Higher cohomologies of modules

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If $\mathbb{C}$ is a small category, then a right $\mathbb{C}$–module is a contravariant functor from $\mathbb{C}$ into abelian groups. The abelian category $\mathrm{Mod}_{\mathbb{C}}$ of right $\mathbb{C}$–modules has enough projective and injective objects, and the groups $\mathrm{Ext}^n_{\mathrm{Mod}_{\mathbb{C}}}(B,A)$ provide the basic cohomology theory for $\mathbb{C}$–modules. We introduce, for each integer $\mathrm{r}\geq 1$, an approach for a level–r cohomology theory for $\mathbb{C}$–modules by defining cohomology groups $H^n_{\mathbb{C},\mathrm{r}}(B,A)$, $n\geq 0$, which are the focus of this article. Applications to the homotopy classification of braided and symmetric $\mathbb{C}$–fibred categorical groups and their homomorphisms are given.

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1 Introduction and summary

Among the Ext groups in the category **Ab** of abelian groups, only $\mathrm{Hom}_{\mathbf{Ab}}(B,A)$ and $\mathrm{Ext}^1_{\mathbf{Ab}}(B,A)$ are relevant since all groups $\mathrm{Ext}^n_{\mathbf{Ab}}(B,A)$ vanish for $n\geq 2$, and there is nothing to say about the latter. In the fifties, however, Eilenberg and Mac Lane [27; 28; 29; 43] introduced what are now known as *higher cohomology theories for abelian groups*: for each integer $\mathrm{r}\geq 1$, the level–r cohomology groups of an abelian group B with coefficients in an abelian group A are defined to be the cohomology groups of the Eilenberg–Mac Lane complex $K(B,\mathrm{r}+1)$ with coefficients in A, up to a dimension shift, that is,

$$H^n_{\mathrm{r}}(B,A)=H^{n+\mathrm{r}}(K(B,\mathrm{r}+1),A),\quad n\geq 0.$$

In the beginning, these higher-level cohomology groups were studied primarily with interest in algebraic topology. For example, Copeland [20, Proposition 9] proved a remarkable classifying fact stating that, for each $k\in H^n_{\mathrm{r}}(B,A)$, $n\geq 3$, there exists a pointed CW-complex $(X,*)$, unique up to homotopy equivalence, such that $\pi_{\mathrm{r}+1}(X,*)=B$, $\pi_{\mathrm{r}+n}(X,*)=A$, $\pi_i(X,*)=0$ for all $i\neq \mathrm{r}+1,\mathrm{r}+n$, and k is the (unique nontrivial) Postnikov invariant of $(X,*)$ (see Whitehead [60, Chapter IX]). But later, these cohomologies of abelian groups found an application in solving purely algebraic problems. Thus, for instance, the classification result for *symmetric categorical*

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groups $\mathbb{P}=(\mathbb{P},\otimes,\boldsymbol{I},\boldsymbol{a},\boldsymbol{r},\boldsymbol{l},\boldsymbol{c})$, which appeared originally in the unpublished thesis of Sinh [54] (where they are called Picard categories), is stated in terms of the associated abelian groups $B=\pi_0\mathbb{P}$, the group of iso-classes of its objects, $A=\pi_1\mathbb{P}$, the group of automorphisms of its unit object, and a second-level cohomology class $k\in H^3_2(B,A)$, canonically deduced from the coherence pentagons and hexagons in $\mathbb{P}$. Previously, as a consequence of having $\mathrm{Ext}^2_{\mathbf{Ab}}(B,A)=0$, Deligne [21] had proved that the classification of *strictly commutative symmetric categorical groups* (ie, when $\boldsymbol{c}_{x,x}=1_{x\otimes x}$) is trivial, in the sense that only the two abelian groups B and A above are a complete invariant for the equivalence type of such a strictly commutative symmetric categorical group. Later, an extension of Sinh's results was proved by Joyal and Street [41], where they stated a classification theorem for *braided categorical groups* (defined similarly to symmetric categorical groups, but where the usual symmetry condition $\boldsymbol{c}_{y,x}\boldsymbol{c}_{x,y}=1_{x\otimes y}$ is not assumed), in terms of elements of the first-level third cohomology groups $H^3_1(B,A)$.

If $\mathbb{C}$ is a small category, then the category $\mathrm{Mod}_{\mathbb{C}}$ of right $\mathbb{C}$–modules has objects the contravariant functors $A\colon \mathbb{C}^{\mathrm{op}}\to\mathbf{Ab}$ from $\mathbb{C}$ into abelian groups, with morphisms the natural transformations. This is an abelian category with enough injectives and projectives, and the abelian groups $\mathrm{Ext}^n_{\mathrm{Mod}_{\mathbb{C}}}(B,A)$ provide the basic cohomology theory for $\mathbb{C}$–modules. For instance, if $\mathbb{Z}\colon\mathbb{C}^{\mathrm{op}}\to\mathbf{Ab}$ is the constant functor with value $\mathbb{Z}$, then the groups $H^n(\mathbb{C},A)=\mathrm{Ext}^n_{\mathrm{Mod}_{\mathbb{C}}}(\mathbb{Z},A)$ are the cohomology groups of the category $\mathbb{C}$ with coefficients in the $\mathbb{C}$–module A, studied by Roos [52], Watts [59], Mitchell [46] and Baues and Wrisching [3], among other authors. In this paper we introduce, for each integer $\mathrm{r}\geq 1$, an approach for a level–r cohomology theory for $\mathbb{C}$–modules by defining cohomology groups

(1) $$H^n_{\mathbb{C},\mathrm{r}}(B,A):=H^{n+\mathrm{r}}(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\,K(B,\mathrm{r}+1),\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\,*;A),\quad n\geq 0,$$

which we believe enjoy many desirable properties and are the focus of this article. In the case where $\mathbb{C}$ is the trivial category with only one arrow, these level–r cohomology groups reduce to those of Eilenberg–Mac Lane above for abelian groups. When $\mathbb{C}$ is a group G (that is, $\mathbb{C}$ has one object and all its morphisms are invertible), then the first-level cohomology groups of a G–module B with coefficients in a G–module A, $H^n_{G,1}(B,A)$ coincide with those "abelian" cohomology groups $H^n_{G,\mathrm{ab}}(B,A)$ studied by Cegarra and Khmaladze [15], while the second-level cohomology groups $H^n_{G,2}(B,A)$ coincide with the "symmetric" cohomology groups $H^n_{G,\mathrm{s}}(B,A)$ treated by same authors in [16].

The results here are mainly of algebraic interest, but they may also be considered within the homotopy theory of diagrams of spaces, as has been studied by various authors such as Dror, Dwyer and Kan [23], Dwyer and Kan [26], Dror Farjoun [24], Dror Farjoun and Zabrodsky [25], Moerdijk and Svensson [47; 48] and Chachólski and Scherer [17]. The contents of the paper can be summarized as follows:

We begin by providing a brief account of the cohomology of simplicial sets. Here, at the same time as fixing notation and terminology, we review necessary aspects and results concerning cohomology groups of small categories, simplicial sets and diagrams of simplicial sets that will be used throughout the paper. The main notion we need to establish is that of the cohomology groups $H^n(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\ X, A)$, for $X\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{S}$ a diagram of simplicial sets, and $A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$ any $\mathbb{C}$–module, since it leads to the definition of the *level–*r *cohomology groups of* $\mathbb{C}$*–modules* (1), $H^n_{\mathbb{C},\mathrm{r}}(B, A)$. A significant component here is the construction of a manageable and lucid cochain complex $C^\bullet_{\mathbb{C},\mathrm{r}}(B, A)$, called the *complex of level–*r *cochains*, for computing higher cohomologies of $\mathbb{C}$–modules:

Theorem 3.3 *For any two* $\mathbb{C}$*–modules* A*,* B*, there are isomorphisms*

$$H^n_{\mathbb{C},\mathrm{r}}(B, A) \cong H^n C^\bullet_{\mathbb{C},\mathrm{r}}(B, A), \quad \mathrm{r} \geq 1.$$

As one of the relevant consequences of the above isomorphisms, we prove the expected isomorphisms and monomorphism for levels r and $\mathrm{r}+1$, namely:

Theorem 3.4 *For* $n \leq \mathrm{r}+1$*, there are natural isomorphisms*

$$H^n_{\mathbb{C},\mathrm{r}+1}(B, A) \cong H^n_{\mathbb{C},\mathrm{r}}(B, A),$$

and a monomorphism

$$H^{\mathrm{r}+2}_{\mathbb{C},\mathrm{r}+1}(B, A) \hookrightarrow H^{\mathrm{r}+2}_{\mathbb{C},\mathrm{r}}(B, A).$$

In this paper, we use mainly the cohomology groups $H^n_{\mathbb{C},\mathrm{r}}(B, A)$ for $n \leq 3$. Hence, we pay particular attention to low-dimensional cocycles and coboundaries. We explicitly describe the cochain complexes $C^\bullet_{\mathbb{C},1}(B, A)$ and $C^\bullet_{\mathbb{C},2}(B, A)$, and specifically analyse their corresponding truncated subcomplexes that, respectively, yield the first- and second-level cocycles and coboundaries in dimensions ≤ 3. This analysis allows us to quickly obtain interpretations for the cohomology groups $H^n_{\mathbb{C},\mathrm{r}}(B, A)$ for $n \leq 2$, in terms of $\mathbb{C}$–module homomorphisms and extensions. More precisely, for any $\mathrm{r} \geq 1$, we prove:

Theorems 5.1 and 5.3 *For any two* $\mathbb{C}$*–modules* A, B*,* $H^0_{\mathbb{C},\mathrm{r}}(B, A) = 0$*, and there are isomorphisms* $H^1_{\mathbb{C},\mathrm{r}}(B, A) \cong \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{C}}}(B, A)$ *and* $H^2_{\mathbb{C},\mathrm{r}}(B, A) \cong \mathrm{Ext}_{\mathrm{Mod}_{\mathbb{C}}}(B, A)$.

The remaining results in the article focus on the cohomology groups $H^3_{\mathbb{C},\mathrm{r}}(B,A)$, to whose elements we give a natural interpretation in terms of equivalence classes of *braided* or *symmetric* $\mathbb{C}$*–fibred categorical groups*, $\mathbb{P}=(\mathbb{P},P,\otimes,I,\boldsymbol{a},\boldsymbol{r},\boldsymbol{l},\boldsymbol{c})$, that is, categories fibred in groupoids, $P\colon\mathbb{P}\to\mathbb{C}$, enriched with a locally compact monoidal $\mathbb{C}$–structure by $\mathbb{C}$–functors $\otimes\colon\mathbb{P}\times_{\mathbb{C}}\mathbb{P}\to\mathbb{P}$ and $I\colon\mathbb{C}\to\mathbb{P}$, and corresponding coherent associativity, unit and commutativity $\mathbb{C}$–fibred constraints. Indeed, a main objective of this paper is to state and prove precise classification theorems for braided and symmetric *fibred* categorical groups by generalizing the aforementioned results for the nonfibred case stated by Joyal and Street [41] and Sinh [54]. To do so, we consider the 2–categories that braided and symmetric $\mathbb{C}$–fibred categorical groups form, respectively denoted by $\mathcal{BCG}_{\downarrow\mathbb{C}}$ and $\mathcal{SCG}_{\downarrow\mathbb{C}}$, and our approach to the issue mainly consists in proving (see Theorems 7.2 and 6.15):

Theorem *There are biequivalences of* 2*–categories*

$$\begin{aligned}\mathcal{Z}^3_{\mathbb{C},1} &\overset{\mathcal{P}_{\mathbb{C}}}{\simeq} \mathbf{Psd}(\mathbb{C}^{\mathrm{op}},\mathcal{BCG}) \overset{\int_{\mathbb{C}}}{\simeq} \mathcal{BCG}_{\downarrow\mathbb{C}},\\ \mathcal{Z}^3_{\mathbb{C},2} &\overset{\mathcal{P}_{\mathbb{C}}}{\simeq} \mathbf{Psd}(\mathbb{C}^{\mathrm{op}},\mathcal{SCG}) \overset{\int_{\mathbb{C}}}{\simeq} \mathcal{SCG}_{\downarrow\mathbb{C}},\end{aligned} \tag{2}$$

where

- $\mathcal{Z}^3_{\mathbb{C},1}$ *(resp.* $\mathcal{Z}^3_{\mathbb{C},2}$*) is the* 2–category of first-level (second-level) 3–cocycles of $\mathbb{C}$–modules, *whose objects are triples* (B,A,h) *with* A *and* B *two* $\mathbb{C}$*–modules and* h *a first-level (second-level)* 3*–cocycle of* B *with coefficients in* A*;*
- $\mathbf{Psd}(\mathbb{C}^{\mathrm{op}},\mathcal{BCG})$ *(resp.* $\mathbf{Psd}(\mathbb{C}^{\mathrm{op}},\mathcal{SCG})$*) is the* 2*–category of pseudofunctors from* $\mathbb{C}^{\mathrm{op}}$ *to the* 2*–category of braided (symmetric) categorical groups.*

With the biequivalences $\mathcal{P}_{\mathbb{C}}$ above, we develop a "factor sets theory" for braided or symmetric fibred categorical groups, like Schreier and Eilenberg–Mac Lane did for ordinary group extensions. The biequivalences denoted by $\int_{\mathbb{C}}$ in (2) are actually monoidal enriched versions of the so-called *Grothendieck construction* (see Grothendieck [37; 36]) biequivalence between the 2–category of contravariant pseudofunctors from a category $\mathbb{C}$ to the 2–category of small 2–categories and the 2–category of fibred categories over $\mathbb{C}$, by Giraud [33; 34] (see also the paper by Vistoli [58], for a recent treatment). In regard to these biequivalences $\int_{\mathbb{C}}$ in (2), we should note that the result is presumably known to experts but, since it does not appear in the literature (to the authors' knowledge), we have included it here with the aim of making this paper as self-contained as possible.

By taking quasi-inverses of the biequivalences (2), we deduce our main classification applications for braided and symmetric fibred categorical groups and their homomorphisms. We introduce the *category of first-level* (resp. *second-level*) 3*–cohomology* *classes*, $\mathcal{H}^3_{\mathbb{C},1}$ ($\mathcal{H}^3_{\mathbb{C},2}$), whose objects are triples (B,A,k) with A and B two $\mathbb{C}$–modules and $k\in H^3_{\mathbb{C},1}(B,A)$ ($k\in H^3_{\mathbb{C},2}(B,A)$). An arrow $(p,q)\colon (B,A,k)\to(B',A',k')$ consists of $\mathbb{C}$–module homomorphisms $p\colon B\to B'$ and $q\colon A\to A'$, such that $p^*(k')=q_*(k)\in H^3_{\mathbb{C},1}(B,A')$. Then, we prove:

Theorem 7.5 *There are classifying functors* $\mathrm{cl}\colon \mathcal{BCG}_{\downarrow\mathbb{C}}\to\mathcal{H}^3_{\mathbb{C},1}$ *and* $\mathrm{cl}\colon \mathcal{SCG}_{\downarrow\mathbb{C}}\to\mathcal{H}^3_{\mathbb{C},2}$, *written*

$$(\mathbb{P}\xrightarrow{F}\mathbb{P}')\overset{\mathrm{cl}}{\mapsto}\big((\pi_0\mathbb{P},\pi_1\mathbb{P},k_3\mathbb{P})\xrightarrow{(\pi_0F,\pi_1F)}(\pi_0\mathbb{P}',\pi_1\mathbb{P}',k_3\mathbb{P}')\big),$$

which have the following properties:

- *Both classifying functors are essentially surjective on objects. That is, for any object* (B,A,k) *of* $\mathcal{H}^3_{\mathbb{C},1}$ *(resp.* $\mathcal{H}^3_{\mathbb{C},2}$*), there is a braided (symmetric)* $\mathbb{C}$*–fibred categorical group* $\mathbb{P}$ *with an isomorphism* $\mathrm{cl}(\mathbb{P})\cong(B,A,k)$.
- *For any isomorphism* $(p,q)\colon \mathrm{cl}(\mathbb{P})\cong\mathrm{cl}(\mathbb{P}')$, *there is a braided* $\mathbb{C}$*–fibred equivalence* $F\colon\mathbb{P}\to\mathbb{P}'$ *such that* $\mathrm{cl}(F)=(p,q)$.
- $\mathrm{cl}(F)$ *is an isomorphism if and only if* F *is a braided* $\mathbb{C}$*–fibred equivalence.*
- *For any given* $(p,q)\colon\mathrm{cl}(\mathbb{P})\to\mathrm{cl}(\mathbb{P}')$, *the set of monoidal* $\mathbb{C}$*–fibred isomorphism classes,* $[F]$, *of braided* $\mathbb{C}$*–fibred functors* $F\colon\mathbb{P}\to\mathbb{P}'$ *with* $\mathrm{cl}(F)=(p,q)$ *is a principal homogeneous space under the abelian group* $H^2_{\mathbb{C},1}(B,A)$. *Hence, there is a (nonnatural) bijection*

$$\{[F]\colon\mathbb{P}\to\mathbb{P}'\mid \mathrm{cl}(F)=(p,q)\}\cong H^2_{\mathbb{C},1}(B,A).$$

(cf Joyal and Street [41, Theorem 3.3] and Cegarra and Khmaladze [15, Theorems 22, 24], [16, Theorem 3.12]).

For any $\mathbb{C}$–modules B,A, let $\mathcal{BCG}_{\downarrow\mathbb{C}}[B,A]$ (resp. $\mathcal{SCG}_{\downarrow\mathbb{C}}[B,A]$) denote the set of braided $\mathbb{C}$–fibred equivalence classes of those braided (symmetric) $\mathbb{C}$–fibred categorical groups $\mathbb{P}$ with $\pi_0\mathbb{P}=B$ and $\pi_1\mathbb{P}=A$. Then, for any integer $\mathrm{r}\geq 2$, we have:

Theorem 7.6 *There are natural bijections*

$$H^3_{\mathbb{C},1}(B,A)\cong\mathcal{BCG}_{\downarrow\mathbb{C}}[B,A],\quad H^3_{\mathbb{C},\mathrm{r}}(B,A)\cong\mathcal{SCG}_{\downarrow\mathbb{C}}[B,A].$$

We should recall here the classical result by Deligne in [21, Proposition 1.4.15], where he states that there is a natural bijection $\mathrm{Ext}^2_{\mathrm{Mod}_{\mathbb{C}}}(B,A)\cong\mathcal{P}ic_{\downarrow\mathbb{C}}[B,A]$, where the latter denotes the set of braided $\mathbb{C}$–fibred equivalence classes of those strictly commutative symmetric $\mathbb{C}$–fibred categorical groups $\mathbb{P}$ with $\pi_0\mathbb{P} = B$ and $\pi_1\mathbb{P} = A$. Hence, the natural inclusions

$$\mathrm{Ext}^2_{\mathrm{Mod}_{\mathbb{C}}}(B, A) \subseteq H^3_{\mathbb{C},2}(B, A) \subseteq H^3_{\mathbb{C},1}(B, A)$$

are, in general, strict. Also, we should mention the work of Breen [8] (see also Aldrovandia and Noohi [1]), where he offers an excellent discussion about the cohomology classification of symmetric and braided categorical group stacks over an arbitrary site $\mathbb{C}$. However, Breen's results are far from being as explicit as ours in this work in terms of cocycles when $\mathbb{C}$ is discrete.

The plan of this paper, briefly, is as follows. After this introductory first section, the paper is organized in seven sections. In Section 2, at the same time as fixing notation and terminology, we review some aspects concerning cohomology groups of small categories, simplicial sets and diagrams of simplicial sets. Section 3 mainly includes the definition of the level–r cohomology groups of $\mathbb{C}$–modules, $H^n_{\mathbb{C},\mathrm{r}}(B, A)$, and the construction of the cochain complexes $C^\bullet_{\mathbb{C},\mathrm{r}}(B, A)$ for computing them. Next, in Section 4, we pay particular attention to the complexes $C^\bullet_{\mathbb{C},1}(B, A)$ and $C^\bullet_{\mathbb{C},2}(B, A)$, with an explicit description of the level–r cochains that are low-dimensional cocycles and coboundaries. Section 5 is devoted to showing natural interpretations for the cohomology groups $H^1_{\mathbb{C},\mathrm{r}}(B, A)$ and $H^2_{\mathbb{C},\mathrm{r}}(B, A)$, in terms of $\mathbb{C}$–module homomorphisms and extensions, respectively. Section 6 is long and very technical, but crucial to our discussions. Here, we mainly show in detail how the 2–category of braided $\mathbb{C}$–fibred categorical groups is biequivalent to the 2–category of pseudofunctors from $\mathbb{C}^{\mathrm{op}}$ to the category of braided categorical groups. In Section 7, we include our theorems on the homotopy classification of braided and symmetric $\mathbb{C}$–fibred categorical groups and their homomorphisms by means of the cohomology groups $H^3_{\mathbb{C},\mathrm{r}}(B, A)$ and $H^2_{\mathbb{C},\mathrm{r}}(B, A)$.

2 Cohomology of diagrams of pointed simplicial sets

The material of this section is fairly standard, so the expert reader may skip most of it. In general, we employ the standard symbolism and nomenclature to be found in texts on the cohomology of simplicial sets, and we refer to Bousfield and Kan [6], Gabriel and Zisman [32], Illusie [40], Mac Lane [44], and mainly to Goerss and Jardine [35] for the background.

2.1 Cohomology of small categories and simplicial sets

If $\mathbb{C}$ is a small category, then a functor $A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of abelian groups, is called a $\mathbb{C}$*–module*. We write $\mathrm{Mod}_{\mathbb{C}}$ for the category of $\mathbb{C}$–modules.

There is a "global sections" functor $\varprojlim_{\mathbb{C}}\colon \mathrm{Mod}_{\mathbb{C}} \to \mathbf{Ab}$, given by

$$\varprojlim_{\mathbb{C}}(A) = \Big\{(a_u) \in \prod_{u\in \mathrm{Ob}\mathbb{C}} A_u \mid \sigma^* a_u = a_v \text{ for every } \sigma\colon v \to u \text{ in } \mathbb{C}\Big\},$$

where we write σ^* for $A(\sigma)$, and, since the category $\mathrm{Mod}_{\mathbb{C}}$ has enough injectives, we can form the right derived functors of $\varprojlim_{\mathbb{C}}$. These yield the *cohomology groups of* $\mathbb{C}$ *with coefficients in a* $\mathbb{C}$*–module* A, due to Roos [52], Watts [59], Mitchell [46] and Baues and Wrisching [3]:

$$H^n(\mathbb{C}, A) = (R^n \varprojlim_{\mathbb{C}})(A), \quad n \geq 0.$$

Cohomology theory of small categories is itself a basis for other cohomology theories such as, for example, the cohomology theory of simplicial sets with twisted coefficients. Recall now that the simplicial category, denoted by Δ, has objects the ordered sets $[n] = \{0, \dots, n\}$, $n \geq 0$, and its arrows are weakly monotone maps $c\colon [m] \to [n]$. Throughout, we shall regard each ordered set $[n]$ as a category with exactly one arrow $j \to i$ whenever $i \leq j$; then, a weakly monotone map $[m] \to [n]$ is the same as a functor, and Δ is viewed as a full subcategory of the category of small categories, denoted by $\mathbf{Cat}$. As is usual, for $X\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ any simplicial set, we write $c^*\colon X_n \to X_m$ for the map induced by a map $c\colon [m] \to [n]$ in the simplicial category, and $d_i\colon X_n \to X_{n-1}$ for its corresponding face maps. The category of simplicial sets is denoted by $\mathbf{S}$.

If X is a simplicial set, then its category of simplices, denoted by $\int_\Delta X$, is the category whose objects are pairs $(x, [m])$, where $[m] \in \Delta$ and $x \in X_m$; an arrow $(x, [m]) \to (y, [n])$ is a weakly monotone map $c\colon [m] \to [n]$ such that $c^* y = x$. A *coefficient system on* X is an $\int_\Delta X$–module, that is, a functor $A\colon (\int_\Delta X)^{\mathrm{op}} \to \mathbf{Ab}$, and the *cohomology groups of* X *with coefficients in* A are, by definition,

$$H^n(X, A) = H^n(\textstyle\int_\Delta X, A), \quad n \geq 0.$$

The functor *nerve* $\mathbf{N}\colon \mathbf{Cat} \to \mathbf{S}$, associates to any small category $\mathbb{C}$ the simplicial set $\mathbf{N}\mathbb{C} = \mathbf{Cat}(-, \mathbb{C})\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$, whose n–simplices are the functors $\sigma\colon [n] \to \mathbb{C}$, or tuples of arrows in $\mathbb{C}$

$$\sigma = \big(\sigma j \xrightarrow{\sigma_{i,j}} \sigma i\big)_{0\leq i\leq j\leq n}$$

such that $\sigma_{i,j}\sigma_{j,k} = \sigma_{i,k}$ and $\sigma_{i,i} = 1_{\sigma i}$. Several times, we shall identify such a simplex $\sigma \in \mathbf{N}_n\mathbb{C}$ with the basic string

$$(\sigma_1, \dots, \sigma_p) \;=\; \sigma 0 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_p} \sigma p,$$

where $\sigma_i = \sigma_{i-1,i}$. There is a functor by Illusie [40, Chaper VI, (3.1.2)]

$$\textstyle\int_\Delta \mathbf{N}\mathbb{C} \to \mathbb{C}, \big((\sigma,[m]) \xrightarrow{c} (\tau,[n])\big) \mapsto (\sigma 0 \xrightarrow{\tau_{0,c0}} \tau 0), \tag{3}$$

that induces, for any $\mathbb{C}$–module A, canonical isomorphisms [40, Chaper VI, (3.4.2)]

$$H^n(\mathbb{C}, A) \cong H^n(\mathbf{N}\mathbb{C}, A), \quad n \geq 0,$$

(see also Gabriel and Zisman [32, Appendix II, Proposition 3.3], Baues and Wrisching [3, Proposition 8.5] and Goerss and Jardine [35, Chaper IV, Lemma 2.11]).

The functor $\mathbf{N}$ fully embeds $\mathbf{Cat}$ in the category of simplicial sets as a reflective subcategory, with reflector the *categorization functor* $\mathbf{C}\colon \mathbf{S} \to \mathbf{Cat}$, Mac Lane [45, II,8], which can be described as follows. For any simplicial set X, one forms the graph whose objects are the 0–simplices of X and whose arrows $x \to x'$ are those 1–simplices $y \in X_1$ such that $d_0 y = x$ and $d_1 y = x'$. Then $\mathbf{C}X$ is the quotient category of the free category of this graph by the relations $s_0 x = 1_x$ if $x \in X_0$, and $(d_2 z)(d_0 z) = d_1 z$ if $z \in X_2$. For each simplicial set X, the unit of the adjunction $\mathbf{u}\colon X \to \mathbf{NC}X$ is the simplicial map carrying a simplex $z \in X_n$ to the simplex $\mathbf{u}z \in \mathbf{N}_n\mathbf{C}X$ with

$$\mathbf{u}z_{i,j} = [d_2^{n-j} d_1^{j-i-1} d_0^i z]\colon d_1^{n-j} d_0^j z \longrightarrow d_1^{n-i} d_0^i z, \quad 0 \leq i < j \leq n,$$

where, for each $y \in X_1$, $[y]\colon d_0 y \to d_1 y$ denotes the arrow in $\mathbf{C}X$ that y defines, and each $d_k^m = d_k \overset{m}{\cdots} d_k\colon X_n \to X_{n-m}$ denotes the m–fold iterated composition of the k–th face operators of X. The counit of the adjunction is an identity, that is, for any small category $\mathbb{C}$, $\mathbf{CN}\mathbb{C} = \mathbb{C}$.

If X is any simplicial set, then, by composition with the canonical composite functor

$$\textstyle\int_\Delta X \xrightarrow{\int_\Delta \mathbf{u}} \int_\Delta \mathbf{NC}X \xrightarrow{(3)} \mathbf{C}X,$$

any $\mathbf{C}X$–module A gives a coefficient system on X, also denoted by A. The *cohomology groups of X with coefficients in the $\mathbf{C}X$–module A* are, by definition, the cohomology groups of X with coefficients in the system A. For computing these cohomology groups $H^n(X, A)$, there is a cosimplicial abelian group $C^\bullet(X, A)\colon \Delta \to \mathbf{Ab}$, which is defined in degree p by

$$C^p(X, A) = \textstyle\prod_{x \in X_p} A_{d_0^p x},$$

so that a *p–cochain of X with coefficients in A* is a map $f\colon X_p \to \bigsqcup_{v \in X_0} A_v$ such that $fx \in A_{d_0^p x}$, for each $x \in X_p$. The homomorphism $c_*\colon C^p(X, A) \to C^q(X, A)$ induced by a map $c\colon [p] \to [q]$ in Δ takes a p–cochain f as above to the q–cochain $c_* f$ given by $(c_* f)x = f(c^* x)$ if $cp = q$ and $(c_* f)x = [d_1^{q-cp-1} d_0^{cp} x]^* f(c^* x)$ if $cp < q$, where $[d_1^{q-cp-1} d_0^{cp} x]^*\colon A_{d_0^p c^* x} \to A_{d_0^q x}$ is the homomorphism induced on coefficients by the 1–simplex $d_1^{q-cp-1}d_0^{cp}x$ of X. In particular, the coboundary operator on the associated complex of *normalized cochains*, equally denoted by $C^\bullet(X,A)$, $\partial^p\colon C^p(X,A)\to C^{p+1}(X,A)$ is given by

$$(\partial^p f)x=\sum_{i=0}^{p}(-1)^i f(d_i x)+(-1)^{p+1}[d_0^p x]^* f(d_{p+1}x).$$

In Gabriel and Zisman [32, Appendix II, Proposition 3.3], there are shown canonical isomorphisms:

$$H^n(X,A)=H^nC^\bullet(X,A),\quad n\geq 0. \tag{4}$$

If $Y\subseteq X$ is any subsimplicial set, then the *relative cohomology groups* are denoted, as usual, by $H^n(X,Y;A)$. Thus, for any $\mathbf{C}X$–module A, $H^n(X,Y;A)=H^nC^\bullet(X,Y;A)$, where $C^\bullet(X,Y;A)\subseteq C^\bullet(X,A)$ is the cochain subcomplex making exact the cochain complex sequence

$$0\to C^\bullet(X,Y;A)\longrightarrow C^\bullet(X,A)\xrightarrow{\text{res}} C^\bullet(Y,A)\to 0.$$

Remark 2.1 Recall from Gabriel and Zisman [32, pages 10, 39], that the *fundamental groupoid*, ΠX, of any simplicial set X, is the localized category of fractions $\mathbf{C}X[\Sigma^{-1}]$ for Σ the set of all morphisms of $\mathbf{C}X$. Those $\mathbf{C}X$–modules that factor through the localization functor $\mathbf{C}X\to\Pi X$ are precisely the *local coefficient systems* on X; that is, a local coefficient system on X is a $\mathbf{C}X$–module $A\colon \mathbf{C}X^{\mathrm{op}}\to\mathbf{Ab}$, such that for any $x\in X_1$, A gives an isomorphism $[x]^*\colon A_{d_1x}\cong A_{d_0x}$. When the simplicial set X satisfies the Kan extension condition, then $\mathbf{C}X\cong\Pi X$ is an isomorphism, whence every $\mathbf{C}X$–module is locally constant.

In general, the cohomology groups (4) are not invariants of the weak homotopy type of the simplicial set X, that is, for an arbitrary $\mathbf{C}X$–module A, the cohomology groups $H^n(X,A)$ cannot be obtained from the geometric realization $|X|$ of the simplicial set. However, if A is a local system of coefficients on a simplicial set Y, then Quillen [51, II, Proposition 4] proves that any weak homotopy equivalence of simplicial sets $f\colon X\to Y$ induces isomorphisms in cohomology $H^n(Y,A)\cong H^n(X,f^*A)$.

2.2 Cohomology of diagrams of simplicial sets

A cohomology theory of diagrams of simplicial sets comes from the cohomology theory of simplicial sets thanks to Bousfield and Kan's *homotopy colimit* construction [6]. Let us recall that, for any small category $\mathbb{C}$, the *simplicial replacement* of a $\mathbb{C}$–diagram of simplicial sets, that is, a functor, $X\colon\mathbb{C}^{\mathrm{op}}\to\mathbf{S}$ is the bisimplicial set

$$\Psi_{\mathbb{C}}(X)\colon \Delta^{\mathrm{op}}\to\mathbf{S},\quad [p]\mapsto \textstyle\coprod_{\sigma\in\mathrm{N}_p\mathbb{C}} X_{\sigma 0}.$$

The homomorphism induced by a map $c\colon [q] \to [p]$, in Δ, maps the simplicial set component at $\sigma\colon [p] \to \mathbb{C}$ into the component at the composite $\sigma c\colon [q] \to \mathbb{C}$, just by the simplicial set map $\sigma^*_{0,c0}\colon X_{\sigma 0} \to X_{\sigma c\, 0}$ attached to the diagram at the morphism $\sigma_{0,c0}\colon \sigma c0 \to \sigma 0$ of $\mathbb{C}$. Then, the homotopy colimit of the $\mathbb{C}$–diagram X, denoted by $\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X$, is the diagonal simplicial set of the simplicial replacement, that is,

$$\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X = \mathrm{diag}\, \Psi_{\mathbb{C}}(X)\colon \Delta^{\mathrm{op}} \to \mathbf{S}.$$

Notice that, for any $X\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{S}$, there is a canonical simplicial projection map $\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X \to \mathbf{N}\mathbb{C}$, defined by $(z,\sigma) \mapsto \sigma$, which, by the adjunction $\mathbf{C} \dashv \mathbf{N}$, gives a canonical functor $\mathbf{C}(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X) \to \mathbb{C}$. Hence, every $\mathbb{C}$–module $A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$ gives rise to a $\mathbf{C}(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X)$–module, also denoted by A, and therefore the corresponding cohomology groups

$$H^n(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X, A)$$

are defined as in (4). We refer to them as the *cohomology groups of the $\mathbb{C}$–diagram X with coefficients in A.*

As we stressed in Remark 2.1, the cohomology groups above cannot be expressed in terms of the geometric realization space $|\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X|$ since A does not generally define a local system of coefficients on it. However, we have the following invariance result (see Moerdijk and Svensson [48, Theorem 2.3, Corollary 2.5] for a very general Invariance Theorem):

Proposition 2.2 *Let $X, Y\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{S}$ be two $\mathbb{C}$–diagrams of simplicial sets, and suppose that $f\colon X \to Y$ is a pointwise weak equivalence between them (ie, a natural transformation such that $f_u\colon X_u \to Y_u$ is a weak homotopy equivalence for each object $u \in \mathbb{C}$). Then, for any $\mathbb{C}$–module A, f induces isomorphisms*

$$H^n(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, Y, A) \cong H^n(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X, A).$$

Proof Let $A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$ be any given $\mathbb{C}$–module. For any two objects $u, v \in \mathbb{C}$ let $C^\bullet(X_u, A_v)$ be the cosimplicial abelian group of cochains of the simplicial set X_u with constant coefficients in the abelian group A_v, and let $C^{\bullet,\bullet}(X, A)$ denote the double cosimplicial abelian group with

$$C^{p,\bullet}(X, A) = \textstyle\prod_{\sigma \in \mathbf{N}_p\mathbb{C}} C^\bullet(X_{\sigma_0}, A_{\sigma p}).$$

The homomorphism $c_*\colon C^{p,\bullet}(X, A) \to C^{q,\bullet}(X, A)$ induced by a map $c\colon [p] \to [q]$ in Δ maps an $f = (f_\sigma) \in C^{p,m}(X, A)$ to $c_* f = ((c_* f)_\tau) \in C^{q,m}(X, A)$, whose component at a q–simplex, say $\tau\colon [q] \to \mathbb{C}$, of $\mathbf{N}\mathbb{C}$ is the dotted map at the top of the commutative square below.

$$\begin{tikzcd} (X_{\tau 0})_m \arrow[r, dotted, "(c_* f)_\tau"] \arrow[d, "\tau^*_{0,c\,0}"'] & A_{\tau q} \\ (X_{\tau c\, 0})_m \arrow[r, "f_{\tau c}"] & A_{\tau c p} \arrow[u, "\tau^*_{cp,q}"'] \end{tikzcd}$$

Since we have an identification $\operatorname{diag} C^{\bullet,\bullet}(X,A) = C^\bullet(\underrightarrow{\operatorname{holim}}_{\mathbb{C}} X, A)$, which is natural both in X and A, the homomorphisms $f^*\colon H^n(\underrightarrow{\operatorname{holim}}_{\mathbb{C}} Y, A) \to H^n(\underrightarrow{\operatorname{holim}}_{\mathbb{C}} X, A)$ are those induced in homology by the map $f^*\colon \operatorname{diag} C^{\bullet,\bullet}(Y,A) \to \operatorname{diag} C^{\bullet,\bullet}(X,A)$. Now, by hypothesis, all maps $f_u\colon X_u \to Y_u$ are weak equivalences, whence they induce cohomology isomorphisms $f^*\colon C^\bullet(Y_u, A_v) \to C^\bullet(X_u, A_v)$. Then, every cosimplicial map $C^{p,\bullet}(Y,A) \to C^{p,\bullet}(X,A)$ is also a cohomology isomorphism, for any $p \geq 0$, and the proposition follows from the generalized Eilenberg–Zilber theorem of Dold and Puppe [22] (see Goerss and Jardine [35, Lemma 2.6]). □

If $\mathbf{S}_*$ denotes the category of pointed simplicial sets, then a $\mathbb{C}$*–diagram of pointed sets* is a functor $X\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{S}_*$. This is the same as a pointed $\mathbb{C}$–diagram of simplicial sets in the natural sense: Let $*\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{S}$ be the $\mathbb{C}$–diagram which is constant the simplicial set with only one simplex. Then a pointed $\mathbb{C}$–diagram of simplicial sets means a $\mathbb{C}$–diagram $X\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{S}$ endowed with a $\mathbb{C}$–diagram morphism $* \to X$. There is an induced simplicial inclusion map $\underrightarrow{\operatorname{holim}}_{\mathbb{C}} * \hookrightarrow \underrightarrow{\operatorname{holim}}_{\mathbb{C}} X$, and the *reduced cohomology groups of the* $\mathbb{C}$*–diagram of pointed simplicial sets*, denoted by $\widetilde{H}^n(\underrightarrow{\operatorname{holim}}_{\mathbb{C}} X, A)$, are defined to be the corresponding relative cohomology groups, that is, for any $\mathbb{C}$–module A,

$$\widetilde{H}^n(\underrightarrow{\operatorname{holim}}_{\mathbb{C}} X, A) = H^n(\underrightarrow{\operatorname{holim}}_{\mathbb{C}} X, \underrightarrow{\operatorname{holim}}_{\mathbb{C}} *; A). \tag{5}$$

Note that there is a simplicial isomorphism $\underrightarrow{\operatorname{holim}}_{\mathbb{C}} * \cong \mathbf{N}\mathbb{C}$, $(*,\sigma) \leftrightarrow \sigma$, and therefore

$$\widetilde{H}^n(\underrightarrow{\operatorname{holim}}_{\mathbb{C}} X, A) = H^n(\underrightarrow{\operatorname{holim}}_{\mathbb{C}} X, \mathbf{N}\mathbb{C}; A).$$

Since, for any $X\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{S}_*$, the retractive diagram $X \leftrightarrows *$ gives a simplicial retraction $\underrightarrow{\operatorname{holim}}_{\mathbb{C}} X \leftrightarrows \mathbf{N}\mathbb{C}$, the restriction homomorphisms in the long exact sequence

$$\cdots \widetilde{H}^n(\underrightarrow{\operatorname{holim}}_{\mathbb{C}} X, A) \to H^n(\underrightarrow{\operatorname{holim}}_{\mathbb{C}} X, A) \to H^n(\mathbf{N}\mathbb{C}, A) \to \widetilde{H}^n(\underrightarrow{\operatorname{holim}}_{\mathbb{C}} X, A) \cdots$$

split. Hence, there are isomorphisms

$$H^n(\underrightarrow{\operatorname{holim}}_{\mathbb{C}} X, A) \cong \widetilde{H}(\underrightarrow{\operatorname{holim}}_{\mathbb{C}} X, A) \oplus H^n(\mathbb{C}, A).$$

Remark 2.3 The category of $\mathbb{C}$–diagrams of pointed simplicial sets, $\mathbf{S}_*^{\mathbb{C}^{\mathrm{op}}}$, has a closed Quillen model structure, where weak equivalences and fibrations are levelwise

(Bousfield and Kan [6, page 313], Quillen [51, II, Theorem 4]). Therefore, for any $X\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{S}_*$ and $A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$, Quillen's homotopical cohomology groups of X with coefficients in A, say $\mathrm{Q}^n(X, A)$, are defined [51, II, 5.1] to be

$$\mathrm{Q}^n(X, A) = \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{S}_*^{\mathbb{C}^{\mathrm{op}}}}(X, K(A,n)),$$

the abelian groups of morphisms in the homotopy category from X to the $\mathbb{C}$–diagram of simplicial abelian groups $K(A,n)\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{SAb}$ obtained by composing A with the Eilenberg–Mac Lane functors $K(-,n)\colon \mathbf{Ab} \to \mathbf{SAb}$.

Also, for any X and A as above, Dwyer and Kan [26, 3.2] defined corresponding cohomology groups, say $\mathrm{DK}^n(X, A)$, as the cohomology groups of the (relative to $*$) complex whose n–cochains are natural transformations $\phi\colon X_n \to A$ (with coboundary $\partial\phi = \sum(-1)^i d_i$ in the standard way) or, equivalently, as

$$\mathrm{DK}^n(X, A) = [X, K(A,n)]_{\mathbf{S}_*^{\mathbb{C}^{\mathrm{op}}}},$$

the abelian groups of homotopy classes of pointed simplicial natural transformations from X to $K(A,n)$; see Piazenza [50, Theorem 2.6]. When X is cofibrant, then there are isomorphisms $\mathrm{Q}^n(X, A) \cong \mathrm{DK}^n(X, A)$, by Quillen [51, I, Corollary 1], since $K(A,n)$ is fibrant in the category of diagrams.

But notice that, in general, the cohomology defined in (5) does not coincide with Quillen's homotopical cohomology. This is due to the fact that the homotopy colimit construction $\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\colon \mathbf{S}_*^{\mathbb{C}^{\mathrm{op}}} \to (\mathbf{S}{\downarrow}\mathbf{N}\mathbb{C})_*$ is not a right Quillen equivalence, between $\mathbf{S}_*^{\mathbb{C}^{\mathrm{op}}}$ and the closed model category of pointed objects over $\mathbf{N}\mathbb{C}$ (ie, of retractive simplicial sets over $\mathbf{N}\mathbb{C}$), except for example, when $\mathbb{C}$ is a groupoid (see Hollander [38, Theorem 2.7], or Dror, Dwyer and Kan [23] for $\mathbb{C} = G$ a group). The closed model structure on $(\mathbf{S}{\downarrow}\mathbf{N}\mathbb{C})_*$ is induced by the usual one of simplicial sets; that is, where a map is a weak equivalence, cofibration or fibration if and only if it is a weak equivalence, cofibration or fibration of simplicial sets, respectively.

Indeed, in the case where $\mathbb{C}$ is a groupoid there are isomorphisms

$$\mathrm{Hom}_{\mathrm{Ho}\,\mathbf{S}_*^{\mathbb{C}^{\mathrm{op}}}}(X, K(A,n)) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{S}\downarrow\mathbf{N}\mathbb{C})_*}\big(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X, \underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(A,n)\big).$$

Now, $\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X$ is cofibrant and, if $\mathbb{C}$ is a groupoid, then $\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(A,n)$ is fibrant in the model category $(\mathbf{S} \downarrow \mathbf{N}\mathbb{C})_*$. Therefore, there are isomorphisms

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{S}\downarrow\mathbf{N}\mathbb{C})_*}\big(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X, \underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(A,n)\big) \cong \big[\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X, \underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(A,n)\big]_{(\mathbf{S}\downarrow\mathbf{N}\mathbb{C})_*}$$

relating morphisms in the homotopy category to homotopy classes of maps in the pointed comma category $(\mathbf{S}{\downarrow}\mathbf{N}\mathbb{C})_*$. Finally, by Goerss and Jardine [35, VI, Lemma 4.13], we have isomorphisms

$$\big[\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X, \underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(A,n)\big]_{(\mathbf{S}\downarrow\mathbf{N}\mathbb{C})_*} \cong \widetilde{H}^n(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X, A),$$

whence $\mathrm{Q}^n(X, A) \cong \widetilde{H}^n(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, X, A)$ whenever $\mathbb{C}$ is a groupoid.

3 Higher cohomologies of $\mathbb{C}$–modules

As above, if B is an abelian group, then, for each integer $m \geq 2$, let $K(B,m)$ denote the Eilenberg–Mac Lane minimal complex having B as its unique nontrivial homotopy group at dimension m, that is,

$$K(B,m)\colon \Delta^{\mathrm{op}} \to \mathbf{Set}, \quad [n] \mapsto Z^m([n], B),$$

is the simplicial abelian group whose n–simplices are normalized m–cocycles of the category $[n]$ with coefficients in B. For any two abelian groups B and A, and each integer $\mathrm{r} \geq 1$, the cohomology groups

$$H^n_{\mathrm{r}}(B, A) = H^{n+\mathrm{r}}(K(B, \mathrm{r}+1), A), \quad n \geq 0,$$

define the level–r Eilenberg–Mac Lane cohomology theory of the abelian group B with coefficients in the abelian group A [27; 28; 29; 43]. Let us now suppose that $\mathbb{C}$ is any given small category. By composing any $\mathbb{C}$–module, say $B\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$, with the functor $K(-,m)\colon \mathbf{Ab} \to \mathbf{S}_*$, we obtain a $\mathbb{C}$–diagram of pointed simplicial sets, denoted by

$$K(B,m)\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{S}_*, \quad u \mapsto K(B_u, m).$$

Then, in a natural way, we establish the following:

Definition 3.1 Let $\mathbb{C}$ be a small category, and $\mathrm{r} \geq 1$ an integer. For any two $\mathbb{C}$–modules $B, A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$, the level–r cohomology groups of B with coefficients in A are defined by

$$H^n_{\mathbb{C},\mathrm{r}}(B, A) = \widetilde{H}^{n+\mathrm{r}}(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(B, \mathrm{r}+1), A).$$

Thus, the cohomology groups $H^n_{\mathbb{C},\mathrm{r}}(B, A)$ are computed from the relative cochain complex

$$C^{\bullet+\mathrm{r}}(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(B, \mathrm{r}+1), \mathbf{N}\mathbb{C}; A).$$

However, both for the theoretical and computational interests, it is desirable to have an explicit description of more appropriate cochain complexes to compute these cohomology groups. To do that, recall that, for any abelian group B, the chain complex of an Eilenberg–Mac Lane space $K(B,m)$ is chain-homotopic to the m–fold iterated bar construction on the group algebra $\bar{B}^m(\mathbb{Z}B)$, which we will denote as $\mathcal{A}(B,m)$, Eilenberg and Mac Lane [28, I, Theorem 20.3]. Thus, the Eilenberg–Mac Lane reduction quasi-isomorphism (ie, cohomology-isomorphism cochain map)

$$\mathbf{k}\colon C^\bullet(K(B,\mathrm{r}+1),A) \to \mathrm{Hom}(\mathcal{A}(B,\mathrm{r}+1),A) \tag{6}$$

provides the possibility to explicitly compute the level–r cohomology groups of an abelian group B with coefficients in an abelian group A, as $H^n_{\mathrm{r}}(B,A) = H^n C^\bullet_{\mathrm{r}}(B,A)$, where

$$C^\bullet_{\mathrm{r}}(B,A) = \mathrm{Hom}(\mathcal{A}(B,\mathrm{r}+1)_{\bullet+\mathrm{r}},A)$$

is the cochain complex

$$\cdots \to \mathrm{Hom}(\mathcal{A}(B,\mathrm{r}+1)_n,A) \xrightarrow{\partial^*_n} \mathrm{Hom}(\mathcal{A}(B,\mathrm{r}+1)_{n+1},A) \to \cdots \tag{7}$$

with dimensions raised by r. One of the main goals of this section is to establish the counterpart result in the diagrammatic setup by means of certain cochain complexes $C^\bullet_{\mathbb{C},\mathrm{r}}(B,A)$, the cohomology groups of which are the level–r cohomology groups $H^n_{\mathbb{C},\mathrm{r}}(B,A)$, given in Definition 3.1. This cochain complex is defined as follows:

Definition 3.2 For any two $\mathbb{C}$–modules $B,A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$, and $\mathrm{r} \geq 1$ an integer, let $C^{\bullet,\bullet}_{\mathbb{C},\mathrm{r}}(B,A)$ be the cochain bicomplex with

$$C^{p,\bullet}_{\mathbb{C},\mathrm{r}}(B,A) = \textstyle\prod_{\sigma\in \mathrm{N}_p\mathbb{C}} \mathrm{Hom}(\mathcal{A}(B_{\sigma 0},\mathrm{r}+1),A_{\sigma p})$$

whose coboundary operator $\partial\colon C^{p-1,\bullet}_{\mathbb{C},\mathrm{r}} \to C^{p,\bullet}_{\mathbb{C},\mathrm{r}}$ is given by the formula

$$(\partial f)_\sigma(x) = f_{d_0\sigma}(\sigma_1^* x) + \textstyle\sum_{i=1}^{p-1}(-1)^i f_{d_i\sigma}(x) + (-1)^p \sigma_p^* f_{d_p\sigma}(x). \tag{8}$$

Then, the complex of *level–*r *cochains* of the $\mathbb{C}$–module B with coefficients in the $\mathbb{C}$–module A, is defined by

$$C^\bullet_{\mathbb{C},\mathrm{r}}(B,A) = \mathrm{KerTot}\big(C^{\bullet,\bullet}_{\mathbb{C},\mathrm{r}}(B,A) \xrightarrow{\mathrm{res}} C^{\bullet,\bullet}_{\mathbb{C},\mathrm{r}}(0,A)\big)^{\bullet+\mathrm{r}}, \tag{9}$$

where Tot is the usual total cochain complex construction on double cochain complexes, the bicomplex homomorphism res is the retraction induced by the zero map $0 \to B$, and the notation "$\bullet + \mathrm{r}$" at the right indicates that the dimensions of the complex are raised by r.

Below is one of the main results of this section.

Theorem 3.3 *For any $B,A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$, two modules over a small category $\mathbb{C}$, and $\mathrm{r} \geq 1$, an integer, there are natural isomorphisms*

$$H^n_{\mathbb{C},\mathrm{r}}(B,A) \cong H^n C^\bullet_{\mathbb{C},\mathrm{r}}(B,A).$$

Proof For any two objects $u, v \in \mathbb{C}$ let $C^\bullet(K(B_u, \mathrm{r}+1), A_v)$ be the cosimplicial abelian group of cochains of $K(B_u, \mathrm{r}+1)$ with coefficients in the abelian group A_v, so that we have the reduction quasi-isomorphism of cochain complexes (6),

$$\mathbf{k}\colon C^\bullet(K(B_u, \mathrm{r}+1), A_v) \to \mathrm{Hom}(\mathcal{A}(B_u, \mathrm{r}+1), A_v),$$

which is natural both on u and v.

Let $C^{\bullet,\bullet}_{\mathbb{C}}(K(B, \mathrm{r}+1), A)$ denote the double cosimplicial abelian group with

$$C^{p,\bullet}_{\mathbb{C}}(K(B, \mathrm{r}+1), A) = \prod\nolimits_{\sigma \in \mathbf{N}_p\mathbb{C}} C^\bullet(K(B_{\sigma 0}, \mathrm{r}+1), A_{\sigma p}) \ .$$

The homomorphism $c_*\colon C^{p,\bullet}(K(B, \mathrm{r}+1), A) \to C^{q,\bullet}(K(B, \mathrm{r}+1), A)$, induced by a map $c\colon [p] \to [q]$ in Δ, maps an $f = (f_\sigma) \in C^{p,m}(K(B, \mathrm{r}+1), A)$ to $c_* f = ((c_* f)_\tau) \in C^{q,m}(K(B, \mathrm{r}+1), A)$, whose component at a q–simplex, say $\tau\colon [q] \to \mathbb{C}$, of $\mathbf{N}\mathbb{C}$ is the composite map

$$K(B_{\tau 0}, \mathrm{r}+1)_m \xrightarrow{\tau^*_{0,c0}} K(B_{\tau c\, 0}, \mathrm{r}+1)_m \xrightarrow{f_{\tau c}} A_{\tau c p} \xrightarrow{\tau^*_{cp,q}} A_{\tau q}.$$

We have a homomorphism of bicomplexes

$$\mathbf{k}\colon C^{\bullet,\bullet}_{\mathbb{C}}(K(B, \mathrm{r}+1), A) \to C^{\bullet,\bullet}_{\mathbb{C},\mathrm{r}}(B, A),$$

inducing a natural quasi-isomorphism on the total cochain complexes

$$\mathbf{k}\colon \mathrm{Tot} C^{\bullet,\bullet}_{\mathbb{C}}(K(B, \mathrm{r}+1), A) \to \mathrm{Tot} C^{\bullet,\bullet}_{\mathbb{C},\mathrm{r}}(B, A),$$

since the cochain complex map $\mathbf{k}\colon C^{p,\bullet}_{\mathbb{C}}(K(B, \mathrm{r}+1), A) \to C^{p,\bullet}_{\mathbb{C},\mathrm{r}}(B, A)$, is a quasi-isomorphism for every p.

Furthermore, since $\mathrm{diag}\, C^{\bullet,\bullet}_{\mathbb{C}}(K(B, \mathrm{r}+1), A) = C^\bullet(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(B, \mathrm{r}+1), A)$, as a result of Dold and Puppe [22, Theorem 2.15], there is a natural quasi-isomorphism of cochain complexes

$$\mathbf{s}\colon \mathrm{Tot} C^{\bullet,\bullet}_{\mathbb{C}}(K(B, \mathrm{r}+1), A) \to C^\bullet(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(B, \mathrm{r}+1), A).$$

By combining the quasi-isomorphisms $\mathbf{k}$ and $\mathbf{s}$, both for B as above and the zero $\mathbb{C}$–module $0\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$, we get the following commutative diagram of cochain complexes, induced by the inclusion of $\mathbb{C}$–modules $0 \hookrightarrow B$,

$$\begin{tikzcd}
C^\bullet(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(B,\mathrm{r}+1), A) \arrow[r, "\mathrm{res}'"] & C^\bullet(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(0,\mathrm{r}+1), A) \\
\mathrm{Tot} C^{\bullet,\bullet}_{\mathbb{C}}(K(B,\mathrm{r}+1), A) \arrow[r, "\mathrm{res}''"] \arrow[u, "\mathbf{s}"] \arrow[d, "\mathbf{k}"'] & \mathrm{Tot} C^{\bullet,\bullet}_{\mathbb{C}}(K(0,\mathrm{r}+1), A) \arrow[u, "\mathbf{s}"'] \arrow[d, "\mathbf{k}"] \\
\mathrm{Tot} C^{\bullet,\bullet}_{\mathbb{C},\mathrm{r}}(B, A) \arrow[r, "\mathrm{res}"] & \mathrm{Tot} C^{\bullet,\bullet}_{\mathbb{C},\mathrm{r}}(0, A)
\end{tikzcd}$$

where the three horizontal restriction maps are retractions.

We now notice the following straightforward five observations: (1) there is an identification $C^\bullet(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(0,\mathrm{r}+1), A) = C^\bullet(\mathbf{N}\mathbb{C}, A)$; (2) the kernel of res', the top retraction in the diagram, is the relative cochain complex $C^\bullet(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(B,\mathrm{r}+1), \mathbf{N}\mathbb{C}; A)$; (3) the bicomplexes $C^{\bullet,\bullet}_{\mathbb{C}}(K(0,\mathrm{r}+1), B)$ and $C^{\bullet,\bullet}_{\mathbb{C},\mathrm{r}}(0, A)$ are both isomorphic to the double cochain complex which is the complex $C^\bullet(\mathbf{N}\mathbb{C}, A)$ constant in the vertical direction; (4) the vertical complex maps $\mathbf{k}$ and $\mathbf{s}$ at the right in the diagram are actually the identity maps on $C^\bullet(\mathbf{N}\mathbb{C}, A)$; (5) the complex $C^\bullet_{\mathbb{C},\mathrm{r}}(B, A)$, by definition, occurs in the diagram as the kernel of the bottom retraction res with its dimensions raised by r, that is, $C^\bullet_{\mathbb{C},\mathrm{r}}(B, A) = \mathrm{Ker}(\mathrm{res})^{\bullet+\mathrm{r}}$.

It follows that the complex maps induced on kernels

$$C^{\bullet+\mathrm{r}}_{\mathbb{C}}(\underrightarrow{\mathrm{holim}}_{\mathbb{C}}\, K(B,\mathrm{r}+1), \mathbf{N}\mathbb{C}; A) \xleftarrow{\mathbf{k}} \mathrm{Ker}(\mathrm{res}'')^{\bullet+\mathrm{r}} \xrightarrow{\mathbf{s}} C^\bullet_{\mathbb{C},\mathrm{r}}(B, A)$$

are both quasi-isomorphisms, whence $H^n_{\mathbb{C},\mathrm{r}}(B, A) \cong H^n C^\bullet_{\mathbb{C},\mathrm{r}}(B, A)$, as claimed. □

One of the relevant consequences of Theorem 3.3 follows:

Theorem 3.4 *Let $B, A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$ be $\mathbb{C}$–modules, and $\mathrm{r} \geq 1$ an integer. For any $n \leq \mathrm{r}+1$, there are natural isomorphisms*

$$H^n_{\mathbb{C},\mathrm{r}+1}(B, A) \cong H^n_{\mathbb{C},\mathrm{r}}(B, A),$$

and a monomorphism

$$H^{\mathrm{r}+2}_{\mathbb{C},\mathrm{r}+1}(B, A) \hookrightarrow H^{\mathrm{r}+2}_{\mathbb{C},\mathrm{r}}(B, A).$$

Proof All the isomorphisms and the monomorphism above are established through the suspension-inclusion maps [28, Section 14] $\mathcal{A}(B,\mathrm{r}+1)_{\bullet+1} \hookrightarrow \mathcal{A}(B,\mathrm{r}+2)$, which are indeed the identity maps between the q–chains of $\mathcal{A}(B,\mathrm{r}+1)$ and the $(q+1)$–chains of $\mathcal{A}(B,\mathrm{r}+2)$, whenever $q \leq 2\mathrm{r}+3$. It follows that the induced bisimplicial map $C^{\bullet,\bullet}_{\mathbb{C},\mathrm{r}+1}(B,A) \to C^{\bullet,\bullet+1}_{\mathbb{C},\mathrm{r}}(B,A)$, on the bicomplexes in Definition 3.2, is an identity on all (p,q)–cochains with $q \leq 2\mathrm{r}+3$. Therefore, the induced suspension map of complexes

$$C^{\bullet}_{\mathbb{C},\mathrm{r}+1}(B,A) \to C^{\bullet}_{\mathbb{C},\mathrm{r}}(B,A) \tag{10}$$

is an identity on n–cochains with $n \leq \mathrm{r}+2$, whence the theorem follows. □

4 Low-dimensional cocycles and coboundaries

In this paper we are going to use only the cohomology groups $H^n_{\mathbb{C},\mathrm{r}}(B,A)$ for $n \leq 3$, and for them Theorem 3.4 can be specified as follows:

Corollary 4.1 *For any $\mathbb{C}$–modules $B, A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$, and $\mathrm{r} \geq 2$ an integer, there are natural identifications*

$$\begin{aligned}
H^0_{\mathbb{C},\mathrm{r}}(B,A) &= H^0_{\mathbb{C},1}(B,A),\\
H^1_{\mathbb{C},\mathrm{r}}(B,A) &= H^1_{\mathbb{C},1}(B,A),\\
H^2_{\mathbb{C},\mathrm{r}}(B,A) &= H^2_{\mathbb{C},1}(B,A),\\
H^3_{\mathbb{C},\mathrm{r}}(B,A) &= H^3_{\mathbb{C},2}(B,A) \subseteq H^3_{\mathbb{C},1}(B,A).
\end{aligned}$$

We next explicitly describe the cochain complexes $C^{\bullet}_{\mathbb{C},1}(B,A)$ and $C^{\bullet}_{\mathbb{C},2}(B,A)$, introduced in Definition 3.2. To do so, we recall the following bar notation: If X is a set, then for $p \geq 1$ any integer,

$$X^p := \{(x_1,\dots,x_p),\ x_i \in X\},$$

as is usual; for a tuple of integers $\mathbf{p} = (p_1,\dots,p_r)$, with $r \geq 1$ and $p_i \geq 1$,

$$X^{\mathbf{p}} = X^{p_1} \mid \cdots \mid X^{p_r} := \big\{(\alpha_1 \mid \cdots \mid \alpha_r),\ \alpha_j \in X^{p_j}\big\};$$

and for a sequence of tuples $(\mathbf{p}_1,\dots,\mathbf{p}_s)$, with $s \geq 1$,

$$X^{\mathbf{p}_1} \| \cdots \| X^{\mathbf{p}_s} := \big\{(\beta_1 \| \cdots \| \beta_s),\ \beta_k \in X^{\mathbf{p}_k}\big\}.$$

Furthermore, for $\pi \in \mathbf{Shuf}(m,n)$ any (m,n)–shuffle, then

$$\begin{aligned}
\pi(x_1,\dots,x_m \mid x_{m+1},\dots,x_{m+n}) &:= (x_{\pi 1},\dots,x_{\pi(m+n)}),\\
\pi(\alpha_1|\cdots|\alpha_m\|\alpha_{m+1}|\cdots|\alpha_{m+n}) &:= (\alpha_{\pi 1}|\cdots|\alpha_{\pi(m+n)}).
\end{aligned}$$

Then, for any two modules $B,A\colon \mathbb{C}^{\mathrm{op}}\to\mathbf{Ab}$ over a small category $\mathbb{C}$, the complex of first-level cochains of B with coefficients in A, $C^{\bullet}_{\mathbb{C},1}(B,A)$, is trivial in dimension 0 and for $n\geq 1$

$$C^n_{\mathbb{C},1}(B,A)=\textstyle\prod_{\sigma\in\mathbf{N}_p\mathbb{C}}\prod_{\mathbf{p}}C^1(B^{\mathbf{p}}_{\sigma 0},A_{\sigma p}),$$

where the product is taken over all $\sigma\in\mathbf{N}_p\mathbb{C}$, with $p\geq 0$, and the tuples $\mathbf{p}=(p_1,\dots,p_r)$ as above, such that

$$p+r+\textstyle\sum_{j=1}^r p_j=n+1.$$

Thus, a cochain in $C^n_{\mathbb{C},1}(B,A)$ is a sequence $f=(f_\sigma)_{\sigma\in\mathbf{N}_p\mathbb{C}}$ of normalized functions

$$f_\sigma\colon \textstyle\coprod_{(p_1,\dots,p_r)}B^{p_1}_{\sigma 0}\mid\cdots\mid B^{p_r}_{\sigma 0}\longrightarrow A_{\sigma p}.$$

For such a sequence, the coboundary operator $\partial\colon C^n_{\mathbb{C},1}(B,A)\to C^{n+1}_{\mathbb{C},1}(B,A)$ is given by the formula below, where the three first terms come from (8), and the two last terms from (7).

$$\begin{aligned}
&(\partial f)_\sigma(\alpha_1\mid\cdots\mid\alpha_r)\\
&=f_{d_0\sigma}(\sigma_1^*\alpha_1\mid\cdots\mid\sigma_1^*\alpha_r)+\textstyle\sum_{m=1}^{p-1}(-1)^m f_{d_m\sigma}(\alpha_1\mid\cdots\mid\alpha_r)\\
&\quad+(-1)^p\sigma_p^* f_{d_p\sigma}(\alpha_1|\cdots|\alpha_r)+\textstyle\sum_{i=1}^r\sum_{j=1}^{p_i}(-1)^{p+\epsilon_{i-1}+j}f_\sigma(\alpha_1|\cdots|d_j\alpha_i|\cdots|\alpha_r)\\
&\qquad+\textstyle\sum_{i=1}^{r-1}\sum_{\pi\in\mathbf{Shuf}(p_i,p_{i+1})}(-1)^{p+\epsilon_i+\epsilon(\pi)}f_\sigma(\alpha_1\mid\cdots\mid\pi(\alpha_i\mid\alpha_{i+1})\mid\cdots\mid\alpha_r),
\end{aligned}$$

where $d_j\colon B^{p_i}_{\sigma 0}\to B^{p_i-1}_{\sigma 0}$ are the face operators of $K(B_{\sigma 0},1)$, $\epsilon_i=p_1+\cdots+p_i+i$, and $\epsilon(\pi)$ is the parity of the shuffle π.

Similarly, the complex $C^{\bullet}_{\mathbb{C},2}(B,A)$ of second-level cochains of B with coefficients in A is trivial in dimension 0, and for $n\geq 1$

$$C^n_{\mathbb{C},2}(B,A)=\textstyle\prod_{\sigma\in\mathbf{N}_p\mathbb{C}}\prod_{(\mathbf{p}_1,\dots,\mathbf{p}_s)}C^1(B^{\mathbf{p}_1}_{\sigma 0}\|\cdots\|B^{\mathbf{p}_s}_{\sigma 0},A_{\sigma p}),$$

where the product is taken over all $\sigma\in\mathbf{N}_p\mathbb{C}$, with $p\geq 0$, and the tuples $(\mathbf{p}_1,\dots,\mathbf{p}_s)$, with $\mathbf{p}_j=(p^j_1,\dots,p^j_{r_j})$ as above, such that

$$p+s+\textstyle\sum_{j=1}^s\big(r_j+\sum_{k=1}^{r_j}p^j_k\big)=n+2.$$

Thus, a cochain in $C^n_{\mathbb{C},2}(B,A)$ is a sequence $f=(f_\sigma)_{\sigma\in\mathbf{N}_p\mathbb{C}}$ of normalized functions

$$f_\sigma\colon \textstyle\coprod_{(\mathbf{p}_1,\dots,\mathbf{p}_s)}B^{\mathbf{p}_1}_{\sigma 0}\|\cdots\|B^{\mathbf{p}_s}_{\sigma 0}\longrightarrow A_{\sigma p}.$$

For such a sequence, the coboundary operator $\partial\colon C^n_{\mathbb{C},2}(B,A)\to C^{n+1}_{\mathbb{C},2}(B,A)$ is given by

$$\begin{aligned}
&(\partial f)_\sigma(\beta_1\|\cdots\|\beta_s)\\
&= f_{d_0\sigma}(\sigma_1^*\beta_1\|\cdots\|\sigma_1^*\beta_s)\\
&\quad+\textstyle\sum_{m=1}^{p-1}(-1)^m f_{d_m\sigma}(\beta_1\|\cdots\|\beta_s)+(-1)^p\sigma_p^* f_{d_p\sigma}(\beta_1\|\cdots\|\beta_s)\\
&\quad+\textstyle\sum_{(A)}(-1)^{\epsilon^j_{k-1}+i} f_\sigma(\beta_1\|\cdots\|\beta_{j-1}\|\alpha_1^j\mid\ldots\mid d_i\alpha_k^j\mid\cdots\mid\alpha^j_{r_j}\|\beta_{j+1}\|\cdots\|\beta_s)\\
&\quad+\textstyle\sum_{(B)}(-1)^{\epsilon^j_k+\epsilon(\pi)} f_\sigma(\beta_1\|\cdots\|\beta_{j-1}\|\alpha_1^j\mid\ldots\mid\pi(\alpha_k^j\mid\alpha_{k+1}^j)\mid\ldots\mid\alpha^j_{r_j}\|\beta_{j+1}\|\cdots\|\beta_s)\\
&\qquad\qquad+\textstyle\sum_{(C)}(-1)^{\epsilon_j+\epsilon(\pi)} f_\sigma(\beta_1\|\cdots\|\beta_{j-1}\|\pi(\beta_j\|\beta_{j+1})\|\beta_{j+2}\|\cdots\|\beta_s),
\end{aligned}$$

in which

$$\textstyle\sum_{(A)}=\sum_{j=1}^s\sum_{k=1}^{r_j}\sum_{i=0}^{p_k^j},\quad \sum_{(B)}=\sum_{j=1}^s\sum_{k=1}^{r_j-1}\sum_{\pi\in\mathbf{Shuf}(p_k^j,p_{k+1}^j)},$$

$$\textstyle\sum_{(C)}=\sum_{j=1}^{s-1}\sum_{\pi\in\mathbf{Shuf}(r_j,r_{j+1})},$$

$$\textstyle\epsilon_j=p+\sum_{l=1}^j\sum_{k=1}^{r_l}p_k^l,\quad \epsilon_{j,k}=\epsilon_{j-1}+\sum_{t=1}^k p_t^j+k,\quad \beta_j=(\alpha_1^j\mid\cdots\mid\alpha^j_{r_j}).$$

For future reference we specify the relevant truncated subcomplexes of $C^\bullet_{\mathbb{C},1}(B,A)$ and $C^\bullet_{\mathbb{C},2}(B,A)$ that, respectively, yield the first- and second-level cocycles and coboundaries at dimensions ≤ 3. We have the induced suspension map of complexes (10)

(11)
$$\begin{array}{ccccccccc}
0\to C^1_{\mathbb{C},2}(B,A) & \xrightarrow{\partial^1} & C^2_{\mathbb{C},2}(B,A) & \xrightarrow{\partial^2} & C^3_{\mathbb{C},2}(B,A) & \xrightarrow{\partial^3} & C^4_{\mathbb{C},2}(B,A)\cdots\\
\| & & \| & & \| & & \downarrow S\\
0\to C^1_{\mathbb{C},1}(B,A) & \xrightarrow{\partial^1} & C^2_{\mathbb{C},1}(B,A) & \xrightarrow{\partial^2} & C^3_{\mathbb{C},1}(B,A) & \xrightarrow{\partial^3} & C^4_{\mathbb{C},1}(B,A)\cdots
\end{array}$$

where

(12)
$$C^1_{\mathbb{C},1}(B,A)=\prod_{u\in\mathrm{Ob}\mathbb{C}}C^1(B_u,A_u),$$

so that a first (=second) level 1–cochain, $f\in C^1_{\mathbb{C},1}(B,A)=C^1_{\mathbb{C},2}(B,A)$, is a normalized function associating

- an element $f_u(x)\in A_u$ to each object u of $\mathbb{C}$ and $x\in B_u$;

(13)
$$C^2_{\mathbb{C},1}(B,A)=\prod_u C^1(B_u\times B_u,A_u)\times\prod_{v\xrightarrow{\sigma}u}C^1(B_u,A_v),$$

thus a first (=second) level 2–cochain $g\in C^2_{\mathbb{C},1}(B,A)=C^2_{\mathbb{C},2}(B,A)$ is a normalized function associating

- an element $g_u(x,y)\in A_u$ to each object u of $\mathbb{C}$ and $x,y\in B_u$,
- an element $g_\sigma(x)\in A_v$ to each arrow $v\xrightarrow{\sigma}u$ of $\mathbb{C}$ and $x\in B_u$;

$$\partial^1\colon C^1_{\mathbb{C},1}(B,A)\to C^2_{\mathbb{C},1}(B,A) \tag{14}$$

is given by (x, y, and σ as above)

$$(\partial^1 f)_u(x,y)=f_u(y)-f_u(x+y)+f_u(x), \tag{15}$$

$$(\partial^1 f)_\sigma(x)=f_v(\sigma^*x)-\sigma^*f_u(x); \tag{16}$$

$$\begin{aligned} C^3_{\mathbb{C},1}(B,A)=\prod_u C^1(B_u\times B_u\times B_u,A_u)\times\prod_u C^1(B_u\mid B_u,A_u)\\ \times\prod_{v\xrightarrow{\sigma}u}C^1(B_u\times B_u,A_v)\times\prod_{w\xrightarrow{\tau}v\xrightarrow{\sigma}u}C^1(B_u,A_w),\end{aligned} \tag{17}$$

thus a first (=second) level 3–cochain $h\in C^3_{\mathbb{C},1}(B,A)=C^3_{\mathbb{C},2}(B,A)$ is a normalized function associating

- an element $h_u(x,y,z)\in A_u$ to each object u of $\mathbb{C}$ and $x,y,z\in B_u$,
- an element $h_u(x\mid y)\in A_u$ to each object u of $\mathbb{C}$ and $x,y\in B_u$,
- an element $h_\sigma(x,y)\in A_v$ to each arrow $v\xrightarrow{\sigma}u$ of $\mathbb{C}$ and $x,y\in B_u$,
- an element $h_{\sigma,\tau}(x)\in A_w$ to each pair of arrows $w\xrightarrow{\tau}v\xrightarrow{\sigma}u$ of $\mathbb{C}$ and $x\in B_u$;

$$\partial^2\colon C^2_{\mathbb{C},1}(B,A)\to C^3_{\mathbb{C},1}(B,A) \tag{18}$$

is given by (x,y,z,σ and τ as above)

$$(\partial^2 g)_u(x,y,z)=g_u(y,z)-g_u(x+y,z)+g_u(x,y+z)-g_u(x,y), \tag{19}$$

$$(\partial^2 g)_u(x\mid y)=g_u(y,x)-g_u(x,y), \tag{20}$$

$$(\partial^2 g)_\sigma(x,y)=\sigma^*g_u(x,y)-g_v(\sigma^*x,\sigma^*y)+g_\sigma(y)-g_\sigma(x+y)+g_\sigma(x), \tag{21}$$

$$(\partial^2 g)_{\sigma,\tau}(x)=\tau^*g_\sigma(x)-g_{\sigma\tau}(x)+g_\tau(\sigma^*x); \tag{22}$$

$$(23)\quad C^4_{\mathbb{C},1}(B,A)=\prod_u C^1(B_u^4,A_u)\times\prod_u C^1(B_u\mid B_u^2,A_u)\times\prod_u C^1(B_u^2\mid B_u,A_u)$$
$$\times\prod_{v\xrightarrow{\sigma}u} C^1(B_u^3,A_v)\times\prod_{v\xrightarrow{\sigma}u} C^1(B_u\mid B_u,A_v)$$
$$\times\prod_{w\xrightarrow{\tau}v\xrightarrow{\sigma}u} C^1(B_u^2,A_w)\times\prod_{m\xrightarrow{\gamma}w\xrightarrow{\tau}v\xrightarrow{\sigma}u} C^1(B_u,A_m),$$

$$(24)\quad C^4_{\mathbb{C},2}(B,A)=\prod_u C^1(B_u^4,A_u)\times\prod_u C^1(B_u\mid B_u^2,A_u)\times\prod_u C^1(B_u^2\mid B_u,A_u)$$
$$\times\prod_u C^1(B_u\mid\mid B_u,A_u)\times\prod_{v\xrightarrow{\sigma}u} C^1(B_u^3,A_v)\times\prod_{v\xrightarrow{\sigma}u} C^1(B_u\mid B_u,A_v)$$
$$\times\prod_{w\xrightarrow{\tau}v\xrightarrow{\sigma}u} C^1(B_u^2,A_w)\times\prod_{m\xrightarrow{\gamma}w\xrightarrow{\tau}v\xrightarrow{\sigma}u} C^1(B_u,A_m),$$

so that a first-level 4–cochain $\psi\in C^4_{\mathbb{C},1}(B,A)$ is a normalized function associating

- an element $\psi_u(x,y,z,t)\in A_u$ to each object u of $\mathbb{C}$ and $x,y,z,t\in B_u$,
- an element $\psi_u(x\mid y,z)\in A_u$ to each object u of $\mathbb{C}$ and $x,y,z\in B_u$,
- an element $\psi_u(x,y\mid z)\in A_u$ to each object u of $\mathbb{C}$ and $x,y,z\in B_u$,
- an element $\psi_\sigma(x,y,z)\in A_v$ to each arrow $v\xrightarrow{\sigma}u$ of $\mathbb{C}$ and $x,y,z\in B_u$,
- an element $\psi_\sigma(x\mid y)\in A_v$ to each arrow $v\xrightarrow{\sigma}u$ of $\mathbb{C}$ and $x,y\in B_u$,
- an element $\psi_{\sigma,\tau}(x,y)\in A_w$ to each arrows $w\xrightarrow{\tau}v\xrightarrow{\sigma}u$ of $\mathbb{C}$ and $x,y\in B_u$,
- an element $\psi_{\sigma,\tau,\gamma}(x)\in A_m$ to each arrows $m\xrightarrow{\gamma}w\xrightarrow{\tau}v\xrightarrow{\sigma}u$ of $\mathbb{C}$ and $x\in B_u$;

while a second-level 4–cochain $\psi\in C^4_{\mathbb{C},2}(B,A)$ is a first-level 4–cochain $\psi\in C^4_{\mathbb{C},1}(B,A)$, as above, together with a normalized function associating in addition

- an element $\psi_u(x\mid\mid y)\in A_u$ to each object u of $\mathbb{C}$ and $x,y\in B_u$.

The suspension epimorphism in (11)

$$(25)\qquad S\colon C^4_{\mathbb{C},2}(B,A)\to C^4_{\mathbb{C},1}(B,A)$$

is the obvious projection map. The coboundary

$$(26)\qquad \partial^3\colon C^3_{\mathbb{C},2}(B,A)\to C^4_{\mathbb{C},2}(B,A)$$

is given by (x,y,z,t,σ,τ and γ as above)

$$(27)\quad (\partial^3h)_u(x,y,z,t)=h_u(y,z,t)-h_u(x+y,z,t)+h_u(x,y+z,t)$$
$$-h_u(x,y,z+t)+h_u(x,y,z),$$

$$(\partial^3 h)_u(x \mid y, z) = h_u(x \mid z) - h_u(x \mid y+z) + h_u(x \mid y) - h_u(x, y, z) + h_u(y, x, z) - h_u(y, z, x), \tag{28}$$

$$(\partial^3 h)_u(x, y \mid z) = h_u(y \mid z) - h_u(x+y \mid z) + h_u(x \mid z) + h_u(x, y, z) - h_u(x, z, y) + h_u(z, x, y), \tag{29}$$

$$(\partial^3 h)_u(x \,||\, y) = -h_u(x \mid y) - h_u(y \mid x), \tag{30}$$

$$(\partial^3 h)_\sigma(x, y, z) = \sigma^* h_u(x, y, z) - h_v(\sigma^* x, \sigma^* y, \sigma^* z) - h_\sigma(y, z) + h_\sigma(x+y, z) - h_\sigma(x, y+z) + h_\sigma(x, y), \tag{31}$$

$$(\partial^3 h)_\sigma(x \mid y) = \sigma^* h_u(x \mid y) - h_v(\sigma^* x \mid \sigma^* y) + h_\sigma(x, y) - h_\sigma(y, x), \tag{32}$$

$$(\partial^3 h)_{\sigma,\tau}(x, y) = \tau^* h_\sigma(x, y) - h_{\sigma\tau}(x, y) + h_\tau(\sigma^* x, \sigma^* y) - h_{\sigma,\tau}(y) + h_{\sigma,\tau}(x+y) - h_{\sigma,\tau}(x), \tag{33}$$

$$(\partial^3 h)_{\sigma,\tau,\gamma}(x) = \gamma^* h_{\sigma,\tau}(x) - h_{\sigma,\tau\gamma}(x) + h_{\sigma\tau,\gamma}(x) - h_{\tau,\gamma}(\sigma^* x). \tag{34}$$

And, finally, the coboundary

$$\partial^3\colon C^3_{\mathbb{C},1}(B, A) \to C^4_{\mathbb{C},1}(B, A) \tag{35}$$

is the composite of (26) with the suspension map (25), that is, it works according to the same formulas as (27)–(34) above, but disregarding (30).

5 $H^1_{\mathbb{C},\mathrm{r}}$ and homomorphisms, $H^2_{\mathbb{C},\mathrm{r}}$ and extensions

Let $\mathbb{C}$ be a given small category, and let $B, A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$ be two $\mathbb{C}$–modules.

Since the group $C^0_{\mathbb{C},1}(B, A)$ is trivial, by Theorem 3.3 and Corollary 4.1,

$$H^0_{\mathbb{C},\mathrm{r}}(B, A) = H^0_{\mathbb{C},1}(B, A) = 0, \tag{36}$$

for any $\mathrm{r} \geq 1$, and there is nothing to say about these cohomology groups.

To analyse the cohomology groups at degree 1, from Theorem 3.3, we have

$$H^1_{\mathbb{C},1}(B, A) = Z^1_{\mathbb{C},1}(B, A) = \mathrm{Ker}(C^1_{\mathbb{C},1}(B, A) \xrightarrow{\partial^1} C^2_{\mathbb{C},1}(B, A)), \tag{37}$$

where $C^1_{\mathbb{C},1}(B, A)$, $C^2_{\mathbb{C},1}(B, A)$ and ∂^1 are described in (12), (13) and (14), respectively. Then, recalling that $\mathrm{Mod}_{\mathbb{C}}$ denotes the abelian category of $\mathbb{C}$–modules with natural transformations as homomorphisms between them, the following is clear from (37) and Corollary 4.1:

Theorem 5.1 *For any $\mathbb{C}$–modules $B, A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$, and any integer $\mathrm{r} \geq 1$, there are natural identifications*

$$H^1_{\mathbb{C},1}(B,A) = H^1_{\mathbb{C},\mathrm{r}}(B,A) = \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{C}}}(B,A). \tag{38}$$

We now deal with the cohomology group $H^2_{\mathbb{C},1}(B,A)$, which, from Theorem 3.3, can be computed as

$$H^2_{\mathbb{C},1}(B,A) = H\big(C^1_{\mathbb{C},1}(B,A) \xrightarrow{\partial^1} C^2_{\mathbb{C},1}(B,A) \xrightarrow{\partial^2} C^3_{\mathbb{C},1}(B,A)\big), \tag{39}$$

where $C^3_{\mathbb{C},1}(B,A)$ and ∂^2 are described in (17) and (18), respectively. Next we shall prove that first-level 2–cocycles in $Z^2_{\mathbb{C},1}(B,A) = \mathrm{Ker}(\partial^2)$ are the appropriate data to construct the set of all $\mathbb{C}$–module extensions of a given $\mathbb{C}$–module B by a given $\mathbb{C}$–module A, up to isomorphism.

Recall that the category $\mathrm{Mod}_{\mathbb{C}}$ is abelian, and that such an extension is a short exact sequence in $\mathrm{Mod}_{\mathbb{C}}$

$$\underline{E}\colon 0 \to A \xrightarrow{i} E \xrightarrow{p} B \to 0, \tag{40}$$

where short exactness means that every sequence $0 \to A_u \xrightarrow{i_u} E_u \xrightarrow{p_u} B_u \to 0$ is an abelian group extension for any object u of $\mathbb{C}$. The extension $\underline{E}$ is equivalent to $\underline{E}'$ if there exists a $\mathbb{C}$–module isomorphism $\Phi\colon E \cong E'$ such that $\Phi i = i'$ and $p'\Phi = p$, and we denote by

$$\mathrm{Ext}_{\mathrm{Mod}_{\mathbb{C}}}(B,A)$$

the set of equivalence classes of extensions of B by A.

Every 2–cocycle $g \in Z^2_{\mathbb{C},1}(B,A)$ gives rise to a $\mathbb{C}$–module extension

$$\underline{E}(g)\colon 0 \to A \xrightarrow{i} E(g) \xrightarrow{p} B \to 0, \tag{41}$$

which is described as follows: for each object u of $\mathbb{C}$, let $E(g)_u$ be the abelian group with the same elements as $A_u \times B_u$ and addition given by

$$(a,x) + (b,y) = (a+b+g_u(x,y), x+y).$$

Note that, so defined, $E(g)_u$ is actually an abelian group thanks to the normal 2–cocycle equalities $(\partial^2 g)_u(x,y,z) = 0$ in (19) for associativity $(\partial^2 g)_u(x\,|\,y) = 0$ in (20) for commutativity and $g_u(0,y) = 0$ for providing $(0,0)$ as the zero element. Inverses in $E(g)_u$ are defined by $-(a,x) = (-a-g_u(x,-x),-x)$. The $\mathbb{C}$–module $E(g)\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$ is then defined as the functor that carries each object u of $\mathbb{C}$ to the abelian group $E(g)_u$, and an arrow $\sigma\colon v \to u$ to the map $\sigma^*\colon E(g)_u \to E(g)_v$ given by

$$\sigma^*(a,x) = (\sigma^* a - g_\sigma(x), \sigma^* x),$$

which is actually a group homomorphism because of the 2–cocycle condition (21) $(\partial^2 g)_\sigma(x,y)=0$. Similarly, it is easy to see the 2–cocycle conditions $(\partial^2 g)_{\sigma,\tau}(x)=0$ in (22) for compositions and $g_1(x)=0$ for identities imply that $E(g)\colon u\mapsto E(g)_u$ is a functor from $\mathbb{C}^{\mathrm{op}}$ into abelian groups, that is, a $\mathbb{C}$–module. Finally, the $\mathbb{C}$–module homomorphisms i and p in the sequence (41), consist of the abelian group homomorphisms $i_u\colon A_u\to E(g)_u$ and $p_u\colon E(g)_u\to B_u$, $u\in\mathrm{Ob}\mathbb{C}$, respectively defined by $i_u(a)=(a,0)$ and $p_u(a,x)=x$. Therefore, $\underline{E}(g)$ is easily recognized as a $\mathbb{C}$–module extension of B by A.

Theorem 5.2 *Let $B, A\colon \mathbb{C}^{\mathrm{op}}\to \mathbf{Ab}$ be any two $\mathbb{C}$–modules.*

(i) *Any $\mathbb{C}$–module extension of B by A is equivalent to an extension of the form $\underline{E}(g)$, for some first-level 2–cocycle $g\in Z^2_{\mathbb{C},1}(B,A)$.*

(ii) *Two 2–cocycles $g,g'\in Z^2_{\mathbb{C},1}(B,A)$ produce equivalent extensions, that is, $\underline{E}(g)\cong\underline{E}(g')$, if and only if they are cohomologous.*

Proof (i) Let $\underline{E}$ be a given $\mathbb{C}$–module extension as in (40). For notational simplicity, there is no loss of generality in assuming that every $i_u\colon A_u\hookrightarrow E_u$ is the inclusion map, $u\in\mathrm{Ob}\mathbb{C}$, and let us choose a function $s_u\colon B_u\to E_u$ with $p_u s_u=1_{B_u}$ and $s_u(0)=0$. Then, a first-level 2–cochain $g\in C^2_{\mathbb{C},1}(B,A)$ is determined by the formulas

- $s_u(x)+s_u(y)=g_u(x,y)+s_u(x+y)$, for each object u of $\mathbb{C}$ and $x,y\in B_u$,
- $s_v(\sigma^*x)=g_\sigma(x)+\sigma^*s_u(x)$, for each arrow $v\xrightarrow{\sigma}u$ of $\mathbb{C}$ and $x\in B_u$.

It is plain to see that, thus defined, g is a normalized function. Moreover, we can prove that g is actually a 2–cocycle, that is, that $\partial^2 g=0$ in (18), as follows: it is well known (see Mac Lane [44, page 121], for example) that the first two cocycle conditions $(\partial^2 g)_u(x,y,z)=0$ and $(\partial^2 g)_u(x\mid y)=0$ are respective consequences of the associative and commutative laws $s_u(x)+(s_u(y)+s_u(z))=(s_u(x)+s_u(y))+s_u(z)$ and $s_u(x)+s_u(y)=s_u(y)+s_u(x)$, in each group E_u. We next prove that the third cocycle condition in $(\partial^2 g)_\sigma(x,y)=0$ follows from the equalities $\sigma^*(s_u(x)+s_u(y))=\sigma^*s_u(x)+\sigma^*s_u(y)$; in fact, on one hand

$$\begin{aligned}\sigma^*(s_u(x)+s_u(y))&=\sigma^*g_u(x,y)+\sigma^*s_u(x+y)\\&=\sigma^*g_u(x,y)-g_\sigma(x+y)+s_v(\sigma^*x+\sigma^*y),\end{aligned}$$

and, on the other hand,

$$\begin{aligned}\sigma^*s_u(x)+\sigma^*s_u(y)&=-g_\sigma(x)+s_v(\sigma^*x)-g_\sigma(y)+s_v(\sigma^*y)\\&=-g_\sigma(x)-g_\sigma(y)+g_v(\sigma^*x,\sigma^*y)+s_v(\sigma^*x+\sigma^*y).\end{aligned}$$

Therefore, comparison gives the claimed third 2–cocycle condition for g. And, similarly, one sees that the fourth and last 2–cocycle condition $(\partial^2 g)_{\sigma,\tau}(x) = 0$ in (22) results from the equalities $(\sigma\tau)^* s_u(x) = \tau^*\sigma^*(s_u(x))$.

We now recognize that $\underline{E}(g)$ and $\underline{E}$ are equivalent extensions due to the existence of the $\mathbb{C}$–module isomorphism $\Phi\colon E(g) \cong E$ defined by $\Phi_u(a,x) = a + s_u(x)$, for any object u of $\mathbb{C}$, $a \in A_u$, and $x \in B_u$.

(ii) Let $\Phi\colon E(g) \cong E(g')$ be a $\mathbb{C}$–module isomorphism making $\underline{E}(g)$ and $\underline{E}(g')$ equivalent extensions. Write $\Phi_u(0,x) = (f_u(x), x)$ where $f_u\colon B_u \to A_u$ is a map, for each object u of $\mathbb{C}$. Then f_u determines Φ_u by the rule

(42) $$\Phi_u(a,x) = (a + f_u(x), x) \quad \text{for } x \in B_u,\ a \in A_u.$$

Because

$$\Phi_u((0,x) + (0,y)) = \Phi_u(g_u(x,y), x+y) = (g_u(x,y) + f_u(x+y), x+y),$$

whereas

$$\Phi_u(0,x) + \Phi_u(0,y) = (f_u(x), x) + (f_u(y), y) = (f_u(x) + f_u(y) + g'_u(x,y), x+y),$$

it follows that $g_u(x,y) = g'_u(x,y) + (\partial^1 f)_u(x,y)$. Since

$$\Phi_v(\sigma^*(0,x)) = \Phi_v(g_\sigma(x), \sigma^* x) = (g_\sigma(x) + f_v(\sigma^* x), \sigma^* x),$$

where $\sigma\colon v \to u$, $x \in B_u$, whereas

$$\sigma^*\Phi_u(0,x) = \sigma^*(f_u(x), x) = (\sigma^* f_u(x) + g'_\sigma(x), \sigma^* x),$$

it follows that $g_\sigma(x) = g'_\sigma(x) + (\partial^1 f)\sigma(x)$. Therefore, $g = g' + \partial^1 f$, where $\partial^1 f$ is the 2–coboundary defined by f as in (14), and therefore g and g' are cohomologous first-level 2–cocycles.

Conversely, if $g = g' + \partial^1 f$, for some $f \in C^1_{\mathbb{C},1}(B,A)$, then g and g' lead to the isomorphic $\mathbb{C}$–module extensions, $\underline{E}(g) \cong \underline{E}(g')$, just by the maps Φ_u, $u \in \mathrm{Ob}\mathbb{C}$, defined by (42), as we see by retracing our steps. □

As a consequence of Theorem 5.2 above and Corollary 4.1, we get the theorem below.

Theorem 5.3 *For any $\mathbb{C}$–modules B, A, and any integer $\mathrm{r} \geq 1$, there are natural identifications*

(43) $$H^2_{\mathbb{C},1}(B,A) = H^2_{\mathbb{C},\mathrm{r}}(B,A) = \mathrm{Ext}_{\mathrm{Mod}_{\mathbb{C}}}(B,A).$$

6 Braided and symmetric fibred categorical groups

As we will state further below, for any small category $\mathbb{C}$, the cohomology groups $H^3_{\mathbb{C},\mathrm{r}}(B, A)$ play a fundamental role in the problem of providing a complete invariant for the classification of braided and symmetric $\mathbb{C}$–fibred categorical groups. These are categories fibred in groupoids, $\mathbb{P} \to \mathbb{C}$, equipped with a compact monoidal $\mathbb{C}$–structure by $\mathbb{C}$–functors $\otimes\colon \mathbb{P} \times_{\mathbb{C}} \mathbb{P} \to \mathbb{P}$ and $I\colon \mathbb{C} \to \mathbb{P}$, and corresponding coherent associativity, unit, and commutativity $\mathbb{C}$–fibred constraints (see Definition 6.1 for the details). For this classification, two braided or symmetric $\mathbb{C}$–fibred categorical groups that are connected by a braided $\mathbb{C}$–fibred equivalence are considered the same. The problem of giving a complete invariant of this relation arises, and we solve it by generalizing the results by Joyal and Street [41] and Sinh [54] for the nonfibred case, and those by Cegarra, García-Calcines and Ortega [13] and Cegarra and Khmaladze [15; 16] for the G–graded case (ie, when $\mathbb{C}$ is a group G). To go further, we recommend to the reader the work of Breen in [8], where he offers an excellent discussion of this issue within the general framework of a Grothendieck topos, that is, of the category of sheaves over a category $\mathbb{C}$ endowed with a Grothendieck topology. Indeed, he shows a cohomological classification for symmetric and braided categorical group stacks over an arbitrary site $\mathbb{C}$. However, the results are far from being as explicit in terms of cocycles as they can be when $\mathbb{C}$ is discrete, such as we impose in this paper.

This technical section concerns setting up the foundations required to handle braided and symmetric $\mathbb{C}$–fibred categorical groups as pseudofunctors, which is crucial to our later discussions and classifying results. By taking into account the well-known relationship between fibred categories over a category and pseudofunctors on itself, by Grothendieck [37; 36] and Giraud [33; 34], most of our work here is dedicated to extending the so-called *Grothendieck construction* to pseudodiagrams of braided (resp. symmetric) categorical groups. We give a detailed proof that our enriched Grothendieck construction process is actually a 2–functor, which makes the 2–category of braided (symmetric) $\mathbb{C}$–fibred categorical groups biequivalent to the 2–category of contravariant pseudofunctors from $\mathbb{C}$ with values in the 2–category of braided (symmetric) categorical groups. But we are not claiming much originality, since extensions of the ubiquitous "Grothendieck construction" have been developed in many general frameworks; see for instance Street [55], Carrasco and Cegarra [10], Cegarra and Garzón [14], Hollander [38; 39], Lurie [42], Carrasco, Cegarra and Garzón [11], Cisinski [18] or Tamaki [57]. The result proven here would probably be considered "folklore" by experts, however, being unable to find a published account of it, we have included it here.

We have organized the section into seven subsections. Section 6.1 contains a minimal amount of notation and terminology on Grothendieck fibrations and 2–categories. In Section 6.2 we mainly include the definitions of *braided and symmetric* $\mathbb{C}$*–fibred categorical groups*, and list some striking examples of them. Section 6.3 is dedicated to describe the 2–categories that braided and symmetric $\mathbb{C}$–fibred categorical groups form, denoted respectively by $\mathcal{BCG}_{\downarrow\mathbb{C}}$ and $\mathcal{SCG}_{\downarrow\mathbb{C}}$, whose 1–cells are *braided* $\mathbb{C}$*–fibred functors*, and whose 2–cells are *monoidal* $\mathbb{C}$*–fibred isomorphisms*. Next, in Section 6.4, we give an explicit description of the 2–categories $\mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG})$ and $\mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{SCG})$, of *pseudofunctors to braided and symmetric categorical groups*, respectively, whose 1–cells are called *braided pseudotransformations*, and whose 2–cells are termed *braided modifications*. The following Sections 6.5 and 6.6 are dedicated to describe in detail the "enriched" Grothendieck construction 2–functors $\int_{\mathbb{C}}\colon \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG}) \to \mathcal{BCG}_{\downarrow\mathbb{C}}$ and $\int_{\mathbb{C}}\colon \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{SCG}) \to \mathcal{SCG}_{\downarrow\mathbb{C}}$, which, in the final Section 6.7, we prove are strong biequivalences of 2–categories.

6.1 Some preliminaries on fibrations and 2–categories

We shall begin by recalling from Grothendieck [36] some needed definitions and terminology about fibred categories. For the general background on 2–categories used in this paper we refer to Street [56] and Borceux [4].

6.1.1 Fibred categories, functors, and natural transformations If $P\colon \mathbb{P} \to \mathbb{C}$ is any given functor, then, for any object u of $\mathbb{C}$, the *fibre category* over u, denoted by $\mathbb{P}_u$, is the subcategory of $\mathbb{P}$ whose objects, called *u–objects*, are those x of $\mathbb{P}$ such that $Px = u$, and whose arrows, called *u–morphisms*, are those arrows f of $\mathbb{P}$ such that $Pf = 1_u$. More generally, if $\sigma\colon u \to v$ is a morphism in $\mathbb{C}$, then a morphism f in $\mathbb{P}$ such that $Pf = \sigma$ is called a *σ–morphism*. A σ–morphism $y_\sigma\colon \sigma^* y \to y$ is deemed *cartesian* if, for any σ–morphism $f\colon x \to y$, there is a unique u–morphism $f'\colon x \to \sigma^* y$ such that $y_\sigma f' = f$. In such a case, the cartesian σ–morphism is unique up to an isomorphism in $\mathbb{P}_u$, and one refers to $\sigma^* y$ as a *pullback of* y *by* σ.

The functor $P\colon \mathbb{P} \to \mathbb{C}$ is called a *fibration* provided that, for any morphism $\sigma\colon u \to v$, and any v–object y, there exists a cartesian σ–morphism with target y, $y_\sigma\colon \sigma^* y \to y$, and, moreover, the composition of cartesian morphisms is also cartesian (existence and transitivity of pullbacks). When a fibration $P\colon \mathbb{P} \to \mathbb{C}$ is given, the category $\mathbb{P}$ is termed a $\mathbb{C}$*–fibred category*. If, moreover, every fibre category $\mathbb{P}_u$, $u \in \mathrm{Ob}\mathbb{C}$, is a groupoid, then $\mathbb{P}$ is called a *category* $\mathbb{C}$*–fibred in groupoids*. It is a fact that in any category $\mathbb{C}$–fibred in groupoids every morphism is cartesian [36, page 21].

If $\mathbb{P} = (\mathbb{P}, P)$ and $\mathbb{P}' = (\mathbb{P}', P')$ are $\mathbb{C}$–fibred categories, then a $\mathbb{C}$*–fibred functor* from $\mathbb{P}$ to $\mathbb{P}'$ is a functor $F\colon \mathbb{P} \to \mathbb{P}'$ such that $P' F = P$ and, moreover, it is *cartesian* in the sense that it carries cartesian morphisms in $\mathbb{P}$ to cartesian morphisms in $\mathbb{P}'$. A $\mathbb{C}$*–fibred homomorphism* or $\mathbb{C}$*–fibred natural transformation* $\Psi\colon F \Rightarrow F'$, between $\mathbb{C}$–fibred functors $F, F'\colon \mathbb{P} \to \mathbb{P}'$ is then a natural transformation such that $P'\Psi = 1_{\boldsymbol{P}}$.

6.1.2 2–Categories A 2–category $\mathcal{X}$ is just a category enriched in the category of small categories. Then, $\mathcal{X}$ is a category in which the hom-set of morphisms $f\colon x \to y$ (which are now also called 1*–cells*) between any two objects $x, y \in \mathcal{X}$ is the set of objects of a category $\mathrm{Hom}_{\mathcal{X}}(x, y)$, whose arrows are called 2*–cells* and are denoted $\psi\colon f \Rightarrow g$. Moreover, the composition is a bifunctor $\mathrm{Hom}_{\mathcal{X}}(y, z) \times \mathrm{Hom}_{\mathcal{X}}(x, y) \to \mathrm{Hom}_{\mathcal{X}}(x, z)$, which is associative and has identities $1_x \in \mathrm{Hom}_{\mathcal{X}}(x, x)$. This bifunctor produces compositions of 1–cells and 2–cells respectively, both denoted here by juxtaposition. On the other hand, composition in each category $\mathrm{Hom}_{\mathcal{X}}(x, y)$ is denoted by " $\cdot$ ".

From now on, we will only consider 2–categories $\mathcal{X}$ such that, for any objects x, y, the category $\mathrm{Hom}_{\mathcal{X}}(x, y)$ is a groupoid. Then, following Gabriel and Zisman's terminology [32, Chapter V, Section 1.2], for a such 2–category $\mathcal{X}$, we will denote by

$$\mathrm{Ho}\,\mathcal{X} \tag{44}$$

the *homotopy category* of $\mathcal{X}$, that is, the quotient category of the underlying category of $\mathcal{X}$, with isomorphism classes of 1–cells as morphisms. Hence, a morphism $f\colon x \to y$ is an *equivalence* in $\mathcal{X}$ if and only if the induced $[f]\colon x \to y$ is an isomorphism in $\mathrm{Ho}\,\mathcal{X}$.

A 2–functor $F\colon \mathcal{X} \to \mathcal{Y}$ between 2–categories is an enriched functor and so it takes objects, arrows and 2–cells in $\mathcal{X}$ to objects, arrows and 2–cells in $\mathcal{Y}$, respectively, in such a way that all the 2–category structure is preserved. A 2–functor $F\colon \mathcal{X} \to \mathcal{Y}$, is a *biequivalence* [56, page 570] when each of the functors $F\colon \mathrm{Hom}_{\mathcal{X}}(x, x') \to \mathrm{Hom}_{\mathcal{Y}}(Fx, Fx')$ is an equivalence, and, for each object y of $\mathcal{Y}$, there is an equivalence $Fx \to y$ in $\mathcal{Y}$, for some object x of $\mathcal{X}$.

If $\mathcal{X}$ is any 2–category, then

$$\mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{X}) \tag{45}$$

denotes the 2–category of (normal) pseudofunctors from $\mathbb{C}^{\mathrm{op}}$ to $\mathcal{X}$, whose 1–cells are the pseudonatural transformations, and whose 2–cells are the modifications. We refer to Street [56] for details.

6.2 Braided (symmetric) $\mathbb{C}$–fibred categorical groups

A *braided categorical group*, Joyal and Street [41, Definition 3.1], is a braided monoidal category in which every arrow is invertible and every object is regular, that is, where, for any object x, the endofunctor $y \mapsto x \otimes y$ is an autoequivalence. In other words, a braided categorical group is a braided monoidal groupoid such that, for each object x, there is an object x' with an arrow $x \otimes x' \to I$. Braided categorical groups are also called *braided Gr-categories* by Breen [7; 8] and *braided (weak) 2–groups* by Baez and Lauda [2] and Aldrovandia and Noohi [1]. A braided categorical group whose braiding is a symmetry is called a *symmetric categorical group* by Joyal and Street [41] and Vitale [5]. Symmetric categorical groups are also termed *Picard categories* by Deligne [21] and Sinh [54], *group-like categories* by Fröhlich and Wall [30], and *symmetric (weak) 2–groups* in [1; 2]. Below we detail the monoidal fibred categories, in the sense of Saavedra [53, Chapter I, Section 4.5], we are mainly going to work with.

Definition 6.1 Let $\mathbb{C}$ be a category. A *braided $\mathbb{C}$–fibred categorical group*

$$\mathbb{P} = (\mathbb{P}, P, \otimes, I, \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{l}, \boldsymbol{c})$$

consists of a category $\mathbb{C}$–fibred in groupoids $\mathbb{P} = (\mathbb{P}, P)$, two $\mathbb{C}$–fibred functors

$$\otimes \colon \mathbb{P} \times_{\mathbb{C}} \mathbb{P} \to \mathbb{P}, \quad I \colon \mathbb{C} \to \mathbb{P},$$

where $\mathbb{P} \times_{\mathbb{C}} \mathbb{P}$ is the $\mathbb{C}$–fibred pullback category, Grothendieck [36, Proposition 6.5], and $\mathbb{C}$–fibred natural equivalences $\boldsymbol{a}$, $\boldsymbol{r}$, $\boldsymbol{l}$, and $\boldsymbol{c}$ (called *associativity, unit and braiding constraints*, respectively), defined by u–isomorphisms

$$(x \otimes y) \otimes z \xrightarrow{\boldsymbol{a}_{x,y,z}} x \otimes (y \otimes z), \quad x \otimes Iu \xrightarrow{\boldsymbol{r}_x} x, \quad Iu \otimes x \xrightarrow{\boldsymbol{l}_x} x, \quad x \otimes y \xrightarrow{\boldsymbol{c}_{x,y}} y \otimes x,$$

for any objects u of $\mathbb{C}$ and x, y, z of $\mathbb{P}_u$, such that the following four coherence conditions hold:

(46) $$\boldsymbol{a}_{x,y,z\otimes t}\, \boldsymbol{a}_{x\otimes y,z,t} = (1_x \otimes \boldsymbol{a}_{y,z,t})\, \boldsymbol{a}_{x,y\otimes z,t}\, (\boldsymbol{a}_{x,y,z} \otimes 1_t),$$

(47) $$(1_x \otimes \boldsymbol{l}_y)\, \boldsymbol{a}_{x,Iu,y} = \boldsymbol{r}_y \otimes 1_y,$$

(48) $$\boldsymbol{a}_{y,z,x}\, \boldsymbol{c}_{x,y\otimes z}\, \boldsymbol{a}_{x,y,z} = (1_y \otimes \boldsymbol{c}_{x,z})\, \boldsymbol{a}_{y,x,z}\, (\boldsymbol{c}_{x,y} \otimes 1_z),$$

(49) $$\boldsymbol{a}^{-1}_{z,x,y}\, \boldsymbol{c}_{x\otimes y,z}\, \boldsymbol{a}^{-1}_{x,y,z} = (\boldsymbol{c}_{x,z} \otimes 1_y)\, \boldsymbol{a}^{-1}_{x,z,y}\, (1_x \otimes \boldsymbol{c}_{y,z}),$$

and, for any u–object x, there is another u–object x' with an u–morphism $x \otimes x' \to Iu$.

A *symmetric $\mathbb{C}$–fibred categorical group* is a braided $\mathbb{C}$–fibred categorical group, as above, whose braiding $\boldsymbol{c}$ is a *symmetry*, that is, it satisfies that, for any objects u of $\mathbb{C}$ and x, y of $\mathbb{P}_u$, the equation below holds.

(50) $$\boldsymbol{c}_{y,x}\, \boldsymbol{c}_{x,y} = 1_{x\otimes y}$$

If $\mathbb{P}$ is any braided $\mathbb{C}$–fibred categorical group, then, for any object u of $\mathbb{C}$, the tensor product and the associativity, commutativity, and unit constraints can be restricted to the fibre category over u so that every fibre inherits a braided categorical group structure $\mathbb{P}_u = (\mathbb{P}_u, \otimes, Iu, \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{l}, \boldsymbol{c})$. Thus, a braided (symmetric) $\mathbb{C}$–fibred categorical group is the same thing as a stack of braided (symmetric) categorical groups over the discrete site $\mathbb{C}$; see Aldrovandia and Noohi [1] and Breen [8].

In any braided monoidal $\mathbb{C}$–fibred category, for any object u of $\mathbb{C}$ and any u–objects x, y, the following equalities hold; see Joyal and Street [41, Propositions 1.1 and 2.1]:

$$\begin{aligned} &\boldsymbol{r}_{Iu} = \boldsymbol{l}_{Iu}, && \boldsymbol{l}_x\, \boldsymbol{c}_{x,Iu} = \boldsymbol{r}_x, \qquad \boldsymbol{r}_x\, \boldsymbol{c}_{Iu,x} = \boldsymbol{l}_x, \\ &\boldsymbol{r}_{x\otimes y} = (1_x \otimes \boldsymbol{r}_y)\, \boldsymbol{a}_{x,y,Iu}, && \boldsymbol{l}_{x\otimes y}\, \boldsymbol{a}_{Iu,x,y} = \boldsymbol{l}_x \otimes 1_y. \end{aligned} \tag{51}$$

6.2.1 Some examples To help motivate the reader we show below some striking examples of braided and symmetric $\mathbb{C}$–fibred categorical groups (see also Example 6.13).

Example 6.2 Let $\mathbb{C}$ be a category. A *braided $\mathbb{C}$–fibred categorical group*, $\mathbb{P} = \mathbb{P}_{\mathbb{C}}(A, B, d, \xi)$, can be constructed from any system of data consisting of

- a pair of $\mathbb{C}$–modules $A, B\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$,
- a $\mathbb{C}$–module homomorphism $d\colon A \to B$,
- a biadditive natural transformation $\xi\colon B \times B \to \mathrm{Ker}(d)$, such that

$$\xi|_{\mathrm{Img}(d)\times B} = 0 = \xi|_{B\times \mathrm{Img}(d)},$$

as follows. For any object $u \in \mathbb{C}$, an u–object of $\mathbb{P}$ is an element $b_u \in B_u$. For any morphism $\sigma\colon u \to v$ in $\mathbb{C}$, a σ–morphism $a_\sigma\colon b_u \to b_v$ is an element $a_\sigma \in A_u$ such that $b_u = d_u(a_\sigma) + \sigma^* b_v$.

If $\sigma\colon u \to v$ and $\tau\colon v \to w$ are two composable arrows in $\mathbb{C}$, the composition of σ–morphisms with τ–morphisms in $\mathbb{P}$ is defined by

$$(b_v \xrightarrow{a_\tau} b_w)(b_u \xrightarrow{a_\sigma} b_v) = (\; b_u \xrightarrow{a_\sigma + \sigma^* a_\tau} b_w \;).$$

The tensor product of two σ–morphisms is given by addition in B_u, B_v, and A_u:

$$(b_u \xrightarrow{a_\sigma} b_v) \otimes (b'_u \xrightarrow{a'_\sigma} b'_v) = (\; b_u + b'_u \xrightarrow{a_\sigma + a'_\sigma} b_v + b'_v \;)$$

The associativity and unit $\mathbb{C}$–fibred constraints are identities, and the braiding, at any u–objects $b_u, b'_u \in B_u$, is given by

$$\xi_u(b_u, b'_u)\colon b_u + b'_u \to b'_u + b_u.$$

So defined, $\mathbb{P}_{\mathbb{C}}(A,B,d,\xi)$ is not symmetric in general. However, when $\xi=0$, then $\mathbb{P}_{\mathbb{C}}(A,B,d)$ becomes a symmetric $\mathbb{C}$–fibred categorical group, whose construction is due to Deligne [21, 1.4.11]. The nonfibred case with $d=0$, $\mathbb{P}(A,B,\xi)$, was considered by Joyal and Street in [41, Example 3.1]. For an instance of $\mathbb{P}_{\mathbb{C}}(A,B,\xi)$, that is, in the fibred case with $d=0$, let T be a topological space and let $\mathbb{C}(T)$ denote its poset of open sets, regarded as a category. Then, if $(T,\mathcal{O}_T)$ is any ringed space (a scheme for example), we can take $\mathbb{C}=\mathbb{C}(T)$, $A=\mathcal{O}_T=B$ under addition, while ξ is given by the ring multiplication operations $\mathcal{O}_T(U)\times\mathcal{O}_T(U)\to\mathcal{O}_T(U)$, for each open set U of T.

Example 6.3 For any topological space T, let $\mathbb{C}(T)$ denote its poset of open sets, regarded as a category. If $\mathcal{F},\mathcal{G}\in \mathrm{Sh}_{\mathrm{Ab}}(T)$ are sheaves of abelian groups on a space T, then the *symmetric $\mathbb{C}(T)$–fibred categorical group of local extensions of $\mathcal{F}$ by $\mathcal{G}$*, denoted by $\mathcal{E}\mathit{XT}(\mathcal{F},\mathcal{G})$, is defined as follows:

For any open set U of T, an U–object of $\mathcal{E}\mathit{XT}(\mathcal{F},\mathcal{G})$ is an extension in $\mathrm{Sh}_{\mathrm{Ab}}(U)$ of $\mathcal{F}|_U$ by $\mathcal{G}|_U$,

$$\underline{\mathcal{E}}\colon 0\to\mathcal{G}|_U\xrightarrow{i}\mathcal{E}\xrightarrow{p}\mathcal{F}|_U\to 0. \tag{52}$$

For any inclusion of open sets $i_{U,U'}\colon U\subseteq U'$, an $i_{U,U'}$–morphism $\underline{\mathcal{E}}\to\underline{\mathcal{E}}'$, from the U–object (52) to the U'–object $\underline{\mathcal{E}}'\colon 0\to\mathcal{G}|_{U'}\to\mathcal{E}'\to\mathcal{F}|_{U'}\to 0$, is an isomorphism $\mathcal{E}\to\mathcal{E}'|_U$ in $\mathrm{Sh}_{\mathrm{Ab}}(U)$, such that the diagram below commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G}|_U & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F}|_U & \longrightarrow & 0\\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{G}|_U & \longrightarrow & \mathcal{E}'|_U & \longrightarrow & \mathcal{F}|_U & \longrightarrow & 0
\end{array}$$

Thus $\mathcal{E}\mathit{XT}(\mathcal{F},\mathcal{G})$ is a category fibred in groupoids over $\mathbb{C}(T)$. The fibre at any open subset U of T, is the groupoid of extensions of $\mathcal{F}|_U$ by $\mathcal{G}|_U$ in the abelian category $\mathrm{Sh}_{\mathrm{Ab}}(U)$, which is pointed by the split extension

$$I(U)=0\to\mathcal{G}|_U\to\mathcal{G}|_U\times\mathcal{F}|_U\to\mathcal{F}|_U\to 0.$$

This describes on objects the $\mathbb{C}(T)$–fibred functor $I\colon\mathbb{C}(T)\to\mathcal{E}\mathit{XT}(\mathcal{F},\mathcal{G})$, and the $\mathbb{C}(T)$–fibred structure functor

$$+\colon\mathcal{E}\mathit{XT}(\mathcal{F},\mathcal{G})\times_{\mathbb{C}(T)}\mathcal{E}\mathit{XT}(\mathcal{F},\mathcal{G})\to\mathcal{E}\mathit{XT}(\mathcal{F},\mathcal{G})$$

is given by the Baer addition of extensions. For $\underline{\mathcal{E}}_1$ and $\underline{\mathcal{E}}_2$ as in (52),

$$\underline{\mathcal{E}}_1+\underline{\mathcal{E}}_2\colon 0\to\mathcal{G}|_U\to\mathcal{E}_1+\mathcal{E}_2\to\mathcal{F}|_U\to 0$$

is the extension in $\mathrm{Sh}_{\mathrm{Ab}}(U)$ given by the pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}|_U \times \mathcal{G}|_U \arrow[r] \arrow[d, "+"'] & \mathcal{E}_1 \times_{\mathcal{F}|_U} \mathcal{E}_2 \arrow[r] \arrow[d] & \mathcal{F}|_U \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{G}|_U \arrow[r] & \mathcal{E}_1 + \mathcal{E}_2 \arrow[r] & \mathcal{F}|_U \arrow[r] & 0. \end{tikzcd}$$

The canonical isomorphisms

$$(\mathcal{E}_1 \times_{\mathcal{F}|_U} \mathcal{E}_2) \times_{\mathcal{F}|_U} \mathcal{E}_3 \cong \mathcal{E}_1 \times_{\mathcal{F}|_U} (\mathcal{E}_2 \times_{\mathcal{F}|_U} \mathcal{E}_3),$$
$$\mathcal{E} \times_{\mathcal{F}|_U} \mathcal{F}|_U \cong \mathcal{E} \cong \mathcal{F}|_U \times_{\mathcal{F}|_U} \mathcal{E}, \quad \mathcal{E}_1 \times_{\mathcal{F}|_U} \mathcal{E}_2 \cong \mathcal{E}_2 \times_{\mathcal{F}|_U} \mathcal{E}_1,$$

induce the $\mathbb{C}(T)$–fibred associativity, unit, and symmetry constraints, respectively.

Example 6.4 The *braided* $\mathbb{C}$*–fibred categorical group of* 2*–loops of a diagram of pointed topological spaces* $(X,*)\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Top}_*$, denoted by $\Pi^2_{\mathbb{C}}(X,*)$, is defined as follows: let us write $(X,*)$ by

$$(u \xrightarrow{\sigma} v) \mapsto ((X_v, *_v) \xrightarrow{\sigma^*} (X_u, *_u)).$$

Then, an object of $\Pi^2_{\mathbb{C}}(X,*)$ is a 2–loop in X_u based on $*_u$, for some object $u \in \mathbb{C}$; that is, a map

$$\omega_u\colon (S^2,*) = (I^2, \partial I^2) \to (X_u, *_u)$$

from the square $I \times I$ into X_u which is constant $*_u$ along the edges. For any morphism $\sigma\colon u \to v$ in $\mathbb{C}$, a σ–morphism, denoted by $[h_\sigma]\colon \omega_u \to \omega_v$, is the homotopy class (relative to ∂I) of a path between 2–loops $h_\sigma\colon \omega_u \Rightarrow \sigma^*\,\omega_v$. That is, it is represented by a relative map

$$h_\sigma\colon (I^3, \partial I^2 \times I) \to (X_u, *_u)$$

with $h_\sigma(s,t,0) = \omega_u(s,t)$ and $h_\sigma(s,t,1) = \sigma^*\omega_v(s,t)$; two such maps h and h'_σ are equivalent whenever there exists a map $H\colon (I^4, \partial I^2 \times I^2) \to (X_u, *_u)$ such that $H(s,t,x,0) = h_\sigma(s,t,x)$, $H(s,t,x,1) = h'_\sigma(s,t,x)$, $H(s,t,0,y) = \omega_u(s,t)$ and $H(s,t,1,y) = \sigma^*\omega_v(s,t)$.

If $\sigma\colon u \to v$ and $\tau\colon v \to w$ are two composable arrows in $\mathbb{C}$, the composition of σ–morphisms with τ–morphisms in $\Pi^2_{\mathbb{C}}(X,*)$ is induced by the usual vertical composition of homotopies, according to the formula

$$(\omega_v \xrightarrow{[h_\tau]} \omega_w)(\omega_u \xrightarrow{[h_\sigma]} \omega_v) = (\ \omega_u \xrightarrow{[\sigma^* h_\tau \circ h_\sigma]} \omega_w\ ),$$

where

$$(\sigma^* h_\tau \circ h_\sigma)(s,t,x) = \begin{cases} h_\sigma(s,t,2x) & 2x \le 1, \\ \sigma^* h_\tau(s,t,2x-1) & 2x \ge 1. \end{cases}$$

Thus, the fibre category, at any object u of $\mathbb{C}$, is the fundamental groupoid of the double loop space of $(X_u, *_u)$, $\Pi^2_{\mathbb{C}}(X)_u = \Pi(\Omega^2(X_u, *_u))$. The fibred tensor product is given on objects by concatenation of 2–loops:

$$\begin{array}{ccc}\star & = & \star\\ \| & \omega_u & \|\\ \star & = & \star\end{array} \otimes \begin{array}{ccc}\star & = & \star\\ \| & \omega'_u & \|\\ \star & = & \star\end{array} = \begin{array}{ccccc}\star & = & \star & = & \star\\ \| & \omega_u & \vdots & \omega'_u & \|\\ \star & = & \star & = & \star\end{array}$$

and on σ–morphisms by the horizontal composition of homotopies. The 1–isomorphisms giving associativity and unit constraints are defined to be the equivalence classes of the respective standard homotopies proving the associativity and unit of the loop composition, and the braiding 1–isomorphisms are the equivalence classes of the ordinary homotopies showing the commutativity of the second homotopy groups of spaces, namely

$$\begin{array}{ccccc}\star & = & \star & = & \star\\ \| & & \vdots & & \|\\ \| & \omega_u & \vdots & \omega'_u & \|\\ \| & & \vdots & & \|\\ \star & = & \star & = & \star\end{array} \simeq \begin{array}{ccccc}\star & = & \star & = & \star\\ \| & \omega_u & \vdots & \star_u & \|\\ \star & \cdots & \star & \cdots & \star\\ \| & \star_u & \vdots & \omega'_u & \|\\ \star & = & \star & = & \star\end{array} \simeq \begin{array}{ccccc}\star & = & \star & = & \star\\ \| & \star_u & \vdots & \omega_u & \|\\ \star & \cdots & \star & \cdots & \star\\ \| & \omega'_u & \vdots & \star_u & \|\\ \star & = & \star & = & \star\end{array} \simeq \begin{array}{ccccc}\star & = & \star & = & \star\\ \| & & \vdots & & \|\\ \| & \omega'_u & \vdots & \omega_u & \|\\ \| & & \vdots & & \|\\ \star & = & \star & = & \star\end{array}.$$

This braided $\mathbb{C}$–fibred categorical group $\Pi^2_{\mathbb{C}}(X,*)$ brings with it all information on the equivariant weak homotopy 3–type of any diagram of pointed spaces X in which all homotopy groups $\pi_i(X_u, *_u)$ vanish for $i \neq 2, 3$ and any $u \in \mathrm{Ob}\mathbb{C}$.

If we consider the induced diagram of pointed loop spaces $(\Omega(X,*),*)\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Top}_*$, then the $\mathbb{C}$–fibred braiding of $\Pi^3_{\mathbb{C}}(X,*) := \Pi^2_{\mathbb{C}}(\Omega(X,*),*)$ is actually a symmetry. This is called the *symmetric* $\mathbb{C}$*–fibred categorical group of* 3*–loops* of $(X,*)$.

Example 6.5 Let G be a group, regarded as a category, and let S be a ring on which an action by ring automorphisms of a group G is given. Then, the *Picard symmetric* G*–fibred categorical group of the* G*–ring* S, $\mathcal{P}ic_G(S)$, is defined as follows the objects of $\mathcal{P}ic_G(S)$ are the invertible (right) S–modules, ie, those finitely generated projective S–modules P with constant rank 1, and, for any $\sigma \in G$, a σ–morphism $P \to Q$ is an isomorphism of abelian groups $f\colon P \to Q$ such that $f(p\,\sigma(s)) = f(p)\,s$, for any $p \in P$ and $s \in S$. The composition is defined by usual composition of maps, and thus $\mathcal{P}ic_G(S)$ becomes a groupoid, which is actually G–fibred thanks to the σ–isomorphism $\mathrm{id}\colon P^\sigma \to P$, for each $\sigma \in G$, where P^σ denotes the S–module which is the same abelian group as P with S–action by $p \cdot s = p\,\sigma^{-1}(s)$. The G–tensor product is given by the tensor product of modules over S, and the unit G–functor $I\colon G \to \mathcal{P}ic_G(S)$ is defined by $I(\sigma) = \sigma^{-1}\colon S \to S$. The associativity, unit and commutativity constraints are as usual for the tensor product of S–modules:

$$\begin{array}{ccc} P\otimes_S(Q\otimes_S R)\cong(P\otimes_S Q)\otimes_S R, & P\otimes_S S\cong\ P\ \cong S\otimes P, & P\otimes_S Q\cong Q\otimes_S P, \\ x\otimes(y\otimes z)\leftrightarrow(x\otimes y)\otimes z & x\otimes s\leftrightarrow xs\leftrightarrow s\otimes x & x\otimes y\leftrightarrow y\otimes x. \end{array}$$

Note that the fibre category (over the unique object of G) is the symmetric categorical group $\mathcal{P}ic(S)=(\mathcal{P}ic(S),\otimes_S)$ of rank one projective S–module, where the dual module $P^*=\mathrm{Hom}_S(P,S)$ is an inverse of each invertible S–module P.

We should note that, for any group G, a symmetric G–fibred categorical group is exactly a *stably G–graded symmetric categorical group* in the sense of Fröhlich and Wall in [30; 31], where we refer the reader for other interesting examples in the study of rings in equivariant situations, such as the *Brauer symmetric G–fibred categorical group*, $\mathcal{B}r_G(S)$.

Example 6.6 Let G be a topological abelian group. Then, the *symmetric fibred categorical group of principal G–bundles, $\mathcal{B}(G)$*, is defined as follows (see Giraud [34] for more details):

The objects of $\mathcal{B}(G)$ are principal G–bundles $p\colon P\to T$. A morphism $\mathcal{B}(G)$ is a cartesian diagram, in the category **Top** of topological spaces,

$$\begin{array}{ccc} P' & \xrightarrow{p'} & T' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\sigma} \\ P & \xrightarrow{p} & T, \end{array}$$

where $\phi\colon P'\to P$ is G–equivariant. The base functor $\mathcal{B}(G)\to\mathbf{Top}$, $(\phi,\sigma)\mapsto\sigma$, makes $\mathcal{B}(G)$ a category fibred in groupoids over **Top**. The fibre at any space T, $\mathcal{B}(G)_T$, is the groupoid of principal G–bundles over T, which is pointed by the *trivial* principal G–bundle $I(T)=T\times G\to T$.

Given two principal G–bundles over the same space, say $p\colon P\to T$ and $q\colon Q\to T$, we can define a new principal G–bundle over T, $P\wedge^G Q\to T$, where $P\wedge^G Q:=P\times_T Q/G$ is the quotient of the pullback space by the antidiagonal action, that is, we identify $(x,y)\sim(x\cdot g,y\cdot g^{-1})$. The G–action on $P\wedge^G Q$ is given by $[x,y]\cdot g=[x\cdot g,y]$. Thus, we have the structure **Top**–fibred functors (this uses that G is abelian)

$$\wedge^G\colon\mathcal{B}(G)\times_{\mathbf{Top}}\mathcal{B}(G)\to\mathcal{B}(G),\qquad I\colon\mathbf{Top}\to\mathcal{B}(G).$$

The associativity, unit, and commutativity constraints

$$(P\wedge^G Q)\wedge^G R\to P\wedge^G(Q\wedge^G R),\qquad P\wedge^G(T\times G)\to P,\qquad P\wedge^G Q\to Q\wedge^G P,$$

are, respectively, given by the maps

$$[[x,y],z]\mapsto[x,[y,z]],\quad [x,(px,g)]\mapsto x\cdot g,\quad [x,y]\mapsto[y,x].$$

For any principal G–bundle $P\to T$, we have the *opposite* principal G–bundle $\check{P}\to T$, which is the same space P with G–action $p\,\check{\cdot}\,g=p\cdot g^{-1}$, and the T–isomorphism $P\wedge^G\check{P}\to T\times G$, $[x\cdot g,x]\mapsto(px,g)$.

6.3 The 2–category of braided (symmetric) $\mathbb{C}$–fibred categorical groups

For any given category $\mathbb{C}$, we shall describe here two striking examples of 2–categories. The 2*–category of braided* $\mathbb{C}$*–fibred categorical groups*, denoted by

$$\mathcal{BCG}_{\downarrow\mathbb{C}}, \tag{53}$$

and its full sub–2–category, the 2*–category of symmetric* $\mathbb{C}$*–fibred categorical groups*,

$$\mathcal{SCG}_{\downarrow\mathbb{C}}, \tag{54}$$

whose 1–cells are *braided* $\mathbb{C}$*–fibred functors*, and whose 2–cells are *monoidal* $\mathbb{C}$*–fibred isomorphisms*, which are defined as follows.

Definition 6.7 If $\mathbb{P}$ and $\mathbb{P}'$ are braided $\mathbb{C}$–fibred categorical groups, then a *braided* $\mathbb{C}$*–fibred functor* $F=(F,\varphi)\colon\mathbb{P}\to\mathbb{P}'$ consists of a $\mathbb{C}$–fibred functor $F\colon\mathbb{P}\to\mathbb{P}'$, between the underlying $\mathbb{C}$–fibred categories, together with two $\mathbb{C}$–fibred natural isomorphisms defined, respectively, by 1–isomorphisms

$$\varphi_{x,y}\colon Fx\otimes Fy\to F(x\otimes y),\quad \varphi_u\colon I'u\to FIu,$$

such that, for any objects u of $\mathbb{C}$ and x,y,z of $\mathbb{P}_u$, the following four coherence equations hold:

$$\varphi_{x,y\otimes z}\,(1_{Fx}\otimes\varphi_{y,z})\,\boldsymbol{a}'_{Fx,Fy,Fz}=F(\boldsymbol{a}_{x,y,z})\,\varphi_{x\otimes y,z}\,(\varphi_{x,y}\otimes 1_{Fz}), \tag{55}$$

$$F(\boldsymbol{r}_x)\,\varphi_{x,Iu}\,(1_{Fx}\otimes\varphi_u)=\boldsymbol{r}'_{Fx},\quad F(\boldsymbol{l}_x)\,\varphi_{Iu,x}\,(\varphi_u\otimes 1_{Fx})=\boldsymbol{l}'_{Fx}, \tag{56}$$

$$F(\boldsymbol{c}_{x,y})\,\varphi_{x,y}=\varphi_{y,x}\,\boldsymbol{c}'_{Fx,Fy}. \tag{57}$$

Definition 6.8 If $F,F'\colon\mathbb{P}\to\mathbb{P}'$ are braided $\mathbb{C}$–fibred functors between braided $\mathbb{C}$–fibred categorical groups, then a *monoidal* $\mathbb{C}$*–fibred isomorphism* between them is a $\mathbb{C}$–fibred isomorphism on the underlying $\mathbb{C}$–fibred functors $\Psi\colon F\Rightarrow F'$, such that, for all objects u of $\mathbb{C}$ and x,y of $\mathbb{P}_u$, the following two coherence equations below hold.

$$\varphi'_{x,y}\,(\Psi_x\otimes\Psi_y)=\Psi_{x\otimes y}\,\varphi_{x,y},\quad \varphi'_u=\Psi_{Iu}\,\varphi_u. \tag{58}$$

If $F\colon \mathbb{P} \to \mathbb{P}'$ and $F'\colon \mathbb{P}' \to \mathbb{P}''$ are braided $\mathbb{C}$–fibred functors, then the structure constraints of the composite $F'F\colon \mathbb{P} \to \mathbb{P}''$, at any objects u of $\mathbb{C}$ and x, y of $\mathbb{P}_u$, are obtained from those φ and φ', of F and F' respectively, by the compositions

$$\varphi^{F'F}_{x,y} = F'(\varphi_{x,y})\, \varphi'_{Fx,Fy}, \quad \varphi^{F'F}_u = F'(\varphi_u)\, \varphi'_u. \tag{59}$$

This composition is associative and unitary, so that the *category of braided $\mathbb{C}$–fibred categorical groups* is defined. Actually, this is the underlying category of the 2–category (53) whose 2–arrows are the monoidal $\mathbb{C}$–fibred isomorphisms. In this 2–category of braided $\mathbb{C}$–fibred categorical groups, the vertical composition of 2–cells $\Psi\colon F \Rightarrow F'$ and $\Psi'\colon F' \Rightarrow F''$, for $F, F', F''\colon \mathbb{P} \to \mathbb{P}'$ braided monoidal $\mathbb{C}$–fibred functors, is given by the ordinary vertical composition of natural transformations $\Psi' \cdot \Psi\colon F \Rightarrow F''$. That is, the component of $\Psi' \cdot \Psi$ at any object x of $\mathbb{P}$ is given by the composition in $\mathbb{P}'$:

$$(\Psi' \cdot \Psi)_x = \Psi'_x\, \Psi_x\colon Fx \xrightarrow{\Psi_x} F'x \xrightarrow{\Psi'_x} F''x \tag{60}$$

Similarly, the horizontal composition $\Psi'\Psi\colon F'F \Rightarrow G'G$, for $\Psi\colon F \Rightarrow G\colon \mathbb{P} \to \mathbb{P}'$ and $\Psi'\colon F' \Rightarrow G'\colon \mathbb{P}' \to \mathbb{P}''$ two $\mathbb{C}$–fibred morphisms is given by the usual horizontal composition of natural transformations:

$$(\Psi'\Psi)_x = G'\Psi_x\, \Psi'_{Fx} = \Psi'_{Gx}\, F'\Psi_x\colon F'Fx \Rightarrow G'Gx \tag{61}$$

The *homotopy category* (44) *of braided $\mathbb{C}$–fibred categorical groups*, $\mathrm{Ho}\,\mathcal{BCG}_{\downarrow\mathbb{C}}$, is then the quotient category of all braided $\mathbb{C}$–fibred categorical groups with monoidal $\mathbb{C}$–fibred isomorphism classes of braided $\mathbb{C}$–fibred functors as morphisms. A *braided $\mathbb{C}$–fibred equivalence* is then a braided $\mathbb{C}$–fibred functor inducing an isomorphism in the homotopy category. It is an easy consequence of a result by Grothendieck [36, Proposition 6.10] that a braided $\mathbb{C}$–fibred functor $F\colon \mathbb{P} \to \mathbb{P}'$ is a braided $\mathbb{C}$–fibred equivalence if and only if, for all objects u of $\mathbb{C}$, the restriction functor $F_u\colon \mathbb{P}_u \to \mathbb{P}'_u$ is an equivalence.

The 2–category (54), $\mathcal{SCG}_{\downarrow\mathbb{C}}$, is the full sub–2–category of $\mathcal{BCG}_{\downarrow\mathbb{C}}$ defined by the symmetric $\mathbb{C}$–fibred categorical groups. Note that the corresponding homotopy category $\mathrm{Ho}\,\mathcal{SCG}_{\downarrow\mathbb{C}}$ is a full subcategory of $\mathrm{Ho}\,\mathcal{BCG}_{\downarrow\mathbb{C}}$.

6.4 Pseudofunctors to braided (symmetric) categorical groups

For any given category $\mathbb{C}$, closely related to the 2–category of braided $\mathbb{C}$–fibred categorical groups is the 2–category (see (45))

$$\mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG})$$

of (normal) pseudofunctors $\mathcal{P}\colon \mathbb{C}^{\mathrm{op}} \to \mathcal{BCG}$ to the 2–category $\mathcal{BCG}$ of braided categorical groups (= $\mathcal{BCG}_{\downarrow_{\mathbf{1}}}$, where $\mathbf{1}$ is the terminal category), whose 1–cells are called *braided pseudotransformations*, and whose 2–cells are termed *braided modifications*. For notational accuracy or emphasis, we state bellow these concepts.

Definition 6.9 A (normal) *pseudofunctor to braided categorical groups*

$$\mathcal{P}\colon \mathbb{C}^{\mathrm{op}} \to \mathcal{BCG}$$

consists of mappings associating

- a braided categorical group $\mathcal{P}_u$, to each object u of $\mathbb{C}$,
- a braided functor $\sigma^*\colon \mathcal{P}_u \to \mathcal{P}_v$, to each arrow $\sigma\colon v \to u$ in $\mathbb{C}$,
- a braided isomorphism $\langle \sigma, \tau\rangle\colon \tau^*\sigma^* \Rightarrow (\sigma\tau)^*$, to each pair of arrows $w \stackrel{\tau}{\to} v \stackrel{\sigma}{\to} u$.

These data are required to satisfy the normalization conditions:

(62) $$1_u^* = 1_{\mathcal{P}_u} \quad \text{for each object } u \text{ of } \mathbb{C},$$

(63) $$\langle \sigma, 1_v\rangle = 1_{\sigma^*} = \langle 1_u, \sigma\rangle \quad \text{for each arrow } v \stackrel{\sigma}{\to} u \text{ of } \mathbb{C},$$

and the coherence condition that, for any triple of composable arrows $t \stackrel{\gamma}{\to} w \stackrel{\tau}{\to} v \stackrel{\sigma}{\to} u$ of $\mathbb{C}$, the following coherence condition holds:

(64) $$\langle \sigma, \tau\gamma\rangle \cdot \langle \tau, \gamma\rangle\sigma^* = \langle \sigma\tau, \gamma\rangle \cdot \gamma^*\langle \sigma, \tau\rangle.$$

Definition 6.10 If $\mathcal{P}, \mathcal{P}'\colon \mathbb{C}^{\mathrm{op}} \to \mathcal{BCG}$ are two pseudofunctors of braided categorical groups, then a *braided pseudotransformation* $T = (T, \Psi)\colon \mathcal{P} \Rightarrow \mathcal{P}'$ consists of

- a braided functor $T_u\colon \mathcal{P}_u \to \mathcal{P}'_u$, for each object u of $\mathbb{C}$,
- a braided isomorphism $\Psi_\sigma\colon T_v\,\sigma^* \Rightarrow \sigma^*\,T_u$, for each arrow $\sigma\colon v \to u$ in $\mathbb{C}$, satisfying that

(65) $$\Psi_{1_u} = 1_{T_u} \quad \text{for each object } u \text{ of } \mathbb{C},$$

and, for any pair $w \stackrel{\tau}{\to} v \stackrel{\sigma}{\to} u$ in $\mathbb{C}$, the following coherence condition holds:

(66) $$\Psi_{\sigma\tau} \cdot T_w\langle \sigma, \tau\rangle = \langle \sigma, \tau\rangle T_u \cdot \tau^*\Psi_\sigma \cdot \Psi_\tau\sigma^*.$$

Definition 6.11 For any two braided pseudotransformations $T, T'\colon \mathcal{P} \Rightarrow \mathcal{P}'$, between pseudofunctors of braided categorical groups as above, a *braided modification* $M\colon T \Rrightarrow T'$ consists of a braided isomorphism $M_u\colon T_u \Rightarrow T'_u\colon \mathcal{P}_u \to \mathcal{P}'_u$, for each object u of $\mathbb{C}$, such that, for each $\sigma\colon v \to u$, the following coherence condition holds

(67) $$\Psi'_\sigma \cdot M_v\sigma^* = \sigma^* M_u \cdot \Psi_\sigma.$$

In the 2–category $\mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG})$, a braided modification $M\colon T \Rrightarrow T'$ composes with a braided modification $M'\colon T' \Rrightarrow T''$ yielding the braided modification $M'\cdot M\colon T \Rrightarrow T''$, which is obtained from M and M' by pointwise vertical composition of braided isomorphisms, that is,

$$(M'\cdot M)_u = M'_u \cdot M_u, \tag{68}$$

for any object u of $\mathbb{C}$. Thus, every such braided modification $M\colon T \Rrightarrow T'$ becomes invertible and therefore, in this 2–category of pseudofunctors of braided categorical groups, the hom-categories are groupoids.

The composition $T'T\colon \mathcal{P} \Rightarrow \mathcal{P}''$ of braided pseudotransformations $T\colon \mathcal{P} \Rightarrow \mathcal{P}'$ and $T'\colon \mathcal{P}' \Rightarrow \mathcal{P}''$ between pseudofunctors of braided categorical groups $\mathcal{P}, \mathcal{P}', \mathcal{P}''\colon \mathbb{C}^{\mathrm{op}} \to \mathcal{BCG}$, is the braided pseudotransformation whose component at any object u of $\mathbb{C}$ is given by the composition of the braided functors T'_u and T_u:

$$(T'T)_u = T'_u T_u, \tag{69}$$

and whose component at any morphism $\sigma\colon v \to u$ of $\mathbb{C}$, $\Psi^{T'T}_\sigma\colon T'_v T_v \sigma^* \Rightarrow \sigma^* T'_u T_u$, is the braided isomorphism obtained as the composite

$$\Psi^{T'T}_\sigma = \Psi'_\sigma T_u \cdot T'_v \Psi_\sigma\colon\ T'_v T_v \sigma^* \xRightarrow{T'_v \Psi_\sigma} T'_v \sigma^* T_u \xRightarrow{\Psi'_\sigma T_u} \sigma^* T'_u T_u. \tag{70}$$

Similarly, the composition $M'M\colon T'T \Rrightarrow S'S$ of braided modifications $M\colon T \Rrightarrow S\colon \mathcal{P} \Rightarrow \mathcal{P}'$ and $M'\colon T' \Rrightarrow S'\colon \mathcal{P}' \Rightarrow \mathcal{P}''$ is given, at any object u of $\mathbb{C}$, by the horizontal composition of the braided isomorphisms M_u and M'_u, that is, by the formula

$$(M'\,M)_u = M'_u\, M_u = S'_u M_u \cdot M'_u T_u = M'_u S_u \cdot T'_u M_u\colon T'_u T_u \Longrightarrow S'_u S_u. \tag{71}$$

The *homotopy category (44) of pseudofunctors of braided categorical groups*,

$$\mathrm{Ho}\mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG}),$$

is then the quotient category of pseudofunctors with isomorphism classes of braided pseudotransformations as morphisms. A *braided pseudoequivalence* is then a braided pseudo transformation $T\colon \mathcal{P} \Rightarrow \mathcal{P}'$ that induces an isomorphism in the homotopy category. It is not hard to see that T is a braided pseudoequivalence if and only if for all objects u of $\mathbb{C}$, the braided functor $T_u\colon \mathcal{P}_u \to \mathcal{P}'_u$ is an equivalence of categories.

Remark 6.12 Let $U\colon \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG}) \to \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathbf{Cat})$ be the forgetful 2–functor. If $\mathcal{P}\colon \mathbb{C}^{\mathrm{op}} \to \mathcal{BCG}$ is any given pseudofunctor to braided categorical groups, then there is a (strict, genuine) functor $\mathcal{P}'\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Cat}$ which is pseudoequivalent to $U\mathcal{P}$ (see Giraud [33, Section 5] and Street [56, Corollary 9.2]). Since one can use any selected pseudoequivalence $T\colon U\mathcal{P} \Rightarrow \mathcal{P}'$ to transport the given braided monoidal structure of $\mathcal{P}$ to one on $\mathcal{P}'$ such that $T\colon \mathcal{P} \Rightarrow \mathcal{P}'$ becomes a braided pseudoequivalence, it follows that *every pseudofunctor* $\mathcal{P}\colon \mathbb{C}^{\mathrm{op}} \to \mathcal{BCG}$ *is pseudoequivalent to a functor* $\mathcal{P}'\colon \mathbb{C}^{\mathrm{op}} \to \mathcal{BCG}$.

The 2–category $\mathcal{SCG}$, of symmetric categorical groups, is a full sub–2–category of the 2–category $\mathcal{BCG}$ of braided categorical groups. Hence, we have

$$\mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{SCG}) \subseteq \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG}),$$

the full sub–2–category of pseudofunctors to symmetric categorical groups, and the corresponding full subcategory $\mathrm{Ho}\,\mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{SCG}) \subseteq \mathrm{Ho}\,\mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG})$, which is the *homotopy category of pseudofunctors to symmetric categorical groups*.

Example 6.13 *Braided crossed modules* (see Brown and Gilbert [9] and Joyal and Street [41, Remark 3.1]), also called *reduced* 2*–crossed modules* by Conduché [19], are systems of data $(H, G, \partial, \{\ ,\ \})$ consisting of groups G, H, a group homomorphism $\partial\colon H \to G$, a (right) group action of G on H, denoted $(h, g) \mapsto h^g$, and a *bracket operation* given by a map $\{\ ,\ \}\colon G \times G \to H$, subject to the following conditions:

- $h_1^{\partial h_2} = h_2^{-1} h_1 h_2$, $\partial(h^g) = g^{-1}\partial(h) g$,
- $\partial\{g_1, g_2\} = g_2^{-1} g_1^{-1} g_2 g_1$, $h\{g, \partial(h)\} = h^g$, $h^g\{\partial(h), g\} = h$.

A braided crossed module $(H, G, \partial, \{\ ,\ \})$ is called a *symmetric crossed module* by Aldrovandia and Noohi [1], or a *stable crossed module* by Conduché [19], whenever $\{g_1, g_2\}\{g_2, g_1\} = 1$, for all $g_1, g_2 \in G$.

These braided (resp. symmetric) crossed modules are the objects of a 2–category (cf Noohi [49, Section 8]), denoted by $\mathcal{BXM}$ ($\mathcal{SXM}$), whose 1–cells

$$(q, p, \varphi)\colon (H, G, \partial, \{\ ,\ \}) \to (H', G', \partial', \{\ ,\ \}')$$

consist of normalized maps $q\colon H \to H'$, $p\colon G \to G'$, and $\varphi\colon G \times G \to H'$, satisfying

- $\partial' q = p\,\partial$,
- $p(g_1 g_2) = p(g_1)\, p(g_2)\, \partial(\varphi(g_1, g_2))$, $q(h_1 h_2) = q(h_1)\, q(h_2)\, \varphi(\partial(h_1), \partial(h_2))$,
- $\varphi(g_1, g_2)^{p(g_3)}\, \varphi(g_1 g_2, g_3) = \varphi(g_2, g_3)\, \varphi(g_1, g_2 g_3)$,
- $\{p(g_1), p(g_2)\}'\, \varphi(g_2, g_1) = \varphi(g_1, g_2)\, q(\{g_1, g_2\})$.

If $(q', p', \varphi')\colon (H, G, \partial, \{\ ,\ \}) \to (H', G', \partial', \{\ ,\ \}')$ is another morphism between braided crossed modules, then a 2–cell $\Psi\colon (q, p, \varphi) \Rightarrow (q', p', \varphi')$ is a map $\Psi\colon G \to H'$, such that

- $p'(g) = p(g)\,\partial(\Psi(g))$, $q'(h) = q(h)\,\Psi(\partial(h))$,
- $\varphi(g_1, g_2)\,\Psi(g_1 g_2) = \Psi(g_1)^{p(g_2)}\,\Psi(g_2)\,\varphi'(g_1, g_2)$.

We should stress that there is an interesting description of the 1–cells in $\mathcal{BXM}$ in terms of *braided butterflies* $(B, \iota, \kappa, \sigma, \rho)\colon (H, G, \partial, \{\ ,\ \}) \to (H', G', \partial', \{\ ,\ \}')$ in Aldrovandia and Noohi [1, Definitions 4.1.3, 7.4.1], that is, commutative diagrams of groups

$$\begin{tikzcd} H \arrow[d, "\partial"'] \arrow[dr, "\kappa"] & & H' \arrow[dl, "\iota"'] \arrow[d, "\partial'"] \\ & B \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\ G & & G' \end{tikzcd}$$

such that the various maps satisfy the following conditions:

- $\rho\kappa = 1$, and the sequence $1 \to H' \overset{\iota}{\to} B \overset{\sigma}{\to} G \to 1$ is short exact,
- $\kappa(h^{\sigma(b)}) = b^{-1}\kappa(h')\,b$, $\iota(h'^{\rho(b)}) = b^{-1}\iota(h')\,b$,
- $\kappa\{\sigma(b_1), \sigma(b_2)\}\,\iota\{\rho(b_1), \rho(b_2)\} = b_2^{-1} b_1^{-1} b_2 b_1$.

Any braided (symmetric) crossed module $(H, G, \partial, \{\ ,\ \})$ gives rise to a braided (symmetric) categorical group $\mathbb{P}(H, G, \partial, \{\ ,\ \})$, which is defined as follows. The objects are the elements $g \in G$. A morphism $h\colon g \to g'$ is an element $h \in H$ with $g' = g\,\partial(h)$, and the composition is multiplication in H. The tensor product is given by

$$(g_1 \xrightarrow{h_1} g_1') \otimes (g_2 \xrightarrow{h_2} g_2') = (\ g_1 g_2 \xrightarrow{h_1^{g_2}\,h_2} g_1' g_2'\ ).$$

The associativity and unit constraints are identities, and the braiding is given by the equation

$$\boldsymbol{c}_{g_1, g_2} = \{g_1, g_2\}\colon g_1 g_2 \to g_2 g_1.$$

Indeed, this construction $(H, G, \partial, \{\ ,\ \}) \mapsto \mathbb{P}(H, G, \partial, \{\ ,\ \})$ is the function on objects of respective biequivalences of 2–categories $\mathcal{BXM} \simeq \mathcal{BCG}$ and $\mathcal{SXM} \simeq \mathcal{SCG}$ (cf Noohi [49, Section 8]). Hence, for any category $\mathbb{C}$, there are biequivalences

$$\mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BXM}) \simeq \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG}), \quad \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{SXM}) \simeq \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{SCG}).$$

6.5 From pseudofunctors to fibrations: The braided Grothendieck construction

Given a pseudofunctor to categories, $\mathcal{P}\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Cat}$, there is a well-known way to form a single $\mathbb{C}$–fibred category $\int_{\mathbb{C}} \mathcal{P}$, called the *Grothendieck construction* on $\mathcal{P}$, due to Grothendieck [37] and Giraud [33; 34]. Below, we enrich the Grothendieck construction for any pseudofunctor to braided categorical groups $\mathcal{P}\colon \mathbb{C}^{\mathrm{op}} \to \mathcal{BCG}$, assembling it into a large braided $\mathbb{C}$–fibred categorical group

$$\int_{\mathbb{C}}\mathcal{P} = (\textstyle\int_{\mathbb{C}}\mathcal{P}, P, \otimes, \boldsymbol{I}, \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{l}, \boldsymbol{c}), \tag{72}$$

which is actually a pseudocolimit of the braided categorical groups $\mathcal{P}_u$, $u \in \mathrm{Ob}\mathbb{C}$ (see Carrasco, Cegarra and Garzón [12, Theorem 3.1]). This braided $\mathbb{C}$–fibred categorical group $\int_{\mathbb{C}}\mathcal{P}$ is defined as follows.

The *objects* are pairs (x, u), where u is an object of $\mathbb{C}$ and x one of the category $\mathcal{P}_u$.

The *arrows* are pairs $(f, \sigma)\colon (y, v) \to (x, u)$ where $\sigma\colon v \to u$ is in $\mathbb{C}$ and $f\colon y \to \sigma^* x$ is in $\mathcal{P}_v$.

The *composition* is defined by

$$((z, w) \xrightarrow{(g,\tau)} (y, v) \xrightarrow{(f,\sigma)} (x, u)) \;\mapsto\; ((z, w) \xrightarrow{(\langle\sigma,\tau\rangle_x\ \tau^*f\ g,\, \sigma\tau)} (x, u)). \tag{73}$$

As is well-known, this composition is associative and unitary owing to the naturality, coherence condition (64), and normalization conditions (62) and (63). For any object (x, u) of $\int_{\mathbb{C}}\mathcal{P}$, the corresponding identity is just $1_{(x,u)} = (1_x, 1_u)\colon (x, u) \to (x, u)$. Thus, $\int_{\mathbb{C}}\mathcal{P}$ is a category.

The *projection functor* $P\colon \int_{\mathbb{C}}\mathcal{P} \to \mathbb{C}$ is given by

$$\big((y, v) \xrightarrow{(f,\sigma)} (x, u)\big) \overset{P}{\mapsto} (v \xrightarrow{\sigma} u).$$

This is actually a fibration since, for any morphism $\sigma\colon v \to u$ in $\mathbb{C}$ and any u–object (x, u) of $\int_{\mathbb{C}}\mathcal{P}$, there is the cartesian σ–morphism $(1_{\sigma^* x}, \sigma)\colon (\sigma^* x, v) \to (x, u)$.

The $\mathbb{C}$*–fibred tensor product*

$$\otimes\colon \textstyle\int_{\mathbb{C}}\mathcal{P} \times_{\mathbb{C}} \int_{\mathbb{C}}\mathcal{P} \longrightarrow \int_{\mathbb{C}}\mathcal{P}, \tag{74}$$

is defined by

$$\begin{aligned} ((y, v) \xrightarrow{(f,\sigma)} (x, u)) \;\otimes\; ((y', v) \xrightarrow{(f',\sigma)} (x', u)) \\ = ((y \otimes y', v) \xrightarrow{(\varphi^{\sigma^*}_{x,x'}(f\otimes f'),\sigma)} (x \otimes x', u)). \end{aligned}$$

So defined, $\otimes$ is actually a functor since, for any morphisms of the form

$$(z, w) \xrightarrow{(g,\tau)} (y, v) \xrightarrow{(f,\sigma)} (x, u), \quad (z', w) \xrightarrow{(g',\tau)} (y', v) \xrightarrow{(f',\sigma)} (x', u),$$

the equality

$$\big((f,\sigma)(g,\tau)\big)\otimes\big((f',\sigma)(g',\tau)\big)=\big((f,\sigma)\otimes(f',\sigma)\big)\big((g,\tau)\otimes(g',\tau)\big),$$

follows (by composing with the morphism $g\otimes g'\colon z\otimes z'\to\tau^*y\otimes\tau^*y'$) from the commutativity of the outside region in the diagram:

$$\begin{tikzcd}
\tau^*y\otimes\tau^*y' \arrow[rr, "\tau^*f\otimes\tau^*f'"] \arrow[d, "\varphi^{\tau^*}_{y,y'}"'] & & \tau^*\sigma^*x\otimes\tau^*\sigma^*x' \arrow[rr, "\langle\sigma,\tau\rangle_x\otimes\langle\sigma,\tau\rangle_{x'}"] \arrow[dl, "\varphi^{\tau^*}_{\sigma^*x,\sigma^*x'}"'] \arrow[dr, "\varphi^{\tau^*\sigma^*}_{x,x'}"] & & (\sigma\tau)^*x\otimes(\sigma\tau)^*x' \arrow[d, "\varphi^{(\sigma\tau)^*}_{x,x'}"] \\
\tau^*(y\otimes y') \arrow[r, "\tau^*(f\otimes f')"'] \arrow[r, phantom, "\text{(A)}", shift left=4ex] & \tau^*(\sigma^*x\otimes\sigma^*x') \arrow[rr, "\tau^*\varphi^{\sigma^*}_{x,x'}"'] \arrow[rr, phantom, "(59)", shift left=4ex] & & \tau^*\sigma^*(x\otimes x') \arrow[r, "\langle\sigma,\tau\rangle_{x\otimes x'}"'] \arrow[r, phantom, "(58)", shift left=4ex] & (\sigma\tau)^*(x\otimes x')
\end{tikzcd}$$

where the region labelled (A) commutes since φ^{τ^*} is natural, and the other two commute by the references given in them.

The *associativity* $\mathbb{C}$*–fibred constraint*, at any u–objects $(x,u),(y,u)$ and (z,u) of $\int_{\mathbb{C}}\mathcal{P}$,

$$\text{(75)}\qquad \boldsymbol{a}_{(x,u),(y,u),(z,u)}\colon ((x,u)\otimes(y,u))\otimes(z,u)\to(x,u)\otimes((y,u)\otimes(z,u))$$

is the morphism $(\boldsymbol{a}_{x,y,z},1_u)\colon ((x\otimes y)\otimes z,u)\to(x\otimes(y\otimes z),u)$, where $\boldsymbol{a}_{x,y,z}$ is the associativity isomorphism in the braided categorical group $\mathcal{P}_u$. This family of 1–isomorphisms (75) actually gives a $\mathbb{C}$–fibred natural equivalence since, for any three morphisms in $\int_{\mathbb{C}}\mathcal{P}$ of the form $(f,\sigma)\colon (x',v)\to(x,u)$, $(g,\sigma)\colon (y',v)\to(y,u)$ and $(h,\sigma)\colon (z',v)\to(z,u)$, the equality

$$\boldsymbol{a}_{(x,u),(y,u),(z,u)}\big((f,\sigma)\otimes(g,\sigma))\otimes(h,\sigma)\big)\\ =\big((f,\sigma)\otimes((g,\sigma)\otimes(h,\sigma))\big)\,\boldsymbol{a}_{(x',v),(y',v),(z',v)}$$

follows from the commutativity of the following diagram in $\mathcal{P}_v$:

$$\begin{tikzcd}
 & x'\otimes(y'\otimes z') \arrow[r, "f\otimes(g\otimes h)"] & \sigma^*x\otimes(\sigma^*y\otimes\sigma^*z) \arrow[dr, "1\otimes\varphi^{\sigma^*}_{y,z}"] & \\
(x'\otimes y')\otimes z' \arrow[ur, "\boldsymbol{a}"] \arrow[d, "(f\otimes g)\otimes h"'] \arrow[r, phantom, "\text{(A)}"] & & & \sigma^*x\otimes\sigma^*(y\otimes z) \arrow[d, "\varphi^{\sigma^*}_{x,y\otimes z}"] \\
(\sigma^*x\otimes\sigma^*y)\otimes\sigma^*z \arrow[uurr, "\boldsymbol{a}"] \arrow[dr, "\varphi^{\sigma^*}_{x,y}\otimes 1"'] \arrow[rrr, phantom, "(55)"] & & & \sigma^*(x\otimes(y\otimes z)) \\
 & \sigma^*(x\otimes y)\otimes\sigma^*z \arrow[r, "\varphi^{\sigma^*}_{x\otimes y,z}"] & \sigma^*((x\otimes y)\otimes z) \arrow[ur, "\sigma^*\boldsymbol{a}"'] &
\end{tikzcd}$$

where the inner subdiagram (A) commutes by the naturality of $\boldsymbol{a}$, and the other is commutative by the reference therein since $\sigma^*\colon \mathcal{P}_u \to \mathcal{P}_v$ is a braided functor.

The $\mathbb{C}$*–fibred unit functor*,

$$I\colon \mathbb{C} \to \textstyle\int_{\mathbb{C}} \mathcal{P}, \tag{76}$$

is defined on objects by $Iu = (I_u, u)$, where I_u is the unit object of the braided categorical group $\mathcal{P}_u$, and it carries a morphism $\sigma\colon v \to u$ of $\mathbb{C}$ to the σ–morphism of $\int_{\mathbb{C}} \mathcal{P}$ given by the structure unit isomorphism of the braided functor $\sigma^*\colon \mathcal{P}_u \to \mathcal{P}_v$, $\varphi_1^{\sigma^*}\colon I_v \to \sigma^* I_u$, that is,

$$I\sigma = (\varphi_1^{\sigma^*}, \sigma)\colon (I_v, v) \to (I_u, u). \tag{77}$$

Note that $I\colon \mathbb{C} \to \int_{\mathbb{C}} \mathcal{P}$ is unitary because of the normalization condition (62). Furthermore, if $\tau\colon w \to v$ and $\sigma\colon v \to u$ is any pair of composable arrows in $\mathbb{C}$, then the equality $I(\sigma)I(\tau) = I(\sigma\tau)$ follows from the commutativity of the following diagram in $\mathcal{P}_w$

$$\begin{tikzcd}
I_w \arrow[r, "\varphi_1^{(\sigma\tau)^*}"] \arrow[d, "\varphi_1^{\tau^*}"'] \arrow[dr, "\varphi_1^{\tau^*\sigma^*}"] & (\sigma\tau)^* I_u \arrow[dl, phantom, "(58)"] \\
\tau^* I_v \arrow[r, "\tau^*\varphi_1^{\sigma^*}"'] \arrow[ur, phantom, "(59)"] & \tau^*\sigma^* I_u, \arrow[u, "\langle\sigma,\tau\rangle_{I_u}"']
\end{tikzcd}$$

where both triangles commute by the references given in them, since $\langle\sigma,\tau\rangle\colon \tau^*\sigma^* \Rightarrow (\sigma\tau)^*$ is a braided isomorphism. Hence, (76) is actually a $\mathbb{C}$–fibred functor.

The *unit* $\mathbb{C}$*–fibred constraints*, at any object (x,u) of $\int_{\mathbb{C}} \mathcal{P}$,

$$\boldsymbol{r}_{(x,u)}\colon (x,u) \otimes (I_u, u) \to (x,u), \quad \boldsymbol{l}_{(x,u)}\colon (I_u,u) \otimes (x,u) \to (x,u), \tag{78}$$

are respectively given by the morphisms

$$(\boldsymbol{r}_x, 1_u)\colon (x \otimes I_u, u) \to (x,u) \quad \text{and} \quad (\boldsymbol{l}_x, 1_u)\colon (I_u \otimes x, u) \to (x,u),$$

where $\boldsymbol{r}_x$ and $\boldsymbol{l}_x$ are the right and left unit isomorphisms in the braided categorical group $\mathcal{P}_u$. These families of 1–isomorphisms (78) actually give $\mathbb{C}$–fibred natural equivalences since, for any morphism of $\int_{\mathbb{C}} \mathcal{P}$, say $(f,\sigma)\colon (y,v) \to (x,u)$, the equalities $\boldsymbol{r}_{(x,u)}\big((f,\sigma) \otimes (\varphi_1^{\sigma^*}, \sigma)\big) = (f,\sigma)\,\boldsymbol{r}_{(y,u)}$ and $\boldsymbol{l}_{(x,u)}\big((\varphi_1^{\sigma^*}, \sigma) \otimes (f,\sigma)\big) = (f,\sigma)\,\boldsymbol{l}_{(y,u)}$

follow from the commutativity of the following diagrams in $\mathcal{P}_v$

$$
\begin{tikzcd}
y\otimes I_v \arrow[r, "\boldsymbol{r}_y"] \arrow[dr, "f\otimes 1"] \arrow[dd, "f\otimes\varphi_1^{\sigma^*}"'] & y \arrow[r, "f"] \arrow[d, phantom, "\text{(B)}"] & \sigma^* x \\
\text{(A)} & \sigma^* x\otimes I_v \arrow[ur, "\boldsymbol{r}_{\sigma^* x}"'] \arrow[dl, "1\otimes\varphi_1^{\sigma^*}"] \arrow[dr, phantom, "(56)"] & \\
\sigma^* x\otimes\sigma^* I_u \arrow[rr, "\varphi^{\sigma^*}_{x,I_u}"] & & \sigma^*(x\otimes I_u), \arrow[uu, "\sigma^*\boldsymbol{r}_x"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
I_v\otimes y \arrow[r, "\boldsymbol{l}_y"] \arrow[dr, "1\otimes f"] \arrow[dd, "\varphi_1^{\sigma^*}\otimes f"'] & y \arrow[r, "f"] \arrow[d, phantom, "\text{(B)}"] & \sigma^* x \\
\text{(A)} & I_v\otimes\sigma^* x \arrow[ur, "\boldsymbol{l}_{\sigma^* x}"'] \arrow[dl, "\varphi_1^{\sigma^*}\otimes 1"] \arrow[dr, phantom, "(56)"] & \\
\sigma^* I_u\otimes\sigma^* x \arrow[rr, "\varphi^{\sigma^*}_{I_u,x}"] & & \sigma^*(I_u\otimes x), \arrow[uu, "\sigma^*\boldsymbol{l}_x"']
\end{tikzcd}
$$

where the regions (A) commute owing to $\otimes$ being a functor, the commutativity of the regions (B) follows from the naturality of the right and left unit constraints of $\mathcal{P}_v$, and the third regions commute by the reference therein.

The *braiding* $\mathbb{C}$*–fibred constraint*,

$$
\boldsymbol{c}_{(x,u),(y,u)}\colon (x,u)\otimes(y,u)\to(y,u)\otimes(x,u), \tag{79}
$$

at any u–objects (x,u) and (y,u), is given by the morphism $(\boldsymbol{c}_{x,y}, 1_u)\colon (x\otimes y,u)\to(y\otimes x,u)$, where $\boldsymbol{c}_{x,y}$ is the braiding in $\mathcal{P}_u$. So defined, $\boldsymbol{c}$ is natural since, for any arrow $\sigma\colon v\to u$ in $\mathbb{C}$ and any pair of σ–morphisms $(f,\sigma)\colon(x',v)\to(x,u)$ and $(g,\sigma)\colon(y',v)\to(y,u)$ in $\int_{\mathbb{C}}\mathcal{P}$, the equality $\boldsymbol{c}_{(x,u),(y,u)}\big((f,\sigma)\otimes(g,\sigma)\big)=\big((g,\sigma)\otimes(f,\sigma)\big)\,\boldsymbol{c}_{(x',v),(y',v)}$ is a direct consequence of the commutativity of the following diagram in $\mathcal{P}_v$

$$
\begin{tikzcd}
x'\otimes y' \arrow[r, "\boldsymbol{c}_{x',y'}"] \arrow[d, "f\otimes g"'] & y'\otimes x' \arrow[r, "g\otimes f"] \arrow[d, phantom, "\text{(A)}"] & \sigma^* y\otimes\sigma^* x \arrow[d, "\varphi^{\sigma^*}_{y,x}"] \\
\sigma^* x\otimes\sigma^* y \arrow[urr, "\boldsymbol{c}_{\sigma^* x,\sigma^* y}"] \arrow[r, "\varphi^{\sigma^*}_{x,y}"'] & \sigma^*(x\otimes y) \arrow[r, "\sigma^*\boldsymbol{c}_{x,y}"'] \arrow[u, phantom, "(57)"] & \sigma^*(y\otimes x),
\end{tikzcd}
$$

where (A) commutes due to the naturality of the braiding $\boldsymbol{c}$ of $\mathcal{P}_v$.

Thus, we conclude that the Grothendieck construction $(\int_{\mathbb{C}}\mathcal{P}, P, \otimes, I, \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{l}, \boldsymbol{c})$ yields a braided $\mathbb{C}$–fibred categorical group since it is straightforward to see that the required coherence conditions in (46)-(49) follow from the corresponding ones in each braided categorical group $\mathcal{P}_u$ for the different objects u of $\mathbb{C}$.

Remark 6.14 By construction, it is clear that if $\mathcal{P}\colon \mathbb{C}^{\mathrm{op}} \to \mathcal{SCG}$ is a pseudofunctor to symmetric categorical groups, then $\int_{\mathbb{C}}\mathcal{P}$ is actually a symmetric $\mathbb{C}$–fibred categorical group.

6.6 The braided Grothendieck construction 2–functor

The ordinary Grothendieck construction is the function on objects of a 2–functor $\int_{\mathbb{C}}\colon \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathbf{Cat}) \to \mathbf{Cat}_{\downarrow\mathbb{C}}$, from the 2–category of pseudofunctors to categories to the 2–category of $\mathbb{C}$–fibred categories (see Giraud [33; 34] and Vistoli [58]). Similarly, the assignment $\mathcal{P} \mapsto \int_{\mathbb{C}}\mathcal{P}$, given by the enriched Grothendieck construction (72), is the function on objects of a 2–functor

$$\int_{\mathbb{C}}\colon \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG}) \to \mathcal{BCG}_{\downarrow\mathbb{C}}\,, \tag{80}$$

described as follows:

On *braided pseudotransformations*, it carries any $T = (T, \Psi)\colon \mathcal{P} \Rightarrow \mathcal{P}'$ to the braided $\mathbb{C}$–fibred functor

$$\int_{\mathbb{C}}T = (\int_{\mathbb{C}}T, \varphi^{\int_{\mathbb{C}}T})\colon \int_{\mathbb{C}}\mathcal{P} \to \int_{\mathbb{C}}\mathcal{P}', \tag{81}$$

which is defined by

$$((y,v) \xrightarrow{(f,\sigma)} (x,u)) \xmapsto{\int_{\mathbb{C}}T} ((T_v y, v) \xrightarrow{(\Psi_\sigma x\ \ T_v f, \sigma)} (T_u x, u)). \tag{82}$$

As is well-known, $\int_{\mathbb{C}}T\colon \int_{\mathbb{C}}\mathcal{P} \to \int_{\mathbb{C}}\mathcal{P}'$ is actually a $\mathbb{C}$–fibred functor thanks to the naturality, coherence condition (66), and normalization condition (65). Indeed, it is a braided $\mathbb{C}$–fibred functor, whose structure $\mathbb{C}$–fibred natural equivalences

$$\varphi^{\int_{\mathbb{C}}T}_{(x,u),(y,u)}\colon \int_{\mathbb{C}}T(x,u) \otimes \int_{\mathbb{C}}T(y,u) \to \int_{\mathbb{C}}T((x,u) \otimes (y,u)), \tag{83}$$

$$\varphi^{\int_{\mathbb{C}}T}_{u}\colon Iu \to \int_{\mathbb{C}}TIu, \tag{84}$$

for any objects u of $\mathbb{C}$ and (x,u) and (y,u) of $\int_{\mathbb{C}}\mathcal{P}$, are respectively defined by

$$\varphi^{\int_{\mathbb{C}}T}_{(x,u),(y,u)} = (\varphi^{T_u}_{x,y}, 1_u)\colon (T_u x \otimes T_u y, u) \to (T_u(x \otimes y), u), \tag{85}$$

$$\varphi^{\int_{\mathbb{C}}T}_{u} = (\varphi^{T_u}_{1}, 1_u)\colon (I_u, u) \to (T_u I_u, u). \tag{86}$$

The four coherence equations in (55), (56), and (57), in order for $\int_{\mathbb{C}}T$ to be a braided $\mathbb{C}$–fibred functor, follow from the corresponding coherence conditions for the various braided functors T_u, $u \in \mathrm{Ob}\mathbb{C}$. To prove that both $\varphi^{\int_{\mathbb{C}}T}$ in (83) and (84) are natural, let $\sigma\colon v \to u$ be any arrow in $\mathbb{C}$ and let $(f,\sigma)\colon (x',v) \to (x,u)$ and

$(g,\sigma)\colon (y',v)\to(y,u)$ be any two σ–morphisms in $\int_{\mathbb{C}}\mathcal{P}$. Then, observe that the commutativity of the diagrams

$$\begin{tikzcd}
\int_{\mathbb{C}}T(x',v)\otimes\int_{\mathbb{C}}T(y',v) \arrow[r, "\varphi^{\int_{\mathbb{C}}T}"] \arrow[d, "\int_{\mathbb{C}}T(f,\sigma)\otimes\int_{\mathbb{C}}T(g,\sigma)"'] & \int_{\mathbb{C}}T((x',v)\otimes(y',v)) \arrow[d, "\int_{\mathbb{C}}T((f,\sigma)\otimes(g,\sigma))"] \\
\int_{\mathbb{C}}T(x,u)\otimes\int_{\mathbb{C}}T(y,u) \arrow[r, "\varphi^{\int_{\mathbb{C}}T}"] & \int_{\mathbb{C}}T((x,u)\otimes(y,u))
\end{tikzcd}
\qquad
\begin{tikzcd}
Iv \arrow[r, "\varphi^{\int_{\mathbb{C}}T}"] \arrow[d, "I\sigma"'] & \int_{\mathbb{C}}TIv \arrow[d, "\int_{\mathbb{C}}TI\sigma"] \\
Iu \arrow[r, "\varphi^{\int_{\mathbb{C}}T}"] & \int_{\mathbb{C}}TIu
\end{tikzcd}$$

follows from the commutativity of the two diagrams below, where the region (A) commutes by naturality, and the other regions by the references given in them.

$$\begin{tikzcd}
 & & T_v(x'\otimes y') \arrow[drr, "T_v(f\otimes g)"] & & \\
T_vx'\otimes T_vy' \arrow[urr, "\varphi^{T_v}_{x',y'}"] \arrow[d, "T_vf\otimes T_vg"'] & & \text{(A)} & & T_v(\sigma^*x\otimes\sigma^*y) \arrow[d, "T_v\varphi^{\sigma^*}_{x,y}"] \\
T_v\sigma^*x\otimes T_v\sigma^*y \arrow[urrrr, "\varphi^{T_v}_{\sigma^*x,\sigma^*y}"] \arrow[rrrr, "\varphi^{T_v\sigma^*}_{x,y}"] \arrow[d, "\Psi_\sigma x\otimes\Psi_\sigma y"'] & & & (59) & T_v\sigma^*(x\otimes y) \arrow[d, "\Psi_\sigma(x\otimes y)"] \\
\sigma^*T_ux\otimes\sigma^*T_uy \arrow[rrrr, "\varphi^{\sigma^*T_u}_{x,y}"] \arrow[drr, "\varphi^{\sigma^*}_{T_ux,T_uy}"'] & & & (58) & \sigma^*T_u(x\otimes y) \\
 & & \sigma^*(T_ux\otimes T_uy) \arrow[urr, "\sigma^*\varphi^{T_u}_{x,y}"'] \arrow[u, phantom, "(59)"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & T_vI_v \arrow[dr, "T_v\varphi^{\sigma^*}_1"] \arrow[d, phantom, "(59)"] & \\
I_v \arrow[ur, "\varphi^{T_v}_1"] \arrow[r, "\varphi^{T_v\sigma^*}_1"] \arrow[d, "\varphi^{\sigma^*}_1"'] \arrow[dr, "\varphi^{\sigma^*T_u}_1"] & T_v\sigma^*I_u \arrow[d, phantom, "(58)"] & \\
\sigma^*I_u \arrow[r, "\sigma^*\varphi^{T_u}_1"'] \arrow[u, phantom, "(59)"] & \sigma^*T_uI_u. &
\end{tikzcd}$$

On *braided modifications*, the 2–functor $\int_{\mathbb{C}}$ acts as follows: For any two braided pseudotransformations $T,T'\colon\mathcal{P}\Rightarrow\mathcal{P}'$, any braided modification $M\colon T\Rrightarrow T'$ gives rise to the braided $\mathbb{C}$–fibred isomorphism

(87) $$\int_{\mathbb{C}}M\colon \int_{\mathbb{C}}T\Rightarrow\int_{\mathbb{C}}T'\colon\int_{\mathbb{C}}\mathcal{P}\to\int_{\mathbb{C}}\mathcal{P}',$$

whose component at an object (x,u) of $\int_{\mathbb{C}}\mathcal{P}$, is defined by

$$\int_{\mathbb{C}} M(x,u)=(M_u x, 1_u)\colon (T_u x,u)\longrightarrow (T'_u x,u). \tag{88}$$

As is well-known, $\int_{\mathbb{C}} M$ is actually a $\mathbb{C}$–fibred natural transformation, thanks to the naturality of the various M_u and the coherence condition (67). Moreover, we conclude that it is actually a braided $\mathbb{C}$–fibred isomorphism since the two coherence conditions in (58), that is, the commutativity of the diagrams

$$\begin{tikzcd}
\int_{\mathbb{C}} T(x,u)\otimes \int_{\mathbb{C}} T(y,u) \arrow[r, "\varphi^{\int_{\mathbb{C}} T}"] \arrow[d, "\int_{\mathbb{C}} M\otimes \int_{\mathbb{C}} M"'] & \int_{\mathbb{C}} T((x,u)\otimes(y,u)) \arrow[d, "\int_{\mathbb{C}} M"] \\
\int_{\mathbb{C}} T'(x,u)\otimes \int_{\mathbb{C}} T'(y,u) \arrow[r, "\varphi^{\int_{\mathbb{C}} T'}"] & \int_{\mathbb{C}} T'((x,u)\otimes(y,u)),
\end{tikzcd}
\qquad
\begin{tikzcd}
& Iu \arrow[dl, "\varphi^{\int_{\mathbb{C}} T}"'] \arrow[dr, "\varphi^{\int_{\mathbb{C}} T'}"] & \\
\int_{\mathbb{C}} TIu \arrow[rr, "\int_{\mathbb{C}} M"] & & \int_{\mathbb{C}} T'Iu,
\end{tikzcd}$$

follows from the commutativity of the corresponding diagrams for the different M_u, $u\in \mathrm{Ob}\mathbb{C}$, that is,

$$\begin{tikzcd}
T_u x\otimes T_u y \arrow[r, "\varphi^{T_u}_{x,y}"] \arrow[d, "M_u x\otimes M_u y"'] & T_u(x\otimes y) \arrow[d, "M_u(x\otimes y)"] \\
T'_u x\otimes T'_u y \arrow[r, "\varphi^{T'_u}_{x,y}"] & T'_u(x\otimes y),
\end{tikzcd}
\qquad
\begin{tikzcd}
& I_u \arrow[dl, "\varphi^{T_u}_1"'] \arrow[dr, "\varphi^{T'_u}_1"] & \\
T_u I_u \arrow[rr, "M_u"] & & T'_u I_u.
\end{tikzcd}$$

For any braided modifications $M\colon T\Rrightarrow T'$ and $M'\colon T'\Rrightarrow T''$, where $T,T',T''\colon \mathcal{P}\Rightarrow\mathcal{P}'$, the equality $\int_{\mathbb{C}}(M'\cdot M)=\int_{\mathbb{C}} M'\cdot \int_{\mathbb{C}} M$ is easily verified (from (60) and (68)), as well as the equality $\int_{\mathbb{C}} 1_T = 1_{\int_{\mathbb{C}} T}$, for any braided pseudotransformation $T\colon \mathcal{P}\Rightarrow\mathcal{P}'$. Furthermore, it is easily seen that $\int_{\mathbb{C}} 1_{\mathcal{P}} = 1_{\int_{\mathbb{C}}\mathcal{P}}$ and, if $T\colon \mathcal{P}\to\mathcal{P}'$ and $T'\colon\mathcal{P}'\to\mathcal{P}''$ are any two composable braided pseudotransformations, then $\int_{\mathbb{C}}(T'T)=\int_{\mathbb{C}} T'\int_{\mathbb{C}} T$ as functors, since

$$\int_{\mathbb{C}}(T'T)(x,u)=(T'_uT_u x,u)=\int_{\mathbb{C}} T'\int_{\mathbb{C}} T(x,u),$$

for any object (x,u) of $\int_{\mathbb{C}}\mathcal{P}$, and, if $(f,\sigma)\colon (y,v)\to(x,u)$ is any morphism of $\int_{\mathbb{C}}\mathcal{P}$, then

$$\begin{aligned}
\int_{\mathbb{C}}(T'T)(f,\sigma) &\overset{(82)}{=} (\Psi^{T'T}_\sigma x\,(T'T)_v f,\sigma) \overset{(69),(70)}{=} (\Psi'_\sigma T_u x\; T'_v\Psi_\sigma x\;\; T'_vT_v f,\sigma)\\
&\overset{(82)}{=} \int_{\mathbb{C}} T'(\Psi_\sigma x\; T_v f,\sigma) \overset{(82)}{=} \int_{\mathbb{C}} T'\int_{\mathbb{C}} T(f,\sigma).
\end{aligned}$$

Indeed, they are the same monoidal functors since, for any objects u of $\mathbb{C}$ and $(x,u),(y,u)$ of $\int_{\mathbb{C}}\mathcal{P}$,

$$\begin{aligned}
\varphi^{\int_{\mathbb{C}}T'\int_{\mathbb{C}}T}_{(x,u),(y,u)} &\overset{(59)}{=} \textstyle\int_{\mathbb{C}}T'\varphi^{\int_{\mathbb{C}}T}_{(x,u),(y,u)}\ \varphi^{\int_{\mathbb{C}}T'}_{\int_{\mathbb{C}}T(x,u),\int_{\mathbb{C}}T(y,u)} = \textstyle\int_{\mathbb{C}}T'(\varphi^{T_u}_{x,y},1_u)\ (\varphi^{T'_u}_{T_ux,T_uy},1_u)\\
&\overset{(82),(65)}{=} (T'_u\varphi^{T_u}_{x,y},1_u)\ (\varphi^{T'_u}_{T_ux,T_uy},1_u) = (T'_u\varphi^{T_u}_{x,y}\ \varphi^{T'_u}_{T_ux,T_uy},1_u)\\
&\overset{(59)}{=} (\varphi^{T'_uT_u}_{x,y},1_u) = \varphi^{\int_{\mathbb{C}}T'T}_{(x,u),(y,u)},\\
\varphi^{\int_{\mathbb{C}}T'\int_{\mathbb{C}}T}_{u} &\overset{(59)}{=} \textstyle\int_{\mathbb{C}}T'\varphi^{\int_{\mathbb{C}}T}_{u}\ \varphi^{\int_{\mathbb{C}}T'}_{u} = \textstyle\int_{\mathbb{C}}T'(\varphi^{T_u}_{1},1_u)\ (\varphi^{T'_u}_{1},1_u)\\
&\overset{(82),(65)}{=} (T'_u\varphi^{T_u}_{1},1_u)\,(\varphi^{T'_u}_{1},1_u) = (T'_u\varphi^{T_u}_{1}\ \varphi^{T'_u}_{1},1_u) \overset{(59)}{=} (\varphi^{T'_uT_u}_{1},1_u) = \varphi^{\int_{\mathbb{C}}T'T}_{u}.
\end{aligned}$$

Therefore, the equality between braided $\mathbb{C}$–fibred functors $\int_{\mathbb{C}}(T'T) = \int_{\mathbb{C}}T'\int_{\mathbb{C}}T$ holds.

Similarly, for $M\colon T\Rrightarrow S\colon \mathcal{P}\Rightarrow\mathcal{P}'$ and $M'\colon T'\Rrightarrow S'\colon \mathcal{P}'\Rightarrow\mathcal{P}''$ braided modifications, we have the equality $\int_{\mathbb{C}}(M'M) = \int_{\mathbb{C}}M'\int_{\mathbb{C}}M$, since, for any object (x,u) of $\int_{\mathbb{C}}\mathcal{P}$,

$$\begin{aligned}
\textstyle\int_{\mathbb{C}}M'\int_{\mathbb{C}}M(x,u) &\overset{(61)}{=} \textstyle\int_{\mathbb{C}}S'(M_ux,1_u)\ \int_{\mathbb{C}}M'(T_ux,u)\\
&\overset{(82),(65)}{=} (S'_uM_ux,1_u)\,(M'_uT_ux,1_u)\\
&\overset{(71)}{=} ((M'M)_ux,1_u) = \textstyle\int_{\mathbb{C}}(M'M)(x,u).
\end{aligned}$$

The above confirms that (80), $\int_{\mathbb{C}}\colon \mathbf{Psd}(\mathbb{C}^{\mathrm{op}},\mathcal{BCG})\to\mathcal{BCG}_{\downarrow\mathbb{C}}$, is a 2–functor, which, from Remark 6.12, restricts to the 2–category of pseudofunctors of symmetric categorical groups. Thus, we have the commutative diagram of 2–functors:

$$\begin{tikzcd}
\mathbf{Psd}(\mathbb{C}^{\mathrm{op}},\mathcal{SCG}) \arrow[r, "\int_{\mathbb{C}}"] \arrow[d, hook] & \mathcal{SCG}_{\downarrow\mathbb{C}} \arrow[d, hook]\\
\mathbf{Psd}(\mathbb{C}^{\mathrm{op}},\mathcal{BCG}) \arrow[r, "\int_{\mathbb{C}}"] & \mathcal{BCG}_{\downarrow\mathbb{C}}
\end{tikzcd} \tag{89}$$

6.7 Braided Grothendieck construction 2–functors are biequivalences

The ordinary Grothendieck construction 2–functor $\int_{\mathbb{C}}\colon \mathbf{Psd}(\mathbb{C}^{\mathrm{op}},\mathbf{Cat})\to\mathbf{Cat}_{\downarrow\mathbb{C}}$ is a biequivalence of 2–categories (see Giraud [33; 34], Hollander [39] or Vistoli [58]). The following theorem is the principal result of this section:

Theorem 6.15 *For any small category $\mathbb{C}$, both Grothendieck construction 2–functors in (89) are strong biequivalences, in the sense that:*

(i) *For any two pseudofunctors of braided categorical groups $\mathcal{P},\mathcal{P}'\colon \mathbb{C}^{\mathrm{op}}\to\mathcal{BCG}$, the functor*

$$\textstyle\int_{\mathbb{C}}\colon \mathrm{Hom}_{\mathbf{Psd}(\mathbb{C}^{\mathrm{op}},\mathcal{BCG})}(\mathcal{P},\mathcal{P}')\to \mathrm{Hom}_{\mathcal{BCG}\downarrow_{\mathbb{C}}}(\int_{\mathbb{C}}\mathcal{P},\int_{\mathbb{C}}\mathcal{P}') \tag{90}$$

is an isomorphism of categories (rather than an equivalence).

(ii) *For any braided (resp. symmetric) $\mathbb{C}$–fibred categorical group $\mathbb{P}$, there exist a pseudofunctor to braided (resp. symmetric) categorical groups $\mathcal{P}$ and a strict braided $\mathbb{C}$–fibred isomorphism (rather than a braided $\mathbb{C}$–fibred equivalence)*

$$\textstyle\int_{\mathbb{C}}\mathcal{P}\xrightarrow{\cong}\mathbb{P}. \tag{91}$$

Proof (i) The functor (90) is plainly recognized to be faithful and injective on objects: Suppose $M,M'\colon T\Rrightarrow T'$ are two braided modifications such that $\int_{\mathbb{C}}M=\int_{\mathbb{C}}M'$, where $T,T'\colon\mathcal{P}\Rightarrow\mathcal{P}'$ are braided pseudotransformations. Then, for any object $(x,u)\in\int_{\mathbb{C}}\mathcal{P}$,

$$(M_u x,1_u)\overset{(88)}{=}\textstyle\int_{\mathbb{C}}M(x,u)=\int_{\mathbb{C}}M'(x,u)\overset{(88)}{=}(M'_u x,1_u),$$

whence $M_u x=M'_u x$, for all objects u of $\mathbb{C}$ and x of $\mathcal{P}_u$. Therefore, $M=M'$, and (90) is faithful. To see that it is also injective on objects, let us suppose that $\int_{\mathbb{C}}T=\int_{\mathbb{C}}T'$. Then, from (82), for the case where $\sigma=1_u$, (85) and (86), we deduce that $T_u=T'_u$ for all objects $u\in\mathbb{C}$. Moreover, again by (82), now for the case where $f=1_{\sigma^*x}$, we conclude that $\Psi^T_\sigma x=\Psi^{T'}_\sigma x$, for any morphism $\sigma\colon v\to u$ in $\mathbb{C}$ and object x of $\mathcal{P}_u$. Therefore, $T=T'$.

To prove that (90) is full, let $\Psi\colon\int_{\mathbb{C}}T\Rrightarrow\int_{\mathbb{C}}T'$ be any braided $\mathbb{C}$–fibred isomorphism. For each object (x,u) of $\int_{\mathbb{C}}\mathcal{P}$, let us write

$$\Psi_{(x,u)}=(M_u x,1_u)\colon(T_u,u)\to(T'_u x,u)$$

for a morphism $M_u x\colon T_u x\to T'_u x$ in $\mathcal{P}_u$. Then, for $(f,\sigma)\colon(y,v)\to(x,u)$ any morphism in $\int_{\mathbb{C}}\mathcal{P}$, the naturality equation $\Psi_{(x,u)}\int_{\mathbb{C}}T(f,\sigma)=\int_{\mathbb{C}}T'(f,\sigma)\,\Psi_{(y,v)}$, implies that the following diagram is commutative:

$$\begin{tikzcd}
T_v y \arrow[r, "T_v f"] \arrow[d, "M_v y"'] & T_v\sigma^* x \arrow[r, "\Psi_\sigma x"] & \sigma^* T_u x \arrow[d, "\sigma^* M_u x"] \\
T'_v y \arrow[r, "T'_v f"] & T'_v\sigma^* x \arrow[r, "\Psi'_\sigma x"] & \sigma^* T'_u x
\end{tikzcd}$$

Then, from the case where $\sigma=1_u$, we deduce that $M_u x\,T_u f=T'_u f\,M_u y$, for any morphism $f\colon y\to x$ in $\mathcal{P}_u$; that is, every $M_u\colon T_u\Rightarrow T'_u$ is natural. And, from the case where $f = 1_{\sigma^* x}$, we have the equality $\Psi'_\sigma x\, M_v \sigma^* x = \sigma^* M_u x\, \Psi_\sigma x$ for any arrow $\sigma\colon v \to u$ in $\mathbb{C}$ and any object x of $\mathcal{P}_u$; that is, the required equation (67) holds.

Furthermore, since it is easy to see that the two coherence conditions in (58) for the different M_u, $u \in \mathrm{Ob}\mathbb{C}$, that is,

$$\varphi^{T'_u}_{x,y}\,(M_u x \otimes M_u y) = M_u(x \otimes y)\,\varphi^{T_u}_{x,y}, \quad M_u\,\varphi^{T_u}_{1} = \varphi^{T'_u}_{1},$$

follow from the corresponding ones for Ψ, that is,

$$\varphi^{\int_{\mathbb{C}} T'}_{(x,u),(y,u)}\,(\Psi_{(x,u)} \otimes \Psi_{(y,u)}) = \Psi_{(x,u)\otimes(y,u)}\,\varphi^{\int_{\mathbb{C}} T}_{(x,u),(y,u)}, \quad \Psi_{Iu}\,\varphi^{\int_{\mathbb{C}} T}_{u} = \varphi^{\int_{\mathbb{C}} T'}_{u},$$

we conclude that $M\colon T \Rrightarrow T'$ is actually a braided modification, and clearly $\int_{\mathbb{C}} M = \Psi$.

We next observe that (90) is surjective on objects: Let $F\colon \int_{\mathbb{C}}\mathcal{P} \to \int_{\mathbb{C}}\mathcal{P}'$ be any given braided $\mathbb{C}$–fibred functor. Then, we obtain a braided functor $T_u = (T_u, \varphi^{T_u})\colon \mathcal{P}_u \to \mathcal{P}'_u$, for any object u of $\mathbb{C}$, if we write

- $F(x,u) = (T_u x, u)$ for each object x of $\mathcal{P}_u$,
- $F(f, 1_u) = (T_u f, 1_u)\colon (T_u x, u) \to (T_u y, u)$ for each morphism $f\colon x \to y$ of $\mathcal{P}_u$,
- $\varphi^F_{(x,u),(y,u)} = (\varphi^{T_u}_{x,y}, 1_u)\colon (T_u x \otimes T_u y, u) \to (T_u(x \otimes y), u)$ for any objects x, y of $\mathcal{P}_u$,
- $\varphi^F_u = (\varphi^{T_u}_1, 1_u)\colon (I_u, u) \to (T_u I_u, u)$.

Furthermore, for each arrow $\sigma\colon v \to u$ in $\mathbb{C}$ and each object x of $\mathcal{P}_u$, let us write

- $F(1_{\sigma^* x}, \sigma) = (\Psi_\sigma x, \sigma)\colon (T_v \sigma^* x, v) \to (T_u x, u)$,

where $\Psi_\sigma x\colon T_v \sigma^* x \to \sigma^* T_u x$ is a morphism in $\mathcal{P}_v$. Note that $\Psi_{1_u} x = 1_{T_u x}$, since F is a functor and preserves identities. Then, for $(f,\sigma)\colon (y,v) \to (x,u)$ any morphism in $\int_{\mathbb{C}}\mathcal{P}$, we have

$$\begin{aligned} F\big((y,v) \xrightarrow{(f,\sigma)} (x,u)\big) &= F\big((y,v) \xrightarrow{(f,1_v)} (\sigma^* x, v) \xrightarrow{(1_{\sigma^* x},\sigma)} (x,u)\big) \\ &= \big((T_v y, v) \xrightarrow{(T_v f, 1_v)} (T_v \sigma^* x, v) \xrightarrow{(\Psi_\sigma x, \sigma)} (T_u x, u)\big) \\ &= ((T_v y, v) \xrightarrow{(\Psi_\sigma x\ T_v f, \sigma)} (T_u x, u)). \end{aligned}$$

If $\tau\colon w \to v$ is any other arrow in $\mathbb{C}$, then the equality

$$F(f,\sigma)F(1_{\tau^* y}, \tau) = F((f,\sigma)(1_{\tau^* y}, \tau))$$

yields the commutativity of the following diagram in $\mathcal{P}_w$,

$$\begin{tikzcd}
T_w\tau^*y \arrow[r, "T_w\tau^*f"] \arrow[d, "\Psi_\tau y"'] & T_w\tau^*\sigma^*x \arrow[rr, "T_w\langle\sigma,\tau\rangle_x"] & & T_w(\sigma\tau)^*x \arrow[d, "\Psi_{\sigma\tau}x"] \\
\tau^*T_vy \arrow[r, "\tau^*T_vf"] & \tau^*T_v\sigma^*x \arrow[r, "\tau^*\Psi_\sigma x"] & \tau^*\sigma^*T_ux \arrow[r, "\langle\sigma,\tau\rangle_{T_ux}"'] & (\sigma\tau)^*T_ux,
\end{tikzcd}$$

which, for the respective cases where $\sigma = 1_v$ or $f = 1_{\sigma^*x}$, tells us that the two diagrams below are commutative.

$$\begin{tikzcd}
T_w\tau^*y \arrow[r, "T_w\tau^*f"] \arrow[d, "\Psi_\tau y"'] & T_w\tau^*x \arrow[d, "\Psi_\tau x"] \\
\tau^*T_vy \arrow[r, "\tau^*T_vf"] & \tau^*T_vx,
\end{tikzcd}
\qquad
\begin{tikzcd}
T_w\tau^*\sigma^*x \arrow[rr, "T_w\langle\sigma,\tau\rangle_x"] \arrow[d, "\Psi_\tau\sigma^*x"'] & & T_w(\sigma\tau)^*x \arrow[d, "\Psi_{\sigma\tau}x"] \\
\tau^*T_v\sigma^*x \arrow[r, "\tau^*\Psi_\sigma x"] & \tau^*\sigma^*T_u \arrow[r, "\langle\sigma,\tau\rangle_{T_ux}"] & (\sigma\tau)^*T_ux
\end{tikzcd}$$

This means that, on one hand, every $\Psi_\tau\colon T_w\tau^* \Rightarrow \tau^*T_v$ is natural, and, on the other hand, that the equation (66) holds.

Moreover, every $\Psi_\sigma\colon T_v\sigma^* \Rightarrow \sigma^*T_u$ is monoidal since, for any objects $x, y \in \mathcal{P}_u$, we have the commutative diagrams

$$\begin{tikzcd}
T_v\sigma^*x \otimes T_v\sigma^*y \arrow[r, "\varphi^{T_v}_{\sigma^*x,\sigma^*y}"] \arrow[rr, bend left, "\varphi^{T_v\sigma^*}_{x,y}"] \arrow[d, "\Psi_\sigma x\otimes\Psi_\sigma y"'] \arrow[rr, bend right=0, phantom, "(\mathrm{A})"', shift right=6ex] & T_v(\sigma^*x\otimes\sigma^*y) \arrow[r, "T_v\varphi^{\sigma^*}_{x,y}"] \arrow[u, phantom, "(59)"] & T_v\sigma^*(x\otimes y) \arrow[d, "\Psi_\sigma(x\otimes y)"] \\
\sigma^*T_ux\otimes\sigma^*T_uy \arrow[r, "\varphi^{\sigma^*}_{T_ux,T_uy}"] \arrow[rr, bend right, "\varphi^{\sigma^*T_u}_{x,y}"'] & \sigma^*(T_ux\otimes T_uy) \arrow[r, "\sigma^*\varphi^{T_u}_{x,y}"] \arrow[d, phantom, "(59)"] & \sigma^*T_u(x\otimes y),
\end{tikzcd}$$

$$\begin{tikzcd}
& I_v \arrow[dl, "\varphi^{T_v}_1"] \arrow[dr, "\varphi^{\sigma^*}_1"'] \arrow[ddl, bend right, "\varphi^{T_v\sigma^*}_1"'] \arrow[ddr, bend left, "\varphi^{\sigma^*T_u}_1"] & \\
T_vI_v \arrow[d, "T_v\varphi^{\sigma^*}_1"] \arrow[r, phantom, "(59)"] & (\mathrm{B}) & \sigma^*I_u \arrow[d, "\sigma^*\varphi^{T_u}_1"'] \arrow[l, phantom, "(59)"] \\
T_v\sigma^*I_u \arrow[rr, "\Psi_\sigma"] & & \sigma^*T_uI_u,
\end{tikzcd}$$

where the commutativity of inner diagrams (A) and (B) hold thanks, respectively, to the commutativity of the two following naturality diagrams:

$$\begin{tikzcd} F(\sigma^*x,v)\otimes F(\sigma^*y,v) \arrow[r, "\varphi^F"] \arrow[d, "{F(1_{\sigma^*x},\sigma)}"', "{\otimes F(1_{\sigma^*y},\sigma)}"'] & F((\sigma^*x,v)\otimes(\sigma^*y,v)) \arrow[d, "{F((1_{\sigma^*x},\sigma)}", "{\otimes(1_{\sigma^*y},\sigma))}"] \\ F(x,u)\otimes F(y,u) \arrow[r, "\varphi^F"] & F((x,u)\otimes(y,u)) \end{tikzcd} \qquad \begin{tikzcd} Iv \arrow[r, "\varphi^F"] \arrow[d, "I\sigma"'] & FIv \arrow[d, "FI\sigma"] \\ Iu \arrow[r, "\varphi^F"] & FIu \end{tikzcd}$$

Hence, $T=(T,\Psi^T)\colon \mathcal{P}\Rightarrow\mathcal{P}'$ is a braided pseudotransformation, and, by construction (recall (82), (85) and (86)), it is clear that $\int_{\mathbb{C}} T = F$. This makes the proof of (i) complete.

(ii) Let $\mathbb{P}=(\mathbb{P},P,\otimes,I,\boldsymbol{a},\boldsymbol{r},\boldsymbol{l},\boldsymbol{c})$ be any given braided $\mathbb{C}$–fibred categorical group. Then, recalling that every fibre inherits a braided categorical group structure $\mathbb{P}_u=(\mathbb{P}_u,\otimes,Iu,\boldsymbol{a},\boldsymbol{r},\boldsymbol{l},\boldsymbol{c})$, a pseudofunctor to braided categorical groups

(92) $$\mathcal{P}\colon \mathbb{C}^{\mathrm{op}}\to\mathcal{BCG} \quad \text{such that } \mathcal{P}_u=\mathbb{P}_u \text{ for each object } u\in\mathbb{C},$$

is defined as follows:

Since $P\colon\mathbb{P}\to\mathbb{C}$ is a fibration, we can choose a *normalized cleavage* for it in the sense of Grothendieck [37, Definition 7.1]; that is, for each arrow $\sigma\colon v\to u$ in $\mathbb{C}$ and each object x of $\mathbb{P}_u$, we select a pullback $x_\sigma\colon\sigma^*x\to x$ of x by σ. Specifically, we choose $x_{1_u}=1_x\colon x\to x$; so that $1_u^*x=x$.

Then, every $\sigma\colon v\to u$ in $\mathbb{C}$ defines a braided functor $\sigma^*\colon\mathcal{P}_u\to\mathcal{P}_v$ by sending each object x of $\mathbb{P}_u$ to σ^*x, and each arrow $f\colon x\to y$ of $\mathbb{P}_u$ to the unique arrow $\sigma^*f\colon\sigma^*x\to\sigma^*y$ in $\mathbb{P}_v$ making the diagram

(93) $$\begin{tikzcd} \sigma^*x \arrow[r, "x_\sigma"] \arrow[d, "\sigma^*f"'] & x \arrow[d, "f"] \\ \sigma^*y \arrow[r, "y_\sigma"] & y \end{tikzcd}$$

commute, and for which the structure isomorphisms $\varphi^{\sigma^*}_{x,y}$ and $\varphi^{\sigma^*}_1$ are respectively those in $\mathbb{P}_v$ causing the triangles below to commute.

(94) $$\begin{tikzcd} \sigma^*x\otimes\sigma^*y \arrow[rr, "\varphi^{\sigma^*}_{x,y}"] \arrow[dr, "x_\sigma\otimes y_\sigma"'] & & \sigma^*(x\otimes y) \arrow[dl, "(x\otimes y)_\sigma"] \\ & x\otimes y, & \end{tikzcd} \qquad \begin{tikzcd} Iv \arrow[rr, "\varphi^{\sigma^*}_1"] \arrow[dr, "I\sigma"'] & & \sigma^*Iu \arrow[dl, "(Iu)_\sigma"] \\ & Iu & \end{tikzcd}$$

To prove the naturality condition for the isomorphisms φ^{σ^*}, let us consider the following diagram, for $f\colon x\to x'$ and $g\colon y\to y'$ any two morphisms of $\mathbb{P}_u$

$$\begin{tikzcd}
\sigma^*x\otimes\sigma^*y \arrow[rr, "\varphi^{\sigma^*}_{x,y}"] \arrow[dr, "x_\sigma\otimes y_\sigma"'] \arrow[ddd, "\sigma^*f\otimes\sigma^*g"'] & \arrow[d, phantom, "(94)"] & \sigma^*(x\otimes y) \arrow[dl, "(x\otimes y)_\sigma"] \arrow[ddd, "\sigma^*(f\otimes g)"] \\
\arrow[r, phantom, "(93)"] & x\otimes y \arrow[d, "f\otimes g"] & \arrow[l, phantom, "(93)"] \\
 & x'\otimes y' \arrow[d, phantom, "(94)"] & \\
\sigma^*x'\otimes\sigma^*y' \arrow[rr, "\varphi^{\sigma^*}_{x',y'}"'] \arrow[ur, "x'_\sigma\otimes y'_\sigma"] & {} & \sigma^*(x'\otimes y'), \arrow[ul, "(x'\otimes y')_\sigma"']
\end{tikzcd}$$

where the inner regions commute by the references in them. Then, the required commutativity of the outside region follows since $(x'\otimes y')_\sigma$ is cartesian. The coherence condition (55) for σ^* is a consequence of the naturality of the associativity constraint of $\mathbb{P}$: for any objects $x,y,z\in\mathbb{P}_u$, we have the following diagram

$$\begin{tikzcd}
(\sigma^*x\otimes\sigma^*y)\otimes\sigma^*z \arrow[rr, "\varphi^{\sigma^*}_{x,y}\otimes 1"] \arrow[dr, "(x_\sigma\otimes y_\sigma)\otimes z_\sigma"'] \arrow[dd, "\boldsymbol{a}"'] & \arrow[d, phantom, "(94)"] & \sigma^*(x\otimes y)\otimes\sigma^*z \arrow[dl, "(x\otimes y)_\sigma\otimes z_\sigma"] \arrow[dd, "\varphi^{\sigma^*}_{x\otimes y,z}"] \\
\arrow[d, phantom, "\text{(A)}"] & (x\otimes y)\otimes z \arrow[dd, "\boldsymbol{a}"] \arrow[r, phantom, "(94)"] & {} \\
\sigma^*x\otimes(\sigma^*y\otimes\sigma^*z) \arrow[dr, "x_\sigma\otimes(y_\sigma\otimes z_\sigma)"'] \arrow[dd, "1\otimes\varphi^{\sigma^*}_{y,z}"'] & & \sigma^*((x\otimes y)\otimes z) \arrow[ul, "((x\otimes y)\otimes z)_\sigma"'] \arrow[dd, "\sigma^*\boldsymbol{a}"] \arrow[dl, phantom, "(93)"] \\
\arrow[r, phantom, "(94)"] & x\otimes(y\otimes z) \arrow[d, phantom, "(94)"] & \\
\sigma^*x\otimes\sigma^*(y\otimes z) \arrow[rr, "\varphi^{\sigma^*}_{x,y\otimes z}"'] \arrow[ur, "x_\sigma\otimes(y\otimes z)_\sigma"] & {} & \sigma^*(x\otimes(y\otimes z)), \arrow[ul, "(x\otimes(y\otimes z))_\sigma"']
\end{tikzcd}$$

where region (A) commutes by the naturality of $\boldsymbol{a}$, and the other inner regions commute by the respective references in them. Then, the required commutativity of the outside region holds since the morphism $((x\otimes y)\otimes z)_\sigma$ is cartesian. Similarly, the coherence conditions (56) and (57) for σ^* hold: for any objects $x, y \in \mathbb{P}_u$, we have the diagrams

$$\begin{tikzcd}
\sigma^*x \otimes Iv \arrow[rrrr, "1\otimes\varphi_1^{\sigma^*}"] \arrow[ddd, "\boldsymbol{r}"'] \arrow[drr, "x_\sigma\otimes I\sigma"'] \arrow[ddr, phantom, "\text{(A)}"] & & & & \sigma^*x \otimes \sigma^*Iu \arrow[ddd, "\varphi^{\sigma^*}_{x,Iu}"] \arrow[dll, "x_\sigma\otimes(Iu)_\sigma"'] \arrow[ddl, phantom, "(94)"] \\
 & & x \otimes Iu \arrow[dl, "\boldsymbol{r}"'] \arrow[u, phantom, "(94)"] & & \\
 & x \arrow[rr, phantom, "(93)"] & & & \\
\sigma^*x \arrow[ur, "x_\sigma"] & & & & \sigma^*(x \otimes Iu), \arrow[uull, "(x\otimes Iu)_\sigma"'] \arrow[llll, "\sigma^*\boldsymbol{r}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\sigma^*x \otimes \sigma^*y \arrow[rr, "\varphi^{\sigma^*}_{x,y}"] \arrow[ddd, "\boldsymbol{c}"'] \arrow[dr, "x_\sigma\otimes y_\sigma"'] \arrow[rr, phantom, "(94)"', shift right=4] & & \sigma^*(x \otimes y) \arrow[ddd, "\sigma^*\boldsymbol{c}"] \arrow[dl, "(x\otimes y)_\sigma"] \\
 & x \otimes y \arrow[d, "\boldsymbol{c}"] \arrow[dl, phantom, "\text{(B)}"] \arrow[dr, phantom, "(93)"] & \\
 & y \otimes x & \\
\sigma^*y \otimes \sigma^*x \arrow[ur, "y_\sigma\otimes x_\sigma"] \arrow[rr, "\varphi^{\sigma^*}_{y,x}"'] \arrow[rr, phantom, "(94)", shift left=4] & & \sigma^*(y \otimes x), \arrow[ul, "(y\otimes x)_\sigma"']
\end{tikzcd}$$

where (A) and (B) commute by the naturality of $\boldsymbol{r}$ and $\boldsymbol{c}$, respectively, and the other inner regions by the references given in them. Then, the required commutativity of the respective outside regions holds since both morphisms x_σ and $(y \otimes x)_\sigma$ are cartesian.

Note that, since the selected cleavage is normalized, it is easily verified that, for any object u of $\mathbb{C}$, 1_u^* is the identity braided functor on $\mathbb{P}_u$. That is, condition (62) holds.

For any pair of composable arrows $w \xrightarrow{\tau} v \xrightarrow{\sigma} u$ of $\mathbb{C}$, the component of the braided isomorphism $\langle\sigma,\tau\rangle\colon \tau^*\sigma^* \Rightarrow (\sigma\tau)^*$, at any object $x \in \mathbb{P}_u$, is the unique morphism in $\mathbb{P}_w$ making the diagram

(95)
$$\begin{tikzcd}
\tau^*\sigma^*x \arrow[r, "(\sigma^*x)_\tau"] \arrow[d, "\langle\sigma,\tau\rangle_x"'] & \sigma^*x \arrow[d, "x_\sigma"] \\
(\sigma\tau)^*x \arrow[r, "x_{\sigma\tau}"] & x
\end{tikzcd}$$

commute. By Grothendieck [37, Proposition 7.4], we know that $\langle\sigma,\tau\rangle$ is a natural isomorphism, as well as that the equalities (63) and (64) hold. Furthermore, to see that it is actually a braided isomorphism, let us consider the following diagrams, where

x, y are any two objects in $\mathbb{P}_u$:

$$\begin{tikzcd}
\tau^*\sigma^*x\otimes\tau^*\sigma^*y \arrow[rrrr, "\varphi^{\tau^*\sigma^*}_{x,y}"] \arrow[drr, "\varphi^{\tau^*}_{\sigma^*x,\sigma^*y}"'] \arrow[ddrr, "(\sigma^*x)_\tau\otimes(\sigma^*y)_\tau"'] \arrow[dddd, "\langle\sigma,\tau\rangle_x\otimes\langle\sigma,\tau\rangle_y"'] & & & & \tau^*\sigma^*(x\otimes y) \arrow[ddl, "(\sigma^*(x\otimes y))_\tau"'] \arrow[dddd, "\langle\sigma,\tau\rangle_{x\otimes y}"] \\
& & \tau^*(\sigma^*x\otimes\sigma^*y) \arrow[urr, "\tau^*\varphi^{\sigma^*}_{x,y}"'] \arrow[d, "(\sigma^*x\otimes\sigma^*y)_\tau"] \arrow[u, phantom, "(59)"] & & \\
& \arrow[u, phantom, "(94)"] \arrow[dd, phantom, "(95)"] & \sigma^*x\otimes\sigma^*y \arrow[r, "\varphi^{\sigma^*}_{x,y}"] \arrow[d, "x_\sigma\otimes y_\sigma"] & \sigma^*(x\otimes y) \arrow[dl, "(x\otimes y)_\sigma"] \arrow[u, phantom, "(93)"] \arrow[dl, phantom, "(94)"] & \arrow[d, phantom, "(95)"] \\
& & x\otimes y & & \\
(\sigma\tau)^*x\otimes(\sigma\tau)^*y \arrow[urr, "x_{\sigma\tau}\otimes y_{\sigma\tau}"] \arrow[rrrr, "\varphi^{(\sigma\tau)^*}_{x,y}"'] & & \arrow[u, phantom, "(94)"] & & (\sigma\tau)^*(x\otimes y) \arrow[ull, "(x\otimes y)_{\sigma\tau}"']
\end{tikzcd}$$

$$\begin{tikzcd}
& Iw \arrow[dl, "\varphi^{\tau^*}_1"'] \arrow[d, "I\tau"] \arrow[dr, "I(\sigma\tau)"] \arrow[ddddl, bend right=60, "\varphi^{\tau^*\sigma^*}_1"'] \arrow[ddddr, bend left=60, "\varphi^{(\sigma\tau)^*}_1"] & \\
\tau^*Iv \arrow[r, "(Iv)_\tau"] \arrow[ddd, "\tau^*\varphi^{\sigma^*}_1"'] \arrow[u, phantom, "(59)"] \arrow[ur, phantom, "(94)"] & Iv \arrow[r, "I\sigma"] \arrow[dd, "\varphi^{\sigma^*}_1"'] \arrow[ddl, phantom, "(93)"] \arrow[dr, phantom, "(94)"] & Iu \arrow[u, phantom, "(94)"] \\
& & \\
& \sigma^*Iu \arrow[uur, "(Iu)_\sigma"'] \arrow[dr, phantom, "(95)"] & \\
\tau^*\sigma^*Iu \arrow[ur, "(\sigma^*Iu)_\tau"'] \arrow[rr, "\langle\sigma,\tau\rangle_{Iu}"'] & & (\sigma\tau)^*Iu \arrow[uuu, "(Iu)_{\sigma\tau}"']
\end{tikzcd}$$

where each inner region is commutative thanks to the reference therein. Then, the required commutativity of the outside regions in the diagrams follows from the fact that both $(x\otimes y)_{\sigma\tau}$ and $(Iu)_{\sigma\tau}$ are cartesian.

Therefore, by (92)–(95) we have a well-defined pseudofunctor to braided categorical groups $\mathcal{P}\colon \mathbb{C}^{\mathrm{op}}\to\mathcal{BCG}$. We now recognize that both braided $\mathbb{C}$–fibred categorical groups $\int_{\mathbb{C}}\mathcal{P}$ and $\mathbb{P}$ are isomorphic by means of the strict braided $\mathbb{C}$–fibred functor $\int_{\mathbb{C}}\mathcal{P}\to\mathbb{P}$ that carries any object (x,u) of $\int_{\mathbb{C}}\mathcal{P}$ to x, and any morphism $(f,\sigma)\colon (y,v)\to(x,u)$ to $x_\sigma f\colon y\to x$, the composite of $f\colon y\to\sigma^*x$ with the cartesian morphism $x_\sigma\colon \sigma^*x\to x$.

Finally, note that, whenever the originally given $\mathbb{P}$ is a symmetric $\mathbb{C}$–fibred categorical group, then the above-constructed pseudofunctor $\mathcal{P}$ is actually of symmetric categorical groups. This completes the proof of part (ii) in the theorem. $\square$

Theorem 6.15 has two key consequences, given below.

Theorem 6.16 *For any small category $\mathbb{C}$, the enriched Grothendieck construction 2–functor $\int_{\mathbb{C}}(-)$ induces equivalences of homotopy categories*

$$\mathrm{Ho}\,\mathbf{Psd}(\mathbb{C}^{\mathrm{op}},\mathcal{BCG}) \xrightarrow{\sim} \mathrm{Ho}\,\mathcal{BCG}_{\downarrow\mathbb{C}}, \qquad \mathrm{Ho}\,\mathbf{Psd}(\mathbb{C}^{\mathrm{op}},\mathcal{SCG}) \xrightarrow{\sim} \mathrm{Ho}\,\mathcal{SCG}_{\downarrow\mathbb{C}}.$$

7 $H^3_{\mathbb{C},\mathrm{r}}$ and braided and symmetric $\mathbb{C}$–fibred categorical groups

Our classification results for braided and symmetric $\mathbb{C}$–fibred categorical groups will be given below by exhibiting biequivalences between their respective 2–categories and corresponding 2–categories of first- and second-level 3–cocycles of $\mathbb{C}$–modules, which are defined as follows:

7.1 The 2–categories of 3–cocycles

For any given small category $\mathbb{C}$, the 2*–category of first-level* 3*–cocycles of* $\mathbb{C}$*–modules*, denoted by

$$\mathcal{Z}^3_{\mathbb{C},1}, \quad (\mathcal{Z}^3_1, \text{ if } \mathbb{C}=\mathbf{1} \text{ is the one arrow trivial category}),$$

has objects triples (B,A,h), where $B,A\colon \mathbb{C}^{\mathrm{op}}\to\mathbf{Ab}$ are $\mathbb{C}$–modules and

$$h\in Z^3_{\mathbb{C},1}(B,A)=\mathrm{Ker}(C^3_{\mathbb{C},1}(B,A)\xrightarrow{\partial^3}C^4_{\mathbb{C},1}(B,A)),$$

is a first-level 3–cocycle of B in A. A 1–cell in $\mathcal{Z}^3_{\mathbb{C},1}$ from (B,A,h) to (B',A',h') is a triple $(p,q,g)\colon (B,A,h)\to(B',A',h')$, consisting of morphisms of $\mathbb{C}$–modules $p\colon B\to B'$ and $q\colon A\to A'$ and a 2–cochain $g\in C^2_{\mathbb{C},1}(B,A')$, such that $q_*(h)=p^*(h')+\partial^2 g$, where

$$C^{\bullet}_{\mathbb{C},1}(B',A')\xrightarrow{p^*}C^{\bullet}_{\mathbb{C},1}(B,A')\xleftarrow{q_*}C^{\bullet}_{\mathbb{C},1}(B,A)$$

are the complex homomorphisms canonically induced by p and q, respectively. The composite of (p,q,g) with the morphism $(p',q',g')\colon (B',A',h')\to(B'',A'',h'')$ is defined by

$$(p',q',g')(p,q,g)=(p'p,q'q,\,p^*(g')+q'_*(g)),$$

and identities are given by $1_{(B,A,h)}=(1_B,1_A,0)$.

For any two morphisms $(p,q,g),(p',q',g')\colon (B,A,h)\to(B',A',h')$, the existence of 2–cells in $\mathcal{Z}^3_{\mathbb{C},1}$ between them requires that $p=p'$ and $q=q'$, and, in such a case, such a 2–cell $f\colon (p,q,g) \Rightarrow (p,q,g')$ consists of a 1–cochain $f \in C^1_{\mathbb{C},1}(B,A')$ such that $g = g' + \partial^1 f$. The vertical composition of f with a 2–cell $f'\colon (p,q,g') \Rightarrow (p,q,g'')$ is given by pointwise addition in A', that is $f' + f\colon (p,q,g) \Rightarrow (p,q,g'')$. The identity 2–cell over a 1–cell $(p,q,g)\colon (B,A,h) \to (B',A',h')$ is given by the zero $\mathbb{C}$–module morphism $0\colon B \to A'$, and every 2–cell $f\colon (p,q,g) \Rightarrow (p,q,g')$ as above is invertible, with the inverse given by the opposite $\mathbb{C}$–module homomorphism $-f\colon B \to A'$. Hence, the hom-categories of $\mathcal{Z}^3_{\mathbb{C},1}$ are all groupoids.

The horizontal composition of 2–cells $f\colon(p,q,g_1) \Rightarrow (p,q,g_2)\colon (B,A,h) \to (B',A',h')$ and $f'\colon (p,q,g'_1) \Rightarrow (p,q,g'_2)\colon (B',A',h') \to (B'',A'',h'')$ is given by the formula

$$f'f = p^*(f') + q'_*(f)\colon (p'p, q'q, p^*(g'_1) + q'_*(g_1)) \Rightarrow (p'p, q'q, p^*(g'_2) + q'_*(g_2)).$$

The corresponding quotient category (44) of isomorphism classes of 1–cells, that is, the *homotopy category of first-level* 3*–cocycles of* $\mathbb{C}$*–modules*, is $\operatorname{Ho}\mathcal{Z}^3_{\mathbb{C},1}$.

The 2*–category of second-level* 3*–cocycles of* $\mathbb{C}$*–modules*, $\mathcal{Z}^3_{\mathbb{C},2} \subseteq \mathcal{Z}^3_{\mathbb{C},1}$, is the full sub–2–category given by those objects (B,A,h) with

$$h \in Z^3_{\mathbb{C},2}(B,A) = \operatorname{Ker}(C^3_{\mathbb{C},2}(B,A) \xrightarrow{\partial^3} C^4_{\mathbb{C},2}(B,A)),$$

a second-level 3–cocycle. The full subcategory $\operatorname{Ho}\mathcal{Z}^3_{\mathbb{C},2} \subseteq \operatorname{Ho}\mathcal{Z}^3_{\mathbb{C},1}$ is then the *homotopy category of second-level* 3*–cocycles of* $\mathbb{C}$*–modules*.

A simple straightforward comparison gives the result below.

Lemma 7.1 *For any small category* $\mathbb{C}$*, there are isomorphisms of* 2*–categories (those labelled with the symbol* $\cong$*) making the diagram below commutative:*

(96)
$$\begin{tikzcd} \mathcal{Z}^3_{\mathbb{C},2} \arrow[r, "\cong"] \arrow[d, hook] & \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{Z}^3_2) \arrow[d, hook] \\ \mathcal{Z}^3_{\mathbb{C},1} \arrow[r, "\cong"] & \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{Z}^3_1) \end{tikzcd}$$

7.2 From 3–cocycles to pseudofunctors

We define here a 2–functor, denoted by

$$\mathcal{P}_{\mathbb{C}}\colon \mathcal{Z}^3_{\mathbb{C},1} \to \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG}),$$

through which we shall state as first-level 3–cocycles are appropriate data for the construction of all braided $\mathbb{C}$–fibred categorical groups, up to braided fibred equivalence.

Every pair of $\mathbb{C}$–modules $A, B\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$, together with a 3–cocycle $h \in Z^3_{\mathbb{C},1}(B, A)$, give rise to a pseudofunctor to braided categorical groups

$$\mathcal{P}_{\mathbb{C}}(B, A, h)\colon \mathbb{C}^{\mathrm{op}} \to \mathcal{BCG}, \tag{97}$$

which is defined as follows:

- At any object u of $\mathbb{C}$, the braided categorical group $\mathcal{P}_{\mathbb{C}}(B, A, h)_u$ has objects the elements of B_u. The hom-sets are given by

$$\mathrm{Hom}_{\mathcal{P}_{\mathbb{C}}(B,A,h)_u}(x, y) = \begin{cases} A_u & \text{if } x = y, \\ \varnothing & \text{otherwise.} \end{cases}$$

Composition is addition in A_u. The tensor product is defined by

$$(x \xrightarrow{a} x) \otimes (x' \xrightarrow{a'} x') = (x + x' \xrightarrow{a+a'} x + x').$$

The associativity isomorphism is

$$h_u(x, y, z)\colon (x + y) + z \to x + (y + z), \tag{98}$$

the braiding isomorphism is

$$h_u(x \mid y)\colon x + y \to y + x \tag{99}$$

and the 0 of B_u is the (strict) unit object.

The equation in (46) follows from the symmetric cocycle condition $(\partial^3 h)_u(x, y, z, t) = 0$ in (27). The coherence conditions in (48) and (49) hold because of the cocycle conditions $(\partial^3 h)_u(x \mid y, z) = 0$ and $(\partial^3 h)_u(x, y \mid z) = 0$, in (28) and (29), respectively. Also, the equation in (47) holds thanks to the normalization condition of h_u. Since both B_u and A_u are groups, objects and morphisms in $\mathcal{P}_{\mathbb{C}}(B, A, h)_u$ are invertible, whence $\mathcal{P}_{\mathbb{C}}(B, A, h)_u$ is actually a braided categorical group for any object u of $\mathbb{C}$.

- For any arrow $\sigma\colon v \to u$ in $\mathbb{C}$, the braided functor $\sigma^*\colon \mathcal{P}_{\mathbb{C}}(B,A,h)_u \to \mathcal{P}_{\mathbb{C}}(B,A,h)_v$ is given by

$$(x \xrightarrow{a} x) \overset{\sigma^*}{\mapsto} (\sigma^* x \xrightarrow{\sigma^* a} \sigma^* x), \tag{100}$$

with the structure constraints

$$\begin{aligned} h_\sigma(x, y)\colon \sigma^* x + \sigma^* y &\to \sigma^*(x + y), \\ 0\colon 0 &\to \sigma^* 0. \end{aligned} \tag{101}$$

The symmetric 3–cocycle condition $(\partial^3 h)_\sigma(x, y, z) = 0$ in (31) implies the coherence condition (55), whereas those in (56) hold thanks to the normalization of h, and (57) holds owing to the cocycle condition $(\partial^3 h)_\sigma(x \mid y) = 0$ in (32).

- The braided isomorphism $\tau^*\sigma^*\Rightarrow(\sigma\tau)^*$ associated to each pair of arrows $w\overset{\tau}{\to}v\overset{\sigma}{\to}u$ of $\mathbb{C}$ is given by

(102) $$h_{\sigma,\tau}(x)\colon \tau^*\sigma^*x\to(\sigma\tau)^*x,$$

at any $x\in B_u$. This actually defines a braided isomorphism since the corresponding conditions in (58) hold by the 3–cocycle equality $(\partial^3 h)_{\sigma,\tau}(x,y)=0$ in (33), and by the normalization condition of h.

The conditions in (62) and (63) are satisfied because the first-level 3–cocycle h is normalized, and the coherence condition in (64) follows from the cocycle condition $(\partial^3 h)_{\sigma,\tau,\gamma}(x)=0$ in (34). Consequently, $\mathcal{P}_{\mathbb{C}}(B,A,h)$ is actually a pseudofunctor of braided categorical groups.

The assignment $(B,A,h)\mapsto\mathcal{P}_{\mathbb{C}}(B,A,h)$ is the function on objects of the 2–functor

(103) $$\mathcal{P}_{\mathbb{C}}\colon \mathcal{Z}^3_{\mathbb{C},1}\to\mathbf{Psd}(\mathbb{C}^{\mathrm{op}},\mathcal{BCG})$$

described below.

On the hom categories, the 2–functor $\mathcal{P}_{\mathbb{C}}$ carries any 1–cell $(p,q,g)\colon(B,A,h)\to(B',A',h')$ to the braided pseudotransformation

(104) $$\mathcal{P}_{\mathbb{C}}(p,q,g)\colon \mathcal{P}_{\mathbb{C}}(B,A,h)\Rightarrow\mathcal{P}_{\mathbb{C}}(B',A',h')$$

defined as follows:

- At any object $u\in\mathbb{C}$, the braided functor

$$\mathcal{P}_{\mathbb{C}}(p,q,g)_u\colon \mathcal{P}_{\mathbb{C}}(B,A,h)_u\to\mathcal{P}_{\mathbb{C}}(B',A',h')_u$$

is given by

(105) $$(x\overset{a}{\to}x)\ \mapsto\ (p_ux\overset{q_ua}{\longrightarrow}p_ux),$$

with the structure constraints

(106) $$g_u(x,y)\colon p_ux+p_uy\to p_u(x+y),0\colon 0\to p_u0.$$

The coherence equations (55) and (57), that is,

$$g_u(x,y)+g_u(x+y,z)+q_uh_u(x,y,z)=h'_u(p_ux,p_uy,p_uz)+g_u(y,z)+g_u(x,y+z),$$
$$g_u(x,y)+q_uh_u(x\mid y)=h'_u(p_ux\mid p_uy)+g_u(y,x),$$

follow from the equality $q_*(h)=p^*(h')+\partial^2 g$ and (19) and (20), respectively. The equation in (56) holds thanks to the normalization of g.

- For each arrow $\sigma\colon v \to u$ in $\mathbb{C}$, the braided isomorphism

$$\begin{tikzcd}
\mathcal{P}_{\mathbb{C}}(B,A,h)_u \arrow[r, "\mathcal{P}_{\mathbb{C}}(p,q,g)_u"] \arrow[d, "\sigma^*"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{P}_{\mathbb{C}}(B',A',h')_u \arrow[d, "\sigma^*"] \\
\mathcal{P}_{\mathbb{C}}(B,A,h)_v \arrow[r, "\mathcal{P}_{\mathbb{C}}(p,q,g)_v"'] & \mathcal{P}_{\mathbb{C}}(B',A',h')_v
\end{tikzcd}$$

is given by

$$g_\sigma(x)\colon p_v\sigma^*x \to \sigma^* p_u x, \tag{107}$$

at any $x \in B_u$. The first coherence condition in (58), that is,

$$g_v(\sigma^*x,\sigma^*y) + q_v h_\sigma(x,y) + g_\sigma(x+y) \\ = g_\sigma(x) + g_\sigma(y) + h'_\sigma(p_u x, p_u y) + \sigma^* g_u(x,y),$$

holds owing to the equality $q_*(h) = p^*(h') + \partial^2 g$ and (21), whereas the second one follows from the normalization condition $g_\sigma(0) = 0$ of g.

The condition in (65), for $\mathcal{P}_{\mathbb{C}}(p,q,g)$, is satisfied because the symmetric 2–cochain g is normalized, and the coherence condition in (66), that is,

$$q_w h_{\sigma,\tau}(x) + g_{\sigma\tau}(x) = g_\tau(\sigma^*x) + \tau^* g_\sigma(x) + h'_{\sigma,\tau}(p_u x),$$

follows from the equality in $q_*(h) = p^*(h') + \partial^2 g$ and (22). Hence, $\mathcal{P}_{\mathbb{C}}(p,q,g)$ is actually a braided pseudotransformation.

If $f\colon (p,q,g) \Rightarrow (p,q,g')\colon (B,A,h) \to (B',A',h')$, is any 2–cell in $\mathcal{Z}^3_{\mathbb{C},1}$, then the associated braided modification

$$\mathcal{P}_{\mathbb{C}}(f)\colon \mathcal{P}_{\mathbb{C}}(p,q,g) \Rrightarrow \mathcal{P}_{\mathbb{C}}(p,q,g') \tag{108}$$

consists of the braided isomorphisms $\mathcal{P}_{\mathbb{C}}(f)_u\colon \mathcal{P}_{\mathbb{C}}(p,q,g)_u \Rightarrow \mathcal{P}_{\mathbb{C}}(p,q,g')_u$ given by

$$f_u(x)\colon p_u x \to p_u x, \tag{109}$$

for any object u of $\mathbb{C}$ and $x \in B_u$. The first coherence equation in (58) for $\mathcal{P}_{\mathbb{C}}(f)_u$, that is,

$$g'_u(x,y) + f_u(x) + f_u(y) = f_u(x+y) + g_u(x,y),$$

holds because of the equality $g = g' + \partial^1 f$ and (15), whereas the second one follows from the normalization condition $f_u(0) = 0$. Thus, every $\mathcal{P}_{\mathbb{C}}(f)_u$ is a braided isomorphism. Furthermore, the required coherence condition (67) for $\mathcal{P}_{\mathbb{C}}(f)$, that is, the equality

$$g'_\sigma(x) + f_v(\sigma^*x) = \sigma^* f_u(x) + g_\sigma(x),$$

follows from the equality $g = g' + \partial^1 f$ and (16).

This makes complete the description of $\mathcal{P}_{\mathbb{C}}$. The verification that it preserves compositions and identities, both vertical and horizontal, is easily done, and we leave it to the reader. Therefore, $\mathcal{P}_{\mathbb{C}}\colon \mathcal{Z}^3_{\mathbb{C},1} \to \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG})$ is a 2–functor.

Let us now observe that, when $(B, A, h) \in \mathcal{Z}^3_{\mathbb{C},2}$, that is, if h is a second-order 3–cocycle, then the cocycle condition $(\partial^3 h)_u(x \,||\, y) = 0$ in (30) implies the relation $\boldsymbol{c}_{y,x}\,\boldsymbol{c}_{x,y} = 1$ for the braiding in every $\mathcal{P}_{\mathbb{C}}(B, A, h)_u$. This means that $\mathcal{P}_{\mathbb{C}}(B, A, h)$ is actually a pseudofunctor of symmetric categorical groups. Hence, the 2–functor $\mathcal{P}_{\mathbb{C}}$ restricts to a 2–functor

$$\mathcal{P}_{\mathbb{C}}\colon \mathcal{Z}^3_{\mathbb{C},2} \to \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{SCG}), \tag{110}$$

and we have the commutative diagram of 2–functors:

$$\begin{tikzcd} \mathcal{Z}^3_{\mathbb{C},2} \arrow[r, "\mathcal{P}_{\mathbb{C}}"] \arrow[d, hook] & \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{SCG}) \arrow[d, hook] \\ \mathcal{Z}^3_{\mathbb{C},1} \arrow[r, "\mathcal{P}_{\mathbb{C}}"] & \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG}) \end{tikzcd} \tag{111}$$

Below is our key result in the matter here, from which the various later results about the homotopy classification for braided and symmetric fibred categorical groups and their homomorphisms are derived.

Theorem 7.2 *For any given small category $\mathbb{C}$, both realization 2–functors $\mathcal{P}_{\mathbb{C}}$ in (111) are biequivalences. Therefore, they induce equivalences of homotopy categories*

$$\mathrm{Ho}\,\mathcal{Z}^3_{\mathbb{C},1} \xrightarrow{\sim} \mathrm{Ho}\,\mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG}), \qquad \mathrm{Ho}\,\mathcal{Z}^3_{\mathbb{C},2} \xrightarrow{\sim} \mathrm{Ho}\,\mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{SCG}).$$

Proof When $\mathbb{C} = \mathbf{1}$, the trivial point category, both 2–functors $\mathcal{P} = \mathcal{P}_{\mathbf{1}}\colon \mathcal{Z}^3_2 \to \mathcal{SCG}$ and $\mathcal{P} = \mathcal{P}_{\mathbf{1}}\colon \mathcal{Z}^3_1 \to \mathcal{BCG}$, are plainly recognized to be biequivalences from the results by Sinh in [54, Section 1], about the classification of symmetric categorical groups (where they are called Picard categories), and Joyal and Street in [41, Section 3], about the classification of braided categorical groups. Then, for the general case where $\mathbb{C}$ is any small category, the theorem follows since there are commutative triangles of 2–functors

$$\begin{tikzcd} \mathcal{Z}^3_{\mathbb{C},1} \arrow[rr, "\cong"] \arrow[dr, "\mathcal{P}_{\mathbb{C}}"'] & & \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{Z}^3_1) \arrow[dl, "\mathcal{P}_*"] \\ & \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{BCG}) & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{Z}^3_{\mathbb{C},2} \arrow[rr, "\cong"] \arrow[dr, "\mathcal{P}_{\mathbb{C}}"'] & & \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{Z}^3_2) \arrow[dl, "\mathcal{P}_*"] \\ & \mathbf{Psd}(\mathbb{C}^{\mathrm{op}}, \mathcal{SCG}) & \end{tikzcd}$$

where the horizontal isomorphisms are those in (96). □

7.3 Classification of braided and symmetric $\mathbb{C}$–fibred categorical groups

Recall, from (53) and (54), that $\mathcal{BCG}_{\downarrow\mathbb{C}}$ and $\mathcal{SCG}_{\downarrow\mathbb{C}}$ denote the 2–categories of braided and symmetric $\mathbb{C}$–fibred categorical groups, respectively. Theorem 6.15 and Theorem 7.2 above jointly give the following main result.

Theorem 7.3 *For any given small category $\mathbb{C}$, both composite 2–functors $\int_{\mathbb{C}}\mathcal{P}_{\mathbb{C}}$ in the commutative diagram*

(112)
$$\begin{tikzcd} \mathcal{Z}^3_{\mathbb{C},2} \arrow[r, "\int_{\mathbb{C}}\mathcal{P}_{\mathbb{C}}"] \arrow[d, hook] & \mathcal{SCG}_{\downarrow\mathbb{C}} \arrow[d, hook] \\ \mathcal{Z}^3_{\mathbb{C},1} \arrow[r, "\int_{\mathbb{C}}\mathcal{P}_{\mathbb{C}}"] & \mathcal{BCG}_{\downarrow\mathbb{C}} \end{tikzcd}$$

are biequivalences. Therefore, they induce equivalences between the corresponding homotopy categories

(113)
$$\begin{tikzcd} Ho\mathcal{Z}^3_{\mathbb{C},2} \arrow[r, "\sim"] \arrow[d, hook] & Ho\mathcal{SCG}_{\downarrow\mathbb{C}} \arrow[d, hook] \\ Ho\mathcal{Z}^3_{\mathbb{C},1} \arrow[r, "\sim"] & Ho\mathcal{BCG}_{\downarrow\mathbb{C}}. \end{tikzcd}$$

It follows that, for any braided $\mathbb{C}$–fibred categorical group $\mathbb{P}=(\mathbb{P},P,\otimes,I,\boldsymbol{a},\boldsymbol{r},\boldsymbol{l},\boldsymbol{c})$, there is an object

(114)
$$(\pi_0\mathbb{P},\pi_1\mathbb{P},h^{\mathbb{P}})\in \mathrm{Ho}\,\mathcal{Z}^3_{\mathbb{C},1},$$

that is, two $\mathbb{C}$–modules $\pi_0\mathbb{P},\pi_1\mathbb{P}\colon \mathbb{C}^{\mathrm{op}}\to\mathbf{Ab}$, and a first-level 3–cocycle

(115)
$$h^{\mathbb{P}}\in Z^3_{\mathbb{C},1}(\pi_0\mathbb{P},\pi_1\mathbb{P})$$

such that there is a braided $\mathbb{C}$–fibred equivalence

(116)
$$\textstyle\int_{\mathbb{C}}\mathcal{P}_{\mathbb{C}}(\pi_0\mathbb{P},\pi_1\mathbb{P},h^{\mathbb{P}})\xrightarrow{\sim}\mathbb{P}.$$

Such an object $(\pi_0\mathbb{P},\pi_1\mathbb{P},h^{\mathbb{P}})$ is unique up to isomorphism in $\mathrm{Ho}\,\mathcal{Z}^3_{\mathbb{C},1}$, and, moreover, it belongs to $\mathrm{Ho}\,\mathcal{Z}^3_{\mathbb{C},2}$ if and only if $\mathbb{P}$ is actually a symmetric $\mathbb{C}$–fibred categorical group.

For completeness, we shall next show how this complete invariant (114) can be built from any given braided $\mathbb{C}$–fibred categorical group $\mathbb{P}$. We start by choosing a cleavage for the fibration $P\colon\mathbb{P}\to\mathbb{C}$, say $\{x_\sigma\colon \sigma^*x\to x\}$, where specifically we choose $x_{1_u}=1_x\colon x\to x$ and, for any arrow $\sigma\colon v\to u$ in $\mathbb{C}$, $(Iu)_\sigma=I\sigma\colon Iv\to Iu$.

Then, for any object u of $\mathbb{C}$,

- $\pi_0\mathbb{P}_u$ is the abelian group of u–isomorphism classes of u–objects of $\mathbb{P}$, where addition is induced by the tensor product, that is, $[x]+[y]=[x\otimes y]$,
- $\pi_1\mathbb{P}_u$ is the abelian group of u–automorphisms in $\mathbb{P}$ of Iu, where the operation is composition.

If $\sigma\colon v\to u$ is any morphism in $\mathbb{C}$, then the homomorphism $\sigma^*\colon \pi_0\mathbb{P}_u\to\pi_0\mathbb{P}_v$ is given by $\sigma^*[x]=[\sigma^*x]$, whereas the homomorphism $\sigma^*\colon \pi_1\mathbb{P}_u\to\pi_1\mathbb{P}_v$ carries any $a\colon Iu\to Iu$ to $\sigma^*a\colon Iv\to Iv$, the (unique) v–morphism in $\mathbb{P}$ making commutative the square

$$\begin{tikzcd} Iv \arrow[r, dotted, "\sigma^*a"] \arrow[d, "I\sigma"'] & Iv \arrow[d, "I\sigma"] \\ Iu \arrow[r, "a"] & Iu. \end{tikzcd}$$

Now, in order to build the 3–cocycle $h^{\mathbb{P}}$, we additionally select

- a representative u–object $F_u(\mathsf{x})\in\mathsf{x}$ for each object u of $\mathbb{C}$ and any $\mathsf{x}\in\pi_0\mathbb{P}_u$,
- a morphism $\varphi_u(\mathsf{x},\mathsf{y})\colon F_u(\mathsf{x})\otimes F_u(\mathsf{y})\to F_u(\mathsf{x}+\mathsf{y})$ in $\mathbb{P}_u$, for each pair $\mathsf{x},\mathsf{y}\in\pi_0\mathbb{P}_u$,
- a morphism in $\mathbb{P}_v$, $\varphi_\sigma(\mathsf{x})\colon F_v(\sigma^*\mathsf{x})\to\sigma^*F_u(\mathsf{x})$, for each arrow $\sigma\colon v\to u$ and $\mathsf{x}\in\pi_0\mathbb{P}_u$,

where, particularly, we take

$$F_u(0)=Iu,\ \ \varphi_u(\mathsf{x},0)=\boldsymbol{r}_{F_u(\mathsf{x})},\ \ \varphi_u(0,\mathsf{x})=\boldsymbol{l}_{F_u(\mathsf{x})},\ \ \varphi_{1_u}(\mathsf{x})=1_{F_u(\mathsf{x})},\ \ \varphi_\sigma(0)=1_{Iv}.$$

Then, by using the group isomorphisms by Saavedra [53, Section 1, (1.3.3.3)]

$$\delta=\delta_{\mathsf{x}}\colon \pi_1\mathbb{P}_u\xrightarrow{\ \cong\ }\mathrm{Aut}_{\mathbb{P}_u}(F_u(\mathsf{x})),\qquad u\in\mathrm{Ob}\mathbb{C},\ \mathsf{x}\in\pi_0\mathbb{P}_u,$$

which carry any $a\colon Iu\to Iu$ to the dotted arrow in the commutative square

$$\begin{tikzcd} F_u(\mathsf{x})\otimes Iu \arrow[r, "1\otimes a"] \arrow[d, "\boldsymbol{r}"'] & F_u(\mathsf{x})\otimes Iu \arrow[d, "\boldsymbol{r}"] \\ F_u(\mathsf{x}) \arrow[r, dotted, "\delta(a)"] & F_u(\mathsf{x}), \end{tikzcd}$$

the first-level 3–cocycle $h^{\mathbb{P}}\in Z^3_{\mathbb{C},1}(\pi_0\mathbb{P},\pi_1\mathbb{P})$ is canonically deduced from the associativity constraint, the braiding, the tensor product, and the composition in $\mathbb{P}$, as follows:

- for each object u of $\mathbb{C}$ and $\mathsf{x},\mathsf{y},\mathsf{z}\in\pi_0\mathbb{P}_u$, the (unique) element $h^{\mathbb{P}}_u(\mathsf{x},\mathsf{y},\mathsf{z})\in\pi_1\mathbb{P}_u$ making commutative the diagram

$$\begin{tikzcd}
(F_u(\mathsf{x})\otimes F_u(\mathsf{y}))\otimes F_u(\mathsf{z}) \arrow[r, "\varphi_u(\mathsf{x},\mathsf{y})\otimes 1"] \arrow[d, "\boldsymbol{a}_{F_u(\mathsf{x}),F_u(\mathsf{y}),F_u(\mathsf{z})}"'] & F_u(\mathsf{x}+\mathsf{y})\otimes F_u(\mathsf{z}) \arrow[r, "\varphi_u(\mathsf{x}+\mathsf{y},\mathsf{z})"] & F_u(\mathsf{x}+\mathsf{y}+\mathsf{z}) \arrow[d, dotted, "\delta(h^{\mathbb{P}}_u(\mathsf{x},\mathsf{y},\mathsf{z}))"] \\
F_u(\mathsf{x})\otimes (F_u(\mathsf{y})\otimes F_u(\mathsf{z})) \arrow[r, "1\otimes\varphi_u(\mathsf{y},\mathsf{z})"] & F_u(\mathsf{x})\otimes F_u(\mathsf{y}+\mathsf{z}) \arrow[r, "\varphi_u(\mathsf{x},\mathsf{y}+\mathsf{z})"] & F_u(\mathsf{x}+\mathsf{y}+\mathsf{z}),
\end{tikzcd}$$

- for each object u of $\mathbb{C}$ and $\mathsf{x},\mathsf{y}\in\pi_0\mathbb{P}_u$, the element $h^{\mathbb{P}}_u(\mathsf{x}\mid\mathsf{y})\in\pi_1\mathbb{P}_u$ making commutative the diagram

$$\begin{tikzcd}
F_u(\mathsf{x})\otimes F_u(\mathsf{y}) \arrow[rr, "\varphi_u(\mathsf{x},\mathsf{y})"] \arrow[d, "\boldsymbol{c}_{F_u(\mathsf{x}),F_u(\mathsf{y})}"'] & & F_u(\mathsf{x}+\mathsf{y}) \arrow[d, dotted, "\delta(h^{\mathbb{P}}_u(\mathsf{x}\mid\mathsf{y}))"] \\
F_u(\mathsf{y})\otimes F_u(\mathsf{x}) \arrow[r, "\varphi_u(\mathsf{y},\mathsf{x})"] & F_u(\mathsf{y}+\mathsf{x}) \arrow[r, equal] & F_u(\mathsf{x}+\mathsf{y}),
\end{tikzcd}$$

- for each arrow $v\xrightarrow{\sigma}u$ of $\mathbb{C}$ and $\mathsf{x},\mathsf{y}\in\pi_0\mathbb{P}_u$, the element $h^{\mathbb{P}}_\sigma(\mathsf{x},\mathsf{y})\in\pi_1\mathbb{P}_u$ making commutative the diagram

$$\begin{tikzcd}
F_v(\sigma^*\mathsf{x})\otimes F_v(\sigma^*\mathsf{y}) \arrow[r, "\varphi_\sigma(\mathsf{x})\otimes\varphi_\sigma(\mathsf{y})"] \arrow[d, "\varphi_v(\sigma^*\mathsf{x},\sigma^*\mathsf{y})"'] & \sigma^*F_u(\mathsf{x})\otimes\sigma^*F_u(\mathsf{y}) \arrow[r, "(F_u(\mathsf{x}))_\sigma\otimes(F_u(\mathsf{y}))_\sigma"] & F_u(\mathsf{x})\otimes F_u(\mathsf{y}) \arrow[d, "\varphi_u(\mathsf{x},\mathsf{y})"] \\
F_v(\sigma^*\mathsf{x}+\sigma^*\mathsf{y}) \arrow[d, equal] & & F_u(\mathsf{x}+\mathsf{y}) \arrow[d, dotted, "\delta(h^{\mathbb{P}}_\sigma(\mathsf{x},\mathsf{y}))"] \\
F_v(\sigma^*(\mathsf{x}+\mathsf{y})) \arrow[r, "\varphi_\sigma(\mathsf{x}+\mathsf{y})"] & \sigma^*F_u(\mathsf{x}+\mathsf{y}) \arrow[r, "(F_u(\mathsf{x}+\mathsf{y}))_\sigma"] & F_u(\mathsf{x}+\mathsf{y}),
\end{tikzcd}$$

- for each pair of arrows $w\xrightarrow{\tau}v\xrightarrow{\sigma}u$ of $\mathbb{C}$ and $\mathsf{x}\in\pi_0\mathbb{P}_u$, the element $h^{\mathbb{P}}_{\sigma,\tau}(\mathsf{x})\in\pi_1\mathbb{P}_u$ making commutative the diagram

$$\begin{tikzcd}
F_w(\tau^*\sigma^*\mathsf{x}) \arrow[r, equal] \arrow[d, "\varphi_\tau(\sigma^*\mathsf{x})"'] & F_w((\sigma\tau)^*\mathsf{x}) \arrow[r, "\varphi_{\sigma\tau}(\mathsf{x})"] & (\sigma\tau)^*F_u(\mathsf{x}) \arrow[r, "(F_u(\mathsf{x}))_{\sigma\tau}"] & F_u(\mathsf{x}) \arrow[d, dotted, "\delta(h^{\mathbb{P}}_{\sigma,\tau}(\mathsf{x}))"] \\
\tau^*F_v(\sigma^*\mathsf{x}) \arrow[r, "(F_v(\sigma^*\mathsf{x}))_\tau"] & F_v(\sigma^*\mathsf{x}) \arrow[r, "\varphi_\sigma(\mathsf{x})"] & \sigma^*F_u(\mathsf{x}) \arrow[r, "(F_u(\mathsf{x}))_\sigma"] & F_u(\mathsf{x}).
\end{tikzcd}$$

Since the composition and tensor in $\mathbb{P}$ are unitary and I is a functor, the normalization of $h^{\mathbb{P}}$ follows from the naturality of the unit constraints, coherent condition (47), and (51). The cocycle condition $\partial^3 h^{\mathbb{P}}=0$ in (31) follows from the associativity law for morphisms in $\mathbb{P}$. That $\partial^3 h^{\mathbb{P}}=0$ in (33) is a consequence of the fibred tensor product $\otimes\colon \mathbb{P}\times_{\mathbb{C}}\mathbb{P}\to\mathbb{P}$ being functorial. The equality $\partial^3 h^{\mathbb{P}}=0$ in (27) holds because of the coherence pentagons (46), and $\partial^3 h^{\mathbb{P}}=0$ in (31) follows from the naturality of the associativity constraints. The cocycle condition $\partial^3 h^{\mathbb{P}}=0$ in (28) and (29) are verified owing to the coherence conditions (48) and (49). And, finally, the naturality of the braiding implies that $\partial^3 h^{\mathbb{P}}=0$ in (32). If $\mathbb{P}$ is symmetric, then the cocycle condition $\partial^3 h^{\mathbb{P}}=0$ in (30) follows from the symmetry equation (50). Hence $h^{\mathbb{P}}\in Z^3_{\mathbb{C},1}(\pi_0\mathbb{P},\pi_1\mathbb{P})$, and $h^{\mathbb{P}}\in Z^3_{\mathbb{C},2}(\pi_0\mathbb{P},\pi_1\mathbb{P})$ if $\mathbb{P}$ is symmetric.

The braided $\mathbb{C}$–fibred equivalence (116) is then realized by the normal braided $\mathbb{C}$–fibred functor $\int_{\mathbb{C}}\mathcal{P}_{\mathbb{C}}(\pi_0\mathbb{P},\pi_1\mathbb{P},h^{\mathbb{P}})\to\mathbb{P}$ carrying an object (x,u) of $\int_{\mathbb{C}}\mathcal{P}_{\mathbb{C}}(\pi_0\mathbb{P},\pi_1\mathbb{P},h^{\mathbb{P}})$ to $F_u(\mathsf{x})$ and a morphism $(a,\sigma)\colon(\mathsf{y},v)\to(\mathsf{x},u)$ to the composite in $\mathbb{P}$ of the morphisms

$$F_v(\mathsf{y})\xrightarrow{\delta(a)}F_v(\mathsf{y})=F_v(\sigma^*\mathsf{x})\xrightarrow{\varphi_\sigma(\mathsf{x})}\sigma^*F_u(\mathsf{x})\xrightarrow{(F_u(\mathsf{x}))_\sigma}F_u(\mathsf{x}),$$

with the structure u–isomorphisms $\varphi_u(\mathsf{x},\mathsf{y})\colon F_u(\mathsf{x})\otimes F_u(\mathsf{y})\to F_u(\mathsf{x}+\mathsf{y})$.

Closely related to the categories of 3–cocycles $\operatorname{Ho}\mathcal{Z}^3_{\mathbb{C},1}$ and $\operatorname{Ho}\mathcal{Z}^3_{\mathbb{C},2}$ are the *categories of* 3*–cohomology classes*, $\mathcal{H}^3_{\mathbb{C},1}$ and $\mathcal{H}^3_{\mathbb{C},2}$ respectively, which play a fundamental role in stating our classification theorem below. These categories are defined as follows:

Definition 7.4 Let $\mathbb{C}$ be any small category. The category $\mathcal{H}^3_{\mathbb{C},1}$ of *first-level* 3*–cohomology classes of* $\mathbb{C}$*–modules* has objects triples (B,A,k), where $B,A\colon\mathbb{C}^{\mathrm{op}}\to\mathbf{Ab}$ are $\mathbb{C}$–modules and $k\in H^3_{\mathbb{C},1}(B,A)$. An arrow $(p,q)\colon(B,A,k)\to(B',A',k')$ consists of $\mathbb{C}$–module homomorphisms $p\colon B\to B'$ and $q\colon A\to A'$, such that

$$p^*(k')=q_*(k)\in H^3_{\mathbb{C},1}(B,A').$$

The composition of arrows is given by componentwise composition of $\mathbb{C}$–module homomorphisms, that is, $(p',q')(p,q)=(p'p,q'q)$.

The category of *second-level* 3*–cohomology classes of* $\mathbb{C}$*–modules*,

$$\mathcal{H}^3_{\mathbb{C},2}\subseteq\mathcal{H}^3_{\mathbb{C},1},$$

is the full subcategory given by the objects (B,A,k) with $k\in H^3_{\mathbb{C},2}(B,A)$.

We are now ready to state the following theorem, where we summarize the classification results for braided and symmetric $\mathbb{C}$–fibred categorical groups (cf Joyal and Street [41, Theorem 3.3], and Cegarra and Khmaladze [15, Theorems 22, 24: 16, Theorem 3.12]:

Theorem 7.5 *For any small category $\mathbb{C}$, there is a commutative diagram of* classifying functors

$$\begin{tikzcd} \mathrm{Ho}\,\mathcal{SCG}\!\downarrow_{\mathbb{C}} \arrow[r, "\mathrm{cl}"] \arrow[d, hook] & \mathcal{H}^3_{\mathbb{C},2} \arrow[d, hook] \\ \mathrm{Ho}\,\mathcal{BCG}\!\downarrow_{\mathbb{C}} \arrow[r, "\mathrm{cl}"] & \mathcal{H}^3_{\mathbb{C},1} \end{tikzcd} \qquad \begin{array}{l} \mathbb{P} \mapsto (\pi_0\mathbb{P}, \pi_1\mathbb{P}, k_3\mathbb{P}), \\ {[F]} \mapsto (\pi_0 F, \pi_1 F), \end{array} \tag{117}$$

where $k_3\mathbb{P}$ denotes the cohomology class of the 3–cocycle $h^{\mathbb{P}}$ in (115), which have the following properties:

(i) *For any object (B, A, k) of $\mathcal{H}^3_{\mathbb{C},1}$ (resp. $\mathcal{H}^3_{\mathbb{C},2}$), there is a braided (resp. symmetric) $\mathbb{C}$–fibred categorical group $\mathbb{P}$ with an isomorphism $(\pi_0\mathbb{P}, \pi_1\mathbb{P}, k_3\mathbb{P}) \cong (B, A, k)$.*

(ii) *For any isomorphism $(p, q)\colon (\pi_0\mathbb{P}, \pi_1\mathbb{P}, k_3\mathbb{P}) \cong (\pi_0\mathbb{P}', \pi_1\mathbb{P}', k_3\mathbb{P}')$, there is an isomorphism $[F]\colon \mathbb{P} \cong \mathbb{P}'$ such that $(\pi_0 F, \pi_1 F) = (p, q)$.*

(iii) *$(\pi_0 F, \pi_1 F)$ is an isomorphism if and only if $[F]$ is an isomorphism.*

(iv) *For any $(p, q)\colon (\pi_0\mathbb{P}, \pi_1\mathbb{P}, k_3\mathbb{P}) \to (\pi_0\mathbb{P}', \pi_1\mathbb{P}', k_3\mathbb{P}')$, there is a bijection*

$$\big\{[F]\colon \mathbb{P} \to \mathbb{P}' \mid \pi_0 F = p,\ \pi_1 F = q\big\} \cong \mathrm{Ext}_{\mathrm{Mod}_{\mathbb{C}}}(\pi_0\mathbb{P}, \pi_1\mathbb{P}').$$

Proof The diagram of classifying functors (117) is obtained from the diagram of equivalences of categories (113), by composing with the following diagram of *cohomology class functors*

$$\begin{tikzcd} \mathrm{Ho}\,\mathcal{Z}^3_{\mathbb{C},2} \arrow[r] \arrow[d, hook] & \mathcal{H}^3_{\mathbb{C},2} \arrow[d, hook] \\ \mathrm{Ho}\,\mathcal{Z}^3_{\mathbb{C},1} \arrow[r] & \mathcal{H}^3_{\mathbb{C},1} \end{tikzcd} \qquad \begin{array}{l} (B, A, h) \mapsto (B, A, [h]), \\ (p, q, g) \mapsto (p, q), \end{array}$$

where $[h]$ denotes the cohomology class of h. Then, it suffices to prove the corresponding statements to (i)–(iv) in the theorem for these cohomology class functors. Now, it is quite obvious to see that both are full, surjective on objects, and reflecting isomorphism functors. However, these cohomology class functors are not faithful. In fact, for any given morphism $(p, q)\colon (B, A, [h]) \to (B', A', [h'])$, there is a bijection

$$\big\{[p, q, g]\colon (B, A, h) \to (B', A', h')\big\} \cong \mathrm{Ext}_{\mathrm{Mod}_{\mathbb{C}}}(B, A').$$

To see this, fix any $(p, q, g)\colon (B, A, h) \to (B', A', h')$. Then, each 2–cocycle $g' \in Z^2_{\mathbb{C},1}(B, A')$ gives rise to a morphism $(p, q, g + g')\colon (B, A, h) \to (B'A', h')$, and any other morphism with the same p and q is necessarily written in such a form for some 2–cocycle g'. Moreover, both (p, q, g) and $(p, q, g + g')$ are homotopic if and only if $g' = \partial^1 f$ for some $f \in C^1_{\mathbb{C},1}(B, A')$. This proves the bijection above, since, by Theorem 5.3, $H^2_{\mathbb{C},1}(B, A') = \mathrm{Ext}_{\mathrm{Mod}_{\mathbb{C}}}(B, A')$. □

If, for any $\mathbb{C}$–modules $B, A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$, we denote by

$$\mathcal{BCG}_{\downarrow\mathbb{C}}[B,A]$$

the set of isomorphism classes in $\mathrm{Ho}\,\mathcal{BCG}_{\downarrow\mathbb{C}}$ of those braided $\mathbb{C}$–fibred categorical groups $\mathbb{P}$ with $\pi_0\mathbb{P} = B$ and $\pi_1\mathbb{P} = A$, and

$$\mathcal{SCG}_{\downarrow\mathbb{C}}[B,A]$$

the set of isomorphism classes in $\mathrm{Ho}\,\mathcal{SCG}_{\downarrow\mathbb{C}}$ of those symmetric $\mathbb{C}$–fibred categorical groups $\mathbb{P}$ with $\pi_0\mathbb{P} = B$ and $\pi_1\mathbb{P} = A$, then, as consequence of Theorem 7.5 above and Corollary 4.1, we have the theorem below.

Theorem 7.6 *For any $\mathbb{C}$–modules B, A, and any integer $\mathrm{r} \geq 2$, there are natural identifications*

$$H^3_{\mathbb{C},1}(B,A) = \mathcal{BCG}_{\downarrow\mathbb{C}}[B,A],$$
$$H^3_{\mathbb{C},2}(B,A) = H^3_{\mathbb{C},\mathrm{r}}(B,A) = \mathcal{SCG}_{\downarrow\mathbb{C}}[B,A].$$

A symmetric $\mathbb{C}$–fibred categorical group $\mathbb{P} = (\mathbb{P}, P, \otimes, \boldsymbol{I}, \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{l}, \boldsymbol{c})$ is called a *strictly commutative Picard $\mathbb{C}$–fibred category* by Deligne [21, Definition 1.4.2] whenever its symmetry constraint satisfies

$$\boldsymbol{c}_{x,x} = 1_{x\otimes x}$$

for any object x of $\mathbb{P}$. If, for any $\mathbb{C}$–modules $B, A\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Ab}$,

$$\mathcal{P}ic_{\downarrow\mathbb{C}}[B,A] \subseteq \mathcal{SCG}_{\downarrow\mathbb{C}}[B,A]$$

denotes the subset defined by those strictly commutative Picard $\mathbb{C}$–fibred categories $\mathbb{P}$ with $\pi_0\mathbb{P} = B$ and $\pi_1\mathbb{P} = A$, then a well-known result by Deligne [21, Proposition 1.4.15] states that there is a natural identification

$$\mathcal{P}ic_{\downarrow\mathbb{C}}[B,A] = \mathrm{Ext}^2_{\mathrm{Mod}_{\mathbb{C}}}(B,A).$$

Hence, in this way, one achieves natural inclusions

$$\mathrm{Ext}^2_{\mathrm{Mod}_{\mathbb{C}}}(B,A) \subseteq H^3_{\mathbb{C},2}(B,A) \subseteq H^3_{\mathbb{C},1}(B,A).$$

We shall end by remarking that these inclusions are, in general, strict despite the equalities $H^1_{\mathbb{C},\mathrm{r}}(B,A) = \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{C}}}(B,A)$ and $H^2_{\mathbb{C},\mathrm{r}}(B,A) = \mathrm{Ext}_{\mathrm{Mod}_{\mathbb{C}}}(B,A)$, stated in Theorem 5.1 and Theorem 5.3, respectively.

In effect, let us take $\mathbb{C} = \mathbf{1}$, the trivial category with only one arrow. Then, the category of $\mathbb{C}$–modules is simply $\mathbf{Ab}$, the category of abelian groups. In the case where B is the cyclic group $\mathbb{Z}_2$ of order 2 and A is the cyclic group $\mathbb{Z}_4$ of order 4, we have $\mathrm{Ext}^2_{\mathbf{Ab}}(\mathbb{Z}_2, \mathbb{Z}_4) = 0$ since, in the category of abelian groups, all groups $\mathrm{Ext}^n_{\mathbf{Ab}}(B,A)$

vanish for $n \geq 2$, while $H^3_2(\mathbb{Z}_2, \mathbb{Z}_4) = \mathbb{Z}_2$, by Eilenberg and Mac Lane, [29, Theorem 27.1], and $H^3_1(\mathbb{Z}_2, \mathbb{Z}_4) = \mathbb{Z}_4$, by [29, Theorem 26.1 and (13.6)].

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