On diffeomorphisms over nonorientable surfaces standardly embedded in the 4–sphere

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For a nonorientable closed surface standardly embedded in the 4–sphere, a diffeomorphism over this surface is extendable if and only if this diffeomorphism preserves the Guillou–Marin quadratic form of this embedded surface.

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1 Introduction

Let S be a closed surface and e be a smooth embedding of S into S^4. A diffeomorphism ϕ over S is *e–extendable* if there is an orientation-preserving diffeomorphism Φ of S^4 such that $\Phi|_S = \phi$. The natural problem to ask is:

> *Find a necessary and sufficient condition for a diffeomorphism over S to be e–extendable.*

For some special embeddings of closed surfaces in 4–manifolds, we have answers to this problem (for example, by Montesinos [11], the author [4] and the author and Yasuhara [6]). An embedding e of the orientable surface Σ_g into S^4 is called standard if $e(\Sigma_g)$ is the boundary of 3–dimensional handlebody embedded in S^4. Montesinos [11] and the author [4] showed:

Theorem 1.1 ([11] for $g = 1$; [4] for $g \geq 2$) *Let Σ_g be standardly embedded in S^4. An orientation-preserving diffeomorphism ϕ over the Σ_g is extendable to S^4 if and only if ϕ preserves the Rokhlin quadratic form of the Σ_g standardly embedded in S^4.*

In this paper, we consider the same kind of problem for nonorientable surfaces embedded in S^4. Let N_g be a connected nonorientable surface constructed from g projective planes by connected sum. We call N_g the *closed nonorientable surface of genus g*. Let $S^3 \times [-1, 1]$ be a closed tubular neighborhood of the equator S^3 in S^4. Then $S^4 - S^3 \times (-1, 1)$ consists of two 4–balls. An embedding $os\colon N_g \hookrightarrow S^4$ is *o–standard* if $os(N_g) \subset S^3 \times [-1, 1]$ and as shown in Figure 1. On the level $t = 0$ in this motion

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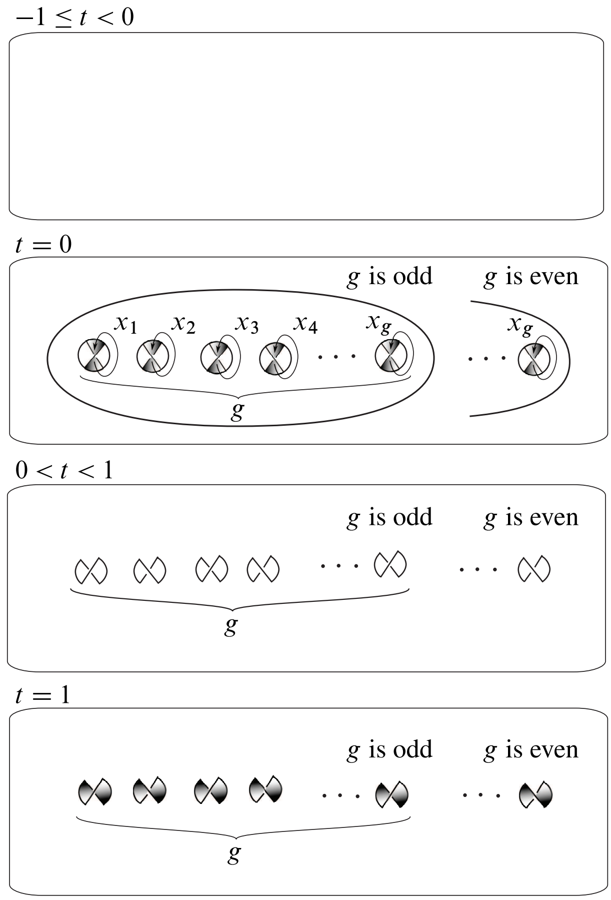

Figure 1. The motion picture of the o–standard embedding of N_g into S^4

Figure 2. Bands attached on the level $t = 0$ in Figure 1

picture, by the numbering of the band from the left to the right, the $(2i-1)$–st band is as shown in the left of Figure 2, and $2i$–th band is as shown in the right of Figure 2. The main result of this paper is:

Theorem 1.2 *The diffeomorphism ϕ over N_g is os–extendable if and only if ϕ preserves the Guillou–Marin quadratic form of the N_g o–standardly embedded in S^4.*

2 Guillou–Marin quadratic form

For a smooth embedding e of the closed nonorientable surface N_g of genus g into S^4, Guillou and Marin [3] (see also Matsumoto [10]) defined a quadratic form $q_e\colon H_1(N_g;\mathbb{Z}_2) \to \mathbb{Z}_4$ as follows: Let C be an immersed circle on N_g, and D be a connected orientable surface immersed in S^4 such that $\partial D = C$, and D is not tangent to N_g. Let ν_D be the normal bundle of D, then $\nu_D|_C$ is a solid torus with a trivialization induced from any trivialization of ν_D. Let $N_{N_g}(C)$ be the tubular neighborhood of C in N_g, then $N_{N_g}(C)$ is a twisted annulus or a Möbius band in $\nu_D|_C$. We denote by $n(D)$ the number of right-hand half-twists of $N_{N_g}(C)$ with respect to the trivialization of $\nu_D|_C$. Let $D \cdot N_g$ be the mod–2 intersection number between the interior of D and N_g, $\mathrm{Self}(C)$ be the mod–2 double points number of C, and $2\times$ be an injection $\mathbb{Z}_2 \to \mathbb{Z}_4$ defined by $2 \times [n]_2 = [2n]_4$. Then the number $n(D) + 2 \times D \cdot N_g + 2 \times \mathrm{Self}(C) \pmod 4$ depend only on the mod–2 homology class $[C]$ of C. Hence, we define

$$q_e([C]) = n(D) + 2 \times D \cdot N_g + 2 \times \mathrm{Self}(C) \pmod 4.$$

This map q_e is called Guillou–Marin *quadratic form*, since q_e satisfies

$$q_e(x + y) = q_e(x) + q_e(y) + 2 \times (x \cdot y)_2,$$

where $(x \cdot y)_2$ is the mod–2 intersection number between x and y. Let $\{x_1, \dots, x_g\}$ be the basis of $H_1(N_g;\mathbb{Z}_2)$ shown in Figure 1. For x_1, let D be a disc such that $\partial D = x_1$ and the interior of D is in $S^2 \times (0, 1]$, then $D \cdot N_g = 0$. Since $N_{N_g}(x_1)$ is a Möbius band with one right-hand half-twist, $q_{os}(x_1) = +1$. By the same way as above, we see that $q_{os}(x_{2i-1}) = +1, q_{os}(x_{2i}) = -1$. This quadratic form q_e is a nonorientable analogy of Rokhlin quadratic form.

A diffeomorphism ϕ over N_g is *e–extendable* if there is an orientation-preserving diffeomorphism Φ of S^4 such that the following diagram is commutative:

$$\begin{array}{ccc} N_g & \xrightarrow{e} & S^4 \\ \phi\big\downarrow & & \big\downarrow\Phi \\ N_g & \xrightarrow{e} & S^4. \end{array}$$

If the diffeomorphisms ϕ_1 over N_g is e–extendable, and ϕ_1 is isotopic to ϕ_2, then ϕ_2 is e–extendable. Therefore, e–extendability is a property about isotopy classes of diffeomorphisms over N_g. The group $\mathcal{M}(N_g)$ of isotopy classes of all diffeomorphisms over N_g is called the *mapping class group of* N_g. An element ϕ of $\mathcal{M}(N_g)$ is *e–extendable* if there is an e–extendable representative of ϕ. By the definition of q_e, we can see that if $\phi \in \mathcal{M}(N_g)$ is e–extendable then ϕ preserves q_e, ie $q_e(\phi_*(x)) = q_e(x)$ for every $x \in H_1(N_g; \mathbb{Z}_2)$. What we would like to know is whether $\phi \in \mathcal{M}(N_g)$ is e–extendable when ϕ preserves q_e. The answer to this problem would be depend on the embedding e. In this paper, we consider the case where e is the o–standard embedding.

3 Generators for $\mathcal{M}(N_g)$

A simple closed curve c on N_g is an A–circle (resp. an M–circle), if the tubular neighborhood of c is an annulus (resp. a Möbius band). We denote by t_c the Dehn twist about an A–circle c on N_g. In each figure, we indicate the direction of a Dehn twist by an arrow. Lickorish [8; 9] showed that Dehn twists and *Y–homeomorphisms* generate $\mathcal{M}(N_g)$. We review the definition of Y–homeomorphism. Let m be an M–circle and a be an oriented A–circle in N_g such that m and a transversely intersect in one point. Let $K \subset N_g$ be a regular neighborhood of $m \cup a$, which is a union of the tubular neighborhoods of m and a and then is homeomorphic to the Klein bottle with a hole. Let M be a regular neighborhood of m. We denote by $Y_{m,a}$ a homeomorphism over N_g which is described as the result of pushing M once along a keeping the boundary of K fixed (see Figure 3). We call $Y_{m,a}$ a Y–homeomorphism.

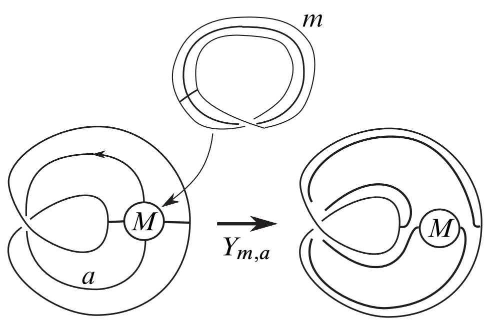

Figure 3. M with circle indicates a place where to attach a Möbius band

Szepietowski [13] showed an interesting results on the proper subgroup of $\mathcal{M}(N_g)$ generated by all Y–homeomorphisms.

Theorem 3.1 [13] $\Gamma_2(N_g) = \{\phi \in \mathcal{M}(N_g) \mid \phi_*\colon H_1(N_g; \mathbb{Z}_2) \to H_1(N_g; \mathbb{Z}_2) = \mathrm{id}\}$ *is generated by Y–homeomorphisms.*

Chillingworth [2] showed that $\mathcal{M}(N_g)$ is finitely generated. In this paper, we use the system of generators of $\mathcal{M}(N_g)$ listed by Szepietowski [13]:

Theorem 3.2 [13, Theorem 3.3] *Let $a_1, \dots, a_{g-1}$, b_j $(1 \leq j \leq g/2)$ and m_{g-1} be circles shown in Figure 4. Then $t_{a_1}, \dots, t_{a_{g-1}}, t_{b_j}$ $(1 \leq j \leq g/2)$, $Y_{m_{g-1},a_{g-1}}$ generate $\mathcal{M}(N_g)$.*

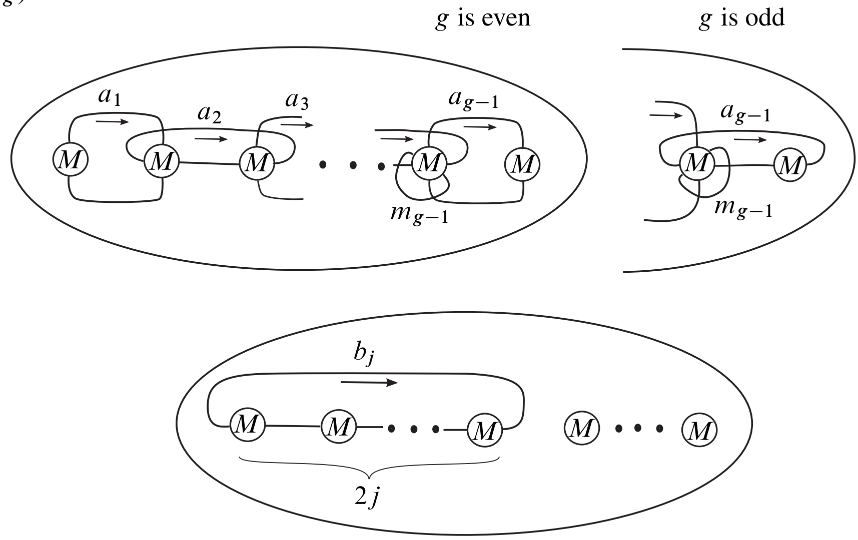

Figure 4. Generators for $\mathcal{M}(N_g)$

Remark 3.3 If $g = 1$, then $\mathcal{M}(N_1)$ is trivial, hence Theorem 1.2 is valid. From here to the end of this paper, we assume $g \geq 2$.

The i–th band is the Möbius band on which the circle x_i in Figure 1 goes across. Let c_i $(i = 1, \dots, g-3)$ be a simple closed curve shown in the top of Figure 5. Let $i, j = 1, \dots, g$ such that $i \neq j$. When $i < j$ (resp. $i > j$), we define $Y_{i,j} = Y_{m,a}$, where m and a are as shown in the bottom left (resp. the bottom right) of Figure 5. Let $\mathcal{YS}_g$ be the subgroup of $\mathcal{M}(N_g)$ generated by all $Y_{i,j}$.

Lemma 3.4 *$\mathcal{YS}_g$ and $t_{a_1}, \dots, t_{a_{g-1}}$, $t_{c_1}, \dots, t_{c_{g-3}}$ generate $\mathcal{M}(N_g)$.*

Proof It suffices to show that t_{b_j} is a product of $t_{a_1}, \dots, t_{a_{g-1}}$, $t_{c_1}, \dots, t_{c_{g-3}}$. When $j = 2$, $t_{b_2} = t_{c_1}$. When $j \geq 3$, by the lantern relation (discovered by Dehn and rediscovered by Johnson [7]), $t_{e_1}t_{e_2}t_{e_3} = t_p t_{a_{2j-3}} t_{a_{2j-1}} t_{b_j}$, where p, e_1, e_2 and e_3 are circles shown in Figure 6, hence $t_{b_j} = (t_p t_{a_{2j-3}} t_{a_{2j-1}})^{-1} t_{e_1}t_{e_2}t_{e_3}$. Since $e_1 = t_{a_{2j-2}} t_{a_{2j-3}} t_{a_{2j-1}} t_{a_{2j-2}}(e_3)$, we see t_{e_3} is the product of $(t_{a_{2j-2}} t_{a_{2j-3}} t_{a_{2j-1}} t_{a_{2j-2}})^{-1}$ and $t_{e_1} t_{a_{2j-2}} t_{a_{2j-3}} t_{a_{2j-1}} t_{a_{2j-2}}$. If $j = 3$, then $e_1 = c_1$, $e_2 = c_3$ and $p = a_1$. Therefore, t_{b_3} is a product of $t_{a_1}, \dots, t_{a_{g-1}}$, $t_{c_1}, \dots, t_{c_{g-3}}$. If $j \geq 4$, then $e_1 = b_{j-1}$, $e_2 = c_{2j-3}$, and $p = b_{j-2}$. By the induction on j, we see that t_{b_j} is a product of $t_{a_1}, \dots, t_{a_{g-1}}$, $t_{c_1}, \dots, t_{c_{g-3}}$. □

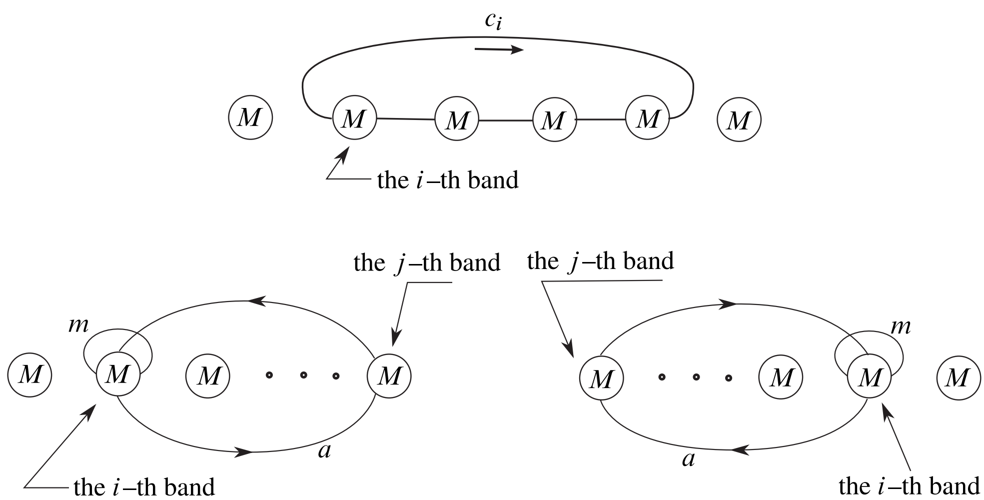

Figure 5. The circle c_i and the Y–homeomorphism $Y_{i,j}$

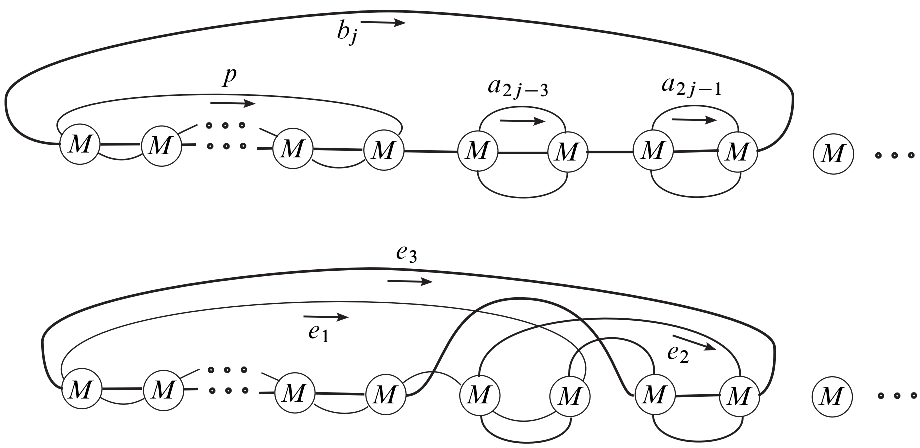

Figure 6. When $j = 3$, we drop the second and third M with circles.

Remark 3.5 If $g = 2, 3$, then this lemma reads $\mathcal{YS}_g$ and $t_{a_1}, \dots, t_{a_{g-1}}$ generate $\mathcal{M}(N_g)$.

4 Generators for subgroup of $\mathcal{M}(N_g)$ preserving q_{os}

In this section, we find a finite system of generators for

$$\mathcal{N}_g(q_{os}) = \left\{\phi \in \mathcal{M}(N_g) \;\middle|\; q_{os}(\phi_*(x)) = q_{os}(x) \text{ for every } x \in H_1(N_g; \mathbb{Z}_2)\right\}$$

and prove the main theorem (Theorem 1.2) of this paper.

We introduce a group

$$\mathcal{O}_g(q_{os}) = \{A \in \mathrm{Aut}(H_1(N_g;\mathbb{Z}_2)) \mid q_{os}(A(x)) = q_{os}(x) \text{ for every } x \in H_1(N_g;\mathbb{Z}_2)\}.$$

Then we have a natural short exact sequence

$$1 \to \Gamma_2(N_g) \to \mathcal{N}_g(q_{os}) \to \mathcal{O}_g(q_{os}) \to 1. \tag{1}$$

Since $\Gamma_2(N_g)$ is a finite index subgroup of $\mathcal{M}(N_g)$ and $\mathcal{O}_g(q_{os})$ is a finite group, there exists a finite system of generators for $\mathcal{N}_g(q_{os})$. We find a system of generators explicitly.

Theorem 4.1 *The group $\mathcal{N}_g(q_{os})$ is generated by $\mathcal{YS}_g$, $t_{a_1}^2, \dots, t_{a_{g-1}}^2$, $t_{c_1}^2, \dots, t_{c_{g-3}}^2$, $t_{d_1}, \dots, t_{d_{g-2}}$, and $t_{a_1}t_{a_3}t_{c_1}, \dots, t_{a_{g-3}}t_{a_{g-1}}t_{c_{g-3}}$, where d_i is illustrated in Figure 7.*

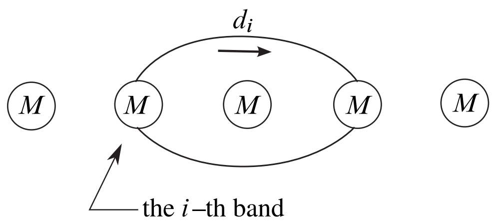

Figure 7. The circle d_i

Remark 4.2 If $g = 2$, this theorem reads $\mathcal{N}_2(q_{os})$ is generated by $\mathcal{YS}_2$ and $t_{a_1}^2 = \mathrm{id}_{N_2}$. If $g = 3$, this theorem reads $\mathcal{N}_3(q_{os})$ is generated by $\mathcal{YS}_3$, $t_{a_1}^2$, $t_{a_2}^2$ and t_{d_1}.

Proof of Theorem 1.2 By the definition of q_{os}, if a diffeomorphism ϕ over N_g is os–extendable then ϕ preserves q_{os}.

Conversely, we assume that ϕ preserves q_{os}. Then ϕ is an element of $\mathcal{N}_g(q_{os})$. Therefore, if each generator of $\mathcal{N}_g(q_{os})$ is os–extendable then ϕ is os–extendable.

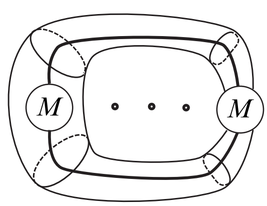

Figure 8. Sliding the left Möbius band along this tube is an extension of $Y_{i,j}$.

Since a sliding of a Möbius band along the tube illustrated in Figure 8 is an extension of $Y_{i,j}$, $Y_{i,j}$ is os–extendable, hence every element of $\mathcal{YS}_g$ is os–extendable. Since the regular neighborhoods of a_i and c_i are annuli trivially embedded in the equator S^3 of S^4, $t_{a_i}^2$, $t_{c_i}^2$ are os–extendable by the same argument as in the introduction of [5]. Since the regular neighborhoods of d_i is a Hopf band embedded in the equator S^3 of S^4, t_{d_i} are os–extendable by the same argument as the proof of [5, Proposition 2.1]. Finally, by using the same argument as showing the extendability of "$C_1C_3C_5$" in the proof of [4, Lemma 2.2], we show that $t_{a_i}t_{a_{i+2}}t_{c_i}$ is os–extendable. □

As shown in the above proof of Theorem 1.2, any elements of $\mathcal{YS}_g$, $t_{a_1}^2, \dots, t_{a_{g-1}}^2$, $t_{c_1}^2, \dots, t_{c_{g-3}}^2$, $t_{d_1}, \dots, t_{d_{g-2}}$, and $t_{a_1}t_{a_3}t_{c_1}, \dots, t_{a_{g-3}}t_{a_{g-1}}t_{c_{g-3}}$ are os–extendable, hence these elements preserve q_{os}. Therefore, in order to prove Theorem 4.1, we should see that every element of $\mathcal{N}_g(q_{os})$ is a product of these elements.

4.1 Short-leg Y–homeomorphisms

For a Y–homeomorphism $Y_{m,a}$, we call m the *leg* of $Y_{m,a}$ and a the *arm* of $Y_{m,a}$. A Y–homeomorphism is called a *short-leg* Y–homeomorphism, if its leg is one of $x_1, \dots, x_g$ illustrated in Figure 1.

Lemma 4.3 *Every short-leg Y–homeomorphism is an element of $\mathcal{YS}_g$.*

Proof We review the *crosscap pushing map* defined in [14]. Fix $p_0 \in N_{g-1}$ and define $\mathcal{M}(N_{g-1}, p_0)$ be the group of isotopy classes of diffeomorphisms over N_{g-1} preserving p_0. Let U be a 2–disk embedded in N_{g-1} such that the center of U is p_0. We parametrize U by $\{(x,y) \mid x^2+y^2 \leq 1\}$ such that $(0,0) = p_0$. Under this parametrization, we define an orientation reversing diffeomorphism $r\colon U \to U$ by $r(x,y) = (x,-y)$. We define a homomorphism j from $\pi_1(N_{g-1}, p_0)$ to $\mathcal{M}(N_{g-1}, p_0)$ such that, for a loop γ in N_g based at x_0 and an element $[\gamma] \in \pi_1(N_{g-1}, x_0)$, $j([\gamma])$ is a diffeomorphism over N_g obtained as the effect of pushing p_0 once along γ. This homomorphism j is a homomorphism in a nonorientable analogy of the Birman exact sequence [1]. We define a homomorphism φ from $\mathcal{M}(N_{g-1}, p_0)$ to $\mathcal{M}(N_g)$ as follows. We represent $h \in \mathcal{M}(N_{g-1}, p_0)$ by a diffeomorphism h over N_g such that $h(U) = U$ and $h|_U = \mathrm{id}_U$ or $h|_U = r$. We construct N_g from $N_{g-1} - \mathrm{int}\, U$ by attaching a Möbius band along ∂U. Here we assume that this Möbius band is the i–th band on N_g. We extend $h|_{N_{g-1}-\mathrm{int}\, U}$ to a diffeomorphism $\varphi(h)$ over N_g naturally. The homomorphism $\psi = \varphi \circ j$ is called a crosscap pushing map.

Every short-leg Y–homeomorphism $Y_{x_i,a}$ is in $\psi(\pi_1(N_{g-1}, p_0))$, $\pi_1(N_{g-1}, p_0)$ is generated by the loops $l_{i,j}$'s indicated in Figure 9, and $\psi(l_{i,j}) = Y_{i,j}$, hence $Y_{x_i,a}$ is a product of $Y_{i,j}$'s. □

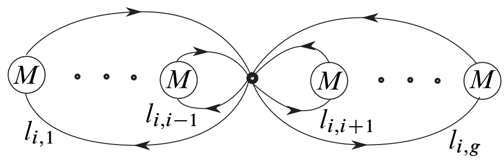

Figure 9. The generators for $\pi_1(N_g, p_0)$

Let G_g be the subgroup of $\mathcal{M}(N_g)$ generated by $\mathcal{YS}_g$, $t_{a_1}^2, \ldots, t_{a_{g-1}}^2$, $t_{c_1}^2, \ldots, t_{c_{g-3}}^2$, $t_{d_1}, \ldots, t_{d_{g-2}}$, $t_{a_1}t_{a_3}t_{c_1}, \ldots, t_{a_{g-3}}t_{a_{g-1}}t_{c_{g-3}}$. We have already shown that $G_g \subset \mathcal{N}_g(q_{os})$, therefore, what we should show is $\mathcal{N}_g(q_{os}) \subset G_g$. Two Y–homeomorphisms Y_1 and Y_2 are G_g*–equivalent* if there is an element ϕ of G_g such that $\phi Y_1 \phi^{-1} = Y_2$. We remark that if $Y_1 = Y_{m,a}$ and $Y_2 = \phi Y_1 \phi^{-1}$ then $Y_2 = Y_{\phi(m),\phi(a)}$. We will show:

Lemma 4.4 *Every Y–homeomorphism is a product of Y–homeomorphisms which are G_g–equivalent to short-leg Y–homeomorphisms.*

By Lemma 4.3 and Lemma 4.4, we see that every Y–homeomorphism is an element of G_g. Therefore, by Theorem 3.1, we conclude:

Corollary 4.5 $\Gamma_2(N_g) \subset G_g$. □

Remark 4.6 While the author was writing this paper, Błażej Szepietowski informed the author that he found a finite system of generators for $\Gamma_2(N_g)$. In the next subsection, we introduce his system of generators and prove Lemma 4.4 by using his result. In this subsection, we show Lemma 4.4 by our original proof.

As shown in Figure 10, we use the symbol $\oplus$ (resp. $\ominus$) to indicate the place where the

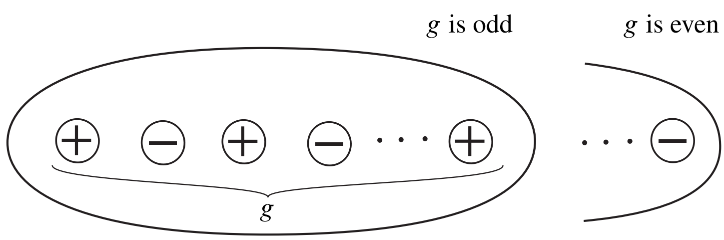

Figure 10. Diagram indicating o–standard N_g in S^4

Möbius band are attached such that $q_{os}(x_i) = +1$ (resp. $q_{os}(x_i) = -1$) for the circle x_i indicated in Figure 1. We denote an element $x = \sum_{i=1}^{g} \epsilon_i x_i \in H_1(N_g; \mathbb{Z}_2)$, where $\epsilon_i = 0$ or 1, by a sequence of symbols $+, -, \oplus, \ominus$ of length g with [,] which are settled by the rule: if $\epsilon_{2i-1}=0$ then the $(2i-1)$–st symbol is $+$, if $\epsilon_{2i-1}=1$ then the $(2i-1)$–st symbol is $\oplus$, if $\epsilon_{2i}=0$ then the $2i$–th symbol is $-$, and if $\epsilon_{2i}=1$ then the $2i$–th symbol is $\ominus$. For example, when $g=7$, we denote an element $x_2+x_3+x_6+x_7$ by $[+\ominus\oplus-+\ominus\oplus]$. This sequence is called the *r–sequence* associated to x. For the r–sequence associated to x, we settle a simple closed curve on N_g by the following rule. For the symbols in this sequence, we put arcs on N_g indicated in the bottom of

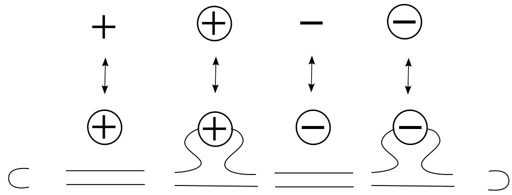

Figure 11. Parts of r–circles

Figure 11, glue them along the boundaries, and cap by the arc indicated on the left of Figure 11 from the left and by the arc indicated on the right of Figure 11 from the right. We call this circle the *r–circle* associated to x and denote by $R(x)$. For an element $x=\sum_{i=1}^{g}\epsilon_i x_i \in H_1(N_g;\mathbb{Z}_2)$, where $\epsilon_i=0$ or 1, we define $\mathrm{supp}(x)=\{x_i \mid \epsilon_i=1\}$.

Two simple closed curves c_1 and c_2 on N_g are *G_g–equivalent* ($c_1\sim_{G_g}c_2$) if there is an element ϕ of G_g such that $\phi(c_1)=c_2$.

Lemma 4.7 *If $g=1$, then every r–circle is G_g–equivalent to $R([+])$ or $R([\oplus])$. If $g=2$, then every r–circle is G_g–equivalent to $R([+-])$, $R([\oplus-])$, $R([+\ominus])$ or $R([\oplus\ominus])$. If $g\geq 3$ is odd, then every r–circle is G_g–equivalent to $R([+-+-\cdots+])$, $R([\oplus-+-\cdots+])$, $R([+\ominus+-\cdots+])$, $R([\oplus\ominus+-\cdots+])$, $R([\oplus-\oplus-\cdots+])$ or $R([\oplus\ominus\oplus\ominus\cdots\oplus])$. If $g\geq 4$ is even, then every r–circle is G_g–equivalent to $R([+-+-\cdots-])$, $R([\oplus-+-\cdots-])$, $R([+\ominus+-\cdots-])$, $R([\oplus\ominus+-\cdots-])$, $R([\oplus-\oplus-\cdots-])$ or $R([\oplus\ominus\oplus\ominus\cdots\ominus])$.*

Proof If $g=1$ or 2, then the conclusion is trivial.

If $g\geq 3$, then

$$
\begin{aligned}
R([\cdots-+\ominus\cdots]) &\sim_{G_g} R([\cdots\ominus+-\cdots]),\\
R([\cdots+-\oplus\cdots]) &\sim_{G_g} R([\cdots\oplus-+\cdots]),\\
R([\cdots-\oplus\ominus\cdots]) &\sim_{G_g} R([\cdots\ominus\oplus-\cdots]),\\
R([\cdots+\ominus\oplus\cdots]) &\sim_{G_g} R([\cdots\oplus\ominus+\cdots]),
\end{aligned}
\tag{2}
$$

where the leftmost symbols are the i–th symbol, since

$$Y_{i+2,i}Y_{i+1,i}t_{d_i}R([\cdots - + \ominus \cdots]) = R([\cdots \ominus + - \cdots]),$$
$$Y_{i+2,i}Y_{i+1,i}t_{d_i}R([\cdots + - \oplus \cdots]) = R([\cdots \oplus - + \cdots]),$$
$$Y_{i+2,i+1}Y_{i+1,i+2}t_{d_i}R([\cdots - \oplus \ominus \cdots]) = R([\cdots \ominus \oplus - \cdots]),$$
$$Y_{i+2,i+1}Y_{i+1,i+2}t_{d_i}R([\cdots + \ominus \oplus \cdots]) = R([\cdots \oplus \ominus + \cdots]).$$

Therefore, when $g = 3$, for two cases $R([+-\oplus])$ and $R([+\ominus\oplus])$ which are not listed in the statement, we see $R([+-\oplus]) \sim_{G_g} R([\oplus-+])$ and $R([+\ominus\oplus]) \sim_{G_g} R([\oplus\ominus+])$.

If $g \geq 4$, then

$$\begin{aligned} R([\cdots - \oplus \ominus \oplus \cdots]) &\sim_{G_g} R([\cdots - \oplus - + \cdots]),\\ R([\cdots + \ominus \oplus \ominus \cdots]) &\sim_{G_g} R([\cdots + \ominus + - \cdots]),\\ R([\cdots \ominus \oplus \ominus + \cdots]) &\sim_{G_g} R([\cdots - + \ominus + \cdots]),\\ R([\cdots \oplus \ominus \oplus - \cdots]) &\sim_{G_g} R([\cdots + - \oplus - \cdots]),\\ R([\cdots - \oplus - \oplus \cdots]) &\sim_{G_g} R([\cdots \ominus + \ominus + \cdots]),\\ R([\cdots + \ominus + \ominus \cdots]) &\sim_{G_g} R([\cdots \oplus - \oplus - \cdots]), \end{aligned} \tag{3}$$

where the leftmost symbols are the i–th symbol, since

$$Y_{i+3,i+1}Y_{i+2,i+1}t_{a_i}^{-2}(t_{a_i}t_{a_{i+2}}t_{c_i})R([\cdots - \oplus \ominus \oplus \cdots]) = R([\cdots - \oplus - + \cdots]),$$
$$Y_{i+3,i+1}Y_{i+2,i+1}t_{a_i}^{-2}(t_{a_i}t_{a_{i+2}}t_{c_i})R([\cdots + \ominus \oplus \ominus \cdots]) = R([\cdots + \ominus + - \cdots]),$$
$$Y_{i,i+2}Y_{i+1,i+2}t_{a_{i+2}}^{2}(t_{a_i}t_{a_{i+2}}t_{c_i})^{-1}R([\cdots \ominus \oplus \ominus + \cdots]) = R([\cdots - + \ominus + \cdots]),$$
$$Y_{i,i+2}Y_{i+1,i+2}t_{a_{i+2}}^{2}(t_{a_i}t_{a_{i+2}}t_{c_i})^{-1}R([\cdots \oplus \ominus \oplus - \cdots]) = R([\cdots + - \oplus - \cdots]),$$
$$Y_{i+1,i}(t_{a_i}t_{a_{i+2}}t_{c_i})Y_{i+2,i+3}^{-1}R([\cdots - \oplus - \oplus \cdots]) = R([\cdots \ominus + \ominus + \cdots]),$$
$$Y_{i+1,i}(t_{a_i}t_{a_{i+2}}t_{c_i})Y_{i+2,i+3}^{-1}R([\cdots + \ominus + \ominus \cdots]) = R([\cdots \oplus - \oplus - \cdots]).$$

When $g \geq 4$, we get our conclusion by the induction on g and G_g–equivalences (2) and (3). When $g \geq 4$ is even, by the induction hypothesis, every r–circle of length $g-1$ is G_{g-1}–equivalent to $R([+-+-\cdots+])$, $R([\oplus-+-\cdots+])$, $R([+\ominus+-\cdots+])$, $R([\oplus\ominus+-\cdots+])$, $R([\oplus-\oplus-\cdots+])$ or $R([\oplus\ominus\oplus\ominus\cdots\oplus])$, therefore every r–circle of length g is G_g equivalent to

(i) $R([+-+-\cdots+-])$,

(ii) $R([+-+-\cdots+\ominus])$,

(iii) $R([\oplus-+-\cdots+-])$,

(iv) $R([\oplus-+-\cdots+\ominus])$,

(v) $R([+\ominus+-\cdots+-])$,

(vi) $R([+\ominus+-\cdots+\ominus])$,

(vii) $R([\oplus\ominus+-\cdots+-])$,

(viii) $R([\oplus\ominus+-\cdots+\ominus])$,

(ix) $R([\oplus-\oplus-\cdots+-])$,

(x) $R([\oplus-\oplus-\cdots+\ominus])$,

(xi) $R([\oplus\ominus\oplus\ominus\cdots\oplus-])$,

(xii) $R([\oplus\ominus\oplus\ominus\cdots\oplus\ominus])$.

By applying G_g–equivalences (2) and (3), we see the elements (ii), (iv), (vi), (viii), (x) and (xi) are G_g–equivalent to one of other six elements. For example,

$$\begin{aligned}(\text{viii}) = R([\oplus\ominus+-\cdots+\ominus]) &\sim_{G_g} R([\oplus\ominus+\ominus\cdots+-])\\ &\sim_{G_g} R([+\ominus\oplus\ominus\cdots+-])\\ &\sim_{G_g} R([+\ominus+-\cdots+-]) = (\text{v}).\end{aligned}$$

By the same method as above, we get our conclusion when $g \geq 4$ is odd. □

If the complement of an M–circle m is orientable, then any circle intersecting m transversely in one point is an M–circle. Therefore the leg of every Y–homeomorphism is an M–circle whose complement is nonorientable. Every element of G_g preserves q_{os}, the r–circles $R([+-\cdots\pm])$, $R([\oplus-\oplus-\cdots\pm])$ and $R([\oplus\ominus+\cdots\pm])$ are A–circles, and the complements of $R([\oplus\ominus\cdots\oplus])$ and $R([\oplus\ominus\cdots\ominus])$ are orientable, hence:

Corollary 4.8 *If an r–circle $R(x)$ is a leg of a Y–homeomorphism, then $R(x)$ is G_g–equivalent to $R([\oplus-+\cdots])$ or $R([+\ominus+\cdots])$.* □

By investigating the action of generators for $\mathcal{M}(N_g)$ listed in Lemma 3.4 on legs of Y–homeomorphisms, we see:

Lemma 4.9 *Every Y–homeomorphism is a product of Y–homeomorphisms whose legs are r–circles.*

Proof Since $\{x_1,\ldots,x_g\}$ are r–circles, $Y_{i,j}$ is a Y–homeomorphism whose leg is an r–circle. For every Y–homeomorphism $Y_{m,a}$, there is an r–circle s and an element $\phi\in\mathcal{M}(N_g)$ such that $\phi(s)=m$, that is, $Y_{m,a}=\phi Y_{s,\phi^{-1}(a)}\phi^{-1}$. Therefore, by Lemma 3.4, it suffices to show that for every r–circle s there are $\phi_i,\phi_i',\psi_i,\psi_i'\in\mathcal{YS}_g$ and r–circles s_i,s_i',t_i,t_i' such that $t_{a_i}(s)=\phi_i(s_i)$, $t_{a_i}^{-1}(s)=\phi_i'(s_i')$, $t_{c_i}(s)=\psi_i(t_i)$

and $t_{c_i}^{-1}(s) = \psi_i'(t_i')$. As observed in the proof of [13, Lemma 3.1], Y_{x_i,a_i} preserves a_i and exchanges the sides of a_i, hence $t_{a_i} = Y_{x_i,a_i} t_{a_i}^{-1} Y_{x_i,a_i}^{-1}$, therefore $t_{a_i}^2 = t_{a_i} Y_{x_i,a_i} t_{a_i}^{-1} Y_{x_i,a_i}^{-1} = Y_{t_{a_i}(x_i),a_i} Y_{x_i,a_i}^{-1} = Y_{i+1,i} Y_{i,i+1}^{-1}$. By the same way as above, we see that $t_{c_i} = Y_{x_i,c_i} t_{c_i}^{-1} Y_{x_i,c_i}^{-1}$, therefore $t_{c_i}^2 = t_{c_i} Y_{x_i,c_i} t_{c_i}^{-1} Y_{x_i,c_i}^{-1} = Y_{t_{c_i}(x_i),c_i} Y_{x_i,c_i}^{-1}$. Since $Y_{g,i+3} \cdots Y_{i+4,i+3} Y_{i,i+1} \cdots Y_{1,i+1} t_{c_i}(x_i)$ is isotopic to $R(x_{i+1}+x_{i+2}+x_{i+3})$, $t_{c_i}^2$ is a product of Y–homeomorphisms whose legs are r–circles. From the above observation, it suffices to show that one of $t_{a_i}(s) = \phi_i(s_i)$, $t_{a_i}^{-1}(s) = \phi_i'(s_i')$, and one of $t_{c_i}(s) = \psi_i(t_i)$, $t_{c_i}^{-1}(s) = \psi_i'(t_i')$.

Since a_i does not intersects $R(x)$ such that $\mathrm{supp}(x) \cap \{x_i, x_{i+1}\} = \varnothing$, we only consider the action of t_{a_i} on $R(x)$ such that $\mathrm{supp}(x) \cap \{x_i, x_{i+1}\} \neq \varnothing$. When we consider the action of t_{a_i} and Y–homeomorphisms, we do not need to take care of the sign on the Möbius bands. Hence, in symbols of r–sequences, we change $+$ and $-$ into $\times$, and $\oplus$ and $\ominus$ into $\otimes$. There are 3 cases to consider: $R([\cdots \otimes \times \cdots])$, $R([\cdots \times \otimes \cdots])$, and $R([\cdots \otimes \otimes \cdots])$, where the i–th and $(i+1)$–st symbols are indicated. The third r–circle does not intersect a_i, hence we ignore this. By drawing figures of r–circles, we see: $Y_{i,i+1}(t_{a_i}^{-1}(R([\cdots \otimes \times \cdots]))) = R([\cdots \times \otimes \cdots])$ and $Y_{i+1,i}(t_{a_i}(R([\cdots \times \otimes \cdots]))) = R([\cdots \otimes \times \cdots])$.

By the same reasons as in the previous paragraph, it suffice to consider the action of t_{c_i} on $R(x)$ such that $\mathrm{supp}(x) \cap \{x_i, x_{i+1}, x_{i+2}, x_{i+3}\} \neq \varnothing$, and, in symbols of r–sequences, we change $+$ and $-$ into $\times$, and $\oplus$ and $\ominus$ into $\otimes$. There are 15 cases to consider:

(1) $R([\cdots \otimes \times \times \times \cdots])$,
(2) $R([\cdots \times \otimes \times \times \cdots])$,
(3) $R([\cdots \otimes \otimes \times \times \cdots])$,
(4) $R([\cdots \times \times \otimes \times \cdots])$,
(5) $R([\cdots \otimes \times \otimes \times \cdots])$,
(6) $R([\cdots \times \otimes \otimes \times \cdots])$,
(7) $R([\cdots \otimes \otimes \otimes \times \cdots])$,
(8) $R([\cdots \times \times \times \otimes \cdots])$,
(9) $R([\cdots \otimes \times \times \otimes \cdots])$,
(10) $R([\cdots \times \otimes \times \otimes \cdots])$,
(11) $R([\cdots \otimes \otimes \times \otimes \cdots])$,
(12) $R([\cdots \times \times \otimes \otimes \cdots])$,
(13) $R([\cdots \otimes \times \otimes \otimes \cdots])$,
(14) $R([\cdots \times \otimes \otimes \otimes \cdots])$,
(15) $R([\cdots \otimes \otimes \otimes \otimes \cdots])$,

where the i–th, $(i+1)$–st, $(i+2)$–nd and $(i+3)$–rd symbols are indicated. Since (3), (6), (12) and (15) do not intersect c_i, t_{c_i} does not change these r–circles. By drawing figures of r–circles (for example, Figure 12 indicates (9)), we see:

(1) $Y_{i,i+1}Y_{i+2,i+3}Y_{i+1,i+3}Y_{i+1,i+2}^{-1}(t_{c_i}^{-1}(R([\cdots\otimes\times\times\times\cdots])))$
$= R([\cdots\times\otimes\otimes\otimes\cdots])$,

(2) $Y_{i+1,i}Y_{i+2,i+3}(t_{c_i}(R([\cdots\times\otimes\times\times\cdots]))) = R([\cdots\otimes\times\otimes\otimes\cdots])$,

(4) $Y_{i+2,i+3}Y_{i+1,i}(t_{c_i}^{-1}(R([\cdots\times\times\otimes\times\cdots]))) = R([\cdots\otimes\otimes\times\otimes\cdots])$,

(5) $Y_{i+1,i+2}Y_{i,i+2}Y_{i+3,i+2}^{-1}Y_{i+2,i+3}^{-1}Y_{i,i+3}^{-1}(t_{c_i}^{-1}(R([\cdots\otimes\times\otimes\times\cdots])))$
$= R([\cdots\otimes\times\otimes\times\cdots])$,

(7) $Y_{i,i+3}Y_{i+1,i+3}Y_{i+2,i+3}(t_{c_i}^{-1}(R([\cdots\otimes\otimes\otimes\times\cdots]))) = R([\cdots\times\times\times\otimes\cdots])$,

(8) $Y_{i+3,i+2}Y_{i+1,i}Y_{i+2,i}Y_{i+2,i+1}^{-1}(t_{c_i}(R([\cdots\times\times\times\otimes\cdots]))) = R([\cdots\otimes\otimes\otimes\times\cdots])$,

(9) $Y_{i+2,i+3}Y_{i+1,i}Y_{i+3,i}Y_{i+3,i+1}^{-1}Y_{i+3,i+2}Y_{i+2,i+3}Y_{i+1,i+3}Y_{i+1,i+2}^{-1}Y_{i,i+3}$
$\cdot Y_{i,i+2}^{-1}Y_{i,i+1}(t_{c_i}^{-1}(R([\cdots\otimes\times\times\otimes\cdots]))) = R([\cdots\otimes\times\times\otimes\cdots])$,

(10) $Y_{i+2,i+1}Y_{i+3,i+1}Y_{i,i+3}^{-1}Y_{i+1,i}^{-1}Y_{i+3,i}^{-1}(t_{c_i}(R([\cdots\times\otimes\times\otimes\cdots])))$
$= R([\cdots\times\otimes\times\otimes\cdots])$,

(11) $Y_{i,i+2}Y_{i+1,i+2}Y_{i+3,i+2}Y_{i+2,i}Y_{i+2,i+1}^{-1}(t_{c_i}(R([\cdots\otimes\otimes\times\otimes\cdots])))$
$= R([\cdots\times\times\otimes\times\cdots])$,

(13) $Y_{i+3,i+1}Y_{i+2,i+1}Y_{i,i+1}Y_{i+1,i+3}Y_{i+1,i+2}^{-1}(t_{c_i}^{-1}(R([\cdots\otimes\times\otimes\otimes\cdots])))$
$= R([\cdots\times\otimes\times\times\cdots])$,

(14) $Y_{i+3,i}Y_{i+2,i}Y_{i+1,i}(t_{c_i}(R([\cdots\times\otimes\otimes\otimes\cdots]))) = R([\cdots\otimes\times\times\times\cdots])$. $\square$

Proof of Lemma 4.4 Let $Y_{m,a}$ be a Y–homeomorphism whose leg is an r–circle. By Corollary 4.8, there is an element $\phi \in G_g$ such that $\phi(m) = R([\oplus - + \cdots])$ or $R([+\ominus+\cdots])$. Therefore $Y_{m,a}$ is G_g–equivalent to a short-leg Y–homeomorphism. By Lemma 4.9, we get our conclusion. $\square$

4.2 Szepietowski's generators for $\Gamma_2(N_g)$

We review the finite system of generators for $\Gamma_2(N_g)$ introduced in [14]. For each nonempty subset $I = \{i_1, i_2, \ldots, i_k\}$ of $\{1, \ldots, g\}$, let α_I be the simple closed curve shown in Figure 13. If $I = \{i\}$, we write α_i instead of $\alpha_{\{i\}}$.

Theorem 4.10 (Szepietowski [14, Theorem 3.2]) *For $g \geq 4$, $\Gamma_2(N_g)$ is generated by the following elements.*

(1) $Y_{\alpha_i,\alpha_{\{i,j\}}}$ *for* $i \neq j$,

(2) $Y_{\alpha_{\{i,j,k\}},\alpha_{\{i,j,k,l\}}}$ *for* $i < j < k < l$.

The group $\Gamma_2(N_3)$ is generated by the elements in (1).

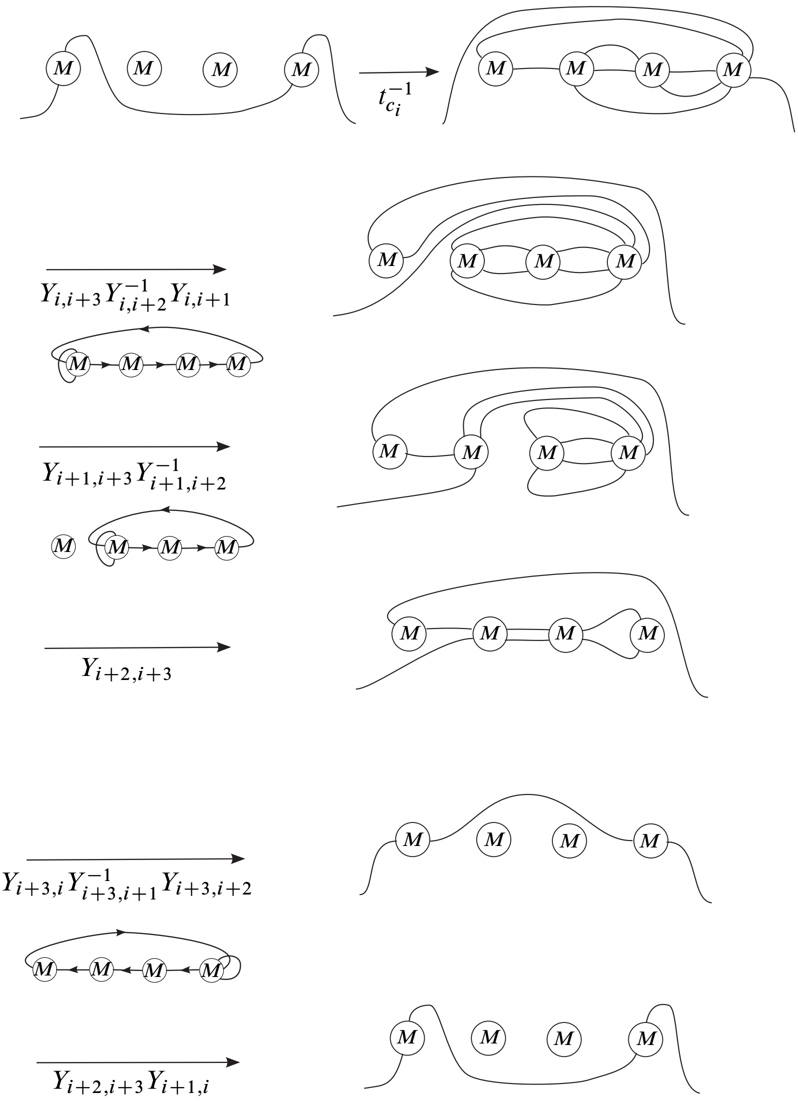

Figure 12

We show:

Lemma 4.11 *For arbitrary $i < j < k < l$, $Y_{\alpha_{\{i,j,k\}},\alpha_{\{i,j,k,l\}}}$ is G_g–equivalent to a short-leg Y–homeomorphism.*

Proof It suffices to show that, for every $i < j < k$, $\alpha_{\{i,j,k\}}$ is G_g–equivalent to α_1 or α_2. By drawing figures, we see when $i > 2$, $Y_{i,i-2}^{-1}Y_{i-1,i-2}^{-1}t_{d_{i-2}}\alpha_{\{i,j,k\}} = \alpha_{\{i-2,j,k\}}$

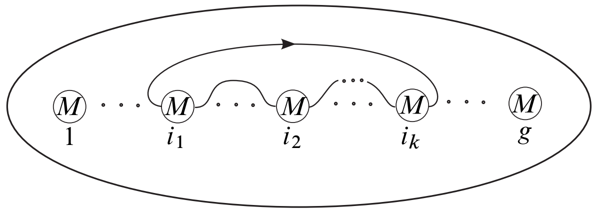

Figure 13. The curve α_I for $I = \{i_1, i_2, \dots, i_k\}$

and when $i < j-2$, $Y_{j,j-2}^{-1} Y_{j-1,j-2}^{-1} t_{d_{j-2}} \alpha_{\{i,j,k\}} = \alpha_{\{i,j-2,k\}}$; when $j < k-2$, $Y_{k,k-2}^{-1} Y_{k-1,k-2}^{-1} t_{d_{k-2}} \alpha_{\{i,j,k\}} = \alpha_{\{i,j,k-2\}}$. By applying the above action of G_g on $\alpha_{\{i,j,k\}}$, we see that $\alpha_{\{i,j,k\}}$ is G_g–equivalent to $\alpha_{\{1,3,4\}}$, $\alpha_{\{1,2,3\}}$, $\alpha_{\{2,3,5\}}$, $\alpha_{\{2,4,6\}}$, $\alpha_{\{1,3,5\}}$, $\alpha_{\{1,2,4\}}$, $\alpha_{\{2,3,4\}}$ or $\alpha_{\{2,4,5\}}$. By drawing figures of the action of G_g on the above 8 circles, we can check that former 4 circles are G_g–equivalent to α_1 and last 4 circles are G_g–equivalent to α_2. For example, Figure 14 indicates that $\alpha_{\{1,3,5\}}$, $\alpha_{\{2,4,5\}}$ and $\alpha_{\{2,3,4\}}$ are G_g–equivalent to α_2. $\square$

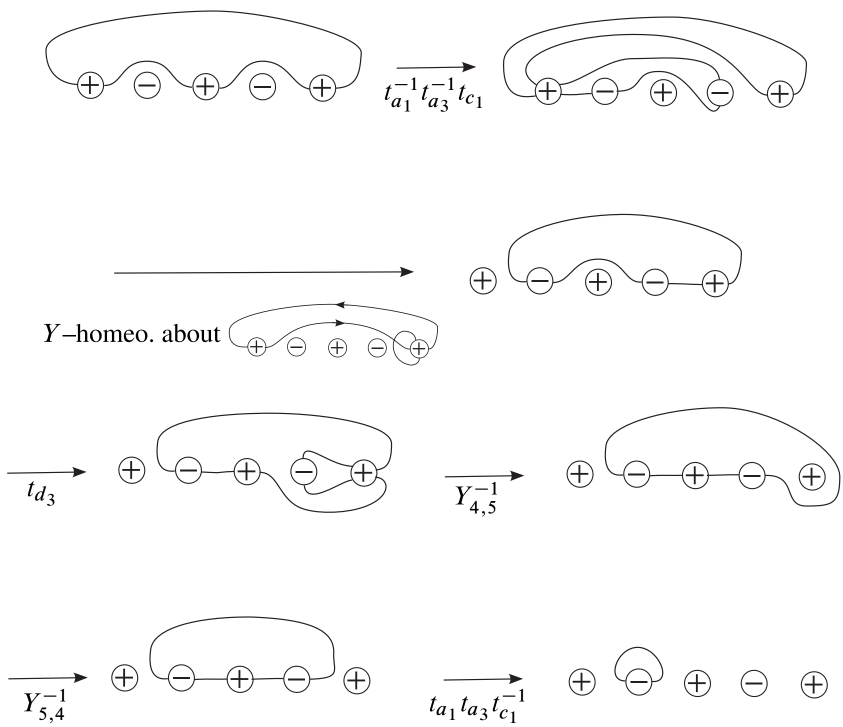

Figure 14. The first circle is $\alpha_{\{1,3,5\}}$, the third circle is $\alpha_{\{2,4,5\}}$, and the sixth circle is $\alpha_{\{2,3,4\}}$.

Every Y–homeomorphism Y is an element of $\Gamma_2(N_g)$, and by Theorem 4.10 and Lemma 4.11, we can express Y as a product of Y–homeomorphisms which are G_g–equivalent to short-leg Y–homeomorphisms. Hence, Lemma 4.4 follows.

4.3 Generators for $\mathcal{O}_g(q_{os})$

For $a \in H_1(N_g;\mathbb{Z}_2)$, we define the *transvection* $T_a\colon H_1(N_g;\mathbb{Z}_2) \to H_1(N_g;\mathbb{Z}_2)$ *about* a by $T_a(x) = x + (x \cdot a)_2\, a$, where $(\cdot)_2$ means the mod–2 intersection form. We remark that if l is an A–circle on N_g such that $[l] = a \in H_1(N_g;\mathbb{Z}_2)$, then $(t_l)_* = T_a$. If we find a system of generators $\{S_1,\dots,S_k\}$ for $\mathcal{O}_g(q_{os})$ and elements $\sigma_1,\dots,\sigma_k$ of $\mathcal{M}(N_g)$ such that $(\sigma_i)_* = S_i$ in $\mathrm{Aut}(H_1(N_g;\mathbb{Z}_2))$, then, by the short exact sequence (1) and Corollary 4.5, we see that $\mathcal{N}_g(q_{os})$ is generated by $G_g \cup \{\sigma_1,\dots,\sigma_k\}$.

Theorem 4.12 (Nowik [12, Theorem 3.2]) *$\mathcal{O}_g(q_{os})$ is generated by the set of elements of the following two forms:*

(1) T_a *for* $a \in H_1(N_g;\mathbb{Z}_2)$ *with* $q_{os}(a) = 2$,

(2) $T_a\, T_b\, T_{a+b}$ *for* $a, b \in H_1(N_g;\mathbb{Z}_2)$ *with* $q_{os}(a) = q_{os}(b) = q_{os}(a+b) = 0$.

Let $\{x_1,\dots,x_g\}$ be the basis of $H_1(N_g;\mathbb{Z}_2)$ which is introduced in Figure 1. We obtain a finite system of generators for $\mathcal{O}_g(q_{os})$ explicitly.

Lemma 4.13 *$\mathcal{O}_g(q_{os})$ is generated by*

(4) $\quad T_{x_i+x_{i+2}}, \qquad i = 1,\dots,g-2,$

(5) $\quad T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}} T_{x_i+x_{i+1}+x_{i+2}+x_{i+3}}, \qquad i = 1,\dots,g-3.$

Proof Write any element v of $H_1(N_g;\mathbb{Z}_2)$ as $v = x_{i_1} + \cdots + x_{i_m}$ such that $i_1 < \cdots < i_m$ and call m the *length* of v, or as $v = (x_{2j_1+1} + \cdots + x_{2j_k+1}) \oplus (x_{2j_{k+1}} + \cdots + x_{2j_m})$ such that $j_1 < \cdots < j_k$, $j_{k+1} < \cdots < j_m$ and call $(x_{2j_1+1} + \cdots + x_{2j_k+1})$ the *odd part* of v, and $(x_{2j_{k+1}} + \cdots + x_{2j_m})$ the *even part* of v. Two elements v, w of $H_1(N_g;\mathbb{Z}_2)$ are *(4)–equivalent* $v \sim_{(4)} w$ if there is a product T of (4) such that $T(v) = w$, and define (5)*–equivalence* $v \sim_{(5)} w$ and (4)*– and* (5)*–equivalence* $v \sim_{(4),(5)} w$ in the same way. We remark that if $v \sim_{(4),(5)} w$ then there is a product T of (4) and (5) such that $T_w = T\,T_v\,T$.

Any element (4) acts only on the odd part of v or only on the even part of v. For example, when $i < j < k$,

$$T_{x_{2j-1}+x_{2j+1}}((\cdots + x_{2i-1} + x_{2j+1} + x_{2k+1} + \cdots) \oplus (\cdots)) \\ = ((\cdots + x_{2i-1} + x_{2j-1} + x_{2k+1} + \cdots) \oplus (\cdots)).$$

Therefore, if we define $l_o(v)$ to be the length of the odd part of v, and $l_e(v)$ to be the length of the even part of v, then $v \sim_{(4)} (x_1 + x_3 + \cdots + x_{2l_o(v)-1}) \oplus (x_2 + x_4 + \cdots + x_{2l_e(v)})$. Hence, $v \sim_{(4)} w$ if and only if $l_o(v) = l_o(w)$ and $l_e(v) = l_e(w)$.

When $p < i, i+3 < s$, $\{i, i+1\} = \{q, q'\}$ and $\{i+2, i+3\} = \{r, r'\}$, the element (5) acts as follows.

$$
\begin{aligned}
&T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}} T_{x_i+x_{i+1}+x_{i+2}+x_{i+3}}(\cdots + x_p + x_q + x_r + x_s + \cdots)\\
&\qquad\qquad = T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}}(\cdots + x_p + x_q + x_r + x_s + \cdots)\\
&\qquad\qquad = \cdots + x_p + x_{q'} + x_{r'} + x_s + \cdots ,\\
&T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}} T_{x_i+x_{i+1}+x_{i+2}+x_{i+3}}(\cdots + x_p + x_q + x_s + \cdots)\\
&\qquad\qquad = T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}}(\cdots + x_p + x_{q'} + x_r + x_{r'} + x_s + \cdots)\\
&\qquad\qquad = \cdots + x_p + x_q + x_r + x_{r'} + x_s + \cdots ,\\
&T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}} T_{x_i+x_{i+1}+x_{i+2}+x_{i+3}}(\cdots + x_p + x_r + x_s + \cdots)\\
&\qquad\qquad = T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}}(\cdots + x_p + x_q + x_{q'} + x_{r'} + x_s + \cdots)\\
&\qquad\qquad = \cdots + x_p + x_q + x_{q'} + x_r + x_s + \cdots .
\end{aligned}
$$

We will show that every element of the first form in Theorem 4.12 is a product of (4) and (5). Let a be an element of $H_1(N_g; \mathbb{Z}_2)$ such that $q_{os}(a) = 2$. Then, $2 \equiv l_o(a) - l_e(a) \mod 4$. Therefore, there are two cases $l_o(a) = l_e(a) + 4t + 2$ or $l_e(a) = l_o(a) + 4t + 2$ $(t \in \mathbb{Z})$. For the first case,

$$
\begin{aligned}
a \sim_{(4)} (x_1 + x_3) + (x_4 + x_5) + \cdots + (x_{2i} + x_{2i+1}) + (x_l + x_{l+2} + x_{l+4} + x_{l+6})&\\
+ \cdots + (x_m + x_{m+2} + x_{m+4} + x_{m+6}).&
\end{aligned}
$$

For the second case,

$$
\begin{aligned}
a \sim_{(4)} (x_2 + x_4) + (x_5 + x_6) + \cdots + (x_{2i+1} + x_{2i+2}) + (x_l + x_{l+2} + x_{l+4} + x_{l+6})&\\
+ \cdots + (x_m + x_{m+2} + x_{m+4} + x_{m+6}).&
\end{aligned}
$$

We see

$$
\begin{aligned}
&(x_2 + x_4) + (x_5 + x_6) + \cdots + (x_{2i+1} + x_{2i+2}) + (x_l + x_{l+2} + x_{l+4} + x_{l+6})\\
&\qquad\qquad + \cdots + (x_m + x_{m+2} + x_{m+4} + x_{m+6})\\
&\quad \sim_{(5)} (x_1 + x_3) + (x_5 + x_6) + \cdots + (x_{2i+1} + x_{2i+2}) + (x_l + x_{l+2} + x_{l+4} + x_{l+6})\\
&\qquad\qquad + \cdots + (x_m + x_{m+2} + x_{m+4} + x_{m+6})\\
&\quad \sim_{(4)} (x_1 + x_3) + (x_4 + x_5) + \cdots + (x_{2i} + x_{2i+1}) + (x_l + x_{l+2} + x_{l+4} + x_{l+6})\\
&\qquad\qquad + \cdots + (x_m + x_{m+2} + x_{m+4} + x_{m+6}),
\end{aligned}
$$

where $\sim_{(5)}$ is by $T_{x_1+x_2}T_{x_3+x_4}T_{x_1+x_2+x_3+x_4}$. Therefore, it suffices to consider the first case. We see

$$(x_l + x_{l+2} + x_{l+4} + x_{l+6}) \sim_{(5)} (x_l + x_{l+2} + x_{l+3} + x_{l+5}) \sim_{(5)} (x_l + x_{l+5}) \\ \sim_{(4)} (x_l + x_{l+1}),$$

where the first $\sim_{(5)}$ is by $T_{x_{l+3}+x_{l+4}}T_{x_{l+5}+x_{l+6}}T_{x_{l+3}+x_{l+4}+x_{l+5}+x_{l+6}}$, and the second $\sim_{(5)}$ is by $T_{x_{l+2}+x_{l+3}}T_{x_{l+4}+x_{l+5}}T_{x_{l+2}+x_{l+3}+x_{l+4}+x_{l+5}}$. Hence, $a \sim_{(4),(5)} (x_1 + x_3) + (x_4 + x_5) + \cdots + (x_{2n} + x_{2n+1})$. Furthermore,

$$\begin{aligned}(x_1 + x_3) + (x_4 + x_5) + (x_6 + x_7) + \cdots + (x_{2n} + x_{2n+1})& \\ &= (x_1 + x_3 + x_4) + x_5 + (x_6 + x_7) + \cdots + (x_{2n} + x_{2n+1}) \\ &\sim_{(5)} x_1 + x_5 + (x_6 + x_7) + \cdots + (x_{2n} + x_{2n+1}) \\ &\sim_{(4)} x_1 + x_3 + (x_4 + x_5) + \cdots + (x_{2(n-1)} + x_{2(n-1)+1}),\end{aligned}$$

where $\sim_{(5)}$ is by $T_{x_1+x_2}T_{x_3+x_4}T_{x_1+x_2+x_3+x_4}$. Therefore, by the induction on n, we see $a \sim_{(4),(5)} x_1 + x_3$. Hence T_a is a product of (4) and (5) if $q_{os}(a) = 2$.

We will show that every element of the second form in Theorem 4.12 is a product of (4) and (5). Let a and b be elements of $H_1(N_g; \mathbb{Z}_2)$ such that $q_{os}(a) = q_{os}(b) = q_{os}(a + b) = 0$. Then $0 = q_{os}(a + b) = q_{os}(a) + q_{os}(b) + (a \cdot b)_2 = (a \cdot b)_2$, $0 = q_{os}(2a) = q_{os}(a) + q_{os}(a) + (a \cdot a)_2 = (a \cdot a)_2$, by the same reason, $0 = (b \cdot b)_2$, hence $(a + b \cdot a)_2 = (a + b \cdot b)_2 = 0$. Therefore, T_a, T_b and T_{a+b} commute each other. For the pairs $[a_1, b_1]$ and $[a_2, b_2]$ of elements of $H_1(N_g; \mathbb{Z}_2)$ which satisfies $q_{os}(a_i) = q_{os}(b_i) = q_{os}(a_i + b_i) = 0$ $(i = 1, 2)$, we define the equivalence $[a_1, b_1] \sim_{(4)} [a_2, b_2]$ if there is a product T of (4) such that $T(a_1) = a_2$ and $T(b_1) = b_2$. The equivalences $[a_1, b_1] \sim_{(5)} [a_2, b_2]$ and $[a_1, b_1] \sim_{(4),(5)} [a_2, b_2]$ are defined in the same way.

Let a, b be elements of $H_1(N_g; \mathbb{Z}_2)$ such that $q_{os}(a) = q_{os}(b) = q_{os}(a + b) = 0$. By the same argument applied for elements of the first from in Theorem 4.12, we see $a \sim_{(4),(5)} (x_1 + x_2) + \cdots + (x_{2n-1} + x_{2n})$.

If $2n \neq g$, then

$$\begin{aligned}(x_1 + x_2) + \cdots + (x_{2n-3} + x_{2n-2}) + (x_{2n-1} + x_{2n})& \\ &= (x_1 + x_2) + \cdots + x_{2n-3} + (x_{2n-2} + x_{2n-1} + x_{2n}) \\ &\sim_{(5)} (x_1 + x_2) + \cdots + x_{2n-3} + x_{2n} \\ &\sim_{(4)} (x_1 + x_2) + \cdots + x_{2n-3} + x_{2n-2} \\ &= (x_1 + x_2) + \cdots + (x_{2(n-1)-1} + x_{2(n-1)}),\end{aligned}$$

where $\sim_{(5)}$ is by $T_{x_{2n-2}+x_{2n-1}}T_{x_{2n}+x_{2n+1}}\ T_{x_{2n-2}+x_{2n-1}+x_{2n}+x_{2n+1}}$. Therefore, by the induction on n, we see $a \sim_{(4),(5)} x_1 + x_2$. Hence, $[a, b] \sim_{(4),(5)} [x_1 + x_2, b']$.

Since $0 = (x_1 + x_2 \cdot b')_2$, $b' = x_1 + x_2 + x_l + \cdots + x_m$ or $b' = x_l + \cdots + x_m$ where $l \geq 3$. For these two cases, $T_{x_1+x_2} T_{b'} T_{x_1+x_2+b'} = T_{x_1+x_2} T_{x_1+x_2+b'} T_{b'}$ are the same. Therefore, we may suppose $b' = x_l + \cdots + x_m$. We see $[a,b] \sim_{(4),(5)} [x_1 + x_2, (x_3 + x_4) + \cdots + (x_{2k-1} + x_{2k})]$, by applying (4) and (5) whose transvections are about v such that $\mathrm{supp}(v)$ contains neither x_1 nor x_2. If $2k \neq g$, by applying to b the same argument as to a, we see $[a,b] \sim_{(4),(5)} [x_1 + x_2, x_3 + x_4]$. If $2k = g$, then $T_a\ T_b\ T_{a+b}$ is equal to $T_{x_1+x_2} T_{x_3+\cdots+x_g} T_{x_1+x_2+x_3+\cdots+x_g}$. In the last paragraph of this proof, we show that $T_{x_1+x_2} T_{x_3+\cdots+x_g} T_{x_1+x_2+x_3+\cdots+x_g}$ is a product of (4) and (5).

If $2n = g$, then $(v \cdot (x_1 + x_2) + \cdots + (x_{2n-1} + x_{2n}))_2 = 0$ for every $v \in H_1(N_g; \mathbb{Z}_2)$ used for the transvections in (4) and (5). Therefore, we see

$$[a,b] \sim_{(4),(5)} [(x_1 + x_2) + \cdots + (x_{g-1} + x_g), (x_{2i+1} + x_{2i+2}) + \cdots + (x_{g-1} + x_g)].$$

This means that $T_a\ T_b\ T_{a+b} = T_{a+b}\ T_b\ T_a$ is conjugate to

$$T_{(x_1+x_2)+\cdots+(x_{2i-1}+x_{2i})}\ T_{(x_{2i+1}+x_{2i+2})+\cdots+(x_{g-1}+x_g)}\ T_{(x_1+x_2)+\cdots+(x_{g-1}+x_g)}$$

by (4) and (5). Moreover,

$$\begin{aligned}
&[((x_1 + x_2) + \cdots + (x_{2i-3} + x_{2i-2}) + (x_{2i-1} + x_{2i})),\\
&\qquad\qquad\qquad\qquad (x_{2i+1} + x_{2i+2}) + \cdots + (x_{g-1} + x_g)]\\
&\sim_{(5)} [(x_1 + x_2) + \cdots + x_{2i-3} + x_{2i},\\
&\qquad\qquad x_{2i-2} + x_{2i-1} + (x_{2i+1} + x_{2i+2}) + \cdots + (x_{g-1} + x_g)]\\
&\sim_{(4)} [(x_1 + x_2) + \cdots + (x_{2i-3} + x_{2i-2}),\\
&\qquad\qquad (x_{2i-1} + x_{2i}) + (x_{2i+1} + x_{2i+2}) + \cdots + (x_{g-1} + x_g)],
\end{aligned}$$

where $\sim_{(5)}$ is by $T_{x_{2i-2}+x_{2i-1}} T_{x_{2i}+x_{2i+1}} T_{x_{2i-2}+x_{2i-1}+x_{2i}+x_{2i+1}}$, and $\sim_{(4)}$ is by $T_{x_{2i-2}+x_{2i}}$. By repeatedly applying the above argument, we see that $T_a\ T_b\ T_{a+b}$ is conjugate to $T_{x_1+x_2} T_{x_3+\cdots+x_g} T_{x_1+x_2+x_3+\cdots+x_g}$ by (4) and (5).

When g is even and $g \geq 6$, by checking the action of transvections on the basis $\{x_1, \ldots, x_g\}$ of $H_1(N_g; \mathbb{Z}_2)$, we show $T_{x_1+x_2} T_{x_3+\cdots+x_g} T_{x_1+x_2+x_3+\cdots+x_g} = T_{x_1+x_2} T_{x_3+x_4}\ T_{x_1+x_2+x_3+x_4} \cdot T_{x_1+x_2} T_{x_5+\cdots+x_g} T_{x_1+x_2+x_5+\cdots+x_g}$. Since we have $[x_1 + x_2, x_5 + \cdots + x_g] \sim_{(4)} [x_1 + x_2, x_3 + \cdots + x_{g-2}]$, we see that the element $T_{x_1+x_2} T_{x_5+\cdots+x_g} T_{x_1+x_2+x_5+\cdots+x_g}$ is a product of (4) and (5) by using the argument for the case where $2k (= g - 2) \neq g$. □

Since $(t_{d_i})_* = T_{x_i+x_{i+2}}$, $(t_{a_i} t_{a_{i+2}} t_{c_i})_* = T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}} T_{x_i+x_{i+1}+x_{i+2}+x_{i+3}}$, and t_{d_i}, $t_{a_i} t_{a_{i+2}} t_{c_i} \in G_g$, we see $G_g = \mathcal{N}_g(q_{os})$, hence Theorem 4.1 follows.

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