Γ); topologically Σ is a pair of pants. Regard the two ends of the cylinder as incoming and the "leak" as outgoing. Consider gluing two incoming sutured background discs (D^2, F_n) to (Σ, Γ); we glue them to the ends of the cylinder along corners as in $\partial(D^2 \times [0, 1])$; after gluing and rounding, we have the vacuum background (D^2, F_1). Choosing a $c \in c(\Gamma) \subset V(\Sigma, \Gamma)$, we obtain a map

$$\langle \cdot | \cdot \rangle = \Phi_{\tau, c} \colon V(D^2, F_n) \otimes V(D^2, F_n) \longrightarrow V(D^2, F_1) = \mathbb{Z}.$$

(There may be several choices for $\langle \cdot | \cdot \rangle$. For now, make an arbitrary choice; we may adjust it later.) Thus $\langle \cdot | \cdot \rangle$ describes "inclusion into a leaky cylinder". If we have sutures Γ_0, Γ_1 on (D^2, F_n) and $c_0 \in c(\Gamma_0)$, $c_1 \in c(\Gamma_1)$, then gluing them into the leaky cylinder at $S^1 \times \{0\}$, $S^1 \times \{1\}$ respectively, and rounding corners, gives a set of sutures on (D^2, F_1). Either this is the vacuum, and $\langle c_0 | c_1 \rangle = \pm 1$, or there is a contractible suture, and $\langle c_0 | c_1 \rangle = 0$. Forgetting the leak, $\langle c_0 | c_1 \rangle = \pm 1$ (resp. 0) if and only if after rounding corners on the cylinder, we have a sutured sphere with connected (resp. disconnected) sutures.

Axiom 9 Suppose two elements $x, y \in V(D^2, F_n)$ have the following property: for any set of sutures Γ on (D^2, F_n), there exists $c \in c(\Gamma)$ such that $\langle x|c \rangle = \pm \langle y|c \rangle$. Then $x = \pm y$.

As mentioned in the introduction, this axiom is essentially equivalent to the bypass relation. The formulation of nondegeneracy is somewhat unorthodox; in Section 3.3 we consider various formulations and their equivalence or nonequivalence.

Note that Axiom 9 is limited to discs; stacking more complicated surfaces does not map into the vacuum background. We can now give a precise version of Theorem 1.1. A much more general statement is given in Theorem 5.1.

Theorem 3.4 *In any sutured TQFT, there is an isomorphism*

$$\left(V(D^2, F_{n+1}), \langle \cdot | \cdot \rangle\right) \cong \left(\mathcal{F}_n, \langle \cdot | \cdot \rangle\right).$$

3.3 Nondegeneracy axioms and bypass relations

Axiom 9 implies certain relations between suture elements. Consider (D^2, F_3), and the three sets of sutures $\Gamma_0, \Gamma_1, \Gamma_2$ shown in Figure 3. Let $c_i \in c(\Gamma_i) \subset V(D^2, F_3)$ be suture elements. We easily obtain
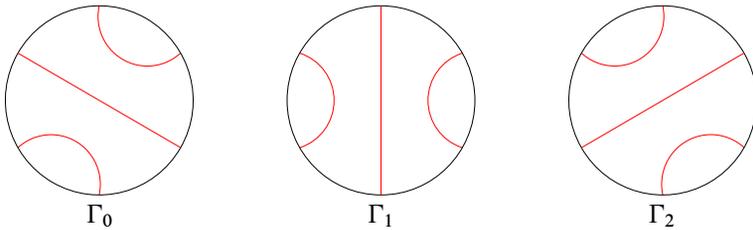

Figure 3: Sutures in the bypass relation

$$\langle c_0|c_0 \rangle = \pm 1, \quad \langle c_0|c_1 \rangle = \pm 1, \quad \langle c_0|c_2 \rangle = 0,$$
$$\langle c_1|c_0 \rangle = 0, \quad\quad \langle c_1|c_1 \rangle = \pm 1, \quad \langle c_1|c_2 \rangle = \pm 1,$$
$$\langle c_2|c_0 \rangle = \pm 1, \quad \langle c_2|c_1 \rangle = 0, \quad\quad \langle c_2|c_2 \rangle = \pm 1.$$

Take $\alpha = c_0 \pm c_1$, with the plus or minus chosen so that $\langle \alpha|c_1 \rangle = 0$. Then we have

$$\langle \alpha|c_0 \rangle = \pm 1, \quad \langle \alpha|c_1 \rangle = 0, \quad \langle \alpha|c_2 \rangle = \pm 1.$$

Thus, by Axiom 9, $\alpha = \pm c_2$; hence $c_2 = \pm c_0 \pm c_1$ for some choice of sign.

On a sutured surface (Σ, Γ), we may consider the operation of removing an embedded disc D in the interior of Σ on which the sutures are isotopic (rel boundary of the smaller

disc) to a set shown in Figure 3 above; and then replacing the sutures on this smaller disc with a different set shown in Figure 3. Such an operation is called *bypass surgery* and comes in two versions: *up*, which takes $\Gamma_0 \mapsto \Gamma_1 \mapsto \Gamma_2 \mapsto \Gamma_0$; and *down*, which takes $\Gamma_0 \mapsto \Gamma_2 \mapsto \Gamma_1 \mapsto \Gamma_0$. Bypass surgery preserves Euler class; bypass-related sutured surfaces naturally come in triples. If $\Gamma_0', \Gamma_1', \Gamma_2'$ are a bypass-related triple of sutures on (Σ, F), consider including the bypass surgery disc into Σ (with fixed sutures); we obtain a map $V(D^2, F_3) \longrightarrow V(\Sigma, F)$ taking each $c(\Gamma_i) \to c(\Gamma_i')$; thus the linear dependency persists.

Lemma 3.5 *Let $\Gamma_0, \Gamma_1, \Gamma_2$ be a bypass related triple of sets of sutures on a sutured background (Σ, F). Choose any $c_i \in c(\Gamma_i)$. Then there exists a choice of signs such that $c_0 = \pm c_1 \pm c_2$.* $\qquad\qquad\square$

In sutured TQFT over $\mathbb{Z}_2$, of course c_0, c_1, c_2 are single elements and $c_0 + c_1 + c_2 = 0$. In [17] we defined groups $SFH_{\mathrm{comb}}(T, n)$, which were generated by chord diagrams of n chords, subject to the relation that bypass related triples sum to zero.

Proposition 3.6 *In any sutured TQFT over $\mathbb{Z}_2$,*

$$V(D^2, F_{n+1}) \cong SFH_{\mathrm{comb}}(T, n+1) \cong \mathbb{Z}_2^{2^n}.$$

This isomorphism takes a chord diagram Γ in $SFH_{\mathrm{comb}}(T, n+1)$ to the suture element $c(\Gamma) \in V(D^2, F_{n+1})$.

The proof is almost immediate from the definition of sutured TQFT, given our previous work; it is given in Section 3.6.

As the formulation of nondegeneracy in Axiom 9 is a little unorthodox, we can consider whether other formulations are equivalent; and the equivalence to various formulations of the bypass relation.

Proposition 3.7 *In the presence of Axioms 1–8 of sutured TQFT, the following axioms are equivalent.*

  (i) *Suppose two elements $x, y \in V(D^2, F_n)$ have the following property: for any set of sutures Γ on (D^2, F_n) and for all $c \in c(\Gamma)$, $\langle x|c \rangle = \pm \langle y|c \rangle$. Then $x = \pm y$.*

  (ii) *(Original Axiom 9) Suppose two elements $x, y \in V(D^2, F_n)$ have the following property: for any set of sutures Γ on (D^2, F_n), there exists $c \in c(\Gamma)$ such that $\langle x|c \rangle = \pm \langle y|c \rangle$. Then $x = \pm y$.*

  (iii) *Suppose two elements $x, y \in V(D^2, F_3)$ have the following property: for any set of sutures Γ on (D^2, F_3) and for all $c \in c(\Gamma)$, $\langle x|c \rangle = \pm \langle y|c \rangle$. Then $x = \pm y$.*

(iv) *Suppose two elements $x, y \in V(D^2, F_3)$ have the following property: for any set of sutures Γ on (D^2, F_3), there exists $c \in c(\Gamma)$ such that $\langle x|c \rangle = \pm \langle y|c \rangle$. Then $x = \pm y$.*

(v) *Suppose $\Gamma_0, \Gamma_1, \Gamma_2$ are a bypass triple of sutures on (D^2, F_n). For any c_0, c_1, c_2 suture elements in $c(\Gamma_0), c(\Gamma_1), c(\Gamma_2)$ respectively, there exist $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ such that $c_0 = \epsilon_1 c_1 + \epsilon_2 c_2$.*

(vi) *Suppose $\Gamma_0, \Gamma_1, \Gamma_2$ are a bypass triple of sutures on (D^2, F_n). Then there exist c_0, c_1, c_2 suture elements in $c(\Gamma_0), c(\Gamma_1), c(\Gamma_2)$ respectively, and $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ such that $c_0 = \epsilon_1 c_1 + \epsilon_2 c_2$.*

(vii) *Suppose $\Gamma_0, \Gamma_1, \Gamma_2$ are a bypass triple of sutures on (D^2, F_3). For any c_0, c_1, c_2 suture elements in $c(\Gamma_0), c(\Gamma_1), c(\Gamma_2)$ respectively, there exist $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ such that $c_0 = \epsilon_1 c_1 + \epsilon_2 c_2$.*

(viii) *Suppose $\Gamma_0, \Gamma_1, \Gamma_2$ are a bypass triple of sutures on (D^2, F_3). Then there exist c_0, c_1, c_2 suture elements in $c(\Gamma_0), c(\Gamma_1), c(\Gamma_2)$ respectively, and $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ such that $c_0 = \epsilon_1 c_1 + \epsilon_2 c_2$.*

Given the foregoing, it's not difficult to see that (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii). The difficult part is to show ((v)–(viii)) $\Rightarrow$ ((i)–(ii)); we do this in Section 3.13.

A more orthodox form of nondegeneracy in fact is not equivalent.

Proposition 3.8 *In the presence of Axioms 1–8, the following two axioms are equivalent.*

(i) *Suppose $x \in V(D^2, F_n)$ has the following property: for any set of sutures Γ on (D^2, F_n) and for any suture element $c \in c(\Gamma)$, $\langle x|c \rangle = 0$. Then $x = 0$.*

(ii) *Suppose $x \in V(D^2, F_n)$ has the following property: for any $y \in V(D^2, F_n)$, $\langle x|y \rangle = 0$. Then $x = 0$.*

There exists a sutured TQFT obeying Axioms 1–8 and these two alternative Axioms 9, such that $V(D^2, F_n) \cong \mathbb{Z}^{C_n}$. Here C_n is the n–th Catalan number; denoting the C_n chord diagrams on (D^2, F_n) by $\{\Gamma_i\}_{i=1}^{C_n}$, each $c(\Gamma_i) = \{\pm c_i\}$ and the c_i form a basis for each $V(D^2, F_n)$.

The proof is given in Section 3.13. We will construct the example explicitly; there will be *no* linear relation between suture elements for distinct chord diagrams, let alone a bypass relation.

3.4 Impossibility of coherent signs

It is possible to have each $c(\Gamma)$ a single element over $\mathbb{Z}_2$. But if $c(\Gamma_\varnothing)$ is nontorsion it is impossible, and it remains impossible even relaxing several axioms. This is a pedantic version of material in [10].

Proposition 3.9 *Consider making assignments:*

(1′) *to each (homeomorphism class of) (D^2, F_n), an abelian group $V(D^2, F_n)$;*

(2′) *to each (isotopy class of) set of sutures Γ on (D^2, F_n), a suture element $c(\Gamma) \in V(D^2, F_n)$;*

(3′) *to an inclusion $(D^2, F_n) \hookrightarrow (D^2, F_m)$ with a set of sutures Γ on the intermediate $(S^1 \times I, F_n \cup F_m)$, a collection of linear maps $\Phi^i \colon V(D^2, F_n) \longrightarrow V(D^2, F_m)$.*

Suppose such assignments satisfy:

(5′) *Each Φ^i is natural with respect to suture elements, ie for a set of sutures Γ' on (D^2, F_n), $c(\Gamma') \mapsto c(\Gamma' \cup \Gamma)$.*

(6) *If Γ contains a closed contractible loop then $c(\Gamma) = 0$.*

(7′) *$c(\Gamma_\varnothing)$ is torsion-free.*

(9′) *Let $\Gamma_0, \Gamma_1, \Gamma_2$ be the three sets of sutures on (D^2, F_3) in Figure 3. Then the three suture elements $c(\Gamma_0), c(\Gamma_1), c(\Gamma_2) \in V(D^2, F_3)$ are linearly dependent over $\mathbb{Z}$.*

Then all suture elements are 0.

Here primed axioms are weaker versions of the original; Axioms 4 and 8 are omitted altogether.

Proof Consider the chord diagrams $\Gamma_0, \Gamma_1, \Gamma_2$ on (D^2, F_3) in Figure 3; and including them into (D^2, F_2) using the sets of sutures $\Gamma_a, \Gamma_b, \Gamma_c$ on the intermediate annulus in Figure 4. From Axiom 3′ then we obtain maps $\Phi_a, \Phi_b, \Phi_c \colon V(D^2, F_3) \longrightarrow V(D^2, F_2)$.

Letting Γ_-, Γ_+ be as shown in Figure 5, we immediately obtain

$$\Phi_a \colon c(\Gamma_0) \mapsto c(\Gamma_+) \quad c(\Gamma_1) \mapsto c(\Gamma_+) \quad c(\Gamma_2) \mapsto 0,$$
$$\Phi_b \colon c(\Gamma_0) \mapsto 0 \quad\quad\ c(\Gamma_1) \mapsto c(\Gamma_-) \quad c(\Gamma_2) \mapsto c(\Gamma_-),$$
$$\Phi_c \colon c(\Gamma_0) \mapsto c(\Gamma_+) \quad c(\Gamma_1) \mapsto 0 \quad\quad\ c(\Gamma_2) \mapsto c(\Gamma_+).$$
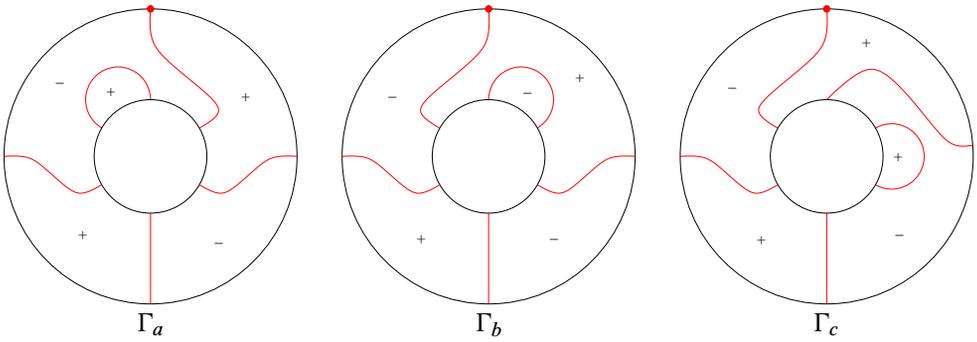
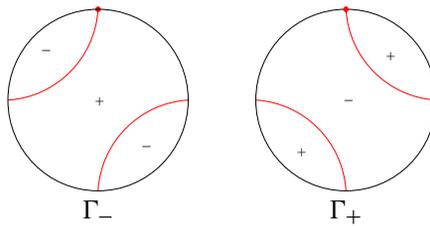Figure 4: Gluing annuli Φ_a, Φ_b, Φ_c

Figure 5: Sutures Γ_-, Γ_+

As $c(\Gamma_0), c(\Gamma_1), c(\Gamma_2)$ are linearly dependent, let $\alpha c(\Gamma_0) + \beta c(\Gamma_1) + \gamma c(\Gamma_2) = 0$ for some $\alpha, \beta, \gamma \in \mathbb{Z}$, not all zero. From Φ_a, Φ_b, Φ_c we then obtain $(\alpha + \beta)c(\Gamma_+) = 0$, $(\beta + \gamma)c(\Gamma_-) = 0$, and $(\gamma + \alpha)c(\Gamma_+) = 0$ respectively. We may then include $(D^2, F_2) \hookrightarrow (D^2, F_1)$ in various ways so that Γ_+ or Γ_- becomes the vacuum $\Gamma_\varnothing$. Thus $(\alpha + \beta)c(\Gamma_\varnothing) = (\beta + \gamma)c(\Gamma_\varnothing) = (\gamma + \alpha)c(\Gamma_\varnothing) = 0$. As $c(\Gamma_\varnothing)$ is nontorsion, we have either $\alpha + \beta = \beta + \gamma = \gamma + \alpha = 0$ or $c(\Gamma_\varnothing) = 0$. In the first case $\alpha = \beta = \gamma = 0$, a contradiction. Thus $c(\Gamma_\varnothing) = 0$; now by inclusion every suture element is 0. $\square$

Return now to our original axioms.

Proposition 3.10 *Let Γ be a set of sutures on (D^2, F_n). If Γ contains a closed loop then $c(\Gamma) = \{0\}$. Otherwise, Γ is a chord diagram, and $c(\Gamma)$ has two distinct elements and is of the form $\{x, -x\}$.*

Proof From Lemma 3.2 for any Γ, $c(\Gamma)$ is either a singleton or is of the form $\{x, -x\}$. If Γ contains a closed loop then it contains a contractible closed loop, so $c(\Gamma) = \{0\}$. Suppose Γ is a chord diagram with $c(\Gamma)$ a singleton. We can find an inclusion map $V(D^2, F_n) \longrightarrow V(D^2, F_1)$ which takes $c(\Gamma) \mapsto c(\Gamma_\varnothing)$ surjectively; thus $c(\Gamma_\varnothing)$ is a nonzero singleton. As the vacuum includes into any sutures then all $c(\Gamma)$ are singletons, contradicting the previous proposition. $\square$

Note that Proposition 3.10 does not rely on Axiom 9; it relies upon the argument of Proposition 3.9, which in turn relies upon a weaker form of Axiom 9. In particular, Axiom $(9')$ of Proposition 3.9 is implied by any of the formulations in Proposition 3.7; to see it is implied by any of Axioms (i)–(iv) we use the argument of Lemma 3.5. Thus, we have the following, which we shall need later.

Lemma 3.11 *Assume Axioms 1–8 of sutured TQFT and any of the 8 alternative formulations of Axiom 9 in Proposition 3.7. Let Γ be any chord diagram. Then $c(\Gamma)$ contains precisely two elements and is of the form $\{x, -x\}$.* □

As for gluing maps, these necessarily have ambiguity too. Let τ be a gluing on a sutured background surface (Σ, F). Axiom 3 gives at least one map Φ_τ; Axiom 7 requires that Φ_τ takes $c(\Gamma) \to c(\#_\tau \Gamma)$ surjectively. By the above, at least on discs, $-\Phi_\tau$ will have the same properties. Thus there is no canonical such Φ_τ; we may freely switch signs.

3.5 Creation and annihilation operators

We now use gluing/inclusion to define various operators in sutured TQFT. We generally follow [17], though with some notational differences.

Creation operators are maps $V(D^2, F_n) \longrightarrow V(D^2, F_{n+1})$; they include a disc into a larger disc and have the effect of inserting a new outermost chord into a chord diagram in a specific place. Annihilation operators are maps $V(D^2, F_n) \longrightarrow V(D^2, F_{n-1})$, including a disc into a larger disc, with the effect of joining two specific adjacent endpoints of a chord diagram. Each operator is defined by giving a specific sutured annulus. To define each operator, we choose a gluing map Φ_τ^i, and a suture element for the sutures on the annulus. This requires careful choices of signs, which we defer to Section 3.8 below; for now we simply consider the effect on sutures.

Recall from Section 3.1 the notation for sutures on (D^2, F_{n+1}), including $n = n_- + n_+$, $e = n_+ - n_-$, base and root, eastside and westside. Annihilation operators $a_{\pm,i}: V(D^2, F_{n+1}) \longrightarrow V(D^2, F_n)$, for $0 \leq i \leq n_\pm + 1$, have the following effect:

   (i) $(-)$–*annihilations*: For $0 \leq i \leq n_- + 1$, $a_{-,i}$ closes off the region between $(-2i, -2i+1)$.

   (ii) $(+)$–*annihilations*: For $0 \leq i \leq n_+ + 1$, $a_{+,i}$ closes off the region between $(2i-1, 2i)$

Note every $(\pm)$–annihilation closes off a $(\mp)$–region; initial annihilations $a_{-,0}, a_{+,0}$ close off regions at the basepoint (Figure 6 shows where the basepoint goes); final

annihilations a_{-,n_-+1}, a_{+,n_++1} close off regions at the root point. Internal $(-)$–annihilations $a_{-,i}$, $1 \leq i \leq n_-$ close off regions on the westside; internal $(+)$–annihilations $a_{+,i}$, $1 \leq i \leq n_+$ close off regions on the eastside. (This numbering of annihilations is different from [17].)
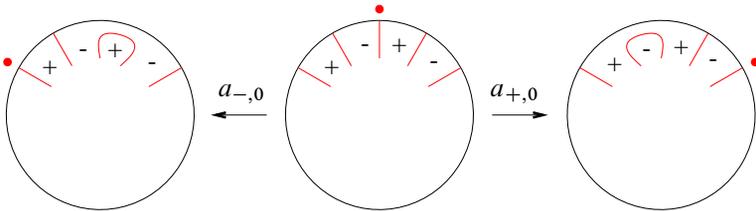

Figure 6: Annihilation maps $a_{\pm,0}$

Similarly creation operators $V(D^2, F_{n+1}) \to V(D^2, F_{n+2})$ are as follows.

(i) $(-)$–*creations*: For $0 \leq i \leq n_- + 1$, $a^*_{-,i}$ creates a new chord which joins $(-2i - 1, -2i)$, between the points previously labelled $(-2i, -2i + 1)$.

(ii) $(+)$–*creations*: For $0 \leq i \leq n_+ + 1$, $a^*_{+,i}$ creates a new chord which joins $(2i, 2i + 1)$, between the points previously labelled $(2i - 1, 2i)$.

Every $(\pm)$–creation creates a new outermost chord enclosing a $\pm$ region; initial creations create new chords at the basepoint (see Figure 7); final creations create new chords at the root point; $(-)$–creations (resp. $(+)$) create new chords on the westside (resp. eastside) of the resulting diagram.
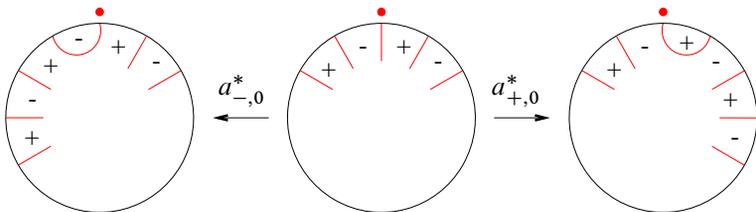

Figure 7: Creation maps $a^*_{\pm,0}$

It's easy to check that these creation and annihilation operators obey the same relations as in Section 2.2, substituting $(-, +)$ for (x, y), at least up to sign. In Sections 3.8–3.9 we will consider signs carefully and show the relations hold.

3.6 Basis, partial order

Let $\mathcal{W}$ denote the set of words on $\{-, +\}$. In a word, let $n_\pm$ denote the number of $\pm$ signs, n the total length, and $e = n_+ - n_-$; so (n, e, n_-, n_+) are related as (n, e, n_x, n_y). Let $\mathcal{W}_n$ be the words of length n, and $\mathcal{W}_n^e = \mathcal{W}_{n_-, n_+}$ for words with $n_\pm \pm$ signs. Note $\mathcal{W} \cong \mathcal{M}$ as graded monoids, identifying $(-, +) \sim (x, y)$. The partial order $\leq$ carries naturally over to $\mathcal{W}$.

In [17], we defined a distinguished chord diagram Γ_w on (D^2, F_{n+1}) for each word $w \in \mathcal{W}_n$; Γ_w is obtained from the vacuum by applying the sequence of initial creation operators $a_{\pm,0}^*$ corresponding to w; its Euler class is $e = n_+ - n_-$. We showed [17, Section 6.3.2] that creation and annihilation operators act on basis diagrams Γ_w exactly as corresponding creation and annihilation operators act on words w. That is, defining $a_{\pm,i}$ and $a_{\pm,i}^*$ to act on words in $\{-, +\}$ in $\mathcal{W}_n^e$, analogously as on words in $\{x, y\}$ in $\mathcal{M}_n^e$, then for a basis diagram Γ_w, we have $a_{\pm,i}^* \Gamma_w = \Gamma_{a_{\pm,i}^* w}$ and $a_{\pm,i} \Gamma_w = \Gamma_{a_{\pm,i} w}$ (provided $a_{\pm,i} w \neq 0$; if $a_{\pm,i} w = 0$ then $a_{\pm,i} \Gamma_w$ contains a closed loop).

We showed [17, Proposition 1.16] that over $\mathbb{Z}_2$ coefficients, the $c(\Gamma_w)$ form a basis for $SFH_{\text{comb}}(T, n+1)$: a basis found by applying creation operators to the vacuum. These arguments immediately carry over to sutured TQFT (over $\mathbb{Z}_2$ or $\mathbb{Z}$), showing that representatives of $c(\Gamma_w)$ form a basis for $V(D^2, F_{n+1})$, and proving Proposition 3.6; we briefly rerun these arguments here.

For every $w \in \mathcal{W}_n$, arbitrarily choose $c_w \in c(\Gamma_w)$ (which has $n + 1$ chords). To see the c_w are linearly independent, suppose some linear combination is zero, $\sum k_i c_{w_i} = 0$. Then we can apply annihilation operators to reduce Γ_{w_1} to $\Gamma_\varnothing$ but every other Γ_{w_i} to a set of sutures containing a closed curve; we obtain $k_1 c(\Gamma_\varnothing) = 0$, a contradiction. To see the c_w span $V(D^2, F_{n+1})$, observe that any chord diagram Γ either has an outermost chord at the basepoint, or is part of a bypass triple in which the other two chord diagrams have outermost chords at the basepoint; thus any element of $c(\Gamma)$ is a linear combination of suture elements for chord diagrams with outermost chords at the basepoint. But a chord diagram Γ with an outermost chord at the basepoint satisfies $c(\Gamma) = a_{\pm,0}^* c(\Gamma')$ for some chord diagram Γ' with fewer chords. Applying this observation repeatedly expresses any suture element of a chord diagram as a linear combination of the c_w. As suture elements span $V(D^2, F_{n+1})$, the c_w span as desired. This argument works over $\mathbb{Z}$ or $\mathbb{Z}_2$. (Over $\mathbb{Z}_2$ it is now easy to construct an isomorphism $SFH_{\text{comb}}(T, n) \longrightarrow V(D^2, F_n)$, proving Proposition 3.6.)

We may refine this argument by Euler class. Since bypass surgery preserves e, every Γ_w in the decomposition of Γ satisfies $w \in \mathcal{W}_n^e$. As in Section 2, write $V(D^2)_n^e$ or $V(D^2)_{n_-, n_+}$ for the submodule of $V(D^2, F_{n+1})$ spanned by suture elements of chord diagrams with Euler class e; also write $V(D^2)_n = V(D^2, F_{n+1})$, so

$V(D^2) = \bigoplus_n V(D^2)_n$ and $V(D^2)_n = \bigoplus_e V(D^2)_n^e$. The c_w, over $w \in \mathcal{W}_n^e$, form a basis for $V(D^2)_n^e$; hence

$$\dim V(D^2)_n^e = \binom{n}{n_-} = \binom{n}{n_+}.$$

Thus, as graded abelian groups, $V(D^2) \cong \mathcal{F}$.

Proposition 3.12 *For any choice of $c_w \in c(\Gamma_w)$ over $w \in \mathcal{W}_n^e$, $\{c_w\}$ forms a basis for $V(D^2)_n^e$.* $\square$

In [17] we considered stacking basis chord diagrams. We showed [17, Proposition 1.28, Section 4.3] that the set of sutures obtained on the sphere (rounded cylinder) by stacking $\Gamma_{w_0}, \Gamma_{w_1}$ is connected if and only if $w_0 \leq w_1$. We thus have $\langle c_{w_0} | c_{w_1} \rangle = \pm 1$ if $w_0 \leq w_1$, and 0 otherwise. (In particular, distinct summands $V(D^2)_n^e$ are orthogonal.) Thus up to sign, the algebraic (in $\mathcal{F}$) and topological (in $V(D^2)$) versions of $\langle \cdot | \cdot \rangle$ agree. In the next sections we consider signs carefully; this will allow us to prove Theorem 3.4, and a much more general isomorphism in Theorem 5.1.

3.7 Previous results: suture elements mod 2

In [17] we effectively considered in detail the structure of suture elements in $V(D^2, F_n)$ over $\mathbb{Z}_2$. All those arguments carry over here, except with sign ambiguities. We briefly recall some of these arguments and results as they are needed here.

We proved several properties of stacking; these now give us information about $\langle \cdot | \cdot \rangle$ up to sign:

- [17, Lemma 3.7] For any chord diagram Γ, $\langle c(\Gamma) | c(\Gamma) \rangle = \pm 1$.
- [17, Lemma 3.9] If Γ' is obtained from Γ by an upwards bypass surgery then $\langle c(\Gamma) | c(\Gamma') \rangle = \pm 1$ and $\langle c(\Gamma') | c(\Gamma) \rangle = 0$.
- [17, Lemma 3.8] If two chord diagrams Γ, Γ' share a common outermost chord γ, then $\langle c(\Gamma) | c(\Gamma') \rangle = \pm \langle c(\Gamma - \gamma) | c(\Gamma' - \gamma) \rangle$.
- Annihilation and creation operators satisfy the adjoint relations of Section 2.5, up to sign: $\langle a_{-,i} c(\Gamma_0) | c(\Gamma_1) \rangle = \pm \langle c(\Gamma_0) | a_{-,i}^* c(\Gamma_1) \rangle$ and $\langle c(\Gamma_0) | a_{+,i} c(\Gamma_1) \rangle = \pm \langle a_{+,i}^* c(\Gamma_0) | c(\Gamma_1) \rangle$.

We [17, Section 5.1] defined the notions of *elementary move* on a word, *generalised elementary move* and *nicely ordered sequence* of generalised elementary moves. All these moves come in *forwards* and *backwards* versions (moving forwards/backwards with respect to $\leq$). A forwards elementary move replaces a substring of the form $(-)^a(+)^b$

with $(+)^b(-)^a$. A generalised forwards elementary move $\mathrm{FE}(i, j)$ moves the i–th $-$ sign (from the left) to the right, past the j–th (from the left) $+$ sign, together with all $-$ signs in between. A sequence of forwards generalised elementary moves $\mathrm{FE}(i_1, j_1), \dots, \mathrm{FE}(i_k, j_k)$ is *nicely ordered* if $i_1 < i_2 < \dots < i_k$ and $j_1 \leq j_2 \leq \dots \leq j_k$. Backwards versions reverse the roles of $-$ and $+$.

Two basis chord diagrams $\Gamma_{w_0}, \Gamma_{w_1}$ are bypass related if and only if w_0, w_1 are related by an elementary move [17, Lemmas 5.7–8]. For any words $w_0 \leq w_1$, there is a nicely ordered sequence of forwards generalised elementary moves taking w_0 to w_1. There is also a set $\mathrm{FBS}(w_0, w_1)$ of disjoint upwards bypass surgeries on Γ_{w_0} which effect the moves, eventually giving w_1. If we instead perform all these bypass surgeries *downwards*, we obtain a chord diagram Γ whose suture element, expressed in terms of the basis, is a sum (mod 2) $\sum c(\Gamma_{w_i})$, where w_0, w_1 occur in the sum, and for every other w_i occurring in the sum, $w_0 \leq w_i \leq w_1$. In fact these Γ_{w_i} can be obtained by performing upwards bypass surgeries along subsets of $\mathrm{FBS}(w_0, w_1)$. Since the number of pairs $w_0 \leq w_1$ in $\mathcal{W}_n^e$ [17, Proposition 1.19], and the number of chord diagrams with $n+1$ chords and Euler class e, are both the Narayana number N_n^e, all suture elements of chord diagrams are of this form [17, Theorem 1.20].

The same argument applies immediately in sutured TQFT over $\mathbb{Z}$: there is still a bijection between comparable pairs $w_0 \leq w_1$ and chord diagrams with minimum and maximum words w_0, w_1 occurring in their suture elements. For Γ as above and $c \in c(\Gamma)$ we have $c = \sum a_i c_{w_i}$, where $c_{w_i} \in c(\Gamma_{w_i})$ and the w_i all satisfy $w_0 \leq w_i \leq w_1$. Following [17], write $\Gamma = [\Gamma_{w_0}, \Gamma_{w_1}]$, and $\pm[w_0, w_1]$ for the corresponding suture elements. All coefficients $a_i \in \{-1, 1\}$, as when we decompose Γ in terms of the basis, using the decomposition algorithm, any basis diagram occurs at most once. If Γ is not a basis diagram, then an *even* number of basis diagrams appears in its decomposition [17, Proposition 1.23].

3.8 Choosing a coherent basis

Although sign ambiguity prevents us choosing canonical representatives for all $c(\Gamma)$ coherently, we will choose representatives for *basis* suture elements. We do this using creation operators and $\langle \cdot | \cdot \rangle$.

Choose a vacuum representative $c_\varnothing \in c(\Gamma_\varnothing)$ arbitrarily. For each pair (n_-, n_+), choose an arbitrary representative for the creation $a_{+,0}^*: V(D^2)_{n_-, n_+} \longrightarrow V(D^2)_{n_-, n_++1}$. For each $(n_-, n_+) = (n_-, 0)$, choose an arbitrary representative for the creation map $a_{-,0}^*: V(D^2)_{n_-, 0} \longrightarrow V(D^2)_{n_-+1, 0}$. For each (n_-, n_+) there is then a unique sequence of these chosen $a_{-,0}^*$ and $a_{+,0}^*$ operators leading from $V(D^2)_{0,0}$ to $V(D^2)_{n_-, n_+}$; this sequence of operators takes $c(\Gamma_\varnothing)$ to $c(\Gamma_{w_{\max}})$, where $w_{\max} = (+)^{n_+}(-)^{n_-}$ is

the maximum word in $\mathcal{W}_{n_-,n_+}$. For each (n_-, n_+), choose $c_{w_{\max}} \in c(\Gamma_{w_{\max}}) \subset V(D^2)_{n_-,n_+}$ to be the image of $c_\varnothing$ under these creations.

We noted (Section 3.6) that the summands $V(D^2)_n^e$ are $\langle \cdot | \cdot \rangle$–orthogonal. Thus we may write $\langle \cdot | \cdot \rangle_n^e$ for stacking in $V(D^2)_n^e$; adjusting $\langle \cdot | \cdot \rangle_n^e$ by a sign preserves all required properties of $\langle \cdot | \cdot \rangle$.

In each $V(D^2)_n^e$ we have chosen $c_{w_{\max}}$. We have $\langle c_{w_{\max}} | c_{w_{\max}} \rangle = \pm 1$. The form $\langle \cdot | \cdot \rangle$ is now fixed by adjusting each $\langle \cdot | \cdot \rangle_n^e$ by a sign if necessary so that

$$\langle c_{w_{\max}} \mid c_{w_{\max}} \rangle = 1.$$

For each $w \in \mathcal{W}_n^e$ and $c_w \in c(\Gamma_w)$, we have $\langle c_w | c_{w_{\max}} \rangle = \pm 1$; we choose, once and for all, the representative c_w such that

$$\langle c_w | c_{w_{\max}} \rangle = 1.$$

We next show this basis is coherent.

3.9 Coherent creation and annihilation

Having chosen signs on some initial creation operators (arbitrarily), we now choose representatives for all creation and annihilation operators, so that they are coherent with respect to the basis c_w.

We have $a_{\pm,i}^* c_w = \pm c_{a_{\pm,i}^* w}$ and $a_{\pm,i} c_w = \pm c_{a_{\pm,i} w}$ (where we set $c_0 = 0$). In $\mathcal{W}_n^e$ we have the minimal and maximal words $w_{\min} = (-)^{n_-}(+)^{n_+}$, $w_{\max} = (+)^{n_+}(-)^{n_-}$. We choose each creation operator $a_{\pm,i}^*$ to take $c_{w_{\max}} \mapsto c_{a_{\pm,i}^* w_{\max}}$; and we choose each annihilation operator $a_{\pm,i}$ to take $c_{w_{\max}} \mapsto c_{a_{\pm,i} w_{\max}}$, unless this is 0; else we define it to take $c_{w_{\min}} \mapsto c_{a_{\pm,i} w_{\min}}$. Clearly this agrees with previous choices of creation operators.

Lemma 3.13 *If $w_0 \le w_1$ are words related by an elementary move, then $c_{w_0} - c_{w_1}$ is a suture element.*

Proof There is an upwards bypass move on Γ_{w_0} which gives Γ_{w_1}; the corresponding downwards bypass move gives a suture element of the form $c_{w_0} \pm c_{w_1}$. Considering stacking these, mod 2 we have $\langle c_{w_0} + c_{w_1} | c_{w_{\max}} \rangle = 0$. By our choice of basis, $\langle c_{w_0} | c_{w_{\max}} \rangle = \langle c_{w_1} | c_{w_{\max}} \rangle = 1$. Thus $\langle c_{w_0} \pm c_{w_1} | c_{w_{\max}} \rangle = 1 \pm 1$; the result must be 0, and $c_{w_0} - c_{w_1}$ is a suture element. $\qquad\square$

Lemma 3.14 *For any w, $\langle c_w | c_w \rangle = 1$. For any two $w_0 \le w_1$ in $\mathcal{W}_n^e$ related by an elementary move, $\langle c_{w_0} | c_{w_1} \rangle = 1$.*

Proof We use the following fact:

- If $w_0 \leq w_1$ are related by an elementary move and $\langle c_{w_1} | c_{w_1} \rangle = 1$, then $\langle c_{w_0} | c_{w_0} \rangle = \langle c_{w_0} | c_{w_1} \rangle = 1$.

To see why the fact is true, from the previous lemma $c_{w_0} - c_{w_1}$ is a suture element. We have mod 2 (hence over $\mathbb{Z}$) $\langle c_{w_0} - c_{w_1} | c_{w_1} \rangle = 0$, so that $\langle c_{w_0} | c_{w_1} \rangle = \langle c_{w_1} | c_{w_1} \rangle = 1$. Then $\langle c_{w_0} | c_{w_0} - c_{w_1} \rangle = 0$ so $\langle c_{w_0} | c_{w_0} \rangle = \langle c_{w_0} | c_{w_1} \rangle = 1$.

By our choice of $\langle \cdot | \cdot \rangle$, $\langle c_{w_{\max}} | c_{w_{\max}} \rangle = 1$. Using the fact repeatedly, we obtain the desired result. $\square$

Proposition 3.15 (Coherence of creation and annihilation) *For any $w \in \mathcal{W}_n^e$, $s \in \{-, +\}$ and $0 \leq i \leq n_s + 1$,*

$$a_{s,i}^* c_w = c_{a_{s,i}^* w} \quad \text{and} \quad a_{s,i} c_w = c_{a_{s,i}.w}.$$

Proof Consider an annihilation $a_{s,i}$, and let $A \subset \mathcal{W}_n^e$ consist of words w for which $a_{s,i} c_w = c_{a_{s,i} w}$. By definition of $a_{s,i}$, A is nonempty ($w_{\max}$ or $w_{\min} \in A$). Now repeatedly apply the following fact.

- If $w \in A$ and w' is obtained from w by an elementary move, then $w' \in A$.

To see why this fact is true, note (by Lemma 3.13) $c_{w'} - c_w$ is a suture element, hence is taken by $a_{s,i}$ to a suture element (possibly 0), $a_{s,i}(c_{w'} - c_w) = \pm c_{a_{s,i} w'} - c_{a_{s,i} w}$. As $a_{s,i} w'$ and $a_{s,i} w$ are related by an elementary move, or are identical, $c_{a_{s,i} w'} - c_{a_{s,i} w}$ is a suture element (possibly 0) and the other alternative is not. Thus $w' \in A$.

The argument for creation operators is similar (and simpler; the 0 case does not arise). $\square$

Proposition 3.16 (Coherence of basis)

(i) *For words $w_0, w_1 \in \mathcal{W}_n^e$ with $w_0 \leq w_1$, $\langle c_{w_0} | c_{w_1} \rangle = 1$.*

(ii) *For any chord diagram Γ and $c \in c(\Gamma)$, $c = \sum_w a_w c_w$ where $a_w \in \{-1, 1\}$. If Γ is not a basis chord diagram then $\sum a_w = 0$.*

(iii) *For each chord diagram Γ and suture element $c \in c(\Gamma) \subset V(D^2)_n^e$, $\langle c | c \rangle = 1$.*

Proof We first prove (ii). In Section 3.7 we showed that all $a_w = \pm 1$; we only need show $\sum a_w = 0$ when Γ is a nonbasis diagram. The easiest way to see this is to use the fact, shown in [17, Section 5.2.1], that if Γ is a nonbasis diagram then Γ, $\Gamma_{w_{\max}}$ are not stackable, $\langle c(\Gamma) | c(\Gamma_{w_{\max}}) \rangle = 0$. Take $\sum_w a_w c_w = c \in c(\Gamma)$. Since the c_w by definition satisfy $\langle c_w | c_{w_{\max}} \rangle = 1$ we obtain $\sum_w a_w = 0$.

Next we prove (iii). We will repeatedly apply this fact:

- Let $x \in c(\Gamma_x)$, $y \in c(\Gamma_y)$, where Γ_y is a chord diagram obtained from Γ_x by bypass surgery (without loss of generality, upwards), so $\langle x|y \rangle = \pm 1$ and $\langle y|x \rangle = 0$. Suppose $\langle x|x \rangle = \langle y|y \rangle = 1$. Change sign of y if necessary so that $\langle x|y \rangle = 1$. Then $x - y$ is a suture element and $\langle x - y|x - y \rangle = 1$.

To see why, note there is a suture element of the form $x \pm y$. We have $\langle x \pm y|x \pm y \rangle = 2 \pm \langle x|y \rangle$; of course we must have $2 - 1 = 1$, and the conclusions follow.

Consider the set A of suture elements c such that $\langle c|c \rangle = 1$. We know ([Lemma 3.14](#)) that A contains all $\pm c_w$, and (by the above fact) if it contains two suture elements for chord diagrams related by a bypass surgery, then it contains the suture element for the third chord diagram in their bypass triple. Repeated application of the above fact (eg considering each step of a decomposition of a chord diagram into basis diagrams) gives (iii).

We now prove (i). Suppose there exist $w_0 \leq w_1$ such that $\langle c_{w_0}|c_{w_1} \rangle \neq 1$, that is, $\langle c_{w_0}|c_{w_1} \rangle = -1$. Fix w_0 and take w_1 minimal among such pairs (so if $w_0 \leq w < w_1$ then $\langle c_{w_0}|c_w \rangle = 1$). Since ([Lemma 3.14](#)) $\langle c_{w_0}|c_{w_0} \rangle = 1$, $w_0 \neq w_1$. Now take the chord diagram $\Gamma = [\Gamma_{w_0}, \Gamma_{w_1}]$ and $c \in c(\Gamma)$. As $w_0 \neq w_1$, Γ is not a basis diagram. Decompose $c = \sum_w a_w c_w$, so all $a_w = \pm 1$ and $\sum_w a_w = 0$. We have

$$\langle c_{w_0}|c \rangle = \left\langle c_{w_0} \,\middle|\, \sum_w a_w c_w \right\rangle = \sum_w a_w \langle c_{w_0}|c_w \rangle.$$

Since all w occurring satisfy $w_0 \leq w \leq w_1$, mod 2 we have $\langle c_{w_0}|c \rangle = \sum_w a_w = 0$; hence the same is true over $\mathbb{Z}$. On the other hand, all $\langle c_{w_0}|c_w \rangle = 1$ except for $\langle c_{w_0}|c_{w_1} \rangle = -1$. So

$$\sum_w a_w = 0 \quad \text{and} \quad -a_{w_1} + \sum_{w \neq w_1} a_w = 0.$$

As these two sums differ by $2a_{w_1} \neq 0$, we have a contradiction. $\qquad\square$

3.10 Multiplication

Given two chord diagrams Γ_0, Γ_1 on $(D^2, F_{n_0+1}), (D^2, F_{n_1+1})$ with Euler classes e_0, e_1, we now *multiply* them. Consider a gluing τ which identifies a neighbourhood of the root point on $\partial(D^2, F_{n_0})$ with a neighbourhood of the basepoint on $\partial(D^2, F_{n_1})$. Taking a gluing map for τ and restricting to e_0, e_1 summands gives a linear operator

$$\times_{(n_0, e_0),(n_1, e_1)} \colon V(D^2)^{e_0}_{n_0} \otimes V(D^2)^{e_1}_{n_1} \longrightarrow V(D^2)^{e_0+e_1}_{n_0+n_1}.$$

Note $\times$ acts on basis suture elements by multiplying the corresponding words:

$$c(\Gamma_{w_0}) \otimes c(\Gamma_{w_1}) \mapsto c(\Gamma_{w_0 w_1}).$$

Taking $\times = \bigoplus_{n_0,e_0,n_1,e_1} \times_{(n_0,e_0),(n_1,e_1)}$ we obtain a $\mathbb{Z}$–bilinear associative multiplication, making $V(D^2)$ a bigraded ring. There are sign ambiguities in $\times$; we now resolve them, adjusting each $\times_{(n_0,e_0),(n_1,e_1)}$ individually. Choose $\times_{(n_0,e_0),(n_1,e_1)}$ to send maximal basis elements to basis elements, ie

$$c_{(+)^{n_{+,0}}(-)^{n_{-,0}}} \otimes c_{(+)^{n_{+,1}}(-)^{n_{-,1}}} \mapsto c_{(+)^{n_{+,0}}(-)^{n_{-,0}}(+)^{n_{+,1}}(-)^{n_{-,1}}}$$

where $n_{+,i}, n_{-,i}$ is the pair n_+, n_- corresponding to (n_i, e_i). All multiplication respects the basis; the method of proof is by now familiar.

Proposition 3.17 *For any $w_0 \in \mathcal{W}_{n_0}^{e_0}$ and $w_1 \in \mathcal{W}_{n_1}^{e_1}$, under $\times_{(n_0,e_0),(n_1,e_1)}$,*

$$c_{w_0} \otimes c_{w_1} \mapsto c_{w_0 w_1}.$$

Proof Consider the set A of words for which this multiplication is coherent: $A = \{(w_0, w_1): c_{w_0} \otimes c_{w_1} \mapsto c_{w_0 w_1}\}$. By our sign choice of $\times_{(n_0,e_0),(n_1,e_1)}$, the pair of maximum words $((+)^{n_{+,0}}(-)^{n_{-,0}}, (+)^{n_{+,1}}(-)^{n_{-,1}}) \in A$. The result now follows obviously from the following two facts.

- If $(w_0, w_1) \in A$ and w_0' is related to w_0 by an elementary move, then $(w_0', w_1) \in A$.
- If $(w_0, w_1) \in A$ and w_1' is related to w_1 by an elementary move, then $(w_0, w_1') \in A$.

To see why the first fact is true, note that $c_{w_0'} - c_{w_0}$ is a suture element, and multiplication must take $(c_{w_0'} - c_{w_0}) \otimes w_1$ to a suture element. The result of this multiplication must be $\pm c_{w_0' w_1} - c_{w_0 w_1}$. But $w_0' w_1$ and $w_0 w_1$ are related by an elementary move, hence $c_{w_0' w_1} - c_{w_0 w_1}$ is a suture element and $-c_{w_0' w_1} - c_{w_0 w_1}$ is not. So $c_{w_0'} \otimes c_{w_1} \mapsto c_{w_0' w_1}$ and $(w_0', w_1) \in A$. The second fact is similar. $\square$

3.11 Temperley–Lieb algebra

Consider the two creation operators $a_{+,i}^*$ and $a_{+,i+1}^*$, acting $V(D^2)_n^e \longrightarrow V(D^2)_{n+1}^{e+1}$, for $0 \leq i \leq n_+$. Both are obtained by gluing a sutured annulus to the exterior of a disc. Observe that these two sets of sutures are bypass-related. Take the third set of sutures of this bypass triple; gluing these to the exterior of a disc gives an operator (see Figure 8):

$$T_{+,i}^*: V(D^2)_n^e \longrightarrow V(D^2)_{n+1}^{e+1}.$$

Similarly on the westside, consider $a^*_{-,i}$ and $a^*_{-,i+1}$, for $0 \le i \le n_-$, which give annuli with bypass-related sutures, and obtain

$$T^*_{-,i}: V(D^2)^e_n \longrightarrow V(D^2)^{e-1}_{n+1}.$$

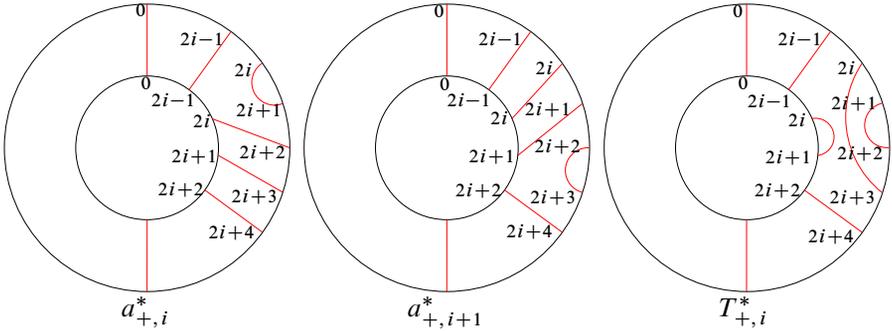Annihilation operators $a_{\pm,i}, a_{\pm,i+1}$ ($0 \le i \le n_\pm$) also give annuli with bypass-related

Figure 8: Operators $a^*_{+,i}, a^*_{+,i+1}$ and $T^*_{+,i}$

sutures, and taking a sutured annulus forming a bypass triple with them, we obtain gluing maps

$$T_{\pm,i}: V(D^2)^e_n \longrightarrow V(D^2)^{e\pm1}_{n-1}.$$

These gluing maps are ambiguous up to sign; we choose coherent signs. First, $T^*_{+,i}$ takes c_w to $\pm c_{a^*_{+,i}w} \pm c_{a^*_{+,i+1}w}$, and $a^*_{+,i}w$, $a^*_{+,i+1}w$ are words related by an elementary move, or are identical; hence $\pm(c_{a^*_{+,i}w} - c_{a^*_{+,i+1}w})$ are suture elements (and the other possibilities are not). Similar considerations apply to $T^*_{-,i}$. Thus we set

$$T^*_{\pm,i} = a^*_{\pm,i} - a^*_{\pm,i+1}.$$

Similarly, $T_{+,i}$ takes c_w to $\pm c_{a_{+,i}w} \pm c_{a_{+,i+1}w}$, and the two words $a_{+,i}w, a_{+,i+1}w$ are either identical or related by an elementary move; thus we define

$$T_{\pm,i} = a_{\pm,i} - a_{\pm,i+1}.$$

These $T_{\pm,i}, T^*_{\pm,i}$ are identical to the operators of the same name in $\mathcal{F}$; we may define

$$U_{\pm,i} = T^*_{\pm,i}\, a_{\pm,i+1}.$$

so that $U_{\pm,i}$ are operators $V(D^2)^e_n \longrightarrow V(D^2)^e_n$ which glue annuli to the exterior of discs, having the effect shown in Figure 9. We see then (Figure 10) that the $U_{\pm,i}$ have sutures which are very similar to the usual generators given for the Temperley–Lieb algebra.
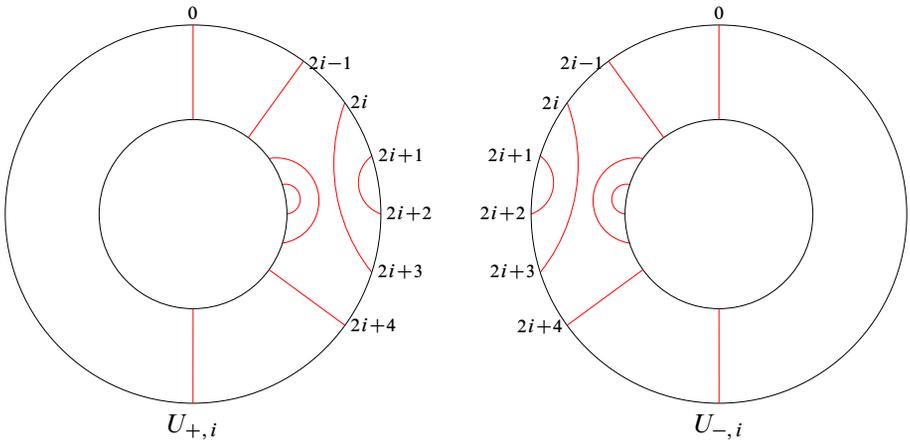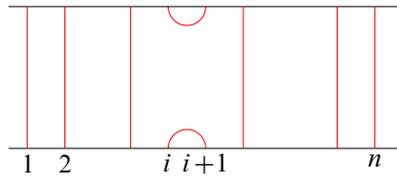
Figure 9: Operators $U_{+,i}$, $U_{-,i}$

Figure 10: Usual generators U_i in Temperley–Lieb algebra

It follows immediately that the $U_{\pm,i}$ obey the relations of the Temperley–Lieb algebra (with $\delta = 0$), up to sign. Since the $U_{\pm,i}$ are identical algebraically to the operators of Section 2.8, we have the "twisted" representation of the Temperley–Lieb algebra described there.

3.12 Rotation

The operation of rotating a chord diagram gives an operation in sutured TQFT. As in [17, Section 7.1], gluing a sutured annulus, with sutures as in Figure 11, to the exterior of a disc has the effect of rotating the basepoint two places (which preserves signs on either side of the basepoint.) We obtain an operator, up to sign:

$$R: V(D^2)_n^e \longrightarrow V(D^2)_n^e.$$

Consider computing $\langle c(\Gamma_0)|c(\Gamma_1)\rangle$, placing two chord diagrams into the ends of a cylinder and rounding sutures. We see that $\langle c(\Gamma_0)|c(\Gamma_1)\rangle = \langle c(\Gamma_1)|Rc(\Gamma_0)\rangle$; the rotation compensates precisely for the difference in rounding corners when the ends of the cylinder are swapped. For $c_0 \in c(\Gamma_0)$, $c_1 \in c(\Gamma_1)$ then $\langle c_0|c_1\rangle = \pm\langle c_1|Rc_0\rangle$, similar to the duality property of H on $\mathcal{F}$.
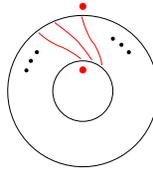
Figure 11: The rotation operator

We choose a sign for R by noting that for any n, e, R takes the basis chord diagram $\Gamma_{w_{\min}}$ to $\Gamma_{w_{\max}}$; we choose a sign on R so that $R c_{w_{\min}} = c_{w_{\max}}$.

Proposition 3.18 *With this choice of sign, $R = H$.*

Proof We first claim that for all $w \in \mathcal{W}_n^e$, $\langle c_w | R c_w \rangle = 1$. Let $A = \{w \in \mathcal{W}_n^e : \langle c_w | R c_w \rangle = 1\}$. Since $R c_{w_{\min}} = c_{w_{\max}}$, $w_{\min} \in A$. The claim then follows obviously from the following fact:

- If $w_0 \in A$, $w_0 \leq w_1$ and w_0, w_1 are related by an elementary move, then $w_1 \in A$.

To see why the fact is true, note $c_{w_0} - c_{w_1}$ is a suture element and $\langle c_{w_0} | c_{w_0} - c_{w_1} \rangle = 0$. Thus $\langle c_{w_0} - c_{w_1} | R c_{w_0} \rangle = 0$. Hence $\langle c_{w_0} | R c_{w_0} \rangle = \langle c_{w_1} | R c_{w_0} \rangle$; as $w_0 \in A$ we have $\langle c_{w_1} | R c_{w_0} \rangle = 1$. Also note $\langle c_{w_0} - c_{w_1} | c_{w_1} \rangle = 0$, so $\langle c_{w_1} | R(c_{w_0} - c_{w_1}) \rangle = 0$. Hence $\langle c_{w_1} | R c_{w_1} \rangle = \langle c_{w_1} | R c_{w_0} \rangle = 1$, so $w_1 \in A$.

Next we claim that for any words w_0, w_1, $\langle c_{w_0} | c_{w_1} \rangle = \langle c_{w_1} | R c_{w_0} \rangle$. This is clearly true when $w_0 \not\leq w_1$; both are 0. So take $w \in \mathcal{W}_n^e$ and let $A_w = \{w' \in \mathcal{W}_n^e : \langle c_w | c_{w'} \rangle = \langle c_{w'} | R c_w \rangle\}$; we will show $A_w = \mathcal{W}_n^e$. From the previous claim, $w \in A_w$; and any w' with $w \not\leq w'$ lies in A_w. Thus we only need consider w' with $w \leq w'$. Such a w' can be reached from w by a sequence of forwards elementary moves, and hence the claim follows from the following fact.

- If $w \leq w_0 \leq w_1$, and $w_0 \in A_w$, and w_0, w_1 are related by an elementary move, then $w_1 \in A_w$.

To see this, note $c_{w_0} - c_{w_1}$ is a suture element, and $\langle c_w | c_{w_0} - c_{w_1} \rangle = 0$, hence $\langle c_{w_0} - c_{w_1} | R c_w \rangle = 0$. Thus $\langle c_{w_0} | R c_w \rangle = \langle c_{w_1} | R c_w \rangle$. As $w_0 \in A_w$ then $\langle c_{w_0} | R c_w \rangle = \langle c_w | c_{w_0} \rangle = 1$; thus $\langle c_{w_1} | R c_w \rangle = 1 = \langle c_w | c_{w_1} \rangle$.

Thus for any words, $\langle c_{w_0} | c_{w_1} \rangle = \langle c_{w_1} | R c_{w_0} \rangle$. As the c_w form a basis for $V(D^2)_n^e$, the result follows. $\square$

It follows immediately that $H^{n+1} c_w = \pm c_w$ for every $w \in \mathcal{W}_n^e$, hence $H^{2n+2} = 1$. The precise sign $H^{n+1} = (-1)^{n-n_+}$ and period (Theorem 2.6) are proved in Section 5.4.

It also follows that R is an isometry: $\langle u | v \rangle = \langle Ru | Rv \rangle$. This is obvious up to sign from the definition of R; the full equality now clear by definition of H.

Recall an $a_{\pm,i}^*$ inserts an outermost chord into a chord diagram at different positions; many different positions, not all. We showed in Section 3.9 that the $a_{\pm,i}^*$ are isometries; they are also clearly injective. We now note that the operation of inserting an outermost chord *anywhere* gives a linear operator which is an injective isometry: for it can be obtained by composing the injective isometries R and $a_{\pm,0}^*$.

3.13 Variations of nondegeneracy axioms

We now prove Propositions 3.7 and 3.8, regarding variations of Axiom 9. Recall (Section 3.7) that given $w_0 \leq w_1$, $\pm[w_0, w_1]$ denotes the two suture elements with minimal and maximal basis elements c_{w_0}, c_{w_1}. Recall also (Section 2.4) the notions of difference $d(w_0, w_1)$, minimum and maximum of words.

We will need the fact that any two words appear together in a suture element.

Lemma 3.19 *For any $w_0, w_1 \in \mathcal{W}_n^e$, both c_{w_0}, c_{w_1} appear in the basis decomposition of $\pm[\min(w_0, w_1), \max(w_0, w_1)]$. Moreover, in the decomposition of Lemma 2.4,*

$$\pm[\min(w_0, w_1), \max(w_0, w_1)]$$
$$= \pm[w_0^0, w_1^0][w_1^1, w_0^1][w_0^2, w_1^2][w_1^3, w_0^3] \cdots [w_1^{2k-1}, w_0^{2k-1}].$$

Proof In each $[w_0^i, w_1^i]$ or $[w_1^i, w_0^i]$, the basis elements $c_{w_0^i}, c_{w_1^i}$ each appear and are minimal/maximal. After multiplying, the basis elements c_{w_0}, c_{w_1}, $c_{\min(w_0, w_1)}$, $c_{\max(w_0, w_1)}$ all appear with $c_{\min(w_0, w_1)}$ minimal and $c_{\max(w_0, w_1)}$ maximal.   □

Proof of Proposition 3.7 The pairs (i)–(ii), (iii)–(iv), (v)–(vi), (vii)–(viii) are equivalent: for sutures Γ with closed components, Axiom 6 gives $c(\Gamma) = \{0\}$; for chord diagrams Γ, Lemma 3.11 (which works for any of the 8 variations) gives $c(\Gamma) = \{x, -x\}$; equivalence of the four pairs follows.

Implications (ii)$\Rightarrow$(iv) and (vi)$\Rightarrow$(viii) are obvious. The Inclusion axiom immediately gives (viii)$\Rightarrow$(vi). Lemma 3.5 is precisely (ii)$\Rightarrow$(v); in fact the argument shows (iv)$\Rightarrow$(v). We have

$$\{(\text{i}) \Leftrightarrow (\text{ii})\} \Rightarrow \{(\text{iii}) \Leftrightarrow (\text{iv})\} \Rightarrow \{(\text{v}) \Leftrightarrow (\text{vi}) \Leftrightarrow (\text{vii}) \Leftrightarrow (\text{viii})\}.$$

It now suffices to show (v) $\Rightarrow$ {(i), (ii)}. Assume (v), a bypass relation. In developing the structure of $V(D^2)$ we only used [Axiom 9](#) to prove (v) ([Lemma 3.5](#)), and then used (v) repeatedly; we have not used (i) or (ii) independently. Thus, we obtain all the structure described in Sections [3.5–3.12](#) above.

Let $\alpha, \beta \in V(D^2, F_n)$ have the property that for every chord diagram Γ and one (hence any) $c \in c(\Gamma)$, $\langle \alpha | c \rangle = \pm \langle \beta | c \rangle$. We will show $\alpha = \pm \beta$, giving (v) $\Rightarrow$ {(i), (ii)} as desired. Write the decompositions of α and β as $\alpha = \sum_{w \in \mathcal{W}_n^e} \alpha_w c_w$, $\beta = \sum_{w \in \mathcal{W}_n^e} \beta_w c_w$.

Using the isomorphism $V(D^2) \cong \mathcal{F}$, and the map Q_- ([Section 2.6](#)), we have $\langle \alpha | c \rangle = (Q_-^{-1} \alpha) \cdot c$ and $\langle \beta | c \rangle = (Q_-^{-1} \beta) \cdot c$. Rewriting $Q_-^{-1} \alpha, Q_-^{-1} \beta$ as α, β, we have two elements $\alpha, \beta \in V(D^2, F_n)$ such that $\alpha \cdot c = \pm \beta \cdot c$ for all suture elements c; it is sufficient to show $\alpha = \pm \beta$. Consider now c to run through basis elements c_w, for $w \in \mathcal{W}_n^e$; we obtain, for every $w \in \mathcal{W}_n^e$, $\alpha_w = \pm \beta_w$.

Partition $\mathcal{W}_n^e$ into three subsets

$$S = \{w \in \mathcal{W}_n^e : \alpha_w = \beta_w = 0\},$$
$$T = \{w \in \mathcal{W}_n^e : 0 \neq \alpha_w = \beta_w\},$$
$$U = \{w \in \mathcal{W}_n^e : 0 \neq \alpha_w = -\beta_w\}.$$

We will show one of T or U is empty. Suppose not. Consider the set A of suture elements $c \in V(D^2, F_n)$ whose basis decomposition contains words from both T and U. As T and U are nonempty, [Lemma 3.19](#) guarantees A is nonempty. For $c \in A$, letting $\pm c = \pm[w_0, w_1]$, define $l(c) = d(w_0, w_1)$ ([Definition 2.3](#)). By definition A contains no basis elements, so $l(c) > 0$.

Take $c \in A$ with $l(c)$ minimal; let $\pm c = \pm[w_0, w_1]$, so $w_0 \leq w_1$, $w_0 \neq w_1$. We claim that in the basis decomposition of c there is only one word from T and one from U. (There may be many from S.) Suppose not, so without loss of generality c contains distinct words $t_0, t_1 \in T$ and $u \in U$.

Let $w_- = \min(t_0, u)$ and let $w_+ = \max(t_0, u)$. By definition of w_0 we have $w_0 \leq t_0, u$ hence $w_0 \leq w_-$; indeed $w_0 \leq w_- \leq w_+ \leq w_1$. Thus $d(w_-, w_+) \leq d(w_0, w_1)$. By [Lemma 3.19](#), $\pm[w_-, w_+]$ contains t_0, u with nonzero coefficients; so $\pm[w_-, w_+] \in A$ and $l(\pm[w_-, w_+]) \leq l(\pm[w_0, w_1]) = l(c)$. By minimality of c then equality must hold; so $w_0 = w_-$ and $w_+ = w_1$. In particular, $w_- \leq t_1 \leq w_+$. As $w_- = \min(t_0, u)$ and $w_+ = \max(t_0, u)$, then, for every $i \in \{1, \ldots, n_x\}$, the i–th x in t_1 lies in a position between the respective positions of the i–th x in t_0 and u:

$$\text{either} \quad h_{t_0}^x(i) \leq h_{t_1}^x(i) \leq h_u^x(i) \quad \text{or} \quad h_u^x(i) \leq h_{t_1}^x(i) \leq h_{t_0}^x(i).$$

But the same argument applies with t_0 and t_1 reversed; then for every $i \in \{1, \ldots, n_x\}$,

$$\text{either} \quad h^x_{t_1}(i) \leq h^x_{t_0}(i) \leq h^x_u(i) \quad \text{or} \quad h^x_u(i) \leq h^x_{t_0}(i) \leq h^x_{t_1}(i).$$

Putting these together, we have that for all i, $h^x_{t_0}(i) = h^x_{t_1}(i)$, ie $t_0 = t_1$, a contradiction.

Thus the suture element $c \in V(D^2, F_n)$ contains precisely one word t from T and one u from U, with coefficients $\epsilon_t, \epsilon_u \in \{-1, 1\}$; the rest must be in S. Then $\alpha \cdot c = \alpha_t \epsilon_t + \alpha_u \epsilon_u$ and $\beta \cdot c = \beta_t \epsilon_t + \beta_u \epsilon_u$. By definition of T and U, $\alpha_t = \beta_t$ and $\alpha_u = -\beta_u$, and recall $\alpha \cdot c = \pm \beta \cdot c$. Hence

$$\alpha_t \epsilon_t + \alpha_u \epsilon_u = \begin{cases} \alpha_t \epsilon_t - \alpha_u \epsilon_u, \\ -\alpha_t \epsilon_t + \alpha_u \epsilon_u. \end{cases}$$

In the first case we have $2\alpha_u \epsilon_u = 0$ so $\alpha_u = 0$; in the second $2\alpha_t \epsilon_t = 0$ so $\alpha_t = 0$. These are both contradictions to the definitions of T and U. Thus one of T or U is empty, and $\alpha = \pm \beta$. □

Next we prove Proposition 3.8, using the idea of the "free presutured TQFT" (see Section 3.2).

Proof of Proposition 3.8 The equivalence of the two alternative axioms is immediate from Axiom 8, that each $V(\Sigma, F)$ is spanned by suture elements.

Define the sutured TQFT as follows: $V(\Sigma, F) = \bigoplus_\Gamma \mathbb{Z} c_\Gamma$ as for the free presutured TQFT; for any set of sutures Γ with contractible components, $c(\Gamma) = \{0\}$; for Γ without contractible components, $c(\Gamma) = \{c_\Gamma, -c_\Gamma\}$. For a gluing τ, gluing maps Φ^i_τ must satisfy $c_\Gamma \mapsto \pm c_{\#_\tau \Gamma}$ or 0 as appropriate; we define *many* gluing maps, covering all sign possibilities. More precisely, each $\Phi^i_\tau \colon V(\Sigma, F) \longrightarrow V(\#_\tau(\Sigma, F))$ takes $c_\Gamma \mapsto 0$, if $\#_\tau \Gamma$ has a contractible component; else takes $c_\Gamma \mapsto \pm c_{\#_\tau \Gamma}$, where the sign can be chosen freely. As the c_Γ form a basis, a choice of signs defines Φ^i_τ. We allow all choices of signs, and these form our collection Φ^i_τ. It is clear that Axioms 1–8 are satisfied.

The bilinear map $\langle \cdot \mid \cdot \rangle \colon V(D^2, F_n) \otimes V(D^2, F_n) \longrightarrow \mathbb{Z}$ obtained from stacking has many sign choices; we will show that these sign choices can be made to satisfy alternative axiom (i), proving the result. For sutures Γ_0, Γ_1 on (D^2, F_n), $\langle c_{\Gamma_0} | c_{\Gamma_1} \rangle$ must be 0, if Γ_0, Γ_1 are not stackable; otherwise $\langle c_{\Gamma_0} | c_{\Gamma_1} \rangle = \pm 1$, and we may choose either sign; as the c_Γ form a basis, signs may be chosen independently.

Ordering arbitrarily, write $\Gamma_1, \Gamma_2, \ldots, \Gamma_{C_n}$ for the chord diagrams on (D^2, F_n). Let $g_{ij} = \langle c_{\Gamma_i} | c_{\Gamma_j} \rangle$. For each pair (i, j), either g_{ij} is forced to be zero, or we can choose $g_{ij} = \pm 1$. The matrix $g = \{g_{ij}\}$ is the matrix for a bilinear map $\mathbb{Z}^{C_n} \otimes \mathbb{Z}^{C_n} \longrightarrow \mathbb{Z}$;

or equally for a map $\mathbb{Q}^{C_n} \otimes \mathbb{Q}^{C_n} \longrightarrow \mathbb{Q}$. We see that (i) holds if the matrix g over $\mathbb{Q}$ has full rank.

For $i \neq j$, choose g_{ij} arbitrarily (if there is a choice!). Now for any chord diagram Γ, $\langle c_\Gamma | c_\Gamma \rangle = \pm 1$; each diagonal element $g_{ii} = \pm 1$. Consider row-reducing g. Successively choose the g_{ii} so that, as row reduction proceeds to the i–th line, the (i,i) element remains nonzero. After row reduction, we have an upper triangular matrix with all diagonal entries nonzero; hence it has full rank, and (i) is satisfied. $\qquad \square$

3.14 An additional axiom

We consider a tenth axiom for sutured TQFT. Note that as gluings can only increase genus, we could simply set $V(\Sigma, F) = 0$ whenever Σ has genus ≥ 1, and obtain a sutured TQFT. Our additional axiom requires certain gluing maps to be isomorphisms.

Axiom 10 Let τ be a sutured gluing map on (Σ, F), identifying two disjoint arcs γ, γ' on $\partial \Sigma$. If $|\gamma \cap F| = |\gamma' \cap F| = 1$, then any associated gluing map Φ_τ^i is an isomorphism.

Any connected sutured background (Σ, F) can be constructed from a (D^2, F_n) by gluing maps of this type; in fact from a disjoint union of discs $\bigsqcup (D^2, F_2)$. So this final axiom gives isomorphisms $V(\Sigma, F) \cong V(\bigsqcup_i (D^2, F_2)) \cong \bigotimes_i V(D^2, F_2)$. Axiom 9 becomes redundant, and Axiom 8 almost redundant.

Lemma 3.20 *In the presence of Axioms 1–8, Axiom 10 implies Axiom 9.*

Proof On (D^2, F_2) there are only two chord diagrams; as in Section 3.6, inclusions $(D^2, F_2) \hookrightarrow (D^2, F_1)$ with intermediate sutures can be found to show that their suture elements are linearly independent. Two such suture elements span $V(D^2, F_2)$; they form a basis which we denote x, y and in fact $V(D^2, F_2) = V(D^2)_1 \cong \mathcal{F}_1$ as graded abelian groups.

Gluing two (D^2, F_2)'s gives an isomorphism $V(D^2, F_3) = V(D^2)_2 \cong (\mathbb{Z}x \oplus \mathbb{Z}y)^{\otimes 2} = \mathcal{F}_2$; the summand $V(D^2)_2^0$ has basis $\{x \otimes y, y \otimes x\}$, which are suture elements (exactly like xy, yx above) for bypass-related chord diagrams. Including $(D^2, F_3) \hookrightarrow (D^2, F_1)$ in various ways, we obtain coordinate projections $\mathbb{Z}^2 \cong V(D^2)_2^0 \longrightarrow V(D^2, F_1) \cong \mathbb{Z}$; from these we see that the third chord diagram in the bypass triple is $\pm x \otimes y \pm y \otimes x$. By Proposition 3.7, this is equivalent to Axiom 9. $\qquad \square$

Lemma 3.21 *Assume Axioms 1–7. Suppose on (D^2, F_2) the two chord diagrams, one each in Euler class -1 and 1, have suture elements $\pm x$, $\pm y$ respectively where $\{x, y\}$ is a basis for $V(D^2, F_2)$. Then Axiom 10 implies Axiom 8. (And hence, by Lemma 3.20, also Axiom 9.)*

Proof Since x, y form a basis for $V(D^2, F_2)$, gluing n (D^2, F_2)'s together, gluing isomorphisms (by Axiom 10) give a basis of suture elements $\{x, y\}^{\otimes n}$ for $V(D^2, F_{n+1})$. Thus each $V(D^2, F_n)$ is spanned by suture elements, and by further gluing we obtain a basis of any $V(\Sigma, F)$ of suture elements. $\qquad\square$

4 Sutured Floer homology and sutured TQFT

We now show how sutured Floer homology [14] defines a sutured TQFT. As noted in [10], $SFH(\Sigma \times S^1, F \times S^1)$, where Σ is a surface with $\partial\Sigma \neq \varnothing$ and $F \subset \partial\Sigma$ is finite, has TQFT-like properties; in fact sutured TQFT is designed to be an axiomatic model for it. Make the following assignments.

- To a sutured background (Σ, F), assign the abelian group $V(\Sigma, F) = SFH(-\Sigma \times S^1, -F \times S^1)$ with $\mathbb{Z}$ coefficients. It is known that SFH splits as a direct sum over spin-c structures.

- A set of sutures Γ on (Σ, F) without contractible components corresponds precisely to an isotopy class of tight contact structures ξ on $\Sigma \times S^1$, such that $\partial\Sigma \times S^1$ is convex with dividing set $F \times S^1$ [7; 8; 9]. Let $c(\Gamma)$ be the *contact invariant* $c(\xi) \subset V(\Sigma, F)$ [18; 12; 11]. This $c(\xi)$ is a subset of the form $\{\pm x\}$. The possible relative Euler classes of ξ correspond to the spin-c structures on (Σ, F); and $c(\xi)$ lies in the corresponding spin-c summand of SFH.

- For a gluing τ of the sutured background surface (Σ, F), let

  $$\Phi_\tau\colon SFH(-\Sigma \times S^1, -F \times S^1) \longrightarrow SFH(-(\#_\tau\Sigma) \times S^1, -(\#_\tau F) \times S^1)$$

  be the map defined in [10] by the obvious inclusion $\Sigma \times S^1 \hookrightarrow \#_\tau\Sigma \times S^1$, together with the canonical contact structure on $\#_\tau\Sigma \times S^1 - \Sigma \times S^1$ as convex neighbourhood of the boundary. In fact we can choose a sign on each Φ_τ on each Euler class summand; let Φ_τ^i be the collection of all maps obtained from Φ_τ by all possible choices of signs.

Proposition 4.1 *These assignments satisfy all axioms of sutured TQFT (including Axiom 10).*

Proof Clearly Axioms 1–3 are satisfied. Axiom 4, $V(\bigsqcup_i (\Sigma_i, F_i)) = \bigotimes_i V(\Sigma_i, F_i)$, is clear from the definition of SFH. Axiom 5, that a gluing map takes contact elements to contact elements, is proved in [10]. If Γ contains a contractible loop then the corresponding contact structure is overtwisted; by [11] then $c(\Gamma) = \{0\}$, so Axiom 6 holds. That $V(D^2, F_1) \cong \mathbb{Z}$ is proved in [10] and follows from [15] or [11]; that $c(\Gamma_\varnothing) = \{\pm 1\}$ is proved in [10] and follows from [11]; so Axiom 7 holds.

Axiom 10 is proved in [10]. Moreover, [10] shows, following [11], that $V(D^2, F_2) \cong \mathbb{Z}^2$, with basis given by contact elements for the two chord diagrams of Euler class -1 and 1. By Lemma 3.21, this implies Axiom 8; then Lemma 3.20 gives Axiom 9. $\qquad\square$

Thus, results about sutured TQFT gives results about SFH. In particular, contact elements in the SFH of sutured solid tori with longitudinal sutures have all the structure of $V(D^2)$.

Corollary 4.2 *A sutured TQFT exists.* $\qquad\square$

Corollary 4.2 could of course also be proved by explicit construction. On discs we could use all the machinery of the foregoing; however on more complicated surfaces, a nonzero construction would take more work.

5 Noncommutative QFT = sutured TQFT of discs

5.1 Main isomorphism and suture elements

We now make a detailed statement that "the sutured TQFT of discs is the QFT of two noncommuting particles"; most of this is already clear. This includes earlier Theorems 1.1 and 3.4.

Theorem 5.1 *Every sutured TQFT obeying Axioms 1–9 above satisfies*

$$V(D^2) \cong \mathcal{F},$$

an isomorphism of bigraded rings. In particular

$$V(D^2)_{n+1} \cong \mathcal{F}_n \cong \mathbb{Z}^{2^n}, \quad V(D^2)_{n_-,n_+} \cong \mathcal{F}_{n_x,n_y} \cong \mathbb{Z}^{\binom{n}{n_x}} \cong \mathbb{Z}^{\binom{n}{n_y}},$$

isomorphisms of abelian groups. Each $c_w \in V(D^2)_n^e$, for $w \in \mathcal{W}_n^e$, corresponds to $w \in \mathcal{M}_n^e \subset \mathcal{F}_n^e$, replacing $(-,+)$ with (x, y). Under this isomorphism:

   (i) *The action of operators $a_{x,i}, a_{y,i}, a_{x,i}^*, a_{y,i}^*$ on words $w \in \mathcal{M}_n^e \subset \mathcal{F}_n^e$ is identical to the action of operators $a_{-,i}, a_{+,i}, a_{-,i}^*, a_{+,i}^*$ on basis suture elements c_w, for $w \in \mathcal{W}_n^e$.*

(ii)   The $\langle\,\cdot\,|\,\cdot\,\rangle$ defined on $\mathcal{F}$ via the partial order $\leq$, and the $\langle\,\cdot\,|\,\cdot\,\rangle$ defined on $V(D^2)$ by stacking, agree.

(iii) The operators $T_{x,i}$, $T_{y,i}$, $T^*_{x,i}$, $T^*_{y,i}$, $U_{x,i}$, $U_{y,i}$ act on each $\mathcal{F}^e_n$ identically to the operators $T_{-,i}$, $T_{+,i}$, $T^*_{-,i}$, $T^*_{+,i}$, $U_{-,i}$, $U_{+,i}$ on $V(D^2)^e_n$.

(iv)   The duality operator H acts on $\mathcal{F}^e_n$ identically to the rotation operator R on $V(D^2)^e_n$.

(v)   The set of suture elements in $V(D^2)^e_n$ maps to the distinguished subset $\mathcal{C}^e_n$.

Most of this theorem has already been proved. The isomorphism of rings, $\langle\,\cdot\,|\,\cdot\,\rangle$, a^*, a, T, T^*, U, H and R is clear (indeed signs were chosen carefully to ensure it). It only remains to prove the last statement (v). But to do this we need to establish that $\mathcal{C}^e_n$ exists, in particular Proposition 2.9 that the three definitions $\mathcal{C}^1$, $\mathcal{C}^2$, $\mathcal{C}^3$ all agree. We prove Proposition 2.9 and Theorem 5.1 together by showing that all of $\mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3$ map to the set of suture elements.

Note that our proof of Proposition 2.9 relies upon an isomorphism to sutured TQFT, and hence upon the existence of a sutured TQFT; thus, at least in our presentation, there is a dependence upon sutured Floer homology.

Proof of Proposition 2.9 and Theorem 5.1(v) We first show $\mathcal{C}^1$ corresponds to suture elements. Clearly suture elements are preserved under $a_{s,i}$, $a^*_{s,i}$, and $T^*_{\pm,i}$; we show every suture element can be created from the vacuum by these operators. Clearly the 0 suture element can be obtained. For any chord diagram Γ with $n+1$ chords, we produce an element of $c(\Gamma)$.

Proof by induction on n. Clearly we can obtain the vacuum. If Γ has an outermost chord at the basepoint, or at the root point, or has an outermost chord enclosing a positive region on the eastside, or enclosing a negative region on the westside, then Γ is obtained from a smaller chord diagram by applying a creation operator, and we reduce to a smaller diagram. So we may assume all outermost regions are negative and on the eastside, or positive and on the westside.

Suppose there is an outermost negative region on the eastside; the other case is similar. Then there is one closest to the basepoint, enclosed by a chord running from $2i+1$ to $2i+2$, where $i \geq 0$ is minimal. (Labelling as in Section 3.1.) Then (Figure 12) Γ is obtained from a chord diagram Γ' by applying $a_{+,i+2}T^*_{+,i}$, where Γ' is identical to Γ, except that the outermost chord at $(2i+1, 2i+2)$ is moved to $(2i-1, 2i)$. The outermost negative region moves closer to the basepoint; repeating, we eventually have an outermost chord at the basepoint, and can reduce to a smaller diagram.
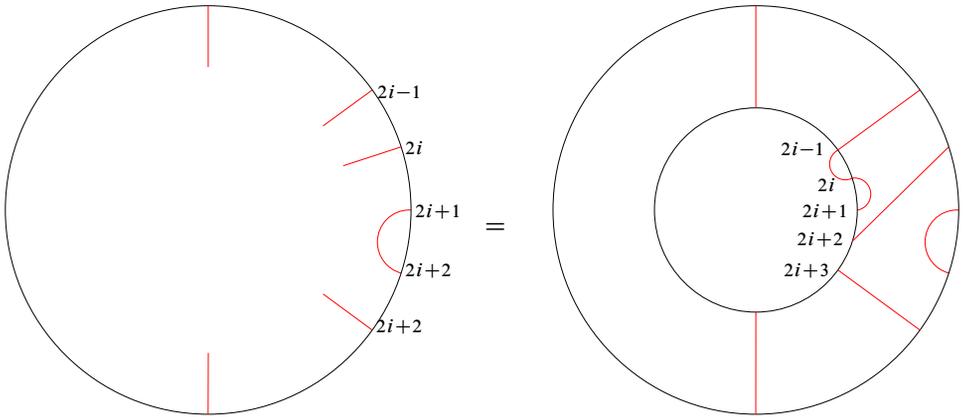
Figure 12: A chord diagram with outermost region at $(2i + 1, 2i + 2)$ is $a_{+,i+2} T_{+,i}^*$ of an otherwise identical chord diagram with outermost region at $(2i - 1, 2i)$.

Thus, for any chord diagram Γ, we can obtain a suture element in $c(\Gamma)$. To obtain both, it suffices to show we can obtain -1, which we do as follows. Hence $\mathcal{C}^1$ coincides with suture elements.

$$1 \overset{a_{x,0}^*}{\longmapsto} x \overset{T_{y,0}^*}{\longmapsto} yx - xy \overset{a_{x,0}}{\longmapsto} -y \overset{a_{y,0}}{\longmapsto} -1.$$

For $\mathcal{C}^2$, suture elements are clearly preserved by $a_{s,0}^*$ and H. We can clearly obtain 0. For a chord diagram Γ of $n + 1$ chords, we show we can produce an element of $c(\Gamma)$; again, induct on number of chords. The vacuum is clear. Now any Γ has an outermost chord; applying H this may be rotated to the basepoint; this diagram may be obtained from a smaller one via $a_{s,0}^*$. To show we can obtain -1, we need $H^3(xy) = -xy$ on $\mathcal{F}_{1,1}$, which can be computed by hand from [Corollary 2.8](we prove [Theorem 2.6], the general result, in [Section 5.4]). So $\mathcal{C}^2$ coincides with suture elements.

$$1 \overset{a_{y,0}^*}{\longmapsto} y \overset{a_{x,0}^*}{\longmapsto} xy \overset{H^3}{\longmapsto} -xy \overset{a_{x,0}}{\longmapsto} -y \overset{a_{y,0}}{\longmapsto} -1.$$

The proof for $\mathcal{C}^3$ is similar, replacing basepoint with root point. □

We now prove the algebraic statements about suture elements of [Theorem 2.10]. (Again, this proof relies on sutured TQFT existence.)

Proof of [Theorem 2.10] Part (i) is now clear; (ii) is a statement of properties of suture elements, as discussed in Sections [3.7] and [3.9]. For (iii), it's easy to check $Q_\pm$ does not preserve $\mathcal{C}$; but it is a bijection (we have computed an inverse), hence

$|Q_\pm \mathcal{C}^e_n| = |\mathcal{C}^e_n| = 2N^e_n$; and since $Q_\pm \circ Q^{-1}_\mp = H^{\pm 1}$ preserves $\mathcal{C}$ we have $Q_+ \mathcal{C} = Q_- \mathcal{C}$. Part (iv) follows from Proposition 3.16, definition of H and $\langle \cdot | \cdot \rangle$.

For (v), if the sum/difference of suture elements $u \in c(\Gamma_u)$, $v \in c(\Gamma_v)$ is also a suture element, then by [17, Proposition 1.10], Γ_u, Γ_v are bypass-related. After possibly reordering and switching signs, $\langle u|v \rangle = 1$, $\langle v|u \rangle = 0$. Then $u - v$ is a suture element for the third diagram. Being bypass-related, Γ_u, Γ_v may be isotoped to be identical, except in a disc $D' \subset D$; and we may choose a basepoint on D' such that $\Gamma_u \cap D' = \Gamma_{xy}$, $\Gamma_v \cap D' = \Gamma_{yx}$; these have suture elements xy, yx respectively, and $\langle xy|yx \rangle = 1$. Now Γ_u, Γ_v can be obtained by applying initial creation operators and rotations to Γ_{xy}, Γ_{yx}, so we obtain an operator A^* taking $xy \mapsto \pm u$, $yx \mapsto \pm v$. Since creations and rotations are isometries, we have $1 = \langle xy|yx \rangle = \langle A^*(xy)|A^*(yx) \rangle = \langle u|v \rangle$; thus under A^*, $(xy, yx) \mapsto \pm(u, v)$. If we get $-(u, v)$, then precompose A^* with $H^3 = -1$ on $\mathcal{F}_{1,1}$.

Part (vi) is essentially [17, Lemma 3.1] without contact geometry and with signs. Given chord diagrams Γ_u, Γ_v with $u \in c(\Gamma_u)$, $v \in c(\Gamma_v)$ and $\langle u|v \rangle = 1$, we find chord diagrams $\Gamma_u = \Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma_v$, and $u = c_0, c_1, \ldots, c_m = v$ where $c_i \in c(\Gamma_i)$, such that each $c_i - c_{i+1}$ is a suture element – in particular, each pair Γ_i, Γ_{i+1} is bypass-related – and for all $i \leq j$, $\langle c_i|c_j \rangle = 1$. Proof by induction on the number of chords in Γ_u, Γ_v. With less than three chords there is nothing to prove; with three chords Γ_u, Γ_v are either identical or bypass-related, and the result is clear.

For general Γ_u, Γ_v, if they share an outermost chord γ then consider $\Gamma_u - \gamma$ and $\Gamma_u - \gamma$; by induction we have a sequence of bypass-related chord diagrams with the desired properties; adding γ to all these gives an isometry (Section 3.12). Hence we obtain desired chord diagrams and suture elements.

Thus we may assume Γ_u, Γ_v have no outermost chords in common. Let γ be an outermost chord of Γ_v; denote its endpoints p, q in clockwise order, and the next marked point clockwise r. On Γ_u there is no outermost chord connecting p, q (by assumption), nor connecting q, r (since then rounding would give $\langle u|v \rangle = 0$). Thus Γ_u, Γ_v appear as in Figure 13. Perform upwards bypass surgery on $\Gamma_u = \Gamma_0$ along the arc δ shown to define Γ_1.

Now, Γ_1, Γ_v share the common outermost chord γ; from above, we obtain a sequence of chord diagrams $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$, where each Γ_i, Γ_{i+1} are bypass-related, and suture elements $c_1, \ldots, c_m$, where $c_i \in c(\Gamma_i)$ satisfy $\langle c_i|c_j \rangle = 1$ for $1 \leq i \leq j \leq m$ and each $c_i - c_{i+1}$ is a suture element. Adjusting the c_i by a sign if necessary, we may assume $c_m = v$. Taking $c_0 = u$ and $\Gamma_0 = \Gamma_u$, we claim that the Γ_i and c_i have the desired properties; we must verify that $c_0 - c_1$ is a suture element and that $\langle c_0|c_i \rangle = 1$ for $0 \leq i \leq m$.
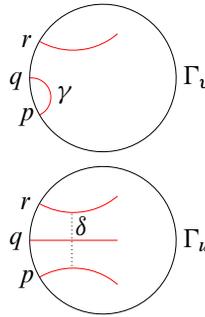
Figure 13: Chord diagrams Γ_u, Γ_v

To see $c_0 - c_1$ is a suture element, note that as Γ_0, Γ_1 are bypass-related, either $c_0 - c_1$ or $c_0 + c_1$ is a suture element; since $\langle c_0 | c_m \rangle = \langle u | v \rangle = 1$ and $\langle c_1 | c_m \rangle = 1$, we have $\langle c_0 \pm c_1 | c_m \rangle = 2$ or 0; thus $c_0 - c_1$ is a suture element.

We next show that for $1 \leq i \leq m$, $\langle c_0 | c_i \rangle = \pm 1$. Since Γ_1 is obtained from Γ_0 by upwards bypass surgery, $\langle c_0 | c_1 \rangle = \pm 1$. Now for $2 \leq i \leq m$, consider the sutured cylinders which are rounded in computing $\langle c_0 | c_i \rangle$ and $\langle c_1 | c_i \rangle$. Note Γ_i contains the outermost chord γ, which can be pushed down by a "finger move" to Γ_0 or Γ_1 to give identical sets of sutures. Since $\langle c_1 | c_i \rangle = 1$ then $\langle c_0 | c_i \rangle = \pm 1$.

Finally we verify that for $0 \leq i \leq m$, $\langle c_0 | c_i \rangle = 1$. Obviously $\langle c_0 | c_0 \rangle = 1$. As $c_0 - c_1$ is a suture element, $\langle c_0 | c_i \rangle = \pm 1$ and $\langle c_1 | c_i \rangle = 1$, we have $\langle c_0 - c_1 | c_i \rangle = \pm 1 - 1 = 0$ or -2. Hence $\langle c_0 | c_i \rangle = 1$. $\square$

The proof of (vi) here effectively gives another proof of [17, Lemma 3.1]. The proof constructs a tight contact structure on a solid cylinder by bypass attachments.

5.2 Duality–rotation explicitly

We now prove some detailed results about the $H = R$ duality/rotation operator. In [17] we obtained (mod 2) recursive and explicit descriptions of H; we prove these here over $\mathbb{Z}$.

Recall the formula of Corollary 2.8, including the exceptional sets E_w^s and the operators ψ_T^s:

$$Hw = Q_+ Q_-^{-1} w = \sum_{w_i \geq w} \sum_{T \subseteq E_{w_i}^y} (-1)^{|T|} \psi_T^y w_i$$

Regard x's as pawns and y's as empty squares, so $\psi_T^x w$ moves some pawns forward (right) one square each; $w \leq \psi_T^x w$. Call any $\psi_T^x w$ *pawn-ahead of* w. Similarly,

any $\psi_T^y w \leq w$; call any $\psi_T^y w$ *pawn-behind* w. Recall $f_w^x(i)$ is the number of y's to the left of the i–th x in w.

Proposition 5.2 *Let $w \in \mathcal{M}_n^e$. The words v occurring in Hw are precisely those such that:*

(i) *If $i \in E_v^x$ then $f_v^x(i) = f_w^x(i) - 1$.*

(ii) *If $i \notin E_v^x$ then $f_v^x(i) \geq f_w^x$.*

The coefficient of v in Hw is $(-1)^{|E_v^x|}$.

Proof From the formula, we have immediately that any v occurring in Hw is pawn-behind some u with $w \leq u$. So for v occurring in Hw, $f_v^x(i) \geq f_w^x(i)$ for $i \notin E_v^x$, and $f_v^x(i) \geq f_w^x(i) - 1$ for $i \in E_v^x$. Partition $\{1, 2, \ldots, n_x\}$ into three sets

$$S_0 = \{1, 2, \ldots, n_x\} - E_v^x,$$
$$S_1 = \left\{ i \in E_v^x : f_v^x(i) = f_w^x(i) - 1 \right\},$$
$$S_2 = \left\{ i \in E_v^x : f_v^x(i) \geq f_w^x(i) \right\}.$$

In fact, v appears once for every $u \geq w$ pawn-ahead of v. Such u are precisely those obtained by moving up all the x–pawns in S_1, and some subset A of the x–pawns in S_2. Thus, terms in the sum for Hw involving v are precisely

$$\sum_{A \subseteq S_2} (-1)^{|A \cup S_1|} v = (-1)^{|S_1|} v \sum_{A \subseteq S_2} (-1)^{|A|}.$$

If $S_2 \neq \varnothing$ this sum is 0; if $S_2 = \varnothing$ then v appears in Hw with coefficient $(-1)^{|S_1|}$.

So, the v occurring in Hw are precisely those with $f_v^x(i) \geq f_w^x(i)$ for $i \notin E_v^x$, and such that $S_2 = \varnothing$, ie $f_v^x(i) = f_w^x(i) - 1$ for all $i \in E_v^x$. The coefficient of such a v is $(-1)^{|S_1|}$, and $S_1 = E_v^x$. $\square$

In general, take a word

$$w = x^{a_1} y^{b_1} x^{a_2} y^{b_2} \cdots x^{a_k} y^{b_k}$$

where $k \geq 2$, possibly $a_1 = 0$, possibly $b_k = 0$, but all other $a_i, b_i \neq 0$. Consider v satisfying the conditions of the above proposition

$$v = y^{\beta_1} x^{\alpha_1} \cdots y^{\beta_l} x^{\alpha_l}$$

for some positive integer l; possibly $\beta_1 = 0$, possibly $\alpha_l = 0$ but all other $\alpha_i, \beta_i \neq 0$. The values of f_v^x are given by the cumulative sums $\beta_1 + \cdots + \beta_i$, and the values of f_w^x by cumulative sums $b_1 + \cdots + b_j$. By condition (i) of Proposition 5.2 we must have

$$\beta_1 + \cdots + \beta_i = b_1 + \cdots + b_{P(i)} - 1$$

for $1 \leq i \leq l-1$, for some increasing sequence $1 \leq P(1) < P(2) < \cdots < P(l-1) \leq k$. And of course $\beta_1 + \cdots + \beta_l = b_1 + \cdots + b_k$, so set $P(l) = k$. To maintain the two conditions of the proposition, we must also have the x's ending each block in v corresponding to the x beginning the next block in w; thus

$$\alpha_1 + \cdots + \alpha_i = a_1 + \cdots + a_{P(i)} + 1.$$

This term appears with sign $(-1)^{l-1}$. So Hw is the sum of the terms

$$(-1)^{l-1} y^{b_1 + \cdots + b_{P(1)} - 1} x^{a_1 + \cdots + a_{P(1)} + 1} y^{b_{P(1)+1} + \cdots + b_{P(2)}} x^{a_{P(1)+1} + \cdots + a_{P(2)}} \cdots$$

$$\cdots y^{b_{P(l-2)+1} + \cdots + b_{P(l-1)}} x^{a_{P(l-2)+1} + \cdots + a_{P(l-1)}}$$

$$\cdot y^{b_{P(l-1)+1} + \cdots + b_{P(l)} + 1} x^{a_{P(l-1)+1} + \cdots + a_{P(l)} - 1}$$

over all $l \leq k$ and all increasing sequences $1 \leq P(1) < P(2) < \cdots < P(l) = k$ (with the exception that for the trivial sequence $l = 1$, $P(1) = k$ the term is $y^{n_y} x^{n_x}$). There are 2^{k-1} terms in the sum. Recall here $k \geq 2$; if $k = 1$ then we can easily check $Hx^n = x^n$, $Hy^n = y^n$, $Hx^{n_x} y^{n_y} = y^{n_y} x^{n_x}$. This gives [17, Proposition 7.2], now with signs.

5.3 Duality–rotation recursively

We prove [17, Proposition 7.1], now with signs. Note the change of notation: A_-, B_-, A_+, B_+ of [17] are now respectively $a_{y,0}, a_{x,0}^*, a_{x,0}, a_{y,0}^*$.

Lemma 5.3

  (i) $a_{y,0} H a_{y,0}^* = H$.

  (ii) $a_{y,0} H (a_{x,0}^*)^j a_{y,0}^* = H(a_{x,0}^*)^j$.

  (iii) $a_{y,0} (a_{x,0})^{j+1} H (a_{x,0}^*)^j a_{y,0}^* a_{x,0}^* = -H$.

  (iv) (a) $a_{y,0} (a_{x,0})^j H (a_{x,0})^j = 0$ for $j = 1, \ldots, n_x$.
       (b) $(a_{x,0})^{j+2} H (a_{x,0}^*)^j a_{y,0}^* = 0$.
       (c) $a_{x,0} H (a_{x,0}^*)^j (a_{y,0}^*)^2 = 0$.

Proof Since all creations and annihilations are initial, we drop 0 from the notation. For (i), we have

$$\langle v|a_y H a_y^* w\rangle = \langle a_y^* v|H a_y^* w\rangle = \langle a_y^* w|a_y^* v\rangle = \langle w|v\rangle = \langle v|Hw\rangle.$$

Here we use the adjoint property; then definition of H; then creation operators are isometries; then definition of H. For (ii),

$$\langle v|a_y H(a_x^*)^j a_y^* w\rangle = \langle a_y^* v|H(a_x^*)^j a_y^* w\rangle = \langle (a_x^*)^j a_y^* w|a_y^* v\rangle$$
$$= \langle (a_x^*)^j w|v\rangle = \langle v|H(a_x^*)^j w\rangle.$$

First we use the adjoint property; second and fourth the definition of H; the third equality follows from the observation that $x^j y w \leq y v$ if and only if $x^j w \leq v$.

Identity (iii) is the most difficult; first observe that in

$$\langle v|a_y (a_x)^{j+1} H(a_x^*)^j a_y^* a_x^* w\rangle,$$

any term of $H(a_x^*)^j a_y^* a_x^* w$ which does not begin with $x^{j+1} y$ is annihilated (to 0!) by the operators $a_y (a_x)^{j+1}$; for each term $x^{j+1} y u$ occurring, the above inner product outputs $\langle v|u\rangle$.

Now note that if $v \leq u$ then $x^{j+1} y v \leq x^{j+1} y u$, and $x^j y x v \nleq x^{j+1} y u$. But, for any word z whose first $j+2$ symbols are not $x^{j+1} y$, we have $x^{j+1} y v \leq z$ if and only if $x^j y x v \leq z$. Thus

$$\langle x^{j+1} y v - x^j y x v | z\rangle = \begin{cases} \langle v|u\rangle & \text{if } z = x^{j+1} y u, \\ 0 & \text{otherwise} \end{cases} = \langle v|a_y (a_x)^{j+1} z\rangle.$$

Hence

$$\langle v|a_y (a_x)^{j+1} H(a_x^*)^j a_y^* a_x^* w\rangle$$
$$= \langle (a_x^*)^{j+1} a_y^* v|H(a_x^*)^j a_y^* a_x^* w\rangle - \langle (a_x^*)^j a_y^* a_x^* v|H(a_x^*)^j a_y^* a_x^* w\rangle$$
$$= \langle (a_x^*)^j a_y^* a_x^* w|(a_x^*)^{j+1} a_y^* v\rangle - \langle (a_x^*)^j a_y^* a_x^* w|(a_x^*)^j a_y^* a_x^* v\rangle$$
$$= -\langle w|v\rangle = -\langle v|Hw\rangle,$$

where we first use the previous observation, then the definition of H, then notice $x^j y x \nleq x^{j+1} y$ and that creations are isometries, and finally the definition of H again. This proves (iii).

We do not need to prove (iv) algebraically. As in [17], it is sufficient to observe that the operators on chord diagrams produce closed loops, so are 0. $\qquad\square$

Consider the matrix of each H_{n_x,n_y}. Following [17], for words w_0, w_1 (with $\leq n_x$ x's and $\leq n_y$ y's), define the $w_0 \times w_1$ minor of this matrix to be the intersection of the rows corresponding to words beginning with w_0, with the columns corresponding to words beginning with w_1. The above lemma gives the following description of H_{n_x,n_y}:

(i) The $y \times y$ minor consists of H_{n_x,n_y-1}.

(ii) The $y \times xy$ minor contains the x–columns of H_{n_x,n_y-1}. More generally, for any $j = 1, \ldots, n_x$, the $y \times x^j y$ minor contains the x^j–columns of H_{n_x,n_y-1}.

(iii) The $xy \times yx$ minor consists of $-H_{n_x-1,n_y-1}$. More generally, for any $j = 0, \ldots, n_x - 1$, the $x^{j+1} y \times x^j yx$ minor consists of $-H_{n_x-j-1,n_y-1}$.

(iv) All other entries are zero.

We may write this recursive structure as a formula, as in [17]: the "fake" commutator (commutators do not mean much mod 2!) there becomes a real commutator here.

Theorem 5.4

$$H = \sum_{i=0}^{\infty} a_{y,0}^* H (a_{x,0}^*)^i a_{y,0} (a_{x,0})^i - (a_{x,0}^*)^{i+1} a_{y,0}^* H a_{x,0} a_{y,0} (a_{x,0})^i$$

$$= \sum_{i=0}^{\infty} [a_{y,0}^* \, H \, a_{x,0}, \, (a_{x,0}^*)^{i+1}] a_{y,0} (a_{x,0})^i. \qquad \square$$

5.4 Periodicity

We now prove Theorem 2.6, $H^{n+1} = (-1)^{n_x n_y}$.

Consider $w \in \mathcal{M}_{n_x,n_y}$ as an $n \times 1$ chessboard; x's are pawns and y's are empty squares. Form a sequence of words / chessboard configurations, which begins at w_1 with all pawns / x's at the extreme right, $w_1 = y^{n_y} x^{n_x}$. Pawns move one square left, if the square to their left is empty. Starting from w_1, form a sequence w_i: on each turn all possible pawns move; we eventually arrive at w_n with all pawns at the extreme left, $w_n = x^{n_x} y^{n_y}$. For example, with $n_x = 2$ and $n_y = 3$,

$$w_1 = yyyxx, \quad w_2 = yyxyx, \quad w_3 = yxyxy, \quad w_4 = xyxyy, \quad w_5 = xxyyy.$$

Obviously $w_1 > w_2 > \cdots > w_n$. We will compute $H[w_i, w_{i-1}]$.

As above, $\pm[u, v]$ denotes the suture elements with first word u and last word v. Adopt the notation that $[u, v]$ is the suture element in which u has coefficient 1. Consider a

suture element $[w_i, w_{i-1}]$, $2 \leq i \leq n$; it must take one of the following four forms:

$$[w_i, w_{i-1}] = x^\alpha(xy - yx)^\beta x^\gamma, \ x^\alpha(xy - yx)^\beta y^\gamma,$$
$$y^\alpha(xy - yx)^\beta x^\gamma \ \text{ or } \ y^\alpha(xy - yx)^\beta y^\gamma.$$

To see why, note that all the above expressions are products of suture elements, hence suture elements; expanding them out we obtain a sum where the lexicographically first is some w_i, and the last is some w_{i-1}; in fact any w_i and w_{i-1} can be obtained in this way.

Examining the explicit formula for $R = H$, we see that in any word w beginning $w = x^\alpha(xy)^k(yx)\cdots$ or $y^\alpha(xy)^k(yx)\cdots$, where $k \geq 0$, the minimum word occurring in Hw begins

$$\min H\big(x^\alpha(xy)^k(yx)\cdots\big) = \begin{cases} x^{\alpha+1}(xy)^k yx \cdots & k \geq 2, \\ yx^{\alpha+2}\cdots & k = 1, \\ x^{\alpha+1}y\cdots & k = 0, \end{cases}$$

$$\min H\big(y^\alpha(xy)^k(yx)\cdots\big) = \begin{cases} y^{\alpha-1}(xy)^k yx \cdots & k \geq 1, \\ y^\alpha x \cdots & k = 0. \end{cases}$$

On the other hand, in Hw_i, the (lexicographically) first word occurring is w_{i+1}, with coefficient $(-1)^{|E^x_{w_{i+1}}|}$. Thus, for any $w \neq w_i$ occurring in $[w_i, w_{i-1}]$, the first word occurring in Hw is lexicographically after w_{i+1}. Hence for $2 \leq i \leq n-1$, $H([w_i, w_{i-1}]) = (-1)^{|E^x_{w_{i+1}}|}[w_{i+1}, v]$ for some v.

We can see what this v is directly by examining chord diagrams. The basis chord diagrams Γ_{w_i} and $[\Gamma_{w_i}, \Gamma_{w_{i-1}}]$ are easily constructed, as in [17, Section 5]. We see $H[\Gamma_{w_i}, \Gamma_{w_{i-1}}] = [\Gamma_{w_{i+1}}, \Gamma_{w_i}]$. Hence

$$H([w_i, w_{i-1}]) = (-1)^{|E^x_{w_{i+1}}|}[w_{i+1}, w_i]$$

for $2 \leq i \leq n-1$. We also easily compute $Hw_1 = (-1)^{|E^x_{w_2}|}[w_2, w_1]$, $H[w_n, w_{n-1}] = w_n$ and $Hw_n = w_1$. It follows that

$$H^{n+1}w_n = H^n w_1 = \left(\prod_{i=1}^n (-1)^{|E^x_{w_i}|}\right) H[w_n, w_{n-1}] = (-1)^{\sum_{i=1}^n |E^x_{w_i}|} w_n$$

Recall that E^x_w is the set of x's in w followed by y's. Now as we run through the set w_i, and pawns move from right to left, we see that for every x and every y, they are adjacent to each other as xy precisely once. Thus $\sum |E^x_{w_i}|$ counts every pair of an x and a y exactly once, and equals $n_x n_y$. So $H^{n+1} x^{n_x} y^{n_y} = (-1)^{n_x n_y} x^{n_x} y^{n_y}$.

Proof of Theorem 2.6 We have shown above that for $w = w_{\min} = x^{n_x} y^{n_y}$, $H^{n+1} w = (-1)^{n_x n_y} w$. Let now $A = \{w \in \mathcal{M}_{n_x, n_y} : H^{n+1} w = (-1)^{n_x n_y} w\}$, so $w_{\min} \in A$. That $H^{n+1} = (-1)^{n_x n_y}$ follows immediately from the following fact:

- If $w \in A$ and w, w' are related by an elementary move, then $w' \in A$.

To see this, let $H^{n+1} w' = \epsilon w'$ where $\epsilon = \pm 1$. As w, w' are related by an elementary move, $w - w'$ is a suture element, and $H^{n+1}(w - w') = \pm(w - w')$. But we have $H^{n+1}(w - w') = (-1)^{n_x n_y} w - \epsilon w' = (-1)^{n_x n_y}(w - (-1)^{n_x n_y} \epsilon w')$. Hence we must have $(-1)^{n_x n_y} \epsilon = 1$, so $\epsilon = (-1)^{n_x n_y}$ and $w' \in A$.

Finally, it's clear from above that the least positive j for which $H^j(x^{n_x} y^{n_y}) = \pm x^{n_x} y^{n_y}$ is $j = n + 1$; and $H^{n+1} = (-1)^{n_x n_y}$. Thus H has period $n + 1$ if $n_x n_y$ is even, and period $2n + 2$ otherwise. □

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