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Studying uniform thickness II: Transversely nonsimple iterated torus knots

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We prove that an iterated torus knot type in (S^3, ξ_{std}) fails the uniform thickness property (UTP) if and only if it is formed from repeated positive cablings, which is precisely when an iterated torus knot supports the standard contact structure. This is the first complete UTP classification for a large class of knots. We also show that all iterated torus knots that fail the UTP support cabling knot types that are transversely nonsimple.

57M25, 57R17; 57M50

1 Introduction

Let K be a knot type in S^3 with the standard tight contact structure ξ_{std}. The *uniform thickness property* (UTP) is fundamental to understanding embeddings of solid tori representing K in (S^3, ξ_{std}); in brief, K satisfies the UTP if every such solid torus thickens to one with convex boundary slope $1/\overline{\text{tb}}(K)$. If there exists a solid torus representing K that does not exhibit thickening, K fails the UTP, and such a solid torus is said to be *nonthickenable*. The UTP was first introduced by Etnyre and Honda [6], who showed that the $(2,3)$–torus knot fails the UTP by identifying such nonthickenable tori. They then used this to show that the $(2,3)$–torus knot supports a transversely nonsimple cabling knot type. In joint work with Etnyre and Tosun [7], we extended this study to show that all positive (p,q)–torus knots fail the UTP and support nonsimple cablings; furthermore, we established a complete Legendrian and transverse classification for cables of positive torus knots through the study of both nonthickenable and *partially thickenable tori*. In [15], we also showed that the general class of knot types K which both satisfy the UTP and are Legendrian simple is closed under the operation of cabling. An application of this was the identification of large classes of Legendrian simple iterated torus knot types.

In this paper we determine precisely which iterated torus knot types satisfy the UTP and which fail the UTP; this is the first complete UTP classification for a large class of knots. We also prove that any iterated torus knot type that fails the UTP supports

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transversely nonsimple cabling knot types. Specifically, we have the following theorems and corollary:

Theorem 1.1 *Let $K_r = ((P_1, q_1), \dots, (P_i, q_i), \dots, (P_r, q_r))$ be an iterated torus knot, where the iterated cablings (P_i, q_i) are measured in the standard Seifert framing, and $q_i > 1$ for all i. Then K_r fails the UTP if and only if $P_i > 0$ for all i, where $1 \le i \le r$.*

In the second theorem, $\chi(K)$ is the Euler characteristic of a minimal genus Seifert surface for a knot K:

Theorem 1.2 *If K_r is an iterated torus knot that fails the UTP, then it supports infinitely many transversely nonsimple cablings K_{r+1}, specifically $(-\chi(K_r), k+1)$–cablings of K_r, where k ranges over an infinite subset of positive integers.*

To state our corollary to Theorem 1.1, recall that if K is a fibered knot, then there is an associated open book decomposition of S^3 that supports a contact structure, denoted ξ_K (see Etnyre [4] and Thurston and Winkelnkemper [16]). Iterated torus knots are fibered knots, and Hedden has shown that the subclass of iterated torus knots where each iteration is a positive cabling, ie $P_i > 0$ for all i, is precisely the subclass of iterated torus knots where ξ_{K_r} is isotopic to ξ_{std} [11]. We thus obtain the following corollary:

Corollary 1.3 *An iterated torus knot K_r fails the UTP if and only if $\xi_{K_r} \cong \xi_{\mathrm{std}}$.*

We make a few remarks about these theorems. First, it will be shown that these transversely nonsimple cablings all have two Legendrian isotopy classes at the same rotation number and maximal Thurston–Bennequin number. Second, in the class of iterated torus knots there are certainly more transversely nonsimple cablings than those in Theorem 1.2, as seen in [6; 7]. However, we present just the class of nonsimple cablings in Theorem 1.2, and leave a more complete classification as an open question.

We now present a conjectural generalization of the above two theorems and corollary. To this end, recall that Hedden has shown that for general fibered knots K in S^3, $\xi_K \cong \xi_{\mathrm{std}}$ precisely when K is a fibered strongly quasipositive knot [12]; he also shows that for knots K with $\xi_K \cong \xi_{\mathrm{std}}$, the maximal self-linking number is $\overline{\mathrm{sl}}(K) = -\chi(K)$ [10]. Furthermore, from the work of Etnyre and Van Horn-Morris [8], we know that for fibered knots K in S^3 that support the standard contact structure there is a unique transverse isotopy class with maximal self-linking number. In the present paper, all of these ideas are brought to bear on the class of iterated torus knots, and this motivates the following conjecture concerning general fibered knots:

Conjecture 1.4 *Let K be a fibered knot in S^3; then K fails the UTP if and only if $\xi_K \cong \xi_{\text{std}}$, and hence if and only if K is fibered strongly quasipositive. Moreover, if a topologically nontrivial fibered knot K fails the UTP, then it supports cablings that are transversely nonsimple.*

Our main tools will be convex surface theory and the classification of tight contact structures on solid tori and thickened tori. Most of the results we use can be found in Etnyre and Honda [6], Etnyre, LaFountain and Tosun [7], Honda [13; 14] and LaFountain [15], and if we use a result from one of these works, it will be specifically referenced. Moreover, Sections 2.2–2.4 of [7] provide a nice summary of much of the needed background.

The plan of the paper is as follows. In Section 2 we recall definitions, notation and identities used in [6; 7; 13; 14; 15]. In Section 3 we outline a strategy of proof of Theorem 1.1 that yields the statement of two key lemmas, one of which is immediately proved. In Section 4 we prove the second lemma and complete the proof of Theorem 1.1. In Section 5 we prove Theorem 1.2.

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2 Definitions, notation and identities

2.1 Iterated torus knots

Iterated torus knots, as topological knot types, can be defined recursively. Let 1–iterated torus knots be simply torus knots (p_1, q_1) with p_1 and q_1 coprime nonzero integers, and $|p_1|, q_1 > 1$. Here, as usual, p_1 is the algebraic intersection with a longitude, and q_1 is the algebraic intersection with a meridian in the preferred Seifert framing for a torus representing the unknot. Then for each (p_1, q_1) torus knot, take a tubular neighborhood $N((p_1, q_1))$; the boundary of this is a torus, and given a framing we can describe simple closed curves on that torus as coprime pairs (p_2, q_2), with $q_2 > 1$. In this way we obtain all 2–iterated torus knots, which we represent as ordered pairs, $((p_1, q_1), (p_2, q_2))$. Recursively, suppose the $(r-1)$–iterated torus knots are defined; we can then take tubular neighborhoods of all of these, choose a framing, and form the r–iterated torus knots as ordered r–tuples $((p_1, q_1), \ldots, (p_{r-1}, q_{r-1}), (p_r, q_r))$, again with p_r and q_r coprime, and $q_r > 1$.

For ease of notation, if we are looking at a general r–iterated torus knot type, we will refer to it as K_r; a Legendrian representative will usually be written as L_r.

We will study iterated torus knots using two framings. The first is the standard Seifert framing for a torus, where the meridian bounds a disc inside the solid torus, and we use the preferred longitude which bounds a surface in the complement of the solid torus. We will refer to this framing as $\mathcal{C}$. The second framing is a nonstandard framing using a different longitude that comes from the cabling torus. More precisely, to identify this nonstandard longitude on $\partial N(K_r)$, we first look at K_r as it is embedded in $\partial N(K_{r-1})$. We take a small neighborhood $N(K_r)$ such that $\partial N(K_r)$ intersects $\partial N(K_{r-1})$ in two parallel simple closed curves. These curves are longitudes on $\partial N(K_r)$ in this second framing, which we will refer to as $\mathcal{C}'$. Note that this $\mathcal{C}'$ framing is well-defined for any cabled knot type. Moreover, for purpose of calculations there is an easy way to change between the two framings, which will be reviewed below.

Given a fixed choice of meridian, m, and longitude, l, on a torus, we may express simple closed curves as homology classes $\mu[m] + \lambda[l]$ on that torus, which we may also denote as (μ, λ). We will say such a curve has *slope* of λ/μ. Therefore we will refer to the longitude in the $\mathcal{C}'$ framing as ∞', and the longitude in the $\mathcal{C}$ framing as ∞. The meridian in both framings will have slope 0. These are the conventions used in [5; 7; 15].

We will also use a convention that curves measured in the standard $\mathcal{C}$ framing will typically be denoted as (P, q), that is, the algebraic intersection with the ∞–longitude will be denoted by upper-case P's. On the other hand, curves in the nonstandard $\mathcal{C}'$ framing will typically be denoted as (p, q), that is, the algebraic intersection with the ∞'–longitude will be denoted by lower-case p's. These are the conventions used in [15]. Given a curve $L = (P, q)$ on a torus ∂N, there is then a relationship between the framings $\mathcal{C}'$ and $\mathcal{C}$ on $\partial N(L)$. In terms of a change of basis, we can represent slopes λ/μ as column vectors and then get from a slope λ/μ', measured in $\mathcal{C}'$ on $\partial N(L)$, to a slope λ/μ, measured in $\mathcal{C}$, by

$$\begin{pmatrix} 1 & Pq \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu' \\ \lambda \end{pmatrix} = \begin{pmatrix} \mu \\ \lambda \end{pmatrix}.$$

In other words, $\mu = \mu' + Pq\lambda$.

Given an iterated torus knot type $K_r = ((p_1, q_1), \dots, (p_r, q_r))$ where the p_i's are measured in the $\mathcal{C}'$ framing, we define two quantities. The two quantities are

$$A_r := \sum_{\alpha=1}^{r} p_\alpha \prod_{\beta=\alpha+1}^{r} q_\beta \prod_{\beta=\alpha}^{r} q_\beta, \quad B_r := \sum_{\alpha=1}^{r} \Bigl(p_\alpha \prod_{\beta=\alpha+1}^{r} q_\beta \Bigr) + \prod_{\alpha=1}^{r} q_\alpha.$$

Note here we use a convention that $\prod_{\beta=r+1}^{r} q_\beta := 1$. Also, if we restrict to the first i iterations, that is, to $K_i = ((p_1, q_1), \ldots, (p_i, q_i))$, we have an associated A_i and B_i. For example,

$$A_i := \sum_{\alpha=1}^{i} p_\alpha \prod_{\beta=\alpha+1}^{i} q_\beta \prod_{\beta=\alpha}^{i} q_\beta.$$

From [15, Section 3] we obtain four useful identities which we will apply extensively throughout this note:

(1) $A_r = q_r^2 A_{r-1} + p_r q_r, \quad B_r = q_r B_{r-1} + p_r, \quad P_r = q_r A_{r-1} + p_r \quad A_r = P_r q_r.$

We conclude with a computation of the Euler characteristic for iterated torus knots obtained through positive cablings (see also [15, Lemma 3.3]).

Lemma 2.1 *Suppose $K_r = ((P_1, q_1), \ldots, (P_r, q_r))$ is an iterated torus knot where $P_i > 0$ for all i. Then $-\chi(K_r) = A_r - B_r$.*

Proof A formula for $\chi(K_r)$ is given at the end of the proof of Corollary 3 in Birman and Wrinkle [2]. In the notation used in that paper, the formula is $\chi(K_r) = \prod_{i=1}^{r} p_i - \sum_{i=1}^{r} q_i (p_i - 1) \prod_{j=i+1}^{r} p_j$, since in our case all the $e_i = 1$ as we are cabling positively at each iteration. However, note that our (P_i, q_i) corresponds to (q_i, p_i) in [2] for $i > 1$. We thus obtain the equation

$$\chi(K_r) = P_1 \prod_{i=2}^{r} q_i - q_1(P_1 - 1) \prod_{i=2}^{r} q_i - \sum_{i=2}^{r} P_i(q_i - 1) \prod_{j=i+1}^{r} q_j.$$

Examination of this formula for $\chi(K_r)$ yields the following recursive expression using our P's and q's:

$$\begin{aligned}\chi(K_r) &= q_r \left(P_1 \prod_{i=2}^{r-1} q_i - q_1(P_1 - 1) \prod_{i=2}^{r-1} q_i - \sum_{i=2}^{r-1} P_i(q_i - 1) \prod_{j=i+1}^{r-1} q_j \right) - P_r(q_r - 1) \\ &= q_r \chi(K_{r-1}) - P_r q_r + P_r.\end{aligned}$$

For a positive torus knot (P_1, q_1), we have $\chi = -P_1 q_1 + P_1 + q_1 = -A_1 + B_1$, so we can assume the lemma holds for K_{r-1}. Thus using the recursive expression we have

$$\begin{aligned}\chi(K_r) &= q_r \chi(K_{r-1}) - P_r q_r + P_r \\ &= q_r(-A_{r-1} + B_{r-1}) - A_r + q_r A_{r-1} + p_{r-1} \\ &= -A_r + B_r.\end{aligned}$$

This last equality uses Equation (1) above. □

2.2 Legendrian knots, convex tori and the UTP

Recall that for Legendrian knots embedded in S^3 with the standard tight contact structure, there are two classical invariants of Legendrian isotopy classes, namely the Thurston–Bennequin number, tb, and the rotation number, r. For a given topological knot type, if the ordered pair $(\mathrm{r}, \mathrm{tb})$ completely determines the Legendrian isotopy classes, then that knot type is said to be *Legendrian simple*. For transverse knots there is one classical invariant, the self-linking number sl; for a given topological knot type, if the value of sl completely determines the transverse isotopy classes, then that knot type is said to be *transversely simple*. For a given topological knot type, if we plot Legendrian isotopy classes at points $(\mathrm{r}, \mathrm{tb})$, we obtain a plot of points that takes the form of a *Legendrian mountain range* for that knot type.

We will be examining Legendrian knots which are embedded in convex tori. Recall that the characteristic foliation induced by the contact structure on a convex torus can be assumed to have a standard form, where there are $2n$ parallel *Legendrian divides* and a one-parameter family of *Legendrian rulings*. Parallel push-offs of the Legendrian divides gives a family of $2n$ *dividing curves*, referred to as Γ. For a particular convex torus, the slope of components of Γ is fixed and is called the *boundary slope* of any solid torus which it bounds; however, the Legendrian rulings can take on any slope other than that of the dividing curves by Giroux's Flexibility Theorem [9]. A *standard neighborhood* of a Legendrian knot L will have two dividing curves and a boundary slope of $1/\operatorname{tb}(L)$.

We can now state the definition of the *uniform thickness property* as given by Etnyre and Honda [6]. For a knot type K, define the *contact width* of K to be

$$w(K) = \sup \frac{1}{\operatorname{slope}(\Gamma_{\partial N})}.$$

In this equation the N are solid tori having representatives of K as their cores; slopes are measured using the Seifert framing where the longitude has slope ∞; the supremum is taken over all solid tori N representing K where ∂N is convex. Any knot type K satisfies the inequality $\overline{\operatorname{tb}}(K) \leq w(K) \leq \overline{\operatorname{tb}}(K) + 1$, where $\overline{\operatorname{tb}}$ is the maximal Thurston–Bennequin number for K. A knot type K then *satisfies the uniform thickness property (UTP)* if the following hold:

1. $\overline{\operatorname{tb}}(K) = w(K)$.
2. Every solid torus N representing K can be thickened to a standard neighborhood of a maximal $\overline{\operatorname{tb}}$ Legendrian knot.

A solid torus N *fails to thicken* if for all $N' \supset N$, we have $\mathrm{slope}(\Gamma_{\partial N'}) = \mathrm{slope}(\Gamma_{\partial N})$. Thus one of the ways a knot type K may fail the UTP is if it is represented by a solid torus N which fails to thicken, and such that $\mathrm{slope}(\Gamma_{\partial N}) \neq 1/\overline{\mathrm{tb}}(K)$.

Given a Legendrian curve $L = (P, q)$ on a convex torus ∂N, we define t to be the twisting of the contact planes along L with respect to the $\mathcal{C}'$ framing on $\partial N(L)$; in this case, [6, Equation 2.1] gives us

$$\mathrm{tb}(L) = Pq + t(L). \tag{2}$$

Observe that $t(L)$ is also the twisting of the contact planes with respect to the framing given by ∂N, and so is equal to $-1/2$ times the geometric intersection number of L with $\Gamma_{\partial N}$. We denote by $\overline{t}$ the maximal twisting number with respect to this framing.

We also had two definitions introduced in [15] that will be useful in this note.

Definition 2.2 Let N be a solid torus with convex boundary in standard form, and with $\mathrm{slope}(\Gamma_{\partial N}) = a/b$ in some framing. If $|2b|$ is the geometric intersection number of the dividing set Γ with a longitude ruling in that framing, then we will call a/b the *intersection boundary slope*.

Note that when we have an intersection boundary slope a/b, then $2\gcd(a, |b|)$ is the number of dividing curves.

Definition 2.3 For $r \geq 1$ and positive integer k, define N_r^k to be any solid torus representing K_r with intersection boundary slope of $-(k+1)/(A_r k + B_r)$, as measured in the $\mathcal{C}'$ framing. Also define the integer $n_r^k := \gcd((k+1), (A_r k + B_r))$. Note that ∂N_r^k has $2n_r^k$ dividing curves. Note also that the above definition is only for $k \geq 1$; however, we will also define N_r^0 to be a standard neighborhood of a $\overline{\mathrm{tb}}(K_r)$ representative, and thus have this as the $k = 0$ case.

Remark 2.4 We will be particularly interested when K_r is an iterated torus knot obtained from positive cablings; in this case, note that after doing a change of coordinates from the $\mathcal{C}'$ framing to the $\mathcal{C}$ framing, one obtains that the intersection boundary slope of N_r^k is $(k+1)/(A_r - B_r)$, or in other words, by Lemma 2.1, $-(k+1)/\chi(K_r)$. Thus $\Gamma_{\partial N_r^k}$ intersects the Seifert longitude exactly $2(-\chi(K_r))$ times, regardless of what k is; this will be vital for our arguments, in particular in Lemma 4.6 below.

Finally, recall that if $\mathcal{A}$ is a convex annulus with Legendrian boundary components, then dividing curves are arcs with endpoints on either one or both of the boundary components. Dividing curves that are boundary parallel are called *bypasses*; an annulus with no bypasses is said to be *standard convex*.

2.3 Twist number lemma and the Farey tessellation

The following lemma, due to Honda [13], will play a role in this work.

Lemma 2.5 (Twist number lemma, Honda) *Let L be a Legendrian knot with twisting n. Let r be the slope of a Legendrian ruling curve on $\partial N(L)$. If there exists a bypass attached along this ruling curve, and $1/r \geq (n+1)$, then passing through the bypass yields a Legendrian curve, with larger twisting, which is isotopic (but not Legendrian isotopic) to L.*

This lemma can be thought of as a corollary to the following proposition, also due to Honda [13], which describes how slopes of dividing curves change due to bypasses attached to convex tori. Recall that fractional slopes can be placed on the boundary of the Poincaré disk $\mathbb{D}$ using the *Farey tessellation*, where two slopes with intersection number one are connected by an arc in the Farey tessellation – see [7, Section 2.2.3] for a complete discussion. In the following proposition, the torus T can be thought of as inheriting an orientation from the solid torus which it bounds.

Proposition 2.6 (Honda) *Let T be a convex torus in standard form with $|\Gamma_T| = 2$, dividing slope s and ruling slope $r \neq s$. Let D be a bypass for T attached to the front of T along a ruling curve. Let T' be the torus obtained from T by attaching the bypass D. Then $|\Gamma_{T'}| = 2$ and the dividing slope s' of $\Gamma_{T'}$ is determined as follows: let $[r, s]$ be the arc on $\partial\mathbb{D}$ running from r counterclockwise to s, then s' is the point in $[r, s]$ closest to r with an edge to s. If the bypass is attached to the back of T then the same algorithm works except one uses the interval $[s, r]$ on $\partial\mathbb{D}$.*

Since the boundary slope of 0 cannot be realized when the contact structure is tight, we focus on $(\partial\mathbb{D}) \setminus \{0\}$, and note that when *thickening* a solid torus N, boundary slopes change in a *clockwise* manner on $(\partial\mathbb{D}) \setminus \{0\}$; and when *thinning* N, slopes change in a *counterclockwise* manner. However, given a tight solid torus N with boundary slope s, and given s' a rational slope somewhere in the interval $(s, 0)$ obtained by going counterclockwise from s to 0, then there exists a solid torus $N' \subset N$ with boundary slope s' (see [13]).

2.4 Imbalance Principle

As we see that bypasses are useful in changing dividing curves on a surface, we mention a standard way to try to find them called the Imbalance Principle [13]. Suppose that T and T' are two disjoint convex tori and $\mathcal{A}$ is a convex annulus whose interior is disjoint from T and T', but whose boundary is Legendrian with one component on each surface. If $|\Gamma_T \cap \partial\mathcal{A}| > |\Gamma_{T'} \cap \partial\mathcal{A}|$ then there will be a bypass on $\mathcal{A}$ along the T–edge.

2.5 Universally tight contact structures

Recall that a contact structure ξ on a 3–manifold M is said to be *overtwisted* if there exists an overtwisted disc, and a contact structure is *tight* if it is not overtwisted. Moreover, one can further analyze tight contact 3–manifolds (M,ξ) by looking at what happens to ξ when pulled back to the universal cover $\widetilde{M}$ via the covering map $\pi\colon \widetilde{M} \to M$. In particular, if the pullback of ξ remains tight, then (M,ξ) is said to be *universally tight*.

The classification of universally tight contact structures on solid tori is known from the work of Honda. Specifically, from [13, Proposition 5.1], we know there are exactly two universally tight contact structures on $S^1 \times D^2$ with boundary torus having two dividing curves and slope $s < -1$ in some framing. These are such that a convex meridional disc has boundary-parallel dividing curves that separate half-discs all of the same sign, and thus the two contact structures differ by $-\mathrm{id}$. (If $s = -1$, there is only one tight contact structure, and it is universally tight.)

Also from the work of Honda, we know that if ξ is a contact structure which is everywhere transverse to the fibers of a circle bundle M over a surface Σ, then ξ is universally tight [14]. Such a contact structure is said to be *horizontal*.

2.6 Transverse push-offs of Legendrian knots

Given a Legendrian knot L, recall that there are well-defined *positive and negative transverse push-offs*, denoted by $T_+(L)$ and $T_-(L)$, respectively (see, for example, Epstein, Fuchs and Meyer [3]). Moreover, the self-linking numbers of these transverse push-offs are given by the formula $\mathrm{sl}(T_\pm(L)) = \mathrm{tb}(L) \mp \mathrm{r}(L)$.

3 Strategy of proof for Theorem 1.1

In this section we present a strategy of proof for Theorem 1.1. We begin with a theorem that in previous works has in effect been proved, but not stated. In this theorem K is a knot type and $K_{(P,q)}$ is the (P,q)–cabling of K.

Theorem 3.1 (Etnyre–Honda, L) *If K satisfies the UTP, then $K_{(P,q)}$ also satisfies the UTP.*

Proof The case where the cabling fraction $P/q < w(K)$ is the content of [6, Theorem 1.3]. For the case where $P/q > w(K)$, we first note that by the proof of [6, Theorem 3.2], we know that $\overline{t}(K_{(P,q)}) < 0$ and $K_{(P,q)}$ achieves $\overline{\mathrm{tb}}(K_{(P,q)})$ as a

Legendrian ruling curve on a convex torus with boundary slope $1/w(K)$ and two dividing curves – we observe that neither of these statements uses the Legendrian simplicity hypothesis in the statement of [6, Theorem 3.2]. Then, we simply observe that in the proof of [15, Section 2, Theorem 1.1], the Legendrian simplicity of K is not needed to prove that $K_{(P,q)}$ satisfies the UTP. □

An immediate application for our current purposes is that if an iterated torus knot $K_r = ((P_1, q_1), \ldots, (P_r, q_r))$ satisfies the UTP, then the iterated torus knot $K_{r+1} = ((P_1, q_1), \ldots, (P_r, q_r), (P_{r+1}, q_{r+1}))$ also satisfies the UTP.

With this theorem in mind, we will prove Theorem 1.1 by way of three lemmas, two of which combine in an induction argument. For this purpose we make the following inductive hypothesis, which from here on we will refer to as the *Inductive Hypothesis*.

Inductive Hypothesis Let $K_r = ((P_1, q_1), \ldots, (P_r, q_r))$ be an iterated torus knot, as measured in the standard $\mathcal{C}$ framing. The Inductive Hypothesis assumes that the following hold:

(1) $P_i > 0$ for all i, where $1 \leq i \leq r$. (Thus $A_i = P_i q_i > 0$ for all i as well.)

(2) $0 < \overline{\text{tb}}(K_r) = w(K_r) \leq A_r$. (Thus $-A_r < \overline{t}(K_r) \leq 0$ – see Equation (2).)

(3) Any solid torus N_r representing K_r thickens to some N_r^k (including N_r^0 which is a standard neighborhood of a $\overline{\text{tb}}$ representative).

(4) If N_r fails to thicken then it is an N_r^k, and it has at least $2n_r^k$ dividing curves.

(5) The candidate nonthickenable N_r^k exist and actually fail to thicken for $k \geq C_r$, where C_r is some positive integer that varies according to the knot type K_r. Moreover, these N_r^k that fail to thicken have contact structures that are universally tight, with convex meridian discs D containing bypasses all of the same sign; ie, the rotation number of meridian curves is $\text{r}(\partial D) = \pm k$. Also, a Legendrian ruling preferred longitude on these ∂N_r^k has rotation number zero for $k > 0$.

Another way of stating items (3) and (4) is that every solid torus N_r is contained in some N_r^k, and if N_r fails to thicken, then boundary slopes do not change in passing to the $N_r^k \supset N_r$, although the number of dividing curves may decrease. Also, note that, by item (5), any K_r which satisfies the Inductive Hypothesis fails the UTP.

We first observe that the Inductive Hypothesis is true for the base case of positive torus knots, as established in [7; 15].

Lemma 3.2 *The Inductive Hypothesis is true for positive torus knots $K_1 = (P, q)$, and thus positive torus knots fail the UTP.*

Proof Clearly item (1) of the Inductive Hypothesis holds. From [5] we know that $0 < \overline{\text{tb}}(K_1) = Pq - P - q < A_1 = Pq$; this proves part of item (2).

The remaining part of (2) follows from [15, Lemma 4.5], and items (3) and (4) hold from [15, Lemma 4.3]. We briefly recall the sketch of the proof of that lemma below, as we will be using similar ideas shortly in the induction step.

The idea in [15, Lemma 4.3] was the following: given a solid torus N_1 representing the positive torus knot K_1, take a neighborhood of a Legendrian Hopf link $N(L_1) \sqcup N(L_2)$ in its complement. Then, in the complement of $N_1 \cup N(L_1) \cup N(L_2)$, join a (P, q)–curve on $\partial N(L_1)$ to a (q, P)–curve on $\partial N(L_2)$ with a standard convex annulus $\mathcal{A}$ having no bypasses (this could be achieved after possibly destabilizing $L_1 \sqcup L_2$). One could then calculate the intersection boundary slope of $-\partial(N(L_1) \cup N(L_2) \cup N(\mathcal{A}))$ to be identical to one of the N_1^k. This established item (3). Then, in that same lemma, item (4) was shown by observing that if N_1 had the same boundary slope as an N_1^k, but with less than $2n_1^k$ dividing curves, then N_1 would in fact thicken.

Construction 3.2 and Lemmas 3.3 and 3.4 in [7] then combine to establish item (5), using $C_1 = 1$. Again, we include the ideas in those results below, as we will use similar arguments shortly in the induction step.

The idea in [7, Construction 3.2] was to take one of the universally tight N_1^k, with convex meridian discs having bypasses all of the same sign, and build S^3 with the tight contact structure around it. Specifically, we joined two ∞'–longitudes on ∂N_1^k by a standard convex annulus $\mathcal{A}$, so that if we then let $R = N_1^k \cup N(\mathcal{A})$, we had that R was diffeomorphic to $T^2 \times [0, 1]$, with a $[0, 1]$–invariant contact structure on $N(\mathcal{A})$. Thinking of R as fibering over an annulus with fibers representing the torus knot, the contact structure on R could then be isotoped to be transverse to the fibers, hence a horizontal contact structure, and therefore universally tight. With appropriate choice of dividing curves on $\mathcal{A}$, we could then assure that the two toric boundaries of R represented those of standard neighborhoods of our desired Legendrian Hopf link, and gluing in such neighborhoods gave us S^3 with the tight contact structure. This showed that the N_1^k exist.

The idea in [7, Lemma 3.3] was to show that the N_1^k are nonthickenable by examining the complement $M_1^k = S^3 \setminus N_1^k$. Specifically, since the positive torus knot (P, q) was a fibered knot (with fiber Σ) with periodic monodromy, M_1^k had a Pq–fold cover $\widetilde{M}_1^k \cong S^1 \times \Sigma$. We then showed that the S^1 fibers in $\widetilde{M}_1^k$ could all be made Legendrian of the same (negative) twisting $-(A_1 k + B_1)$. We then assumed, for contradiction, that N_1^k thickened, and showed this resulted in a new Legendrian, topologically isotopic to the S^1 fibers, with twisting $-t' > -(A_1 k + B_1)$. We then cut Σ into a polygon P to obtain a solid torus $S^1 \times P$ which we showed was in fact a standard neighborhood of a Legendrian of twisting $-(A_1k+B_1)$; crucial to this calculation was the fact that on ∂N_1^k, the Seifert longitude intersected the dividing set exactly $2(-\chi(K_1))$ times. Finally, we showed we could tile enough copies of $S^1 \times P$ together to enclose the Legendrian with twisting $-t'$ inside a standard neighborhood of a Legendrian with twisting $-(A_1k+B_1)$. This was a contradiction, and showed that the N_1^k failed to thicken.

Finally, [7, Lemma 3.4] computed rotation numbers. □

Our second key lemma used in proving Theorem 1.1 is the following induction step, which, along with the base case of positive torus knots, will show that if the iterated torus knot $K_r = ((P_1,q_1),\dots,(P_r,q_r))$ is such that $P_i > 0$ for all i, then K_r fails the UTP.

Lemma 3.3 *Suppose K_r satisfies the Inductive Hypothesis, and K_{r+1} is a cabling where $P_{r+1} > 0$; then K_{r+1} satisfies the Inductive Hypothesis, and thus fails the UTP.*

The main idea in the argument used to prove this lemma will be that since K_r satisfies the Inductive Hypothesis, there is an infinite collection of nonthickenable solid tori whose boundary slopes form an increasing sequence converging to $-1/A_r$ in the $\mathcal{C}'$ framing (which is ∞ in the $\mathcal{C}$ framing). As a consequence, cabling slopes with $P_{r+1} > 0$ in the $\mathcal{C}$ framing are clockwise from infinitely many nonthickenable boundary slopes; we will use this to show that such cabling knot types with $P_{r+1} > 0$ have a similar sequence of nonthickenable solid tori.

Our third key lemma is the following, which along with Theorem 3.1 and the fact that negative torus knots satisfy the UTP (see [6]), will show that if at least one of the $P_i < 0$, then K_r satisfies the UTP.

Lemma 3.4 *Suppose K_r satisfies the Inductive Hypothesis, and K_{r+1} is a cabling where $P_{r+1} < 0$; then K_{r+1} satisfies the UTP.*

Proof This is the case where $q_{r+1}/p_{r+1} \in (-1/A_r, 0)$ in the $\mathcal{C}'$ framing, we know K_r satisfies the Inductive Hypothesis, and we wish to show that K_{r+1} satisfies the UTP. The proof is identical to that of Steps 1 and 2 in the proof of [15, Section 6, Theorem 1.5], the key being that since $-1/A_r < q_{r+1}/p_{r+1} < 0$, this cabling slope is shielded (in the Farey tessellation by an arc from $-1/A_r$ to 0) from any N_r^k that fail to thicken. □

In the following Section 4 we prove Lemma 3.3; this will then complete the proof of Theorem 1.1.

4 Positive cablings that fail the UTP

Now that we know that the base case holds for positive torus knots, we begin to prove Lemma 3.3 – for the whole of this section we will thus have that $P_{r+1} > 0$, K_r satisfies the Inductive Hypothesis, and we work to show that K_{r+1} satisfies the Inductive Hypothesis. It will be convenient to break the proof of Lemma 3.3 into two cases, Case I being where $P_{r+1}/q_{r+1} > w(K_r)$, and Case II being where $w(K_r) > P_{r+1}/q_{r+1} > 0$. However, we first note the following.

Lemma 4.1 *Let K_r be an iterated torus knot with $P_i > 0$ for all i. If $0 \leq k_1 < k_2$, then $-(k_1+1)/(A_r k_1 + B_r)$ is clockwise from $-(k_2+1)/(A_r k_2 + B_r)$ in the Farey tessellation.*

Proof Following Lemma 2.1 and Remark 2.4, in the standard $\mathcal{C}$ framing we have that $(k_1+1)/(A_r - B_r) < (k_2+1)/(A_r - B_r)$; changing coordinates to the $\mathcal{C}'$ framing yields the result. □

We now directly address the two different cases in two different subsections.

4.1 Case I: $P_{r+1}/q_{r+1} > w(K_r)$

We work through proving items (2)–(5) in the Inductive Hypothesis via a series of lemmas. The following lemma begins to address item (2).

Lemma 4.2 *If $P_{r+1}/q_{r+1} > w(K_r)$, then*

$$\overline{\mathrm{tb}}(K_{r+1}) = A_{r+1} - (P_{r+1} - q_{r+1} w(K_r)) > 0.$$

Proof The proof is similar to that of [6, Lemma 3.3] (note that our $A_{r+1} = P_{r+1} q_{r+1}$). We first claim that $\overline{t}(K_{r+1}) < 0$. If not, there exists a Legendrian L_{r+1} with $t(L_{r+1}) = 0$ and a solid torus N_r with L_{r+1} as a Legendrian divide. But then we would have a boundary slope of $P_{r+1}/q_{r+1} > w(K_r)$ in the $\mathcal{C}$ framing, which cannot occur.

So since $\overline{t}(K_{r+1}) < 0$, any Legendrian L_{r+1} must be a ruling on a convex ∂N_r with slope $0 > s \geq 1/\overline{t}(K_r)$ in the $\mathcal{C}'$ framing. But then if $s = -\lambda/\mu > 1/\overline{t}(K_r)$, we have that $t(L_r) = -(p_{r+1}\lambda + q_{r+1}\mu) < -\lambda(p_{r+1} - \overline{t}(K_r) q_{r+1}) \leq -(p_{r+1} - \overline{t}(K_r) q_{r+1})$. This shows that $\overline{\mathrm{tb}}(K_{r+1})$ is achieved by a Legendrian ruling on a convex torus having slope $1/w(K_r)$ in the standard $\mathcal{C}$ framing; thus $\overline{\mathrm{tb}}(K_{r+1}) = A_{r+1} - (P_{r+1} - q_{r+1} w(K_r))$, using Equation (2) and recalling that the twisting of the Legendrian ruling is $-1/2$ times the geometric intersection number of the ruling with the dividing set.

Finally, note $A_{r+1}-(P_{r+1}-q_{r+1}w(K_r)) = A_{r+1}-(q_{r+1}(A_r-w(K_r))+p_{r+1}) > A_{r+1}-(q_{r+1}^2 A_r + p_{r+1}q_{r+1}) = 0$. For the first equality we use Equation (1), and for the inequality we use the fact that $w(K_r) > 0$ via the Inductive Hypothesis. □

With the following lemma we prove that items (3) and (4) of the Inductive Hypothesis hold for K_{r+1}; we refer the reader to the comment following the Inductive Hypothesis for the precise meaning of the case when N_{r+1} both *can be thickened*, and *fails to thicken*.

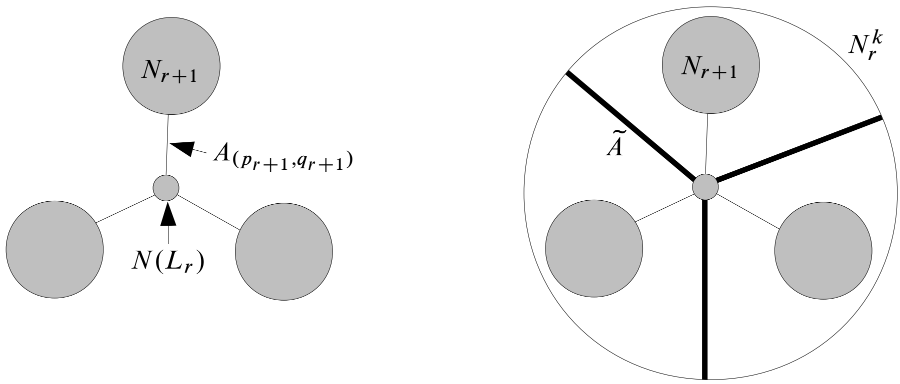
Lemma 4.3 *If $P_{r+1}/q_{r+1} > w(K_r)$, let N_{r+1} be a solid torus representing K_{r+1}, for $r \geq 1$. Then N_{r+1} can be thickened to an $N_{r+1}^{k'}$ for some nonnegative integer k'. Moreover, if N_{r+1} fails to thicken, then it has the same boundary slope as some $N_{r+1}^{k'}$, as well as at least $2n_{r+1}^{k'}$ dividing curves.*

Proof In this case, for the $\mathcal{C}'$ framing, we have either $p_{r+1} > 0$ or $q_{r+1}/p_{r+1} < 1/\overline{t}(K_r)$ (the latter being relevant only if $\overline{t}(K_r) < 0$); in other words, q_{r+1}/p_{r+1} is clockwise from $1/\overline{t}(K_r)$ in the Farey tessellation. The proof in this case is nearly identical to the proof of [15, Lemma 4.4]; we will include the details, however, as certain particular calculations differ. Moreover, we will use modifications of this argument in Case II and thus will be able to refer to the details here.

Let N_{r+1} be a solid torus representing K_{r+1}. Let L_r be a Legendrian representative of K_r in $S^3 \backslash N_{r+1}$ and such that we can join $\partial N(L_r)$ to ∂N_{r+1} by a convex annulus $\mathcal{A}_{(p_{r+1},q_{r+1})}$ whose boundaries are (p_{r+1}, q_{r+1}) and ∞' rulings on $\partial N(L_r)$ and ∂N_{r+1}, respectively. Then topologically isotop L_r in the complement of N_{r+1} so that it maximizes tb over all such isotopies; this will induce an ambient topological isotopy of $\mathcal{A}_{(p_{r+1},q_{r+1})}$, where we still can assume $\mathcal{A}_{(p_{r+1},q_{r+1})}$ is convex. A picture is shown in (a) in Figure 1. In the $\mathcal{C}'$ framing we will have $\text{slope}(\Gamma_{\partial N(L_r)}) = -1/m$ where $m \geq 0$, since $\overline{t}(K_r) \leq 0$. Now if $m = \overline{t}(K_r)$, then there will be no bypasses on the $\partial N(L_r)$–edge of $\mathcal{A}_{(p_{r+1},q_{r+1})}$, since the (p_{r+1}, q_{r+1}) ruling would be at maximal twisting. On the other hand, if $m < \overline{t}(K_r)$, then there will still be no bypasses on the $\partial N(L_r)$–edge of $\mathcal{A}_{(p_{r+1},q_{r+1})}$, since such a bypass would induce a destabilization of L_r, thus increasing its tb by one – here we are using the twist number lemma, Lemma 2.5 above. To satisfy the conditions of this lemma, we are using the fact that either $p_{r+1} > 0$ or $q_{r+1}/p_{r+1} < 1/\overline{t}(K_r)$. Furthermore, we can thicken N_{r+1} through any bypasses on the ∂N_{r+1}–edge, and thus assume $\mathcal{A}_{(p_{r+1},q_{r+1})}$ is standard convex.

Now let $N_r := N_{r+1} \cup N(\mathcal{A}_{(p_{r+1},q_{r+1})}) \cup N(L_r)$. By the Inductive Hypothesis we can thicken N_r to an N_r^k with intersection boundary slope $-(k+1)/(A_r k + B_r)$

where k is minimized over all such thickenings (if we have $k = 0$, then we will have N_{r+1} thickening to a standard neighborhood of a knot at maximal Thurston–Bennequin number – see the proof of [15, Section 2, Theorem 1.1]; so we can assume $k > 0$). Then consider a convex annulus $\widetilde{\mathcal{A}}$ from $\partial N(L_r)$ to ∂N_r^k, such that $\widetilde{\mathcal{A}}$ is in the complement of N_r and $\partial\widetilde{\mathcal{A}}$ consists of (p_{r+1}, q_{r+1}) rulings. A picture is shown in (b) in Figure 1. By an argument identical to that used in [15, Lemma 4.4], $\widetilde{\mathcal{A}}$ is standard convex; we briefly recall the details below for completeness.



(a) (b)

Figure 1: N_{r+1} is the larger solid torus in gray; $N(L_r)$ is the smaller solid torus in gray.

Certainly there are no bypasses on the $\partial N(L_r)$–edge of $\widetilde{\mathcal{A}}$; furthermore, any bypasses on the ∂N_r^k–edge must pair up via dividing curves on ∂N_r^k and cancel each other out as in part (a) of Figure 2, for otherwise a bypass on $\partial N(L_r)$ would be induced via the annulus $\widetilde{\mathcal{A}}$ as in part (b) of Figure 2. As a consequence, allowing N_r^k to thin inward through such bypasses does not change the boundary slope, but just reduces the number of dividing curves to less than $2n_r^k$. But then by the Inductive Hypothesis we can thicken this new N_r^k to a smaller k–value, contradicting the minimality of k. Thus $\widetilde{\mathcal{A}}$ is standard convex.

Now four annuli compose the boundary of a solid torus $\widetilde{N}_{r+1}$ containing N_{r+1}: the two sides of a thickened $\widetilde{\mathcal{A}}$; $\partial N_r^k \backslash \partial\widetilde{\mathcal{A}}$; and $\partial N(L_r)\backslash\partial\widetilde{\mathcal{A}}$. We can compute the intersection boundary slope of this solid torus. To this end, recall that $\mathrm{slope}(\Gamma_{\partial N(L_r)}) = -1/m$ where $m > 0$ ($m = 0$ would be the $\overline{t}$ case which we have taken care of above). To determine m we note that the geometric intersection number of (p_{r+1}, q_{r+1}) with Γ on ∂N_r^k and $\partial N(L_r)$ must be equal, yielding the equality

$$p_{r+1} + mq_{r+1} = p_{r+1}k + p_{r+1} + q_{r+1}(A_r k + B_r). \tag{3}$$

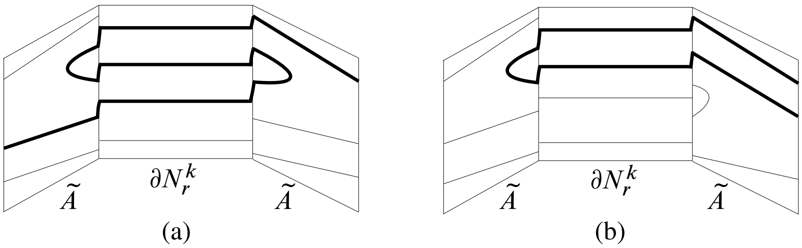


Figure 2: Part (a) shows bypasses that cancel each other out after edge-rounding. Part (b) shows a bypass induced on $\partial N(L_r)$ via $\widetilde{A}$.

These equal quantities are greater than zero, since q_{r+1}/p_{r+1} is clockwise from $-1/m$ (and $-(k+1)/(A_r k + B_r)$) in the Farey tessellation – we note here that this will yield $(A_{r+1}k' + B_{r+1}) > 0$ for the calculations below. In the meantime, however, the above equation gives

$$m = p_{r+1}\frac{k}{q_{r+1}} + A_r k + B_r.$$

We define the integer $k' := k/q_{r+1}$. We now choose (p'_{r+1}, q'_{r+1}) to be a curve on these two tori such that $p_{r+1}q'_{r+1} - p'_{r+1}q_{r+1} = 1$, and we change coordinates to a framing $\mathcal{C}''$ via the map $((p_{r+1}, q_{r+1}), (p'_{r+1}, q'_{r+1})) \mapsto ((0,1), (-1,0))$. Under this map we obtain

$$\begin{aligned} \operatorname{slope}(\Gamma_{\partial N_r^k}) &= \frac{q'_{r+1}(A_r k + B_r) + p'_{r+1}(q_{r+1}k' + 1)}{A_{r+1}k' + B_{r+1}}, \\ \operatorname{slope}(\Gamma_{\partial N(L_r)}) &= \frac{q'_{r+1}(p_{r+1}k' + A_r k + B_r) + p'_{r+1}}{A_{r+1}k' + B_{r+1}}. \end{aligned}$$

We then obtain in the $\mathcal{C}'$ framing, after edge-rounding (see [13, Section 3.3.2]), that the intersection boundary slope of $\widetilde{N}_{r+1}$ is

$$\begin{aligned} \operatorname{slope}(\Gamma_{\partial \widetilde{N}_{r+1}}) &= \operatorname{slope}(\Gamma_{\partial N_r^k}) - \operatorname{slope}(\Gamma_{\partial N(L_r)}) - \frac{1}{A_{r+1}k' + B_{r+1}} \\ &= -\frac{k'+1}{A_{r+1}k' + B_{r+1}}. \end{aligned}$$

The first two summands in the equation have opposite signs since, to form $\partial\widetilde{N}_{r+1}$, we use the same orientation coming from ∂N_r^k for the annulus $\partial N_r^k \backslash \partial\widetilde{\mathcal{A}}$, and the opposite orientation coming from $\partial N(L_r)$ for the annulus $\partial N(L_r)\backslash\partial\widetilde{\mathcal{A}}$. Also, the third summand, $-1/(A_{r+1}k' + B_{r+1})$, comes from the fact that, after edge-rounding, the two annuli coming from the two sides of a thickened $\widetilde{A}$ each contribute one intersection of $\Gamma_{\partial\widetilde{N}_{r+1}}$ with the new meridian curve.

This shows that any N_{r+1} representing K_{r+1} can be thickened to one of the $N^{k'}_{r+1}$, and if N_{r+1} fails to thicken, then it has the same boundary slope as some $N^{k'}_{r+1}$. We now show that if N_{r+1} fails to thicken, and if it has the minimum number of dividing curves over all such N_{r+1} which fail to thicken and have the same boundary slope as $N^{k'}_{r+1}$, then N_{r+1} is actually an $N^{k'}_{r+1}$.

To see this, as above we can choose a Legendrian L_r that maximizes tb in the complement of such a nonthickenable N_{r+1}, and such that we can join $\partial N(L_r)$ to ∂N_{r+1} by a convex annulus $\mathcal{A}_{(p_{r+1},q_{r+1})}$ whose boundaries are (p_{r+1}, q_{r+1}) and ∞' rulings on $\partial N(L_r)$ and ∂N_{r+1}, respectively. Again we have no bypasses on the $\partial N(L_r)$–edge, and in this case we have no bypasses on the ∂N_{r+1}–edge since N_r fails to thicken and is at minimum number of dividing curves.

As above, let $N_r := N_{r+1} \cup N(\mathcal{A}_{(p_{r+1},q_{r+1})}) \cup N(L_r)$. We claim this N_r fails to thicken. To see this, take a convex annulus $\widetilde{\mathcal{A}}$ from $\partial N(L_r)$ to ∂N_r, such that $\widetilde{\mathcal{A}}$ is in the complement of N_{r+1} and $\partial\widetilde{\mathcal{A}}$ consists of (p_{r+1}, q_{r+1}) rulings. We know $\widetilde{\mathcal{A}}$ is standard convex since the twisting is the same on both edges and there are no bypasses on the $\partial N(L_r)$–edge. A picture is shown in Figure 3.

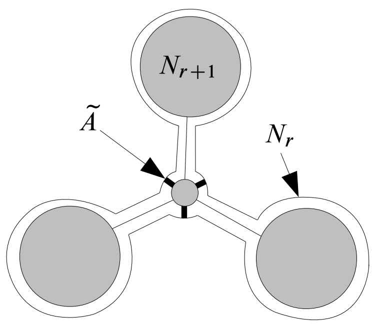


Figure 3: Shown is a meridional cross-section of N_r. The larger gray solid torus represents N_{r+1}; the smaller gray solid torus is $N(L_r)$.

Now four annuli compose the boundary of a solid torus containing N_{r+1}: the two sides of the thickened $\widetilde{\mathcal{A}}$, which we will call $\widetilde{\mathcal{A}}_+$ and $\widetilde{\mathcal{A}}_-$; $\partial N_r \backslash \partial\widetilde{\mathcal{A}}$, which we will call $\mathcal{A}_r$; and $\partial N(L_r)\backslash\partial\widetilde{\mathcal{A}}$, which we will call $\mathcal{A}_{L_r}$. Any thickening of N_r will induce a thickening of N_{r+1} to $\widetilde{N}_{r+1}$ via these four annuli.

Suppose, for contradiction, that N_r thickens outward so that slope($\Gamma_{\partial N_r}$) changes. Note that during the thickening, $\mathcal{A}_{L_r}$ stays fixed. We examine the rest of the annuli by breaking into two cases.

Case 1 After thickening, suppose $\widetilde{\mathcal{A}}$ is still standard convex; that means both $\widetilde{\mathcal{A}}_+$ and $\widetilde{\mathcal{A}}_-$ are standard convex. Since we can assume that after thickening $\mathcal{A}_r$ is still standard convex, this means that in order for slope$(\Gamma_{\partial N_r})$ to change, the holonomy of $\Gamma_{\mathcal{A}_r}$ must have changed. But this will result in a change in slope$(\Gamma_{\partial N_{r+1}})$, since $\mathcal{A}_{L_r}$ stays fixed and any change in holonomy of $\Gamma_{\widetilde{\mathcal{A}}_+}$ and $\Gamma_{\widetilde{\mathcal{A}}_-}$ cancels each other out and does not affect slope$(\Gamma_{\partial N_{r+1}})$. Thus we would have a slope-changing thickening of N_{r+1}, which by hypothesis cannot occur.

Case 2 After thickening, suppose $\widetilde{\mathcal{A}}$ is no longer standard convex. Now note that there are no bypasses on the $\partial N(L_r)$–edge of $\widetilde{\mathcal{A}}$; furthermore, any bypass for $\widetilde{\mathcal{A}}_+$ on the ∂N_r–edge must be cancelled out by a corresponding bypass for $\widetilde{\mathcal{A}}_-$ on the ∂N_r–edge as in part (a) of Figure 2, so as not to induce a bypass on the $\partial N(L_r)$–edge as in part (b) of the same figure. But then again, in order for slope$(\Gamma_{\partial N_{r+1}})$ to remain constant, the holonomy of $\Gamma_{\mathcal{A}_r}$ must remain constant, and thus slope$(\Gamma_{\partial N_r})$ must also have remained constant, with just an increase in the number of dividing curves.

This proves the claim that N_r does not thicken, and we therefore know that its boundary slope is $-(k+1)/(A_r k + B_r)$. Furthermore, we know the number of dividing curves is $2n$ where $n \geq n_r^k$. Suppose, for contradiction, that $n > n_r^k$. Then we know we can thicken N_r to an N_r^k, and if we take a convex annulus from ∂N_r to ∂N_r^k whose boundaries are (p_{r+1}, q_{r+1}) rulings, by the Imbalance Principle there must be bypasses on the ∂N_r–edge. But these would induce bypasses off of ∞' rulings on N_{r+1}, which by hypothesis cannot exist. Thus $n = n_r^k$, and by a calculation as above we obtain that the intersection boundary slope of N_{r+1} must be $-(k'+1)/(A_{r+1}k' + B_{r+1})$ for the integer $k' = k/q_{r+1}$. □

We now finish the proof of item (2) of the Inductive Hypothesis.

Lemma 4.4 *If $P_{r+1}/q_{r+1} > w(K_r)$, then $w(K_{r+1}) = \overline{\mathrm{tb}}(K_{r+1})$.*

Proof Using Lemma 4.3, we need to show $1/\overline{t}(K_{r+1}) < -(k'+1)/(A_{r+1}k' + B_{r+1})$ for any candidate $N_{r+1}^{k'}$. But changing to standard $\mathcal{C}$ coordinates, this means we need to show that $1/\overline{\mathrm{tb}}(K_{r+1}) < (k'+1)/(A_{r+1} - B_{r+1})$. So, to this end, take an $N_{r+1}^{k'}$ with boundary slope $(k'+1)/(A_{r+1} - B_{r+1})$; as in the proof of Lemma 4.3, this $N_{r+1}^{k'}$ sits inside an N_r^k, and we know by the Inductive Hypothesis that $1/\overline{\mathrm{tb}}(K_r) < (k+1)/(A_r - B_r)$. Thus, since by the Inductive Hypothesis N_r^k is universally tight, we can embed N_r^k contactomorphically into an abstract N_r^0; so in this abstract tight solid torus, N_r^k thickens to a standard neighborhood of a Legendrian at $\overline{\mathrm{tb}}(K_r)$. But then, just as in the proof of [15, Theorem 1.1], this induces a thickening of $N_{r+1}^{k'}$ (in the abstract tight solid torus) to a standard neighborhood of a Legendrian at $\overline{\mathrm{tb}}(K_{r+1})$; as in that proof, this uses the fact that $P_{r+1}/q_{r+1} > w(K_r)$. Thus, to prevent overtwisting, we must have that $1/\overline{\mathrm{tb}}(K_{r+1}) < (k'+1)/(A_{r+1} - B_{r+1})$. □

We conclude this subsection by proving item (5) of the Inductive Hypothesis, using a construction and two lemmas. We begin with the construction, which shows that the candidate $N^{k'}_{r+1}$ exist for $k' \geq C_{r+1}$, where C_{r+1} is some positive integer.

Construction 4.5 We know by the Inductive Hypothesis that there exists a C_r such that if $k \geq C_r$, then the N^k_r exist and fail to thicken, and have convex meridian discs with bypasses all of the same sign. So suppose $k/q_{r+1} \in \mathbb{N}$ for some $k \geq C_r$. We will show that $N^{k'}_{r+1}$ exists for $k' := k/q_{r+1}$. Then C_{r+1} will be the least such $k/q_{r+1} \in \mathbb{N}$.

The idea is to build S^3. We first take one of the two universally tight candidate $N^{k'}_{r+1}$, with intersection boundary slope $-(k'+1)/(A_{r+1}k' + B_{r+1})$, and with convex meridian discs having bypasses all of the same sign; thus the two possible contact structures on $N^{k'}_{r+1}$ differ by $-\mathrm{id}$. We then show that we can use such a $N^{k'}_{r+1}$ to build $N^{k'q_{r+1}}_r$, essentially running backwards the decomposition from Lemma 4.3 above – since the $N^{k'q_{r+1}}_r$ inductively exists in S^3, we will then be done. To this end, let $\mathcal{A}$ be a standard convex annulus joining two ∞'_{r+1}–longitudes on $\partial N^{k'}_{r+1}$, so that if we then let $R = N^{k'}_{r+1} \cup N(\mathcal{A})$, we have that R is diffeomorphic to $T^2 \times [0,1]$, with a $[0,1]$–invariant contact structure on $N(\mathcal{A})$. Furthermore, we can think of R as containing a horizontal annulus joining $T^2 \times \{0\}$ to $T^2 \times \{1\}$, and such that the original ∞'_{r+1}–longitudes on $\partial N^{k'}_{r+1}$ intersect this horizontal annulus q_{r+1} times; thus, with an appropriate choice of ∞'_r–longitude for $T^2 \times \{i\}$, the original ∞'_{r+1}–longitudes on $\partial N^{k'}_{r+1}$ are now (p_{r+1}, q_{r+1}) curves on $T^2 \times \{i\}$.

We will thus think of R as fibering over the horizontal annulus with fiber circles representing the knot type K_{r+1}. For either choice of the two universally tight contact structures on $N^{k'}_{r+1}$, the contact structure on R can be isotoped to be transverse to the fibers of R, while preserving the dividing set on R. Hence the contact structure is horizontal, and therefore universally tight. Furthermore, with appropriately chosen dividing curves on $\mathcal{A}$, we can obtain intersection boundary slopes (on the two boundary tori $T^2 \times \{0\}$ and $T^2 \times \{1\}$) of $-(k'q_{r+1}+1)/(A_r k' q_{r+1} + B_r)$ and $-1/(p_{r+1}k' + A_r k + B_r)$; ie, the intersection boundary slopes of a $N^{k'q_{r+1}}_r$ and a Legendrian of twisting $-(p_{r+1}k' + A_r k + B_r)$.

We now glue, onto the $T^2 \times \{1\}$ side of R, a standard neighborhood of a Legendrian L_r of twisting $-(p_{r+1}k' + A_r k + B_r)$; we claim the resulting solid torus N_r is one of the $N^{k'q_{r+1}}_r$. To see this, look at a q_{r+1}–fold cover of N_r, and examine its convex meridian disc D_r (which is also the same convex meridian disc D_r for the N_r downstairs). The disc D_r is formed by taking q_{r+1} meridian discs from the q_{r+1} copies of lifts of $N^{k'}_{r+1}$, and first banding them together via bands coming from the $[0,1]$–invariant $N(\mathcal{A})$, and then finally gluing in the convex meridian disc for the standard neighborhood of a

Legendrian. But now evaluating the relative Euler class (of ξ) on D_r, we note that these bands and the meridian disc for the standard neighborhood yield no obstruction, and thus we obtain $\pm k' q_{r+1}$, as each of the q_{r+1} meridian discs from $N^{k'}_{r+1}$ yields $\pm k'$. We then know inductively that this $N_r^{k' q_{r+1}}$ (and hence $N^{k'}_{r+1}$) exists in S^3. □

We now show that the $N^{k'}_{r+1}$ coming from the above construction in fact fail to thicken.

Lemma 4.6 *The $N^{k'}_{r+1}$ from Construction 4.5 fail to thicken for $k' \geq C_{r+1}$.*

Proof To show that $N^{k'}_{r+1}$ fails to thicken, by Lemmas 4.1 and 4.3 it suffices to show $N^{k'}_{r+1}$ does not thicken to any $N^{k''}_{r+1}$, where $k'' < k'$. Inductively, we can assume that N_r^k fails to thicken for $k \geq C_r$; in particular, the $N_r^{k' q_{r+1}}$ that contains $N^{k'}_{r+1}$ fails to thicken. So let $k = k' q_{r+1}$. Then define $M_r^k = S^3 \backslash N_r^k$, and define $M^{k'}_{r+1} = S^3 \backslash N^{k'}_{r+1}$.

We first make some purely topological observations, which in the rest of this proof we will refer to as *the topological observations*. We begin by observing that K_{r+1} is a fibered knot, and has periodic monodromy – see, for example, [1]. One way to see this is as follows. We think of K_{r+1} embedded on ∂N_r, and let Σ_{r+1} be a Seifert surface for K_{r+1}. Furthermore, we note that Σ_{r+1} can be formed by taking q_{r+1} copies of the Seifert surface Σ_r for the Seifert longitude on ∂N_r, and P_{r+1} copies of a meridian disc D_r for N_r, and banding them together with $P_{r+1} q_{r+1}$ positive (half-twist) bands. We then observe that if we take a slightly larger $N_r' \supset N_r$, there will be q_{r+1} separating simple closed curves on Σ_{r+1} that are in fact preferred Seifert longitudes for $\partial N_r'$, and thus bound Seifert surfaces Σ_r for the knot K_r in the complement of N_r' (all q_{r+1} of which are subsurfaces of Σ_{r+1}). In fact, the monodromy for Σ_{r+1} is reducible along these q_{r+1} curves; that is, if we call $\sigma_{r+1} := \Sigma_{r+1} \cap N_r'$, the monodromy will take σ_{r+1} to itself, and sweep out the interior of N_r'. Moreover, the monodromy will restrict to being periodic on σ_{r+1}, of period $P_{r+1} q_{r+1}$, as repeated application of the monodromy cycles through the $P_{r+1} q_{r+1}$ bands. Then, since positive torus knots have periodic monodromy, inductively we can assume that the exterior of N_r' fibers periodically with the q_{r+1} copies of the Σ_r's. As a result, there is a positive integer m_{r+1} such that $\phi^{m_{r+1}} = \mathrm{id}$ (where here ϕ is the Σ_{r+1}–monodromy), and such that $P_{r+1} q_{r+1}$ divides m_{r+1}.

We return to contact topology, and now we specifically let Σ_{r+1} be a Seifert surface for a preferred longitude on $\partial N^{k'}_{r+1}$; so Σ_{r+1} is a surface of genus g' with one boundary component. As noted in the topological observations, there are q_{r+1} separating simple closed curves on Σ_{r+1} that are in fact preferred longitudes for ∂N_r^k, and thus bound Seifert surfaces Σ_r for the knot K_r. We will let g denote the genus of such a Seifert surface Σ_r. Also we will call $\sigma_{r+1} := \Sigma_{r+1} \cap N_r^k$; so $\Sigma_{r+1} = \sigma_{r+1} \cup (\bigcup_{j=1}^{q_{r+1}} \Sigma_r^j)$; see Figure 4.

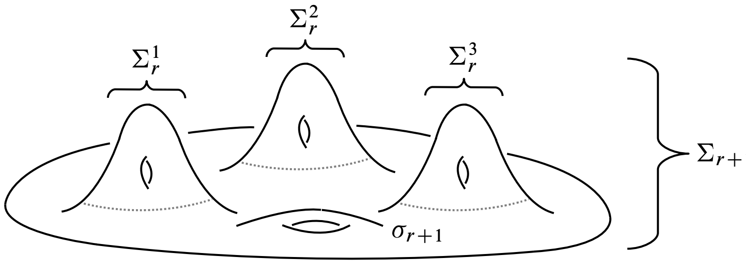


Figure 4: Shown is a Seifert surface Σ_{r+1} for a preferred longitude on $\partial N_{r+1}^{k'}$. In the figure, $q_{r+1} = 3$, and thus the three separating simple closed curves, indicated by dashed gray lines, separate three Seifert surfaces for preferred longitudes on ∂N_r^k; these Seifert surfaces are indicated as Σ_r^1, Σ_r^2 and Σ_r^3. The complement of $\Sigma_r^1 \cup \Sigma_r^2 \cup \Sigma_r^3$ in Σ_{r+1} is σ_{r+1}; it is formed by banding together (using $P_{r+1}q_{r+1}$ bands) P_{r+1} meridian discs for N_r with $q_{r+1} = 3$ collar annuli of the $\partial \Sigma_r^j$.

We now look at various finite covers of $M_{r+1}^{k'}$ that are obtained by cutting $M_{r+1}^{k'}$ along Σ_{r+1} and then cyclically stacking copies of these split-open $M_{r+1}^{k'}$. We first look at a $P_{r+1}q_{r+1}$–fold cover obtained in this fashion, and, due to the topological observations above, focus in on the lift of the space $N_r^k \setminus N_{r+1}^{k'}$ which contains σ_{r+1}. If we arrange that downstairs ∂N_r^k has Legendrian rulings that are (P_{r+1}, q_{r+1}) cables (which are ∞'_{r+1}–rulings on $\partial N_{r+1}^{k'}$), then upstairs, in the $P_{r+1}q_{r+1}$–fold cover, the lift of $N_r^k \setminus N_{r+1}^{k'}$ can be fibered by Legendrian fibers all with twisting $-(A_{r+1}k' + B_{r+1})$. The reason for this is as follows. First of all, the ∞'_{r+1}–rulings have twisting $-(A_{r+1}k' + B_{r+1})$ on $\partial N_{r+1}^{k'}$, and intersect the ∞_{r+1}–longitude positively $P_{r+1}q_{r+1}$ times; hence upstairs in the $P_{r+1}q_{r+1}$–fold cover they will lift to Legendrians of twisting $-(A_{r+1}k'+B_{r+1})$. As a result, the standard convex annulus $\mathcal{A}$ from Construction 4.5 will be fibered by Legendrians of twisting $-(A_{r+1}k' + B_{r+1})$ upstairs in the cover as well. Moreover, the (P_{r+1}, q_{r+1}) rulings on $\partial N(L_r)$ in Construction 4.5 also have twisting $-(A_{r+1}k' + B_{r+1})$, and in the cover will become longitudinal $(P_{r+1}, 1)$ rulings, but still with twisting $-(A_{r+1}k'+B_{r+1})$. Furthermore, the lift of $N(L_r)$ will have convex boundary with two longitudinal dividing curves (a different longitude, of course). Thus we see that the contact structure on this lift of $N(L_r)$ is just a standard neighborhood of one of the ruling curves (pushed into the interior of the solid torus), and thus the solid torus can be fibered by Legendrians of twisting $-(A_{r+1}k' + B_{r+1})$.

Note that the rest of the cover (outside the lift of $N_r^k \setminus N_{r+1}^{k'}$) is fibered (horizontally) by the copies of the Σ_r's. We now make an inductive hypothesis, which one may think of as part of item (5) of the main Inductive Hypothesis, which specifies part of why the candidate N_r^k fail to thicken. Specifically, using the proof of [7, Lemma 3.3] for the case of positive torus knots as our base case (see also the discussion in Lemma 3.2 above), inductively we may assume that the monodromy for the fibered space M_r^k is periodic, with period that divides a positive integer m_r, and such that a resulting m_r–fold product cover can be fibered by Legendrian fibers that all have twisting $-s_r(A_r k + B_r)$, where s_r is again some positive integer (for the base case of positive torus knots, $m_1 = P_1 q_1$ and $s_1 = 1$). As we proceed with the current proof of the present lemma, we will show that this inductive hypothesis holds for $M_{r+1}^{k'}$ as well (for appropriate m_{r+1} and s_{r+1}). It will be convenient for us, however, to take m_r, and multiply it by $-\chi(K_r)$ to get a new m_r; in other words, we can assume that $-\chi(K_r)$ divides m_r and s_r, and we will still have the m_r–fold product cover of M_r^k being fibered by Legendrians all having twisting $-s_r(A_r k + B_r)$.

As a consequence of this and the above topological observations, we can now pass to another finite cover, and cyclically stack m_r copies of our $P_{r+1}q_{r+1}$–fold cover of $M_{r+1}^{k'}$ to obtain $\widetilde{M}_{r+1}^{k'} = S^1 \times \Sigma_{r+1}$. Furthermore, if we restrict to $S^1 \times \sigma_{r+1} \subset S^1 \times \Sigma_{r+1}$, the space $S^1 \times \sigma_{r+1}$ can be fibered by Legendrians all of twisting $-s_{r+1}(A_{r+1}k' + B_{r+1})$, for some positive integer s_{r+1}, with respect to the product framing. However, at the moment all we know is that the q_{r+1} copies of $S^1 \times \Sigma_r$ can be fibered by *topological* copies of these Legendrian fibers in $S^1 \times \sigma_{r+1}$; what we will show is that in fact $S^1 \times \Sigma_{r+1}$ can be fibered by Legendrian copies of the fibers in $S^1 \times \sigma_{r+1}$. (The topological picture is shown in Figure 5.)

To this end, we first establish some notation; downstairs let $T = \partial N_r^k$. As just mentioned, we may assume that the rulings on T are copies of ∞'_{r+1} (ie, (P_{r+1}, q_{r+1}) cables on T), and assume the space $N_r^k \setminus N_{r+1}^{k'}$ bounded by T lifts to $S^1 \times \sigma_{r+1}$, where all the S^1 fibers are Legendrian isotopic to lifts of ∞'_{r+1} and have twisting $-s_{r+1}(A_{r+1}k' + B_{r+1})$ for some positive integer s_{r+1}. We will call these S^1 fibers S^1_{r+1}, and note that they are topologically isotopic to the S^1 fibers in the product space $S^1 \times \Sigma_{r+1}$. We also have that if we think of M_r^k as bounded by T, then M_r^k lifts to q_{r+1} copies of $S^1 \times \Sigma_r$ for a *different* S^1, where all the S^1 fibers are Legendrian isotopic to lifts of ∞'_r, and have twisting $-s_r P_{r+1}(A_r k + B_r)$. We will call these S^1 fibers S^1_r, and emphasize that these are not the same as the S^1_{r+1}'s. However, we will show that in fact, all of $\widetilde{M}_{r+1}^{k'}$ can be fibered by Legendrian S^1_{r+1}'s.

On the Seifert surface Σ_{r+1}, recall that we have labelled the q_{r+1} Σ_r's as Σ_r^j. Now let α^i_{r+1} be $2g'$ disjoint arcs on Σ_{r+1}, each with endpoints on $\partial\Sigma_{r+1}$, and such that if we cut along the α^i_{r+1} we obtain a polygon P_{r+1}. Also, let $\alpha^i_{r,j}$ be $2g$ disjoint arcs on Σ_r^j that, when we cut along them, yield polygons P_r^j. Thus we have solid tori $S^1_r \times P_r^j$ embedded in $\widetilde{M}_{r+1}^{k'}$. We can calculate the boundary slopes

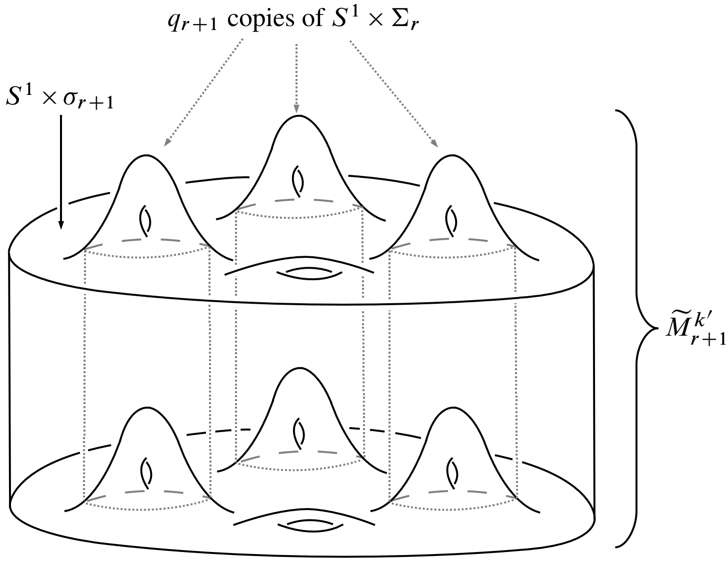


Figure 5: Shown is $\widetilde{M}^{k'}_{r+1}$, with the q_{r+1} copies of $S^1 \times \Sigma_r$ indicated by the gray dashed cylinders; the complement of the gray dashed cylinders is $S^1 \times \sigma_{r+1}$. The top and bottom of the figure are identified.

of these solid tori using the framing coming from the lifts of ∞'_r; this calculation is similar to that in [7, Lemma 3.3]. Specifically, note that a longitude for this torus intersects Γ, $2s_r P_{r+1}(A_r k + B_r)$ times, and a meridian for this torus is composed of 2 copies each of the associated $2g$ arcs $\alpha^i_{r,j}$, as well as $4g$ arcs β_i from $\partial\Sigma^j_r$. Now since $\partial\Sigma^j_r$ is a preferred longitude downstairs in M^k_r, we know that Γ intersects these β_i, $2(-\chi(K_r)) = 2(2g-1)$ times positively; see Remark 2.4 above. But then the edge-rounding that results at each intersection of an $S^1_r \times \beta_i$ with an $S^1_r \times \alpha^i_{r,j}$ yields $4g$ negative intersections with Γ. Thus we obtain after edge-rounding that the boundary slope is $-1/(s_r P_{r+1}(A_r k + B_r))$; as a consequence, we see that the solid torus $S^1_r \times P^j_r$ is simply a standard neighborhood of a Legendrian of twisting $-(s_r P_{r+1}(A_r k + B_r))$. We will use this shortly.

Now we switch our attention to the S^1_{r+1}'s, and note that the arcs α^i_{r+1} that stay in σ_{r+1} represent an interval's worth of S^1_{r+1} fibers of twisting $-s_{r+1}(A_{r+1}k' + B_{r+1})$, and hence represent standard convex annuli in the space $\widetilde{M}^{k'}_{r+1}$. The arcs α^i_{r+1} that leave σ_{r+1} represent convex annuli that are fibered by Legendrian S^1_{r+1}'s only when restricted to their intersection with the lift of the space $N^k_r \setminus N^{k'}_{r+1}$ bounded by T. So what is of interest is a convex annulus $\mathcal{A}_i$ with boundary components on T that both have twisting $-s_{r+1}(A_{r+1}k' + B_{r+1})$, and which is fibered by topological copies of the S^1_{r+1}'s, but which is embedded in one of the q_{r+1} lifts of M^k_r.

So suppose, for contradiction, that there exists a bypass on one of the $\mathcal{A}_i$'s. We look at what passing through this bypass will do on the lift of T to which $\mathcal{A}_i$ is attached; we use the framing on the lift of T that comes from the lifts of ∞_r'. First, recall that we know that q_{r+1}/p_{r+1} is clockwise from $-(k+1)/(A_r k + B_r)$ in the Farey tessellation; as a result, we know that the bypass of interest is on a ruling with slope $1/t'$ that is clockwise (in the Farey tessellation) from the dividing slope s of the lift of T. Moreover, we know what this dividing slope s is; it is $-\chi(K_r)/(-s_r P_{r+1}(A_r k+B_r))$, since the original preferred Seifert longitude on T lifts to the meridian on the lift of T. But, since $-\chi(K_r)$ divides s_r, this means in lowest terms, $s = -1/t$ for some t. As a result, passing through the bypass would yield a new torus T', on which is a longitudinal curve γ topologically isotopic to the S^1_r's, but with twisting greater than $-s_r P_{r+1}(A_r k + B_r)$. But if we then split the $S^1_r \times \Sigma^j_r$ that contains the $\mathcal{A}_i$ along arcs $\alpha^i_{r,j}$ to obtain $S^1_r \times P^j_r$, and then pass to a finite cover of the base by tiling copies of $S^1_r \times P^j_r$ (similar to what we did in [7, Lemma 3.3]), we will enclose γ in a standard neighborhood of a Legendrian of twisting $-s_r P_{r+1}(A_r k + B_r)$, which is a contradiction. Thus $\mathcal{A}_i$ must be standard convex.

By similar reasoning, if we look at the convex annuli inside one of the lifts of M^k_r that are in fact $S^1_{r+1} \times \alpha^i_{r,j}$, they will be standard convex. Splitting the lift of M^k_r along these annuli then yields an $S^1_{r+1} \times P^j_r$; the resulting boundary torus will have a characteristic foliation that matches that of the standard neighborhood of a Legendrian with twisting $-s_{r+1}(A_{r+1}k'+B_{r+1})$, by a similar edge-rounding calculation as above. Now since the contact structure on the lift of M^k_r can be isotoped to be horizontal, and hence tight, we thus know that the contact structure on $S^1_{r+1} \times P^j_r$ can be isotoped so that all the S^1_{r+1} fibers are Legendrian with twisting $-s_{r+1}(A_{r+1}k' + B_{r+1})$. As a result, the contact structure can be isotoped so that all of the S^1_{r+1} fibers in $\widetilde{M}^{k'}_{r+1}$ are Legendrian of twisting $-s_{r+1}(A_{r+1}k' + B_{r+1})$.

Thus the argument that $N^{k'}_{r+1}$ fails to thicken proceeds exactly as in [7, Lemma 3.3]; specifically, if $N^{k'}_{r+1}$ thickens, then there exists a curve γ' upstairs in $\widetilde{M}^{k'}_{r+1}$ which is topologically isotopic to the S^1_{r+1}'s but has greater twisting. However, if we then split the whole Σ_{r+1} along arcs α^i_{r+1} to obtain $S^1_{r+1} \times P_{r+1}$, the resulting boundary torus will have a characteristic foliation that matches that of the standard neighborhood of a Legendrian with twisting $-s_{r+1}(A_{r+1}k'+B_{r+1})$, since the dividing curves on the lift of $\partial N^{k'}_{r+1}$ intersect $\partial\Sigma_{r+1}$ exactly $2(-\chi(K_{r+1}))$ times and hence a similar edge-rounding calculation applies as above. If we then pass to a finite cover of the base by tiling copies of $S^1_{r+1} \times P_{r+1}$ (similar to what we did in [7, Lemma 3.3]), we will enclose γ' in a standard neighborhood of a Legendrian of twisting $-s_{r+1}(A_{r+1}k' + B_{r+1})$, which is a contradiction. □

We conclude with the following lemma that calculates rotation numbers; this will complete the induction step for Case I.

Lemma 4.7 *Any nonthickenable $N^{k'}_{r+1}$ have contact structures that are universally tight and have convex meridian discs D whose bypasses bound half-discs all of the same sign; ie,* $\mathrm{r}(\partial D) = \pm k'$*. Also, the preferred longitude on $\partial N^{k'}_{r+1}$ has rotation number zero for $k' > 0$.*

Proof We first prove that the contact structure on a candidate $N^{k'}_{r+1}$ which fails to thicken is universally tight. To see this note that from Lemma 4.3 above, and the Inductive Hypothesis, such a candidate $N^{k'}_{r+1}$ is embedded inside a N^k_r with a universally tight contact structure. Now there is a q_{r+1}–fold cover of N^k_r that maps a total of q_{r+1} lifts (say, $\widetilde{N}^{k'}_{r+1}$) to $N^{k'}_{r+1}$, the lifts themselves each being an $S^1 \times D^2$. This cover in turn has a universal cover $\mathbb{R} \times D^2$ that contains q_{r+1} copies of a universal cover $\mathbb{R} \times D^2$ of $N^{k'}_{r+1}$. Since, by the Inductive Hypothesis, the universal cover of N^k_r has a tight contact structure, a tight contact structure is thus induced on the universal cover of $N^{k'}_{r+1}$.

Now recall that N^k_r is formed from $N^{k'}_{r+1}$ by first joining ∞'–longitudes on $\partial N^{k'}_{r+1}$ with an annulus $\mathcal{A}$ to get a thickened torus $R = T^2 \times [0,1]$, and then gluing in a standard neighborhood of a Legendrian knot. Thus, since N^k_r has bypasses all of the same sign, by similar reasoning as that in Construction 4.5, it follows that a horizontal annulus in R has bypasses all of the same sign. We will thus have that $N^{k'}_{r+1}$ must have convex meridian discs all of the same sign. The computation of rotation numbers for the meridian curve follows.

To show that the preferred longitude on $\partial N^{k'}_{r+1}$ has rotation number zero, we use an argument similar to that used in [7, Lemma 3.4]. We call the meridian disc D_r for N^k_r and the Seifert surface Σ_r for the preferred longitude on ∂N^k_r. If we then look at the (P_{r+1}, q_{r+1}) cable on ∂N^k_r, we can calculate its rotation number as

$$\mathrm{r}((P_{r+1}, q_{r+1})) = P_{r+1}\,\mathrm{r}(\partial D_r) + q_{r+1}\,\mathrm{r}(\partial \Sigma_r) = P_{r+1}(\pm q_{r+1} k').$$

But then since this same knot is a $(P_{r+1} q_{r+1}, 1)$ cable on $\partial N^{k'}_{r+1}$, we have that $\mathrm{r}((P_{r+1}, q_{r+1})) = P_{r+1} q_{r+1}(\pm k') + q_{r+1}\,\mathrm{r}(\partial \Sigma)$, where Σ is a Seifert surface for the preferred longitude on $\partial N^{k'}_{r+1}$. This implies that $\mathrm{r}(\partial \Sigma) = 0$. □

4.2 Case II: $w(K_r) > P_{r+1}/q_{r+1} > 0$

As in Case I, we work through proving items (2)–(5) in the Inductive Hypothesis via a series of lemmas.

We begin by proving item (2) in the Inductive Hypothesis.

Lemma 4.8 *If $w(K_r) > P_{r+1}/q_{r+1} > 0$, then $\overline{\mathrm{tb}}(K_{r+1}) = w(K_{r+1}) = A_{r+1}$.*

Proof The proof is almost identical to that of Step 1 in the proof of from [15, Section 6, Theorem 1.5]; we will include the details, though, since certain aspects differ. We first examine representatives of K_{r+1} at maximal Thurston–Bennequin number $\overline{\mathrm{tb}}$. Since there exists a convex torus representing K_r with Legendrian divides that are (p_{r+1}, q_{r+1}) cablings (inside of the solid torus representing K_r with $\mathrm{slope}(\Gamma) = 1/\overline{t}(K_r)$) we know that $\overline{\mathrm{tb}}(K_{r+1}) \geq P_{r+1}q_{r+1} = A_{r+1}$. To show that $\overline{\mathrm{tb}}(K_{r+1}) = A_{r+1}$, we show that $\overline{t}(K_{r+1}) = 0$ by showing that the contact width $w(K_{r+1}, \mathcal{C}') = 0$, since this will yield $\overline{\mathrm{tb}}(K_{r+1}) \leq w(K_{r+1}) = A_{r+1}$. So suppose, for contradiction, that some N_{r+1} has convex boundary with $\mathrm{slope}(\Gamma_{\partial N_{r+1}}) = s > 0$, as measured in the $\mathcal{C}'$ framing, and two dividing curves. After shrinking N_{r+1} if necessary, we may assume that s is a large positive integer. Then let $\mathcal{A}$ be a convex annulus from ∂N_{r+1} to itself having boundary curves with slope ∞'. Taking a neighborhood of $N_{r+1} \cup \mathcal{A}$ yields a thickened torus R with boundary tori T_1 and T_2, arranged so that T_1 is inside the solid torus N_r representing K_r bounded by T_2.

Now there are no boundary parallel dividing curves on $\mathcal{A}$, for otherwise, we could pass through the bypass and increase s to ∞', yielding excessive twisting inside N_{r+1}. Hence $\mathcal{A}$ is in standard form, and consists of two parallel nonseparating arcs. We now choose a new framing $\mathcal{C}''$ for N_r where $(p_{r+1}, q_{r+1}) \mapsto (0,1)$; then choose $(p'', q'') \mapsto (1,0)$ so that $p''q_{r+1} - q''p_{r+1} = 1$ and such that $\mathrm{slope}(\Gamma_{T_1}) = -s$ and $\mathrm{slope}(\Gamma_{T_2}) = 1$. As mentioned in [6], this is possible since Γ_{T_1} is obtained from Γ_{T_2} by $s+1$ right-handed Dehn twists. Then note that in the $\mathcal{C}'$ framing, we have that $q_{r+1}/p_{r+1} > \mathrm{slope}(\Gamma_{T_2}) = (q'' + q_{r+1})/(p'' + p_{r+1}) > q''/p''$, and q_{r+1}/p_{r+1} and q''/p'' are connected by an arc in the Farey tessellation of the hyperbolic disc (see [7, Section 2.2.3]). Thus, since $1/\overline{t}(K_r)$ is connected by an arc to $0/1$ in the Farey tessellation, we must have that $(q'' + q_{r+1})/(p'' + p_{r+1}) > 1/\overline{t}(K_r)$. Thus we can thicken N_r to one of the solid tori with $\mathrm{slope}(\Gamma) = -(k+1)/(A_r k + B_r)$ which fails to thicken. Then, just as in [6, Claim 4.2], we have the following:

(i) Inside R there exists a convex torus parallel to T_i with slope q_{r+1}/p_{r+1}.

(ii) R can thus be decomposed into two layered *basic slices*.

(iii) The tight contact structure on R must have *mixing of sign* in the Poincaré duals of the relative half-Euler classes for the layered basic slices.

(iv) This mixing of sign cannot happen inside the universally tight solid torus which fails to thicken.

This last statement is due to the proof of [13, Proposition 5.1], where it is shown that mixing of sign will imply an overtwisted disc in the universal cover of the solid torus. Thus we have contradicted $s > 0$. So $\overline{\mathrm{tb}}(K_{r+1}) = P_{r+1}q_{r+1} = A_{r+1}$. □

With the following lemma we prove that items (3) and (4) of the Inductive Hypothesis hold for K_{r+1}.

Lemma 4.9 *If $w(K_r) > P_{r+1}/q_{r+1} > 0$, let N_{r+1} be a solid torus representing K_{r+1}, for $r \geq 1$. Then N_{r+1} can be thickened to an $N^{k'}_{r+1}$ for some nonnegative integer k'. Moreover, if N_{r+1} fails to thicken, then it has the same boundary slope as some $N^{k'}_{r+1}$, as well as at least $2n^{k'}_{r+1}$ dividing curves.*

Proof This is the case where $p_{r+1} < 0$ but $q_{r+1}/p_{r+1} \in (1/\overline{t}(K_r), -1/A_r)$; we have that $\overline{t}(K_{r+1}) = 0$. We begin as we did in Case I. If N_{r+1} is a solid torus representing K_{r+1}, as before choose L_r in $S^3 \backslash N_{r+1}$ such that $\partial N(L_r)$ is joined to ∂N_{r+1} by an annulus $\mathcal{A}_{(p_{r+1},q_{r+1})}$, and with $\mathrm{tb}(L_r)$ maximized over topological isotopies in the space $S^3 \backslash N_{r+1}$. We will then have three cases to establish the first part of the lemma, namely that N_{r+1} can be thickened to an $N^{k'}_{r+1}$ for some nonnegative integer k', and if N_{r+1} fails to thicken, then it has the same boundary slope as some $N^{k'}_{r+1}$:

Case 1 Suppose $\mathrm{slope}(\Gamma_{\partial N(L_r)}) = -1/m$ where $-1/m < q_{r+1}/p_{r+1}$; we know $m \geq 0$. Then inside $N(L_r)$ is an N_r with boundary slope q_{r+1}/p_{r+1}. But then we can extend $\mathcal{A}_{(p_{r+1},q_{r+1})}$ to an annulus that has no twisting on one edge, and we can thus thicken N_{r+1} so it has boundary slope ∞'. Moreover, since there is twisting inside $N(L_r)$, we can assure there are two dividing curves on the thickened N_{r+1} (see [6, Claim 4.1]). So this situation yields no nontrivial solid tori N_{r+1} which fail to thicken; in other words, N_{r+1} can be thickened to an N^0_{r+1}.

Case 2 Alternatively, suppose $-1/m > q_{r+1}/p_{r+1}$; note here we must have $m > 0$ – furthermore, in this case we further restrict to the situation where $-1/(m-1) > q_{r+1}/p_{r+1}$. Then we can use the twist number lemma (Lemma 2.5 above) to conclude that there are no bypasses on the $\partial N(L_r)$–edge of $\mathcal{A}_{(p_{r+1},q_{r+1})}$, and so we can thicken N_{r+1} through bypasses so that $\mathcal{A}_{(p_{r+1},q_{r+1})}$ is standard convex.

Then, as in Lemma 4.3, we let $N_r := N_{r+1} \cup N(\mathcal{A}_{(p_{r+1},q_{r+1})}) \cup N(L_r)$. We know that $w(K_{r+1}, \mathcal{C}') = 0$, and we know that the geometric intersection number of the ∞'_{r+1}–rulings on ∂N_{r+1} with $\Gamma_{\partial N_{r+1}}$ equals $p_{r+1} + mq_{r+1} > 0$. As a result, we know that ∞'_{r+1} is clockwise from $\Gamma_{\partial N_{r+1}}$ in the Farey tessellation. Thus, we must have that the (p_{r+1}, q_{r+1})–rulings intersect $\Gamma_{\partial N_r}$ positively; ie, q_{r+1}/p_{r+1} is clockwise from $\mathrm{slope}(\Gamma_{\partial N_r})$ in the Farey tessellation. As a result, when we thicken to N^k_r as in Lemma 4.3, we must also have q_{r+1}/p_{r+1} clockwise from $-(k+1)/(A_r k+B_r)$ in the Farey tessellation, for otherwise we could destabilize L_r (in the complement of N_{r+1}) via an annulus with (p_{r+1}, q_{r+1})–ruling boundary on $\partial N(L_r)$, and (p_{r+1}, q_{r+1})–Legendrian divide boundary on a torus N'_r with boundary slope q_{r+1}/p_{r+1} in the thickened torus cobounded by ∂N_r and ∂N^k_r.

Thus, since q_{r+1}/p_{r+1} is clockwise from both $-1/m$ and $-(k+1)/(A_r k + B_r)$ in the Farey tessellation, the calculation of the boundary slope goes through as above in Lemma 4.3 – see the comment after Equation (3) above, and note that such N_r^k exist since $-(k+1)/(A_r k + B_r) \to -1/A_r$ as k increases. We conclude that N_{r+1} thickens to some $N_{r+1}^{k'}$.

Case 3 For the remaining case, suppose $-1/m > q_{r+1}/p_{r+1}$ and m is the least positive integer satisfying this inequality. Thus $-1/(m-1) < q_{r+1}/p_{r+1}$. Again look at the $\partial N(L_r)$–edge of $\mathcal{A}_{(p_{r+1},q_{r+1})}$. We claim that this edge has no bypasses. So, for contradiction, suppose it does. Then we can thicken $N(L_r)$ to a solid torus where the (efficient) geometric intersection number of (p_{r+1}, q_{r+1}) with dividing curves is less than $p_{r+1} + mq_{r+1}$. Suppose the slope of this new solid torus is $-\lambda/\mu < -1/m$, where $\lambda > 1$ since m is minimized in the complement of N_{r+1}.

We do some calculations. Note first that if $m/\mu > 1$, then $m > \mu$, which means $m-1 \geq \mu$. This implies $-1/(m-1) \geq -1/\mu > -\lambda/\mu$, which cannot happen, again since m is minimized in the complement of N_{r+1}. Thus we must have $m/\mu \leq 1$. But then the geometric intersection number of (p_{r+1}, q_{r+1}) with $(-\mu, \lambda)$ is $\lambda p_{r+1} + \mu q_{r+1} > (\mu/m) p_{r+1} + \mu q_{r+1} \geq (m/\mu)[(\mu/m) p_{r+1} + \mu q_{r+1}] = p_{r+1} + mq_{r+1}$. This is a contradiction.

Thus there are no bypasses on the $\partial N(L_r)$–edge of $\mathcal{A}_{(p_{r+1},q_{r+1})}$, and we can thicken N_{r+1} through any bypasses so that $\mathcal{A}_{(p_{r+1},q_{r+1})}$ is standard convex. The calculations that show N_{r+1} thickens to $N_{r+1}^{k'}$ go through as above in Lemma 4.3; in particular, as above, the nonthickenable N_r^k that is used will be such that $q_{r+1}/p_{r+1} < -(k+1)/(A_r k + B_r)$.

This concludes the three cases, which together show that any N_{r+1} representing K_{r+1} can be thickened to one of the $N_{r+1}^{k'}$, and if N_{r+1} fails to thicken, then it has the same boundary slope as some $N_{r+1}^{k'}$. We now show that if N_{r+1} fails to thicken, and if it has the minimum number of dividing curves over all such N_{r+1} which fail to thicken and have the same boundary slope as $N_{r+1}^{k'}$, then N_{r+1} is actually an $N_{r+1}^{k'}$.

To see this, as above we can choose a Legendrian L_r that maximizes tb in the complement of N_{r+1} and such that we can join $\partial N(L_r)$ to ∂N_{r+1} by a convex annulus $\mathcal{A}_{(p_{r+1},q_{r+1})}$ whose boundaries are (p_{r+1}, q_{r+1}) and ∞' rulings on $\partial N(L_r)$ and ∂N_{r+1}, respectively. Now since N_{r+1} fails to thicken, we can assume that $q_{r+1}/p_{r+1} < -1/m$ and that there are no bypasses on the $\partial N(L_r)$–edge, and in this case we have no bypasses on the ∂N_{r+1}–edge since N_{r+1} fails to thicken and is at minimum number of dividing curves.

As above, let $N_r := N_{r+1} \cup N(\mathcal{A}_{(p_{r+1},q_{r+1})}) \cup N(L_r)$. We claim this N_r fails to thicken – the proof proceeds identically as above in Lemma 4.3, as does the proof that N_{r+1} is in fact an $N_{r+1}^{k'}$. □

The following proof of item (5) of the Inductive Hypothesis is similar to that of Case I.

Lemma 4.10 *If $w(K_r) > P_{r+1}/q_{r+1} > 0$, the candidate $N_{r+1}^{k'}$ exist and actually fail to thicken for $k' \geq C_{r+1}$, where C_{r+1} is some positive integer. Moreover, these $N_{r+1}^{k'}$ have contact structures that are universally tight and have convex meridian discs whose bypasses bound half-discs all of the same sign. Also, the preferred longitude on $\partial N_{r+1}^{k'}$ has rotation number zero for $k' > 0$.*

Proof The proof that the contact structure on a candidate $N_{r+1}^{k'}$ which fails to thicken is universally tight is identical to the argument in Case I, as are the calculations of the rotation numbers.

Now we know by the Inductive Hypothesis that there exists a C_r such that if $k \geq C_r$, then the N_r^k exist and fail to thicken. So suppose $k/q_{r+1} \in \mathbb{N}$ for some $k \geq C_r$. Also assume that $q_{r+1}/p_{r+1} < -(k+1)/(A_r k + B_r)$; we know such a k exists since $-(k+1)/(A_r k + B_r) \to -1/A_r$ as k increases. Then $N_{r+1}^{k'}$ exists and fails to thicken as in the argument for Case I for $k' := k/q_{r+1}$, and C_{r+1} will be the least such $k/q_{r+1} \in \mathbb{N}$. □

This completes the induction step for Case II, and hence the proof of Theorem 1.1.

5 Transversely nonsimple iterated torus knots

We have completed the UTP classification of iterated torus knots; it now remains to show that if an iterated torus knot fails the UTP, then it supports transversely nonsimple cablings. To this end, in this section we prove Theorem 1.2; we do so by working through two lemmas. The first lemma will give us information about just a piece of the Legendrian mountain range for $K_r = ((P_1, q_1), \ldots, (P_r, q_r))$ where $P_i > 0$ for all i; in the second lemma, we will then use this information to obtain enough information about the Legendrian mountain ranges of certain cables K_{r+1} to conclude that these cables are transversely nonsimple. We will therefore not be completing the Legendrian or transverse classification for these iterated torus knots. The particular cables K_{r+1} we focus in on will be knot types corresponding to boundary slopes of dividing curves on certain nonthickenable tori representing K_r; we will thus establish the transverse nonsimplicity of K_{r+1} by examining dividing curves on tori that thicken, versus those that fail to thicken.

Lemma 5.1 *Suppose $K_r = ((P_1, q_1), \dots, (P_r, q_r))$ is an iterated torus knot where $P_i > 0$ for all i. Then there exists Legendrian representatives $L_r^{\pm}$ with $\mathrm{tb}(L_r^{\pm}) = 0$ and $\mathrm{r}(L_r^{\pm}) = \pm(A_r - B_r)$; also, $L_r^{\pm}$ destabilizes.*

Proof The lemma is true for positive torus knots [5], so we inductively assume it is true for K_{r-1}. Then look at Legendrian rulings $\widetilde{L}_r^{\pm}$ (representing K_r) on standard neighborhoods of the inductive $L_{r-1}^{\pm}$. In the $\mathcal{C}'$ framing the boundary slope of these $N(L_{r-1}^{\pm})$ is $-1/A_{r-1}$, and so $t(\widetilde{L}_r^{\pm}) = -(p_r + q_r A_{r-1}) = -P_r$, using Equation (1); hence $\mathrm{tb}(\widetilde{L}_r^{\pm}) = A_r - P_r$.

To calculate the rotation number of $\widetilde{L}_r^{\pm}$, we use the following formula from [6, Lemma 2.2], where D is a convex meridian disc for $N(L_{r-1}^{\pm})$ and Σ is a Seifert surface for the preferred Seifert longitude on $\partial N(L_{r-1}^{\pm})$:

$$\begin{aligned}\mathrm{r}(\widetilde{L}_r^{\pm}) &= P_r\,\mathrm{r}(\partial D) + q_r\,\mathrm{r}(\partial\Sigma)\\ &= \pm q_r(A_{r-1} - B_{r-1})\\ &= \pm(P_r - B_r),\end{aligned}$$

where for the last equality we are using Equation (1). This gives us

$$\begin{aligned}\mathrm{sl}(T_-(\widetilde{L}_r^{+})) &= (A_r - P_r) + (P_r - B_r) = A_r - B_r,\\ \mathrm{sl}(T_+(\widetilde{L}_r^{-})) &= (A_r - P_r) - (-(P_r - B_r)) = A_r - B_r.\end{aligned}$$

This, along with Lemma 2.1, shows us that $\widetilde{L}_r^{+}$ is on the right-most slope of the Legendrian mountain range of K_r, and $\widetilde{L}_r^{-}$ is on the left-most edge. To the former we can perform positive stabilizations to reach L_r^{+} at $\mathrm{tb} = 0$ and $\mathrm{r} = A_r - B_r$; to the latter we can perform negative stabilizations to reach L_r^{-} at $\mathrm{tb} = 0$ and $\mathrm{r} = -(A_r - B_r)$ – we know such stabilizations can be performed since $A_r - P_r = P_r q_r - P_r > 0$. □

So suppose K_r is an iterated torus knot that fails the UTP (which is precisely when $P_i > 0$ for all i). Then we know that for $k \geq C_r$ there exist nonthickenable solid tori N_r^k having intersection boundary slopes of $-(k+1)/(A_r k + B_r)$, where these slopes are measured in the $\mathcal{C}'$ framing. Switching to the standard $\mathcal{C}$ framing, these intersection boundary slopes are $(k+1)/(A_r - B_r) = -(k+1)/\chi(K_r)$. Now as $k \to \infty$, there are infinitely many values of $k+1$ which are prime and greater than $A_r - B_r$. As a consequence, there are infinitely many N_r^k with two dividing curves. Based on this observation, we make the following definition:

Definition 5.2 Suppose $K_r = ((P_1, q_1), \dots, (P_r, q_r))$ is an iterated torus knot where $P_i > 0$ for all i, and let $-(k+1)/(A_r k + B_r)$ be such that $-1/(A_r - 1) < -(k+1)/(A_r k + B_r) < -1/A_r$, and such that there exists an associated N_r^k with two dividing curves that fails to thicken. We then define $\widehat{K}_{r+1}$ to be the knot type that is a $(-\chi(K_r), k+1)$–cabling of K_r.

So given K_r, there are infinitely many such cabling knot types $\widehat{K}_{r+1}$, all of these being cablings of slope $-(k+1)/(A_r k + B_r)$ as measured in the nonstandard $\mathcal{C}'$ framing (for different values of $k \to \infty$). The following lemma will then prove Theorem 1.2.

Lemma 5.3 $\widehat{K}_{r+1}$ *is a transversely nonsimple knot type.*

Proof We first calculate $\chi(\widehat{K}_{r+1})$. Using the recursive expression from Lemma 2.1 we obtain

$$\begin{aligned}\chi(\widehat{K}_{r+1}) &= q_{r+1}\chi(K_r) - P_{r+1}q_{r+1} + P_{r+1}\\ &= (k+1)(-A_r + B_r) - (A_r - B_r)(k+1) + (A_r - B_r)\\ &= (2k+1)(-A_r + B_r).\end{aligned}$$

We now look at the two universally tight nonthickenable N_r^k that have representatives of $\widehat{K}_{r+1}$ as Legendrian divides. These Legendrian divides have $\mathrm{tb} = A_{r+1} = q_{r+1}P_{r+1} = (k+1)(A_r - B_r)$. To calculate rotation numbers, we have two possibilities, depending on which boundary of the two universally tight N_r^k the Legendrian divides reside. Using the formula from [6, Lemma 2.2], we obtain

$$\begin{aligned}\mathrm{r}(\widehat{K}_{r+1}) &= q_{r+1}\,\mathrm{r}(\partial\Sigma) + P_{r+1}\,\mathrm{r}(\partial D)\\ &= P_{r+1}(\pm k)\\ &= \pm k(A_r - B_r).\end{aligned}$$

We will call the two Legendrian divides corresponding to $\mathrm{r} = \pm k(A_r - B_r)$, $L^{\pm}_{r+1}$ respectively. We can calculate the self-linking number for the negative transverse pushoff of L^{+}_{r+1} to be $sl = (2k+1)(A_r - B_r) = -\chi(\widehat{K}_{r+1})$. This shows that L^{+}_{r+1} is on the right-most edge of the Legendrian mountain range and is at maximal Thurston–Bennequin number $\overline{\mathrm{tb}}$. Similarly, L^{-}_{r+1} is on the left-most edge of the Legendrian mountain range and is at $\overline{\mathrm{tb}}$.

We now look at solid tori $\widehat{N}_r$ with intersection boundary slope $-(k+1)/(A_r k + B_r)$, but which *thicken* to solid tori with intersection boundary slopes $-1/(A_r - 1)$. Such tori $\partial\widehat{N}_r$ are embedded in universally tight basic slices bounded by tori with dividing curves of slope $-1/(A_r - 1)$ and $-1/A_r$. Legendrian divides on such $\widehat{N}_r$ have $\mathrm{tb} = (k+1)(A_r - B_r)$; to calculate possible rotation numbers for these Legendrian divides, we recall the procedure used in the proof of [15, Section 6, Theorem 1.5]. There, in Equation (14), we used a formula for the rotation numbers from [6, Lemma 3.8], where the range of rotation numbers was given by (substituting $A_r - 1$ for n)

$$\mathrm{r}(L_{r+1}) \in \left\{\pm(p_{r+1} + (A_r - 1)q_{r+1} + q_{r+1}\,\mathrm{r}(L_r)) \,|\, \mathrm{tb}(L_r) = A_r - (A_r - 1) = 1\right\}.$$

Now from Lemma 5.1 we know that there is an L_r with $\mathrm{tb}(L_r) = 1$ and $\mathrm{r}(L_r) = -(A_r - B_r) + 1$. Plugging this value of the rotation number into the expression above, along with $p_{r+1} = -(A_r k + B_r)$ and $q_{r+1} = k+1$, yields $\mathrm{r}(L_{r+1}) = \pm k(A_r - B_r)$. We will call the Legendrian divides having these rotation numbers $\widehat{L}^{\pm}_{r+1}$, respectively. Important for our purposes is that $\widehat{L}^{\pm}_{r+1}$ have the same values of tb and r as $L^{\pm}_{r+1}$.

We focus in on L^{-}_{r+1} and $\widehat{L}^{-}_{r+1}$, and we show that $T_-(L^{-}_{r+1})$ is not transversely isotopic to $T_-(\widehat{L}^{-}_{r+1})$, despite having the same self-linking number. (Although it is not needed, and we shall not present it here, we note that there is a similar symmetric argument showing that $T_+(L^{+}_{r+1})$ is not transversely isotopic to $T_+(\widehat{L}^{+}_{r+1})$.)

Consider first $T_+(L^{-}_{r+1})$. Being a transverse push-off of a Legendrian divide, it is in fact one of the dividing curves on ∂N^k_r, and is also at maximal self-linking number for $\widehat{K}_{r+1}$. Similarly, $T_+(\widehat{L}^{-}_{r+1})$ is one of the dividing curves on $\partial \widehat{N}_r$, and is also at maximal self-linking number. Now from [11] we know that $\widehat{K}_{r+1}$ is a fibered knot that supports the standard contact structure, since it is an iterated torus knot obtained by cabling positively at each iteration. As a consequence, from [8], we also know that $\widehat{K}_{r+1}$ has a unique transverse isotopy class at maximal self-linking number. Hence we know that $T_+(L^{-}_{r+1})$ and $T_+(\widehat{L}^{-}_{r+1})$ are transversely isotopic. Thus there is a transverse isotopy (inducing an ambient contact isotopy) that takes these two dividing curves on the two different tori to each other. Thus we may assume that ∂N^k_r and $\partial \widehat{N}_r$ intersect along one component of the two dividing curves; we call this component γ_+.

Now suppose, for contradiction, that $T_-(L^{-}_{r+1})$ is transversely isotopic to $T_-(\widehat{L}^{-}_{r+1})$. These transverse knots are represented by the other two dividing curves on ∂N^k_r and $\partial \widehat{N}_r$, respectively, and we are therefore assuming that there is a transverse isotopy taking one to the other. This transverse isotopy will induce an ambient contact isotopy of S^3, including a contact isotopy of the two tori ∂N^k_r and $\partial \widehat{N}_r$, with γ_+ sitting on both of them. Since ∂N^k_r and $\partial \widehat{N}_r$ are incompressible in $S^3 \setminus N(\gamma_+)$, we may assume that after this contact isotopy of S^3, ∂N^k_r and $\partial \widehat{N}_r$ intersect along their two dividing curves, which we denote as γ_+ and γ_-. We now observe that there is an isotopy (although, a priori, not necessarily a contact isotopy) of ∂N^k_r to $\partial \widehat{N}_r$ relative to γ_+ and γ_-. We claim that as a result $\widehat{N}_r$ cannot thicken, thus obtaining our contradiction. We do this by noting that the isotopy of ∂N^k_r to $\partial \widehat{N}_r$ relative to γ_+ and γ_- may be accomplished by the attachment of successive bypasses, beginning with ∂N^k_r and ending at $\partial \widehat{N}_r$; thus $\partial \widehat{N}_r$ is fixed throughout the process. Since these bypasses are attached in the complement of the two dividing curves, none of these bypass attachments can change the boundary slope. However, they may increase or decrease the number of dividing curves. Starting with $T = \partial N_r^k$, we make the following inductive hypothesis, which we will prove is maintained after bypass attachments:

(1) T is a convex torus which contains γ_+ and γ_-, and therefore has slope $-(k+1)/(A_r k + B_r)$.

(2) T is a boundary-parallel torus in a $[0,1]$–invariant $T^2 \times [0,1]$ with $\mathrm{slope}(\Gamma_{T_0}) = \mathrm{slope}(\Gamma_{T_1}) = -(k+1)/(A_r k + B_r)$, where the boundary tori have two dividing curves.

(3) There is a contact diffeomorphism $\phi\colon S^3 \to S^3$ which takes $T^2 \times [0,1]$ to a standard I–invariant neighborhood of ∂N_r^k and matches up their complements.

The argument that follows is similar to [6, Lemma 6.8]. First note that item (1) is preserved after a bypass attachment, since such a bypass is in the complement of γ_+ and γ_-, and thus cannot change the slope of the dividing curves. To see that items (2) and (3) are preserved, suppose that T' is obtained from T by a single bypass. Since the slope was not changed, such a (nontrivial) bypass must either increase or decrease the number of dividing curves by 2. We distinguish between two cases below.

Case 1 Suppose first that the bypass is attached from the inside, so that $T' \subset N$, where N is the solid torus bounded by T. For convenience, suppose $T = T_{0.5}$ inside the $T^2 \times [0,1]$ satisfying items (2) and (3) of the inductive hypothesis. Then we form the new $T^2 \times [0.5, 1]$ by taking the old $T^2 \times [0.5, 1]$ and adjoining the thickened torus between T and T'. Now T' bounds a solid torus N', and, by the classification of tight contact structures on solid tori, we can factor a nonrotative layer which is the new $T^2 \times [0, 0.5]$.

Case 2 Alternatively, if $T' \subset (S^3 \backslash N)$, then we know that N' thickens to an N_r^k, and thus there exists a nonrotative outer layer $T^2 \times [0.5, 1]$ for $S^3 \backslash N'$, where T_1 has two dividing curves. Thus the proof is done, for after enough bypass attachments we will obtain $T = \partial \widehat{N}_r$, with $\widehat{N}_r$ nonthickenable. But this is a contradiction, since $\widehat{N}_r$ does thicken. □

This completes the proof of Theorem 1.2.

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