A note on Gornik's perturbation of Khovanov–Rozansky homology

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We show that the information contained in the associated graded vector space to Gornik's version of Khovanov–Rozansky knot homology is equivalent to a single even integer $s_n(K)$. Furthermore we show that s_n is a homomorphism from the smooth knot concordance group to the integers. This is in analogy with Rasmussen's invariant coming from a perturbation of Khovanov homology.

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1 Introduction and statement of results

In the last few years there have been associated to a knot $K \subset S^3$ various multiply-graded modules, each one exhibiting a classical knot polynomial as its graded Euler characteristic. It now seems likely that such *knot homologies* exist for each polynomial arising from the Reshetikhin–Turaev construction.

It has been observed that there sometimes exist spectral sequences converging from one knot homology to another (see Rasmussen [11] for a slew of these). One of the first examples was due to Lee [6].

1.1 Khovanov homology and Lee's spectral sequence

From here on we shall work over the complex numbers $\mathbb{C}$. The E_2 page of Lee's spectral sequence is standard Khovanov homology [3]. With a one-component knot K as input, the E_∞ page is a 2–dimensional complex vector space supported in homological degree 0. The E_∞ page also has another integer grading (the quantum grading); we write $\widetilde{H}^{i,j}(K)$ for this E_∞ page where i is the homological grading and j is the quantum grading. Another way to think of $\widetilde{H}^{i,j}(K)$ is as the associated graded vector space to the homology of a filtered chain complex defined by Lee.

Rasmussen [12] showed that $\widetilde{H}^{i,j}(K)$ is supported in bidegrees $i = 0, j = s - 1$ and $i = 0, j = s + 1$ where $s(K) \in 2\mathbb{Z}$. Hence the information contained in $\widetilde{H}^{i,j}(K)$ is equivalent to a single even integer. Rasmussen further showed:

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Theorem 1.1 (Rasmussen [12]) *If $g_*(K)$ is the smooth slice genus of the knot K, then*

$$g_*(K) \geq \frac{|s(K)|}{2}.$$

This bound is sufficient to recover the Milnor conjecture on the slice genus of torus knots, a result previously only accessible through gauge theory. Furthermore Rasmussen showed:

Theorem 1.2 (Rasmussen [12]) *The map $s\colon K \mapsto s(K) \in 2\mathbb{Z}$ is a homomorphism from the smooth concordance group of knots to the integers.*

1.2 Khovanov–Rozansky homology and Gornik's spectral sequence

In the case of Khovanov–Rozansky homology $H_n^{i,j}(K)$ $(n \geq 2)$ (which has the quantum $sl(n)$ knot polynomial as its Euler characteristic), a spectral sequence with E_2 page $H_n^{i,j}(K)$ was defined by Gornik [1]. He showed that the E_∞ page of this spectral sequence is a complex vector space of dimension n supported in homological degree $i = 0$. The invariance of this spectral sequence under the Reidemeister moves was first shown by Wu [14].

Again there is also a quantum grading on this vector space, and the vector space can be thought of as the associated graded vector space to the homology of a filtered chain complex $\mathscr{F}^j \widetilde{C}_n^i(D)$ defined by Gornik for any diagram D of a knot K. We shall write $\mathscr{F}^j \widetilde{H}_n^i(K)$ for the filtered homology groups $\cdots \subseteq \mathscr{F}^{j-1} \widetilde{H}_n^i(K) \subseteq \mathscr{F}^j \widetilde{H}_n^i(K) \subseteq \cdots$ of this chain complex and

$$\widetilde{H}_n^{i,j}(K) = \mathscr{F}^j \widetilde{H}_n^i(K) / \mathscr{F}^{j-1} \widetilde{H}_n^i(K)$$

for the associated graded vector space.

It was shown by the author [7] and independently by Wu [14] that one can extract a lower bound on the slice genus from the quantum j–grading of each nonzero vector space $\widetilde{H}_n^{0,j}(K)$ (in fact in these cited papers this was done also for more general perturbations of Khovanov–Rozansky homology than Gornik's). Again, these lower-bounds are enough to imply the Milnor conjecture on the slice genus of torus knots. The highest nonzero quantum grading in this setup has been called $g_n^{\max}$ and the lowest $g_n^{\min}$ by Wu. In [13] Wu asks for a relation between $g_n^{\max}$ and $g_n^{\min}$, and we provide an answer with our Theorem 1.3.

1.3 New results

We first show that the information contained in $\widetilde{H}_n^{i,j}(K)$ is equivalent to a single even integer $s_n(K)$.

Theorem 1.3 *For K a knot, define the polynomial*

$$\widetilde{P}_n(q) = \sum_{j=-\infty}^{j=\infty} \dim_{\mathbb{C}}(\widetilde{H}_n^{0,j}(K))q^j.$$

Then there exists $s_n(K) \in 2\mathbb{Z}$ such that

$$\widetilde{P}_n(q) = q^{s_n(K)} \frac{(q^n - q^{-n})}{(q - q^{-1})}.$$

In other words, this theorem says that the Gornik homology of any knot K is isomorphic to that of the unknot, but shifted by quantum degree $s_n(K)$.

The results of the author and of Wu's on the slice genus are then immediately stated as the following:

Corollary 1.4 (Lobb [7]; Wu [14]) *Writing $g_*(K)$ for the smooth slice genus of a knot, we have*

$$g_*(K) \geq \frac{|s_n(K)|}{2(n-1)}.$$

Furthermore, if K admits a diagram D with only positive crossings then

$$\begin{aligned} g_*(K) &= \frac{-s_n(K)}{2(n-1)} \\ &= \frac{1}{2}(1 - \#O(D) + w(D)), \end{aligned}$$

where $\#O(D)$ is the number of circles in the oriented resolution of D and $w(D)$ is the writhe of D.

It is a question of much interest whether the $s_n(K)$ are in fact all equivalent to each other. We hope that this is not true, and do not know whether to expect it to be true. Nevertheless, let us formulate this as a conjecture.

Conjecture 1.5 *For any knot K and $m, n \geq 2$, we have*

$$\frac{s_m(K)}{s_n(K)} = \frac{m-1}{n-1}.$$

Note that $s_2(K) = -s(K)$, so every s_n is equivalent to Rasmussen's original $s(K)$.

The falsity of this conjecture would have consequences for the nondegeneracy of the spectral sequences defined by Rasmussen [11] on the triply-graded Khovanov–Rozansky homology [5]. One can also make a weaker conjecture:

Conjecture 1.6 *For any knot K and $n \geq 2$, we have*

$$s_n(K) \in 2(n-1)\mathbb{Z}.$$

This has the cosmetic appeal that it would rule out the possibility of fractional bounds on the slice genus coming from Corollary 1.4, but again we have no expectations either way on the truth of this conjecture.

By analogy with Rasmussen's Theorem 1.2 we might anticipate that each s_n is a concordance homomorphism. We show that this is in fact the case:

Theorem 1.7 *For each $n \geq 2$, the map $s_n\colon K \mapsto s_n(K) \in 2\mathbb{Z}$ is a homomorphism from the smooth concordance group of knots to the integers.*

This theorem tells us that we have a concordance homomorphism for each integer ≥ 2. It is a fascinating problem to try and understand if and how these homomorphisms are related to each other; we hope that this paper will stimulate some activity towards this goal.

We conclude by noting that there are many properties of Rasmussen's concordance homomorphism s from Khovanov homology and of the homomorphism τ coming from Heegaard Floer knot homology (see Rasmussen [10] and Ozsváth and Szabó [9]) which follow formally from the properties of s and τ analogous to Corollary 1.4 and Theorem 1.7. Rescaled versions of these results can now be seen to hold for s_n. We restrict ourselves to mentioning one of these which is not well-known as following from these formal properties.

Corollary 1.8 *If K is an alternating knot then*

$$s_n(K) = \frac{1}{1-n}\sigma(K),$$

where $\sigma(K)$ is the classical knot signature of K.

We sketch the proof of this at the end of the next section.

2 Proofs of results

We assume in this section some familiarity with Khovanov and Rozansky [4]. We fix an integer $n \geq 2$ and let K be a 1–component knot. In [4] the polynomial $w = x^{n+1}$ is called the *potential*. Gornik's key insight [1] was that it made sense to take a perturbation $\widetilde{w} = x^{n+1} - (n+1)x$ of this potential and much of [4] goes through as before. Gornik showed that for his choice of potential $\widetilde{w}$, a knot diagram D determines a chain complex that no longer has a quantum grading but instead a quantum filtration respected by the differential.

$$\cdots \subseteq \mathcal{F}^{j-1}\widetilde{C}_n^i(D) \subseteq \mathcal{F}^j\widetilde{C}_n^i(D) \subseteq \cdots,$$
$$d\colon \mathcal{F}^j\widetilde{C}_n^i(D) \to \mathcal{F}^j\widetilde{C}_n^{i+1}(D).$$

It was immediate from his definitions that there exists a spectral sequence with E_2 page the original Khovanov–Rozansky homology $H_n^{i,j}(K)$ converging to the associated graded vector space

$$E_\infty^{i,j}(K) = \widetilde{H}_n^{i,j}(K) = \mathcal{F}^j\widetilde{H}_n^i(K)/\mathcal{F}^{j-1}\widetilde{H}_n^i(K)$$

to the filtered homology groups $\mathcal{F}^j\widetilde{H}_n^i(K)$.

Given a knot diagram D for K, Gornik gave a basis at the chain level generating the homology; we now describe this basis. We write $O(D)$ for the oriented resolution of D, and write r for the number of components of $O(D)$. The oriented resolution $O(D)$ gives rise to a summand of the chain group $\widetilde{C}_n^0(D) = \bigcup_j \mathcal{F}^j\widetilde{C}_n^0(D)$, isomorphic in a natural way to

$$\mathbb{C}[x_1, x_2, \ldots, x_r]/(x_1^n - 1, x_2^n - 1, \ldots, x_r^n - 1)\,[(1-n)(w(D)+r)],$$

where we have indicated a shift in the quantum filtration depending on r and on the writhe $w(D)$ of the diagram.

Definition 2.1 Let $\xi = e^{2\pi i/n}$. For each $p = 0, 1, \ldots, n-1$ we define an element $g_p \in \widetilde{C}_n^0(D)$ that lies in this summand by

$$g_p = \prod_{k=1}^{r} \frac{(x_k^n - 1)}{(x_k - \xi^p)}.$$

Theorem 2.2 (Gornik [1]) *Each g_p is a cycle and $\{[g_0], [g_1], \ldots, [g_{n-1}]\}$ is a basis for the homology $\widetilde{H}_n^i(K) = \bigcup_j \mathcal{F}^j\widetilde{H}_n^i(K)$. Consequently $\widetilde{H}_n^i(K)$ is a vector space of dimension n supported in homological degree $i = 0$.*

Our first observation is that we can find a good basis for the subspace of $\widetilde{C}_n^0(D)$ spanned by $g_0, g_1, \dots, g_{n-1}$. What we mean here by "good" requires another definition.

Definition 2.3 A monomial $\prod_{i=1}^s x_i^{a_i} \in \mathbb{C}[x_1, x_2, \dots, x_s]$ is said to be of n–degree d if and only if
$$\sum_{i=1}^s a_i = d \pmod n.$$
A polynomial is said to have be n–homogenous of n–degree d if and only if it is a linear combination of monomials of n–degree d.

We note that projection extends the notion of n–degree unambiguously to elements lying in the ring
$$\mathbb{C}[x_1, x_2, \dots, x_s]/(x_1^n - 1, x_2^n - 1, \dots, x_s^n - 1)$$
since the quotient ideal is generated by n–homogeneous polynomials.

Next we give a basis consisting of n–homogeneous elements for the vector space spanned by the elements $g_0, g_1, \dots, g_{n-1} \in \widetilde{C}_n^0(D)$.

Lemma 2.4 *Let $g_0, g_1, \dots, g_{n-1}$ be given as in Definition 2.1, and consider the n–dimensional complex vector space*
$$V = \langle g_0, g_1, \dots, g_{n-1} \rangle \subseteq \mathbb{C}[x_1, x_2, \dots, x_r]/(x_1^n - 1, x_2^n - 1, \dots, x_r^n - 1).$$
For $p = 0, 1, \dots, n-1$ let
$$h_p \in \mathbb{C}[x_1, x_2, \dots, x_r]/(x_1^n - 1, x_2^n - 1, \dots, x_r^n - 1)$$
be the unique n–homogeneous element of n–degree p such that
$$g_0 = h_0 + h_1 + \cdots + h_{n-1}.$$
Then we have
$$V = \langle h_0, h_1, \dots, h_{n-1} \rangle.$$

Proof For dimensional reasons it is enough to show that for each $t = 0, 1, \dots, n-1$ we have
$$g_t \in \langle h_0, h_1, \dots, h_{n-1} \rangle.$$
So let us fix such a t and let $\overline{h}_p$ be the unique n–homogeneous element of n–degree p such that
$$g_t = \overline{h}_0 + \overline{h}_1 + \cdots + \overline{h}_{n-1}.$$
We will show that $\overline{h}_p$ is a multiple of h_p and then we will be done.

Consider a monomial of n–degree p

$$\prod_{i=1}^{r} x_i^{a_i} \quad \text{where} \quad \sum_{i=1}^{r} a_i = p \pmod n \quad \text{and} \quad 0 \le a_i \le n-1 \text{ for all } i.$$

The coefficient of this monomial in h_p (or, equivalently, in g_0) is clearly 1. The coefficient c of this monomial in g_t is expressible as a product $c = c_1 c_2 \cdots c_r$ where c_i is the coefficient of x^{a_i} in the expansion of

$$\frac{x^n - 1}{x - \xi^t} = \frac{x^n - (\xi^t)^n}{x - \xi^t}.$$

We leave it to the reader to check that $c_i = \xi^{-(a_i+1)t}$, so that

$$c = \xi^{-t(\sum_{i=1}^{r}(a_i+1))} = \xi^{-t(p+r)}.$$

Hence we see that

$$\overline{h}_p = \xi^{-t(p+r)} h_p \quad \text{so} \quad g_t \in \langle h_0, h_1, \dots, h_{n-1} \rangle. \qquad \square$$

To put our new n–homogeneous basis to use, we require a proposition telling us how we might expect it to behave with respect to the filtration. In what follows, since we are assuming some familiarity with [4], we allow ourselves to refer to a matrix factorization as just a letter, M. We begin with a definition.

Definition 2.5 If V is some filtered vector space

$$\cdots \subseteq \mathscr{F}^i V \subseteq \mathscr{F}^{i+1} V \subseteq \cdots,$$

and we have a nonzero $x \in V$, we shall define the *quantum grading* $\mathrm{qgr}(x) \in \mathbb{Z}$ by the requirement that x is nonzero in

$$\mathscr{F}^{\mathrm{qgr}(x)} V / \mathscr{F}^{\mathrm{qgr}(x)-1} V.$$

The reason for the word "quantum" in the definition is that in this paper the only vector spaces we shall worry about are those coming from chain groups or homology groups carrying a "quantum" filtration.

Proposition 2.6 *If M is a matrix factorization whose homology $H(M)$ appears as a summand of the chain group $\widetilde{C}^i(D)$, then there is a natural $(\mathbb{Z}/2n\mathbb{Z})$–grading on $H(M)$ which we write as*

$$\mathrm{Gr}^{\alpha}\, H(M) \quad \text{for } \alpha \in \mathbb{Z}/2n\mathbb{Z}.$$

This grading extends to a grading on the chain groups $\widetilde{C}^i_n(D)$*, which is respected by the differential*

$$d\colon \mathrm{Gr}^\alpha\,\widetilde{C}^i_n(D) \longrightarrow \mathrm{Gr}^\alpha\,\widetilde{C}^{i+1}_n(D) \quad \text{for } \alpha \in \mathbb{Z}/2n\mathbb{Z},$$

thus giving a $(\mathbb{Z}/2n\mathbb{Z})$*–grading on the homology groups* $\mathrm{Gr}^\alpha\,\widetilde{H}^i_n(K)$ *for* $\alpha \in \mathbb{Z}/2n\mathbb{Z}$*.*

Furthermore, if $a \in \mathrm{Gr}^\alpha\,\widetilde{C}^0_n(D)$ *and* $b \in \mathrm{Gr}^\beta\,\widetilde{C}^0_n(D)$ *represent nonzero classes* $[a]$*,* $[b]$ *in homology* $\widetilde{H}^0_n(K)$ *then we have*

$$\begin{aligned}\alpha - \beta &= \mathrm{qgr}(a) - \mathrm{qgr}(b) \pmod{2n}\\ &= \mathrm{qgr}([a]) - \mathrm{qgr}([b]) \pmod{2n}.\end{aligned}$$

Proof The matrix factorization M consists of two "internal" graded vector spaces V_0, V_1 and pair of "internal" differentials

$$d_0\colon V_0 \to V_1 \quad \text{and} \quad d_1\colon V_1 \to V_0, \quad d_1 d_0 = d_0 d_1 = 0.$$

If we were working with Khovanov and Rozansky's potential $w = x^{n+1}$ then we would know that these internal differentials d_0, d_1 were both graded of degree $n+1$. But with Gornik's potential $\widetilde{w} = x^{n+1} - (n+1)x$ the internal differentials cease to respect the grading. So instead we take the filtration associated to the grading of the internal vector spaces and we observe that the internal differentials are then *filtered* of degree $n+1$. This gives rise to a filtered homology $H(M)$ and so to filtered chain groups.

The crux of this proposition is recognizing that the polynomials appearing as matrix entries in Gornik's internal differentials are all n–homogenous. Since the various x_i appearing in the definition of M are assigned grading 2, this means that the homology $H(M)$ inherits a $(\mathbb{Z}/2n\mathbb{Z})$–grading from the $(\mathbb{Z}/2n\mathbb{Z})$–grading on the internal vector spaces of M coming from collapsing their $\mathbb{Z}$–grading.

Similarly the differentials on the chain complex $\widetilde{C}^i_n(D)$ have n–homogeneous matrix entries. It needs to be checked that these entries are graded of degree $0 \in \mathbb{Z}/2n\mathbb{Z}$ – we leave this to the reader. Hence we inherit a $(\mathbb{Z}/2n\mathbb{Z})$–grading on homology

$$\mathrm{Gr}^\alpha\,\widetilde{H}^i_n(K) \quad \text{where } \alpha \in \mathbb{Z}/2n\mathbb{Z}.$$

The first equality of the final part of the proposition follows from the observation that both the filtration and the $(\mathbb{Z}/2n\mathbb{Z})$–grading on $\widetilde{C}^i_n$ are induced from the same $\mathbb{Z}$–grading on the matrix factorizations. The second equality follows from the fact that the differential on $\widetilde{C}^i_n$ respects the $(\mathbb{Z}/2n\mathbb{Z})$–grading. □

In Proposition 2.6 we restricted ourselves to *relative* quantum gradings, but we did this simply as a matter of convenience, so that we did not have to worry about the various grading shifts happening in the definition of the chain complex. It is of course possible to be more precise. The content of the next proposition is that we have computed the grading shifts to give a precise statement of Proposition 2.6 applied to the case of the n–homogenous generators $h_0, h_1, \dots, h_{n-1}$.

Proposition 2.7 *For $p = 0, 1, \dots, n-1$, each h_p of Lemma 2.4 can be considered as a cycle of the chain group $\widetilde{C}_n^0(D)$, each lying in the summand of this chain group corresponding to the oriented resolution $O(D)$.*

Then each $[h_p]$ is a nonzero class in homology lying in the graded part $\widetilde{H}_n^{0,j_p}(K)$ for some j_p satisfying

$$j_p = 2p + (1-n)(w(D)+r) \pmod{2n}.$$

Proof Certainly each h_p lies in a unique $(\mathbb{Z}/2n\mathbb{Z})$–grading. We note that the writhe of the diagram $w(D)$ and the number r of components of $O(D)$ appear in Proposition 2.7 because of the grading shift of the chain group summand. The factors of 2 appear since the various x_i appearing in the definition of the homology are assigned grading 2. We note also that $w(D)+r$ is always an odd number. □

Definition 2.8 For K a knot let

$$s_n^{\max}(K) = \max\{j : \widetilde{H}_n^{0,j}(K) = \mathbb{C}\},$$
$$s_n^{\min}(K) = \min\{j : \widetilde{H}_n^{0,j}(K) = \mathbb{C}\}.$$

It is now clear that Theorem 1.3 follows immediately from Proposition 2.7 and the following:

Proposition 2.9 *For any knot K we have*

$$s_n^{\max}(K) - s_n^{\min}(K) \leq 2(n-1).$$

To verify Proposition 2.9 we need to appeal to the results of [7], specifically those of Section 3.3 which explains how, given a link L, $\widetilde{H}_n^{i,j}(L)$ may change under elementary 1–handle addition to L. We do not need these results in full generality; the relevant picture for this paper is that of Figure 1.

We state the next proposition without proof and refer interested readers to [7, Section 3.3] for more details.

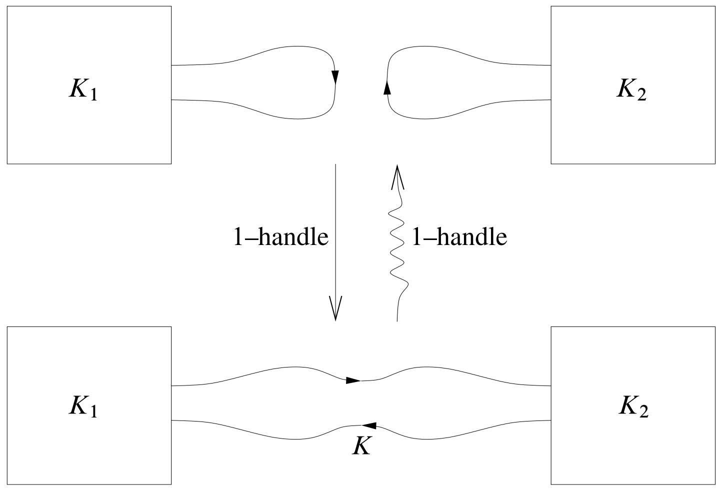

Figure 1. In this figure we show how the connect sum $K = K_1 \# K_2$ of two knots K_1, K_2 is obtained from the disjoint union of the knots by a knot cobordism consisting of a single 1–handle attachment (the straight arrow), and likewise the reverse direction (the bendy arrow).

Proposition 2.10 *Consider the setup of Figure 1 where $K = K_1 \# K_2$. Associated to the straight arrow is a map*

$$F\colon \mathcal{F}^{j_1} \widetilde{H}^i_n(K_1) \otimes \mathcal{F}^{j_2} \widetilde{H}^i_n(K_2) \longrightarrow \mathcal{F}^{j_1+j_2+n-1} \widetilde{H}^i_n(K),$$

and associated to the bendy arrow is a map

$$G\colon \mathcal{F}^{j} \widetilde{H}^i_n(K) \longrightarrow \bigcup_{\substack{j_1,j_2 \\ j_1+j_2=j+n-1}} \mathcal{F}^{j_1} \widetilde{H}^i_n(K_1) \otimes \mathcal{F}^{j_2} \widetilde{H}^i_n(K_2).$$

For $p = 0, 1, \ldots, n-1$ we write $[g_p], [g^1_p], [g^2_p]$ for Gornik's basis elements of $\widetilde{H}^0_n(K)$, $\widetilde{H}^0_n(K_1)$, $\widetilde{H}^0_n(K_2)$ respectively. We have

$$F([g^1_{p_1}] \otimes [g^2_{p_2}]) = \alpha [g_{p_1}],$$

where $\alpha \neq 0$ if and only if $p_1 = p_2$, and

$$G([g_p]) = \beta([g^1_p] \otimes [g^2_p]),$$

where $\beta \neq 0$. □

With this proposition in hand we are almost ready to prove Proposition 2.9 and hence Theorem 1.3. We just need one more easy lemma.

Lemma 2.11 *If $g \in \widetilde{C}^0_n(D)$ is one of Gornik's basis elements of $\widetilde{H}^0_n(K)$ then*

$$\mathrm{qgr}([g]) = s_n^{\max}(K).$$

Proof This follows from the observation that the quantum grading of exactly one of the $[h_p]$ must be $s_n^{\max}(K)$, and that g is a linear combination of the h_p, with all coefficients nonzero. □

Proof of Proposition 2.9 In Figure 1, let $K = K_1$ and let $K_2 = U$, the unknot. Choose $p \in \{0, 1, \dots, n-1\}$ so that $[h^1_p]$ is nonzero in $\widetilde{H}^{0,s_n^{\min}}_n(K_1)$. Now h^1_p is expressible as a linear combination of Gornik's generators $g^1_0, g^1_1, \dots, g^1_{n-1}$. Assume without loss of generality that the coefficient of g^1_0 in this linear combination is nonzero. Then we have

$$\begin{aligned} s_n^{\max}(K) &= \mathrm{qgr}([g_0]) \\ &= F([h^1_p] \otimes [g^2_0]) \\ &\leq \mathrm{qgr}([h^1_p]) + \mathrm{qgr}([g^2_0]) + n - 1 \\ &= s_n^{\min}(K) + n - 1 + n - 1 \\ &= s_n^{\min}(K) + 2n - 2. \end{aligned}$$

□

Now Theorem 1.3 follows easily.

Proof of Theorem 1.3 Propositions 2.7 and 2.9 combine to imply Theorem 1.3 □

We can use the same technique from the proof of Proposition 2.9 to prove Theorem 1.7.

Proof of Theorem 1.7 To check we have a homomorphism, it is enough to show that s_n respects the group operations. In other words if $K = K_1 \# K_2$ we wish to see

$$s_n(K) = s_n(K_1) + s_n(K_2).$$

Again we refer to Figure 1 and choose $p \in \{0, 1, \dots, n-1\}$ so that $[h^1_p]$ is nonzero in $\widetilde{H}^{0,s_n^{\min}}_n(K_1)$ and assume without loss of generality that the coefficient of g^1_0 in the expansion of h^1_p is nonzero.

We observe

$$\begin{aligned}
s_n(K_1) + s_n(K_2) &= s_n^{\min}(K_1) + s_n^{\max}(K_2) \\
&= \mathrm{qgr}([h_p^1] \otimes [g_0^2]) \\
&\geq \mathrm{qgr}(F([h_p^1] \otimes [g_0^2])) - n + 1 \\
&= \mathrm{qgr}([g_0]) - n + 1 \\
&= s_n^{\max}(K) - n + 1 \\
&= s_n(K),
\end{aligned}$$

$$\begin{aligned}
s_n(K_1) + s_n(K_2) &= s_n^{\max}(K_1) + s_n^{\max}(K_2) - 2n + 2 \\
&= \mathrm{qgr}([g_0^1] \otimes [g_0^2]) - 2n + 2 \\
&= \mathrm{qgr}(G([g_0])) - 2n + 2 \\
&\leq \mathrm{qgr}([g_0]) + n - 1 - 2n + 2 \\
&= s_n^{\max}(K) - n + 1 \\
&= s_n(K).
\end{aligned}$$

□

Finally we indicate the proof of Corollary 1.8.

Proof of Corollary 1.8 The main tool is due to Kawamura [2] in which she gives an explicit estimate of $s(K)$ and $\tau(K)$ depending on a diagram D of K. In deriving this estimate she only makes use of the formal properties of s and τ analogous to Corollary 1.4 and Theorem 1.7, hence her arguments also apply to s_n.

In [8], the author independently derives this estimate for $s(K)$, using an algebraic argument rather than the formal properties of s. Proposition 1.5 of [8] shows that the estimates are tight given an alternating diagram D of K, but the proof of this Proposition does not use the definition of s and hence also shows that the bounds on $s_n(K)$ are tight for alternating knots.

Therefore since we know appropriately rescaled versions of this Corollary hold for s and for τ, they also hold for s_n. □

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