

Representation stability for the cohomology of the pure string motion groups

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The cohomology of the pure string motion group $P\Sigma_n$ admits a natural action by the hyperoctahedral group W_n . In [6], Church and Farb conjectured that for each $k \geq 1$, the cohomology groups $H^k(P\Sigma_n; \mathbb{Q})$ are uniformly representation stable; that is, the description of the decomposition of $H^k(P\Sigma_n; \mathbb{Q})$ into irreducible W_n -representations stabilizes for $n \gg k$. We use a characterization of $H^*(P\Sigma_n; \mathbb{Q})$ given by Jensen, McCammond and Meier to prove this conjecture. Using a transfer argument, we further deduce that the rational cohomology groups of the string motion group $H^k(\Sigma_n; \mathbb{Q})$ vanish for $k \geq 1$. We also prove that the subgroup of $\Sigma_n^+ \subseteq \Sigma_n$ of orientation-preserving string motions, also known as the braid-permutation group, is rationally cohomologically stable in the classical sense.

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1 Introduction

Let $C_n = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_n$ be the disjoint union of n smoothly embedded, oriented, unlinked, unknotted circles γ_i in \mathbb{R}^3 . We define Σ_n to be the group of motions of C_n , as follows: A *motion* of C_n is a path of diffeomorphisms $f_t \in \text{Diff}(\mathbb{R}^3)$ such that f_0 is the identity and f_1 stabilizes C_n setwise. Two motions $f_{t,0}$ and $f_{t,1}$ are considered equivalent if they are smoothly isotopic through an isotopy $f_{t,s}$ such that $f_{0,s}$ and $f_{1,s}$ stabilize C_n . The product of two motions f_t and g_t is given by

$$(g \cdot f)_t = \begin{cases} f_{2t} & 0 \leq t \leq \frac{1}{2}, \\ g_{2t-1} \circ f_1 & \frac{1}{2} \leq t \leq 1. \end{cases}$$

With this product, the set of these homotopy classes of motions forms the *string motion group*, also called the *circle-braid group*, with identity the *stationary motion*.

The string motion group Σ_n is a generalization of the classical braid group B_n , the motions of n points in the plane, though the circles defining Σ_n may additionally pass over and through each other. The analogue of the pure braid group P_n is the *pure string motion group* $P\Sigma_n \subseteq \Sigma_n$, the subgroup of motions in which all circles return to their original positions and orientations.

The groups Σ_n and $P\Sigma_n$ may alternatively be defined in terms of spaces of smooth embeddings of circles into \mathbb{R}^3 , or of configurations of ‘rings’, as described in Section 2. The structure of the group Σ_n is described in more detail in Section 3, where it is identified with a subgroup of automorphisms of the free group on n letters.

The quotient $\Sigma_n/P\Sigma_n$ is the *hyperoctahedral group* W_n , the wreath product

$$W_n = \mathbb{Z}/2\mathbb{Z} \wr S_n := (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n,$$

where S_n conjugates $(\mathbb{Z}/2\mathbb{Z})^n$ by permuting the coordinates. The hyperoctahedral group is also called the *signed permutation group*, since it may be described as the symmetries of the set

$$S = \{-1, 1\}, \{-2, 2\}, \dots, \{-n, n\},$$

where the k^{th} factor of $(\mathbb{Z}/2\mathbb{Z})^n$ transposes the elements in the block $\{-k, k\}$, and S_n permutes the n blocks. The hyperoctahedral group $W_n = \Sigma_n/P\Sigma_n$ acts by conjugation on the pure string motion group $P\Sigma_n$ by relabelling and reversing the orientations of the n circles.

The cohomology of $P\Sigma_n$ is described in Section 4. It has been studied by Collins [7], Brownstein and Lee [5], Brady, McCammond, Meier and Miller [2], Pettet [20], Jensen, McCammond and Meier [17] and others. As with the pure braid group, the pure string motion group is not cohomologically stable in the classical sense: Jensen, McCammond and Meier determined the formula

$$\dim_{\mathbb{Q}} H^k(P\Sigma_n; \mathbb{Q}) = \binom{n-1}{k} n^k.$$

For each fixed $k \geq 1$, the dimension grows unboundedly with n .

In [6], Church and Farb define an alternate notion of stability, which they call representation stability. We may consider the group $H^k(P\Sigma_n; \mathbb{Q})$ as a rational representation of the hyperoctahedral group W_n under the action of W_n induced by its action on $P\Sigma_n$. The irreducible representations of W_n over \mathbb{Q} are classified by pairs of partitions $\lambda = (\lambda^+, \lambda^-)$ with $|\lambda^+| + |\lambda^-| = n$, a classification described in more detail in Section 5. Church and Farb conjectured (see [6, Conjecture 4.8]) that, in analogy with the pure braid group, the cohomology groups $H^k(P\Sigma_n; \mathbb{Q})$ stabilize in terms of their decompositions into irreducible representations of W_n . We prove this conjecture in Section 6.

To make sense of this statement, we need a system of classifying the irreducible representations of W_n that is independent of n . As in [6], we write

$$V((\lambda_1^+, \lambda_2^+, \dots, \lambda_p^+), (\lambda_0^-, \lambda_1^-, \dots, \lambda_q^-))_n$$

to denote the irreducible representation of W_n associated to the double partition (λ^+, λ^-) of n with

$$\lambda^+ = (n - k, \lambda_1^+, \lambda_2^+, \dots, \lambda_p^+), \quad \lambda^- = (\lambda_0^-, \lambda_1^-, \dots, \lambda_q^-),$$

where $k = (\sum_{i=1}^p \lambda_i^+) + (\sum_{i=0}^q \lambda_i^-)$. In this notation, for example, the decomposition of $H^1(P\Sigma_n; \mathbb{Q})$ into irreducible W_n -representations is:

$$\begin{aligned} H^1(P\Sigma_2; \mathbb{Q}) &= V((0), (1))_2, \\ H^1(P\Sigma_n; \mathbb{Q}) &= V((0), (1))_n \oplus V((1), (1))_n \quad \text{for all } n \geq 3. \end{aligned}$$

This decomposition stabilizes once $n = 3$.

A complete definition of representation stability is given in Section 5. Using this framework, we adapt Church and Farb’s proof of the uniform representation stability of the cohomology of P_n to a description of $H^k(P\Sigma_n; \mathbb{Q})$ given by Jensen, McCammond and Meier in [17] to prove our main result:

Theorem 6.1 *For each fixed $k \geq 0$, the sequence of W_n -representations*

$$\{H^k(P\Sigma_n; \mathbb{Q})\}_{n \in \mathbb{N}}$$

is uniformly representation stable with respect to the maps

$$\phi_n: H^k(P\Sigma_n; \mathbb{Q}) \rightarrow H^k(P\Sigma_{n+1}; \mathbb{Q})$$

induced by the ‘forgetful’ map $P\Sigma_{n+1} \rightarrow P\Sigma_n$. The sequence stabilizes once $n \geq 4k$.

It is generally true that the restriction of a uniformly stable sequence of W_n -representations to the action of $S_n \subseteq W_n$ yields a uniformly representation stable sequence of S_n -representations, a result that we prove in a forthcoming paper. However, by using the same method of proof as for Theorem 6.1, we prove uniform representation stability of the sequences $H^k(P\Sigma_n; \mathbb{Q})$ with respect to the action of S_n with an improved stable range.

Theorem 6.4 *For each fixed $k \geq 0$, the sequence of groups*

$$\{H^k(P\Sigma_n; \mathbb{Q})\}_{n \in \mathbb{N}},$$

considered as S_n -representations, is uniformly representation stable with respect to the maps

$$\phi_n: H^k(P\Sigma_n; \mathbb{Q}) \rightarrow H^k(P\Sigma_{n+1}; \mathbb{Q})$$

induced by the ‘forgetful’ map $P\Sigma_{n+1} \rightarrow P\Sigma_n$. The sequence stabilizes once $n \geq 4k$.

If we define

$$V(\lambda_1, \dots, \lambda_d)_n$$

to be the irreducible S_n -representation associated to the partition

$$\lambda = (n - k, \lambda_1, \dots, \lambda_d),$$

with $k = \sum_{i=1}^d \lambda_i$, then the decomposition of $H^1(P\Sigma_n; \mathbb{Q})$ into irreducible S_n -representations is

$$\begin{aligned} H^1(P\Sigma_2; \mathbb{Q}) &= V(0)_2 \oplus V(1)_2, \\ H^1(P\Sigma_3; \mathbb{Q}) &= V(0)_3 \oplus V(1)_3^{\oplus 2} \oplus V(1, 1)_3, \\ H^1(P\Sigma_n; \mathbb{Q}) &= V(0)_n \oplus V(1)_n^{\oplus 2} \oplus V(1, 1)_n \oplus V(2)_n \quad \text{for all } n \geq 4. \end{aligned}$$

The decomposition of $H^2(P\Sigma_n; \mathbb{Q})$ is:

$$\begin{aligned} H^2(P\Sigma_2) &= 0 \\ H^2(P\Sigma_3) &= V(1, 1)_3^{\oplus 2} \oplus V(1)_3^{\oplus 3} \oplus V(0)_3 \\ H^2(P\Sigma_4) &= V(1, 1, 1)_4^{\oplus 2} \oplus V(1, 1)_4^{\oplus 7} \oplus V(2)_4^{\oplus 3} \oplus V(1)_4^{\oplus 6} \oplus V(0)_4 \\ H^2(P\Sigma_5) &= V(1, 1, 1)_5^{\oplus 4} \oplus V(2, 1)_5^{\oplus 5} \oplus V(1, 1)_5^{\oplus 9} \oplus V(2)_5^{\oplus 6} \oplus V(1)_5^{\oplus 6} \oplus V(0)_5 \\ H^2(P\Sigma_6) &= V(2, 1, 1)_6^{\oplus 2} \oplus V(1, 1, 1)_6^{\oplus 4} \oplus V(2, 1)_6^{\oplus 7} \oplus V(3)_6^{\oplus 3} \\ &\quad \oplus V(1, 1)_6^{\oplus 9} \oplus V(2)_6^{\oplus 6} \oplus V(1)_6^{\oplus 6} \oplus V(0)_6 \\ H^2(P\Sigma_n) &= V(2, 1, 1)_n^{\oplus 2} \oplus V(3, 1)_n^{\oplus 2} \oplus V(1, 1, 1)_n^{\oplus 4} \oplus V(2, 1)_n^{\oplus 7} \oplus V(3)_n^{\oplus 3} \\ &\quad \oplus V(1, 1)_n^{\oplus 9} \oplus V(2)_n^{\oplus 6} \oplus V(1)_n^{\oplus 6} \oplus V(0)_n \quad \text{for all } n \geq 7. \end{aligned}$$

These results provide a new perspective on the structure and growth of the cohomology groups $H^k(P\Sigma_n; \mathbb{Q})$ for large values of n .

The structure of the $H^k(P\Sigma_n; \mathbb{Q})$ provides insight into the cohomology of the string motion group Σ_n . A transfer argument identifies the rational cohomology $H^k(\Sigma_n; \mathbb{Q})$ with the W_n -invariants $H^k(P\Sigma_n; \mathbb{Q})^{W_n}$. In Section 7, using a combinatorial argument, we show that the trivial W_n -representation does not occur in $H^k(P\Sigma_n; \mathbb{Q})$ for $k > 0$, and so prove:

Theorem 7.1 *For $k \geq 1$, the rational cohomology groups $H^k(\Sigma_n; \mathbb{Q}) = 0$. In particular, the integral homology and cohomology groups of Σ_n are finite.*

The subgroup $\Sigma_n^+ \subseteq \Sigma_n$ of string motions that preserve the orientations of the circles has been called the *braid-permutation group*. The quotient $\Sigma_n^+ / P\Sigma_n = S_n$, and

so the structure of $H^k(P\Sigma_n; \mathbb{Q})$ as an S_n -representation encodes the structure of $H^k(\Sigma_n^+; \mathbb{Q})$. In Section 8, we use a transfer argument to conclude the following corollary.

Corollary 8.1 *The braid-permutation group Σ_n^+ of orientation-preserving string motions is rationally cohomologically stable in the classical sense. For each $k \geq 0$, $H^k(\Sigma_n^+; \mathbb{Q}) \cong H^k(\Sigma_{n+1}^+; \mathbb{Q})$ once $n \geq 4k$.*

By analyzing the combinatorics of $H^k(P\Sigma_n; \mathbb{Q})$, we gain some insights into the structure of $H^k(\Sigma_n^+; \mathbb{Q})$, and we compute the stable dimensions in small degrees.

Remark We have recently learned that the integral (co)homology of Σ_n and Σ_n^+ could be computed using the techniques of Griffin in [14], which gives a general approach to computing the (co)homology of the symmetric automorphism groups of certain free products of groups.

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2 Some background on the theory of motion groups

The string motion group Σ_n was first defined by Dahm in his 1962 thesis [8], and the theory of motion groups was further developed by Goldsmith [13] and others. These groups were originally studied in the topological category, as follows: Given a manifold M and a compact subspace N in the interior of M , let $H(M)$ be the space of self-homeomorphisms of M with the compact-open topology. Let $H(M, N)$ be the subspace of homeomorphisms that stabilize N setwise. Let $H_C(M) \subseteq H(M)$ and $H_C(M, N) \subseteq H(M, N)$ be those homeomorphisms with compact support. Let \mathbb{I}_M be the identity map on M . The *motion group* $\mathcal{M}(M, N)$ is then defined as the relative fundamental group $\pi_1(H_C(M), H_C(M, N), \mathbb{I}_M)$. Although relative fundamental groups are not groups in general, a group structure on $\mathcal{M}(M, N)$ derives from the fact that $H_C(M, N) \subseteq H_C(M)$ is a subgroup.

These motion groups were reconceived in the differential category by Wattenberg [23]. Let M be a smooth, connected manifold without boundary and $i: N \hookrightarrow M$ the smooth embedding of a compact submanifold N . Let $E(N, M)$ be the space of smooth embeddings $f: N \rightarrow M$ with the C^∞ topology, and $P(N, M)$ the subspace of maps $f \in E(N, M)$ such that $f(N) = i(N)$. Let $D(M)$ be the space of self-diffeomorphisms of M with the C^∞ topology, $D(M, N)$ the subspace stabilizing $i(N)$, and $D_C(M)$ and $D_C(M, N)$ the diffeomorphisms with compact support. Wattenberg proved that

$$\pi_1(D(M), D(M, N), \mathbb{I}_M) \cong \pi_1(E(N, M), P(N, M), i).$$

When $M = \mathbb{R}^3$ and $N = C_n$, this is the string motion group Σ_n . Wattenberg proved, in this case, that the group is isomorphic to $\pi_1(D_C(\mathbb{R}^3), D_C(\mathbb{R}^3, C_n), \mathbb{I}_{\mathbb{R}^3})$, and moreover coincides with Dahm's motion group $\mathcal{M}(\mathbb{R}^3, C_n)$.

In the introduction, we defined Σ_n as the group $\pi_1(D(\mathbb{R}^3), D(\mathbb{R}^3, C_n), \mathbb{I}_{\mathbb{R}^3})$, though by Wattenberg's results we may equivalently define Σ_n as the fundamental group of the space of smooth unlinks of n components, that is, smooth embeddings $E(C_n, \mathbb{R}^3)$ relative embeddings stabilizing C_n . In this framework, the pure string motion group $P\Sigma_n$ is then the fundamental group of the space of smooth, labelled, oriented unlinks of n components, relative those embeddings preserving the order and orientation of each circle. We note that the hypothesis of smoothness is necessary: the analogous construction with continuous embeddings encounters problems due to the existence of 'pathological' isotopies, for example, continuous paths that pull a knot through to an unknot. Such isotopies do not extend to motions of the ambient space.

Recently, Brendle and Hatcher have shown in [3] that it suffices to consider configuration spaces of 'rings', that is, embeddings of Euclidean circles in \mathbb{R}^3 . They define $\mathcal{PUR}(n)$ to be the configuration space of n labelled, oriented, disjoint, unlinked rings. Each ring can be specified by vector determining its centre, and a 'normal' vector that indicates its radius and orientation. Thus $\mathcal{PUR}(n)$ can be realized as an open subset of \mathbb{R}^{6n} . The configuration space $\mathcal{R}(n)$ of disjoint, unlinked rings is also a $6n$ -manifold, the quotient of $\mathcal{PUR}(n)$ by the action of the hyperoctahedral group W_n . Brendle and Hatcher prove that $\Sigma_n = \pi_1(\mathcal{R}(n))$ and that $P\Sigma_n = \pi_1(\mathcal{PUR}(n))$. This characterization of these groups facilitates the study of their algebraic structure.

3 The (pure) string motion group

A valuable technique in the study of the string motion group is to embed the group in the automorphism group of a free group.

If we take the one-point compactification $S^3 = \mathbb{R}^3 \cup \{\infty\}$ of \mathbb{R}^3 , then the fundamental group $\pi_1(S^3 \setminus C_n, \infty)$ is the free group F_n on n generators. Let x_i denote the generator linked once (in the positive sense) with γ_i , and unlinked with the other circles. By extending a string motion f_i to S^3 and taking the induced map $(f_i)_*$ on the fundamental group $\pi_1(S^3 \setminus C_n, \infty)$, we get a map

$$D: \Sigma_n \rightarrow \text{Aut}(F_n)$$

where $\text{Aut}(F_n)$ denotes the automorphism group of F_n . The map D is a particular instance of the *Dahm homomorphism* on motion groups, developed by Dahm in [8] and described in [13] by Goldsmith. In the case of the string motion group Σ_n , D is injective, and its image is the group of *symmetric automorphisms*, the automorphisms that map each generator x_i to a conjugate of a generator or of the inverse of a generator. This group is generated by elements $\alpha_{i,j}$, τ_i , and ρ_i , defined as follows:

$$\alpha_{i,j} = \begin{cases} x_i \mapsto x_j x_i x_j^{-1} \\ x_k \mapsto x_k \end{cases} \quad (k \neq i) \quad \tau_i = \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_i \\ x_k \mapsto x_k \end{cases} \quad (k \neq i, i+1)$$

$$\rho_i = \begin{cases} x_i \mapsto x_i^{-1} \\ x_k \mapsto x_k \end{cases} \quad (k \neq i)$$

The automorphism $\alpha_{i,j}$, $i \neq j$, corresponds to a string motion where the i^{th} circle γ_i passes through the j^{th} circle γ_j , and returns to its original position and orientation. The automorphism τ_i is induced by a string motion that transposes the circles γ_i with γ_{i+1} while preserving their orientations, and ρ_i is induced by a string motion that reverses the orientation of the circle γ_i . The algebraic structure of this group was described by authors including McCool [19] and Rubinfeld [21].

We note that the classical braid group B_n embeds in Σ_n as the (infinite-index) subgroup generated by elements of the form $\alpha_{i,i+1} \tau_i$, for $i = 1, \dots, (n-1)$. This is equivalently the group of symmetric automorphisms that fix the product $x_n x_{n-1} \cdots x_2 x_1$.

The pure string motion group $P\Sigma_n$ is the group of all automorphisms that map each generator x_i to a conjugate of itself, also called the *basis-conjugating automorphisms* or the *pure symmetric automorphisms*. This group is torsion-free, and is generated by the elements $\alpha_{i,j}$, $i \neq j$. It is the intersection of Σ_n with the subgroup $\text{IA}_n \subseteq \text{Aut}(F_n)$ of automorphisms that induce the trivial map on the homology group $H^1(F_n; \mathbb{Z}) \cong \mathbb{Z}^n$.

The quotient $\Sigma_n / P\Sigma_n = \langle \rho_i, \tau_i \rangle$ is the hyperoctahedral group W_n described in the introduction. As explained in by Brownstein and Lee in [5], these groups fit in to a

commutative diagram of short exact sequences:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & P\Sigma_n & \longrightarrow & \Sigma_n & \longrightarrow & W_n \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \text{IA}_n & \longrightarrow & \text{Aut}(F_n) & \longrightarrow & \text{GL}_n(\mathbb{Z}) \longrightarrow 1
 \end{array}$$

The top exact sequence splits, giving a decomposition of Σ_n as a semidirect product $\Sigma_n \cong P\Sigma_n \rtimes W_n$. The group W_n acts on $P\Sigma_n$ as follows:

$$\begin{aligned}
 \rho_i(\alpha_{j,k})\rho_i^{-1} &= \begin{cases} \alpha_{j,k}^{-1} & (k = i) \\ \alpha_{j,k} & (k \neq i) \end{cases} \\
 \tau_i(\alpha_{j,k})\tau_i^{-1} &= \alpha_{\sigma(j),\sigma(k)} \quad \text{where } \sigma = (i \ i+1) \in S_n.
 \end{aligned}$$

4 The cohomology of the pure string motion group

Define $[n] := \{1, 2, \dots, n\}$. In [19], McCool gave a presentation for $P\Sigma_n$ in which all relations are commutators, and so the abelianization $H_1(P\Sigma_n; \mathbb{Z})$ of $P\Sigma_n$ is the free abelian group $\mathbb{Z}[\alpha_{i,j}]$, with $i, j \in [n], i \neq j$. The cohomology ring $H^*(P\Sigma_n; \mathbb{Z})$ is generated by the dual basis $\{\alpha_{i,j}^*\}$.

In [5, Conjecture 4.6], Brownstein and Lee conjectured a presentation for the cohomology ring $H^*(P\Sigma_n; \mathbb{Z})$, which was established by Jensen, McCammond and Meier in [17]. Their result is:

Theorem 4.1 (Jensen, McCammond and Meier [17, Theorem 6.7]) *The cohomology ring $H^*(P\Sigma_n; \mathbb{Z})$ is the exterior algebra generated by the degree-one classes $\alpha_{i,j}^*$, with $i, j \in [n], i \neq j$, modulo the relations*

- (1) $\alpha_{i,j}^* \wedge \alpha_{j,i}^* = 0$
- (2) $\alpha_{k,j}^* \wedge \alpha_{j,i}^* - \alpha_{k,j}^* \wedge \alpha_{k,i}^* + \alpha_{i,j}^* \wedge \alpha_{k,i}^* = 0$

These relations, computed in [5], all pull back from $H^*(P\Sigma_3; \mathbb{Z})$, under the map induced by the map $P\Sigma_n \rightarrow P\Sigma_3$ determined by ‘forgetting’ all but the $i^{\text{th}}, j^{\text{th}}$, and k^{th} circle.

The cohomology groups are computed in [17] by Jensen, McCammond and Meier using the Lyndon–Hochschild–Serre spectral sequence associated with the short exact sequence,

$$1 \longrightarrow \text{Inn}(F_n) \longrightarrow P\Sigma_n \longrightarrow P\Sigma_n/\text{Inn}(F_n) \longrightarrow 1$$

In the proof of [17, Lemma 6.6], Jensen, McCammond and Meier determine that any cyclic product

$$\alpha_{i,j}^* \wedge \alpha_{j,l}^* \wedge \alpha_{l,m}^* \wedge \dots \wedge \alpha_{r,s}^* \wedge \alpha_{s,i}^*$$

is trivial. They argue that the group $H^k(P\Sigma_n; \mathbb{Z})$ is generated by elements of the form $\{\alpha_{i_1,j_1}^* \wedge \alpha_{i_2,j_2}^* \wedge \dots \wedge \alpha_{i_k,j_k}^*\}$ that have distinct indices i_m and that contain no cyclic products. Using the anticommutativity of the generators, we may insist that $i_1 < i_2 < \dots < i_k$. Using a rank argument, Jensen, McCammond and Meier infer that this generating set is minimal. We conclude the following lemma.

Lemma 4.2 (Jensen, McCammond and Meier [17]) *The group $H^k(P\Sigma_n; \mathbb{Z})$ has basis*

$$\{\alpha_{i_1,j_1}^* \wedge \alpha_{i_2,j_2}^* \wedge \dots \wedge \alpha_{i_k,j_k}^*\}$$

where $i_1 < i_2 < \dots < i_k$, and where no permutations of the factors of

$$\alpha_{i_1,j_1}^* \wedge \alpha_{i_2,j_2}^* \wedge \dots \wedge \alpha_{i_k,j_k}^*$$

contains a cyclic product.

Jensen, McCammond and Meier count these basis elements by observing that they are indexed by directed graphs on $[n]$, where the graph contains an edge $i \leftarrow j$ whenever the basis element includes the factor $\alpha_{i,j}^*$. Two examples for $n = 6$ and $k = 4$ are given in Figure 1.

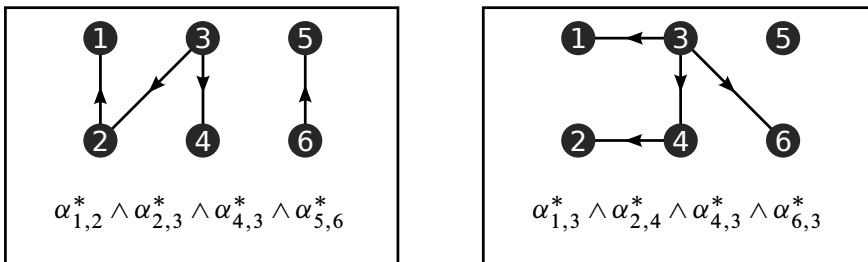


Figure 1. Two directed graphs and their corresponding basis elements in $H^4(P\Sigma_6; \mathbb{Q})$

Jensen, McCammond and Meier observe that, because each i_m is distinct, each vertex is the target of at most one edge. This implies that graph cannot contain any cycles, as these would correspond to cyclic products. Thus, the connected components of the graph are trees, that is, the graph is a forest. Moreover, since each vertex is the target of at most one edge, each tree's digraph structure is determined by a choice of root. Since there are k edges, the forest has $(n - k)$ connected components. In summary:

Lemma 4.3 (Jensen, McCammond and Meier [17]) *The group $H^k(P\Sigma_n; \mathbb{Z})$ is the free abelian group on the set of forests on $[n]$ of $(n - k)$ rooted trees.*

This description will be a key ingredient in our proofs of Theorem 6.1 and Theorem 6.4.

5 Representation stability

The concept of representation stability was introduced by Church and Farb in [6]. Given a sequence of groups $\{G_n\}_{n \in \mathbb{N}}$ with inclusions $G_n \hookrightarrow G_{n+1}$, and a sequence $\{V_n\}_{n \in \mathbb{N}}$ of G_n -representations, we can describe a form of stability for $\{V_n\}_{n \in \mathbb{N}}$ in terms of its decomposition into irreducible representations. Before giving a precise definition of representation stability, we introduce some notation used in [6].

We will specialize to the cases that the group G_n is either the symmetric group S_n , or the hyperoctahedral group W_n . The irreducible representations of S_n over \mathbb{Q} are classified by partitions λ of n , where $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_d)$ with $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_d$ and $\lambda_0 + \lambda_1 + \dots + \lambda_d = n$. There is a natural identification of the irreducible representations of S_n with certain irreducible representations of S_{n+1} , attained by associating the partition $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_d)$ of n with the partition $(\lambda_0 + 1, \lambda_1, \dots, \lambda_d)$ of $(n + 1)$. To emphasize this identification, from here on, we will describe a partition of n by a partition $\lambda \vdash k$, $\lambda = (\lambda_1, \dots, \lambda_d)$, with the first part $\lambda_0 = (n - k)$ left implicit. This notation is now independent of n and gives a well-defined partition of n whenever $n \geq (\lambda_1 + k)$. We call

$$\lambda[n] := (n - k, \lambda_1, \dots, \lambda_d)$$

the *padded partition* of λ .

The rational irreducible representations of the hyperoctahedral group W_n are described in (for example) Geck and Pfeiffer [12]. They are classified by *double partitions* of n , that is, ordered pairs of partitions $\lambda = (\lambda^+, \lambda^-)$ with $|\lambda^+| + |\lambda^-| = n$. These representations are constructed as follows: Given $\lambda \vdash n$ and $V(\lambda)$ the associated representation of S_n , we define $V(\lambda, (0))$ to be the W_n -representation pulled back from $V(\lambda)$ under the surjection $\pi: W_n \twoheadrightarrow S_n$. Let ε be the one-dimensional ‘‘sign’’ representation of W_n , where the canonical generators of $(\mathbb{Z}/2\mathbb{Z})^n$ act by (-1) , and elements of S_n act trivially. Define

$$V((0), \lambda) := V(\lambda, (0)) \otimes \varepsilon.$$

Next, we recall the following definition:

Definition 5.1 Let G, H be groups. Let V be a rational representation of G , and U a rational representation of H . Then the *external tensor product* $V \boxtimes U$ denotes

the $(G \times H)$ -representation given by the action of $G \times H$ on the vector space $V \otimes U$ given by linearly extending the map

$$(g, h) \cdot (v \otimes u) = (g \cdot v) \otimes (h \cdot u)$$

for all $(g, h) \in G \times H$ and $v \otimes u \in V \otimes U$.

Recall additionally that, given a subgroup H of a finite group G and an H -representation V , the *induced representation* $\text{Ind}_H^G(V)$ is the $\mathbb{Q}G$ -module $\mathbb{Q}G \otimes_{\mathbb{Q}H} V$. Equivalently, $\text{Ind}_H^G(V)$ is the G -representation constructed as a direct sum $\text{Ind}_H^G(V) = \bigoplus_{\sigma \in G/H} \sigma V$ of a translate of V for each coset σH .

Then, generally, for $\lambda^+ \vdash k$ and $\lambda^- \vdash (n - k)$, we define

$$V(\lambda^+, \lambda^-) := \text{Ind}_{W_k \times W_{n-k}}^{W_n} V(\lambda^+, (0)) \boxtimes V((0), \lambda^-).$$

Again, given a double partition $\lambda = (\lambda^+, \lambda^-)$ of m with $|\lambda^-| = l$, we define the padded partition

$$\lambda[n] := (\lambda^+[n - l], \lambda^-).$$

For either S_n or W_n , denote the irreducible representation associated to $\lambda[n]$ by $V(\lambda)_n$, and denote the multiplicity of $V(\lambda)_n$ in a representation V_n by $c_{\lambda,n}(V_n)$. Using this notation, Church and Farb make the following definitions:

Definition 5.2 A sequence $\{V_n\}_{n \in \mathbb{N}}$ of G_n -representations with maps $\phi_n: V_n \rightarrow V_{n+1}$ is called *consistent* if ϕ_n is equivariant with respect to the G_n -action on V_n and the G_n -action on V_{n+1} induced by the natural inclusion $G_n \hookrightarrow G_{n+1}$.

Definition 5.3 A consistent sequence $\{V_n\}_{n \in \mathbb{N}}$ of G_n -representations with maps $\phi_n: V_n \rightarrow V_{n+1}$ is called *representation stable* if it satisfies the following properties:

- (1) *Injectivity* The maps $\phi_n: V_n \rightarrow V_{n+1}$ are injective, for n sufficiently large.
- (2) *Surjectivity* The space V_{n+1} is the G_{n+1} -span of $\phi_n(V_n)$, for n sufficiently large.
- (3) *Multiplicities* In the decomposition

$$V_n = \bigoplus_{\lambda} c_{\lambda,n} V(\lambda)_n$$

of V_n into irreducible G_n -representations, for each λ , the coefficient $c_{\lambda,n}$ is eventually independent of n .

Definition 5.4 A representation stable sequence $\{V_n\}_{n \in \mathbb{N}}$ is *uniformly* representation stable if the multiplicities $c_{\lambda,n}(V_n)$ become constant at some $N \in \mathbb{N}$ not depending on λ .

Our goal is now to prove that the cohomology groups $H^*(P\Sigma_n, \mathbb{Q})$ are uniformly representation stable as W_n -representations.

6 The cohomology groups of the pure string motion group are uniformly representation stable

The action of W_n on $P\Sigma_n$ by conjugation induces an action on the cohomology:

$$\begin{aligned} \rho_i \cdot \alpha_{j,k}^* &= \begin{cases} -\alpha_{j,k}^* & (k = i), \\ \alpha_{j,k}^* & (k \neq i). \end{cases} \\ \tau_i \cdot \alpha_{j,k}^* &= \alpha_{\sigma(j),\sigma(k)}^* \quad \text{where } \sigma = (i \ i+1) \in S_n. \end{aligned}$$

With this action, we will consider the rational cohomology group $H^k(P\Sigma_n; \mathbb{Q})$ as a W_n -representation.

The maps $P\Sigma_{n+1} \rightarrow P\Sigma_n$ obtained by ‘forgetting’ the circle γ_{n+1} induce W_n -equivariant maps

$$\begin{aligned} \phi_n: H^*(P\Sigma_n; \mathbb{Q}) &\longrightarrow H^*(P\Sigma_{n+1}; \mathbb{Q}) \\ \alpha_{i,j}^* &\longmapsto \alpha_{i,j}^* \end{aligned}$$

Thus, $\{(H^k(P\Sigma_n; \mathbb{Q}), \phi_n)\}_{n \in \mathbb{N}}$ is a consistent sequence.

We will use the strategy of Church and Farb for the proof of [6, Theorem 4.1], the uniform representation stability of the cohomology groups of the pure braid group, to prove the following:

Theorem 6.1 For each fixed $k \geq 0$, the sequence of W_n -representations

$$\{H^k(P\Sigma_n; \mathbb{Q})\}_{n \in \mathbb{N}}$$

is uniformly representation stable with respect to the maps

$$\phi_n: H^k(P\Sigma_n; \mathbb{Q}) \rightarrow H^k(P\Sigma_{n+1}; \mathbb{Q})$$

induced by the ‘forgetful’ map $P\Sigma_{n+1} \rightarrow P\Sigma_n$. The sequence stabilizes once $n \geq 4k$.

Injectivity follows easily from the definition of the maps ϕ_n , since a basis element

$$\alpha_{i_1, j_1}^* \wedge \dots \wedge \alpha_{i_k, j_k}^* \in H^k(P\Sigma_n; \mathbb{Q})$$

maps to the basis element

$$\alpha_{i_1, j_1}^* \wedge \dots \wedge \alpha_{i_k, j_k}^* \in H^k(P\Sigma_{n+1}; \mathbb{Q}).$$

For the surjectivity criterion, observe that for $n \geq 2k$ and any basis element $v = \alpha_{i_1, j_1}^* \wedge \dots \wedge \alpha_{i_k, j_k}^*$ of $H^k(P\Sigma_n; \mathbb{Q})$, there must exist some number in $[n + 1]$ that does not appear among the indices of v . Then, for some suitable choice of permutation $\sigma \in S_{n+1} \subseteq W_{n+1}$, the basis element $\sigma \cdot v$ does not have $(n + 1)$ as an index, and so is contained in the image $\phi_n(H^k(P\Sigma_n; \mathbb{Q}))$.

To prove the multiplicities criterion, we will use a combinatorial result concerning certain induced representations. In [16], Hemmer proved the following for the symmetric group:

Lemma 6.2 (Hemmer [16, Theorem 2.4]) *Fix $r \geq 1$. Let H be any subgroup of S_r and V an H -representation. Let \mathbb{Q} denote the trivial representation of S_{n-r} . Then the sequence of S_n -representations $\{\text{Ind}_{H \times S_{n-r}}^{S_n} (V \boxtimes \mathbb{Q})\}_{n \in \mathbb{N}}$ is uniformly representation stable, stabilizing for $n \geq 2r$.*

In [6, Theorem 4.6], Church and Farb adapt the proof to the hyperoctahedral group. We reproduce their proof, with more explicit attention paid to the bounds on the stable range.

Lemma 6.3 (Church and Farb [6]) *Fix $r \geq 1$. Let H be any subgroup of W_r , and V an H -representation. Let \mathbb{Q} denote the trivial representation of W_{n-r} . Then the sequence of W_n -representations $\{\text{Ind}_{H \times W_{n-r}}^{W_n} (V \boxtimes \mathbb{Q})\}_{n \in \mathbb{N}}$ is uniformly representation stable, stabilizing for $n \geq 2r$.*

Proof We first observe that

$$\text{Ind}_{H \times W_{n-r}}^{W_n} (V \boxtimes \mathbb{Q}) = \text{Ind}_{W_r \times W_{n-r}}^{W_n} ((\text{Ind}_H^{W_r} V) \boxtimes \mathbb{Q}),$$

and so it would suffice to prove the theorem for an irreducible representation $V(\lambda^+, \lambda^-)$ of W_r , and then apply the result to each irreducible factor of $\text{Ind}_H^{W_r} V$. Throughout this proof, for notational simplicity, we refer to irreducible representations by the full padded partitions.

By a lemma of Geck and Pfeiffer [12, Lemma 6.1.3],

$$\text{Ind}_{W_r \times W_{n-r}}^{W_n} (V(\lambda^+, \lambda^-) \boxtimes V(\mu^+, \mu^-)) = \bigoplus_{(v^+, v^-)} C_{\lambda^+, \mu^+}^{v^+} C_{\lambda^-, \mu^-}^{v^-} V(v^+, v^-)$$

where $C_{\lambda,\mu}^{\nu}$ denotes the Littlewood–Richardson coefficient. According to the classification of irreducible hyperoctahedral representations, the trivial representation of W_{n-r} is $V((n-r), (0))$, and so

$$\text{Ind}_{W_r \times W_{n-r}}^{W_n} (V(\lambda^+, \lambda^-) \boxtimes \mathbb{Q}) = \bigoplus_{(\nu^+, \nu^-)} C_{\lambda^+, (n-r)}^{\nu^+} C_{\lambda^-, (0)}^{\nu^-} V(\nu^+, \nu^-).$$

It is a standard result that

$$C_{\lambda^-, (0)}^{\nu^-} = \begin{cases} 1 & \text{if } \nu^- = \lambda^- \\ 0 & \text{if } \nu^- \neq \lambda^- \end{cases}.$$

(See, for example, Fulton [11] for the combinatorics of the Littlewood–Richardson coefficients.)

Then,

$$\begin{aligned} \text{Ind}_{W_r \times W_{n-r}}^{W_n} (V(\lambda^+, \lambda^-) \boxtimes \mathbb{Q}) &= \bigoplus_{\nu^+} C_{\lambda^+, (n-r)}^{\nu^+} V(\nu^+, \lambda^-) \\ &= \bigoplus_{\nu^+} V(\nu^+, \lambda^-) \end{aligned}$$

where the final sum is taken over all Young diagrams ν^+ that are constructed by adding $(n-r)$ boxes to the Young diagram λ^+ , with no two boxes added to the same column. When $n > 2r$, then $n-r > r$, so $(n-r)$ is necessarily greater than the number of columns of λ^+ , which implies that at least one box must be added to the first row. Thus, when $n > 2r$, the function ‘removing a box from the first row’ gives a bijection between the admissible partitions of n and those of $(n-1)$, a bijection which identifies a double partition $\nu[n]$ with $\nu[n-1]$. It follows that the decomposition stabilizes once $n \geq 2r$. □

Having proved Lemma 6.3, our objective now is to decompose the representations $H^k(P\Sigma_n; \mathbb{Q})$ into a sum of subrepresentations of the form $\text{Ind}_{H \times W_{n-r}}^{W_n} (V \boxtimes \mathbb{Q})$.

Proof of Theorem 6.1 By Lemma 4.3, $H^k(P\Sigma_n; \mathbb{Q})$ is the rational vector space over a basis indexed by the forests on $[n]$ with $(n-k)$ connected components. Let \mathcal{P} be a partition of $[n]$ into $(n-k)$ subsets; we call its subsets *blocks*. Define $H^{\mathcal{P}}(P\Sigma_n; \mathbb{Q}) \subseteq H^k(P\Sigma_n; \mathbb{Q})$ to be the \mathbb{Q} -span of the set of those forests whose connected components are the blocks of \mathcal{P} . Define $\bar{\mathcal{P}}$ to be the partition of n associated to \mathcal{P} , given by the block sizes.

Consider, for example, the case where $n = 6$, $k = 2$, and $\mathcal{P} = \{\{1\}, \{2, 4\}, \{3, 5\}, \{6\}\}$. Then $H^{\mathcal{P}}(P\Sigma_n; \mathbb{Q})$ is the four-dimensional subspace spanned by the basis elements shown in Figure 2, and $\bar{\mathcal{P}} = (2, 2, 1, 1)$.

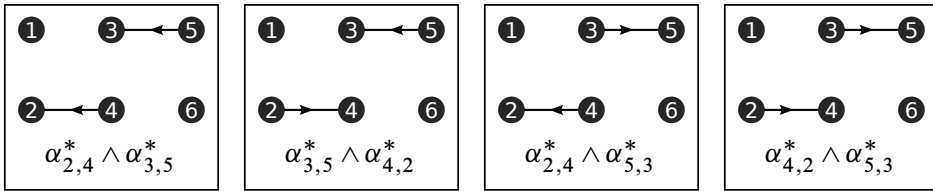


Figure 2. A basis for $H^{\{\{1\},\{2,4\},\{3,5\},\{6\}\}}(P\Sigma_6; \mathbb{Q})$

Given any n and any partition \mathcal{P} of $[n]$, we can compute the action of W_n on $H^{\mathcal{P}}(P\Sigma_n; \mathbb{Q})$. An element $\rho_i \in (\mathbb{Z}/2\mathbb{Z})^n \subseteq W_n$ either fixes or negates each basis element of $H^k(P\Sigma_n; \mathbb{Q})$, and so stabilizes $H^{\mathcal{P}}(P\Sigma_n; \mathbb{Q})$. An element $\tau_i \in S_n \subseteq W_n$ permutes the labels on the vertices, and so the S_n -orbit of $H^{\mathcal{P}}(P\Sigma_n; \mathbb{Q})$ are all subspaces $H^{\mathcal{P}'}(P\Sigma_n; \mathbb{Q})$ of $H^k(P\Sigma_n; \mathbb{Q})$ with $\bar{\mathcal{P}}' = \bar{\mathcal{P}}$. We therefore have a decomposition:

$$\begin{aligned} H^k(P\Sigma_n; \mathbb{Q}) &= \bigoplus_{\substack{\text{partitions } \mathcal{P} \text{ of } [n] \\ \text{with } (n-k) \text{ blocks}}} H^{\mathcal{P}}(P\Sigma_n) \\ &= \bigoplus_{\substack{\mu \vdash n \\ |\mu| = (n-k)}} \bigoplus_{\{\mathcal{P} \mid \bar{\mathcal{P}} = \mu\}} H^{\mathcal{P}}(P\Sigma_n) \\ &= \bigoplus_{\substack{\mu \vdash n \\ |\mu| = (n-k)}} \text{Ind}_{\text{Stab}(\mathcal{P}_\mu)}^{W_n} H^{\mathcal{P}_\mu}(P\Sigma_n) \end{aligned}$$

for some fixed \mathcal{P}_μ with $\bar{\mathcal{P}}_\mu = \mu$.

For any μ , the stabilizer $\text{Stab}(\mathcal{P}_\mu) \subseteq W_n$ is generated by all elements ρ_i , by the permutations that permute the numbers within a block of \mathcal{P}_μ , and by the permutations that interchange two blocks of \mathcal{P}_μ of the same size. If \mathcal{P}_μ has $(n-r)$ blocks of size one, then we may decompose the stabilizer as $\text{Stab}(\mathcal{P}_\mu) \cong H \times W_{n-r}$, where H is a subgroup of W_r , and W_{n-r} acts by permuting the singleton blocks. Since the singleton blocks of \mathcal{P}_μ are those numbers that do not appear as an index in any basis element of $H^{\mathcal{P}_\mu}(P\Sigma_n; \mathbb{Q})$, this direct factor W_{n-r} acts trivially on $H^{\mathcal{P}_\mu}(P\Sigma_n; \mathbb{Q})$.

We note moreover that, whenever $k > 0$, not all vertices can be singletons, and so we satisfy the hypothesis of Lemma 6.3 that $r \geq 1$.

The maps

$$\phi_{n-1}: H^*(P\Sigma_{n-1}; \mathbb{Q}) \rightarrow H^*(P\Sigma_n; \mathbb{Q})$$

map $\alpha_{i_1, j_1}^* \wedge \dots \wedge \alpha_{i_k, j_k}^* \in H^k(P\Sigma_{n-1}; \mathbb{Q})$ to $\alpha_{i_1, j_1}^* \wedge \dots \wedge \alpha_{i_k, j_k}^* \in H^k(P\Sigma_n; \mathbb{Q})$, and accordingly, they map a rooted forest of $((n-1)-k)$ trees on $[n-1]$ to the rooted

forest of $(n - k)$ trees on $[n]$ obtained by adding the singleton vertex n . For each partition \mathcal{P} of $[n - 1]$, the map ϕ_{n-1} induces an isomorphism of W_{n-1} -representations

$$H^{\mathcal{P}}(P\Sigma_{n-1}; \mathbb{Q}) \cong H^{\mathcal{P} \cup \{\{n\}\}}(P\Sigma_n; \mathbb{Q}).$$

When $n > 2k$, any partition \mathcal{P}^+ of $[n]$ into $(n - k)$ blocks must have at least one singleton block. Thus $H^{\mathcal{P}^+}(P\Sigma_n; \mathbb{Q})$ is in the W_n -orbit of $(\mathcal{P} \cup \{\{n\}\})$ for some partition \mathcal{P} of $[n - 1]$. It follows that

$$\begin{aligned} H^k(P\Sigma_n; \mathbb{Q}) &= \bigoplus_{\substack{\mu \vdash (n) \\ |\mu| = (n-k)}} \text{Ind}_{\text{Stab}(\mathcal{P}_\mu)}^{W_n} H^{\mathcal{P}_\mu}(P\Sigma_n) \\ &= \bigoplus_{\substack{\nu \vdash (n-1) \\ |\nu| = (n-1-k)}} \text{Ind}_{\text{Stab}(\mathcal{P}_\nu \cup \{\{n\}\})}^{W_{n-1}} H^{\mathcal{P}_\nu \cup \{\{n\}\}}(P\Sigma_n) \end{aligned}$$

for some choices of \mathcal{P}_μ and \mathcal{P}_ν with $\bar{\mathcal{P}}_\mu = \mu$, $\bar{\mathcal{P}}_\nu = \nu$. The number of direct summands in this decomposition therefore stabilizes once $n \geq 2k$.

For given ν , we may choose \mathcal{P}_ν so that its $((n-1)-r)$ singleton blocks are

$$\{\{r+1\}, \{r+2\}, \dots, \{n-1\}\},$$

and denote the complement, a partition of $[r]$, by \mathcal{P}^c . Since $\text{Stab}(\mathcal{P}_\nu \cup \{\{n\}\}) \cong H \times W_{n-r}$ as above, we can iteratively write

$$\begin{aligned} \text{Ind}_{\text{Stab}(\mathcal{P}_\nu \cup \{\{n\}\})}^{W_n} H^{\mathcal{P}_\nu \cup \{\{n\}\}}(P\Sigma_n) &= \text{Ind}_{H \times W_{n-r}}^{W_n} H^{\mathcal{P}_\nu \cup \{\{n\}\}}(P\Sigma_n) \boxtimes \mathbb{Q} \\ &= \text{Ind}_{H \times W_{n-r}}^{W_n} H^{\mathcal{P}_\nu}(P\Sigma_{n-1}) \boxtimes \mathbb{Q} \\ &= \dots \\ &= \text{Ind}_{H \times W_{n-r}}^{W_n} H^{\mathcal{P}^c}(P\Sigma_r) \boxtimes \mathbb{Q}. \end{aligned}$$

We may conclude that the sequence $\{H^k(P\Sigma_n; \mathbb{Q})\}_{n \geq 2k}$ of W_n -representations decomposes into a finite sum of sequences of the form described in Lemma 6.3.

Among all partitions of $[n]$ into $(n - k)$ blocks, the number of singleton blocks is minimal in the partition with k blocks of two elements, and $(n - 2k)$ singleton blocks. This implies that the maximal value of r is $2k$, and so we can ensure that $n \geq 2r$ whenever $n \geq 4k$. Theorem 6.1 now follows from Lemma 6.3. □

We notice that we may use this same method of proof to prove representation stability when we restrict the action of W_n to its subgroup S_n .

Theorem 6.4 For each fixed $k \geq 0$, sequence of groups

$$\{H^k(P\Sigma_n; \mathbb{Q})\}_{n \in \mathbb{N}},$$

considered as S_n -representations, is uniformly representation stable with respect to the maps

$$\phi_n: H^k(P\Sigma_n; \mathbb{Q}) \rightarrow H^k(P\Sigma_{n+1}; \mathbb{Q})$$

induced by the ‘forgetful’ map $P\Sigma_{n+1} \rightarrow P\Sigma_n$. The sequence stabilizes once $n \geq 4k$.

Proof Since this proof is so similar to that of Theorem 6.1, we merely give a sketch. We may use the same decomposition of the cohomology groups as above, noting that

$$H^k(P\Sigma_n; \mathbb{Q}) = \bigoplus_{\substack{\mu \vdash n \\ |\mu| = (n-k)}} \text{Ind}_{\text{Stab}(\mathcal{P}_\mu)}^{S_n} H^{\mathcal{P}_\mu}(P\Sigma_n)$$

for some \mathcal{P}_μ with $\bar{\mathcal{P}}_\mu = \mu$. Here, if \mathcal{P}_μ has m_j blocks of size j , then its stabilizer $\text{Stab}(\mathcal{P}_\mu) \subseteq S_n$ decomposes as

$$\text{Stab}(\mathcal{P}_\mu) = \bigoplus_{j=1}^n S_j \wr S_{m_j}$$

where S_j permutes the numbers within a block of size j , and S_{m_j} permutes the m_j blocks. In particular, if \mathcal{P}_μ has $(n-r)$ blocks of size one, then we can write $\text{Stab}(\mathcal{P}_\mu) \cong H \times S_{n-r}$, where S_{n-r} permutes the singleton blocks. Then, using Lemma 6.2 as above, we may conclude the theorem. \square

These results raise the question of whether it is true in general that a uniformly representation stable sequence of W_n -representations will restrict to a uniformly representation stable sequence of S_n -representations. This is indeed the case, and will be shown in a forthcoming paper. However, Theorem 6.4 gives a better stable range than the general case.

7 The cohomology of the string motion group

Consider a short exact sequence of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

with finite quotient Q . Recall that, by a transfer argument, the restriction map $H^k(G; \mathbb{Q}) \rightarrow H^k(N; \mathbb{Q})$ identifies the cohomology of G with the ring of invariants

$H^k(N; \mathbb{Q})^Q$ under the natural action of Q . (See, for example, Brown [4, Chapter 3.9].) It therefore follows from the exact sequence

$$1 \longrightarrow P\Sigma_n \longrightarrow \Sigma_n \longrightarrow W_n \longrightarrow 1$$

that the cohomology of $H^k(\Sigma_n; \mathbb{Q})$ is isomorphic to the isotypic component of trivial W_n -representations $V((0), (0))_n$ in $H^k(P\Sigma_n; \mathbb{Q})$.

The representation stability of $H^k(P\Sigma_n; \mathbb{Q})$ implies, in particular, that the multiplicity of the trivial representation $V((0), (0))_n$ stabilizes for n sufficiently large, and therefore that the dimension of $H^k(\Sigma_n; \mathbb{Q})$ is eventually constant. In fact, even the integral homology groups of Σ_n are eventually independent of n , a result proved by Hatcher and Wahl [15, Corollary 1.2].

Hatcher and Wahl realize Σ_n as follows: Let $P := S^1 \times D^2$ be the product of a 1-sphere S^1 and the 2-disk D^2 . Let $N_n := D^3 \# P \# \dots \# P$ be the connected sum of the 3-disk D^3 with n copies of P . Then Σ_n is the quotient of the mapping class group $\Gamma(N_n, \partial D^3) := \pi_0 \text{Diff}(N_n \text{ rel } \partial D^3)$ by the subgroup generated twists along spheres and disks with boundary in $\partial N_n - \partial D^3$. Homological stability for Σ_n is a consequence of their stability theorems for mapping class groups of 3-manifolds.

From our description of $H^k(P\Sigma_n; \mathbb{Q})$, we can say more: for $k \geq 1$, the trivial representation does not occur, and so the rational cohomology groups of Σ_n vanish.

Theorem 7.1 *For $k \geq 1$, the rational cohomology groups $H^k(\Sigma_n; \mathbb{Q}) = 0$. In particular, the integral homology and cohomology groups of Σ_n are finite.*

Proof For a G -representation V , the averaging map

$$\begin{aligned} \varphi: V &\rightarrow V \\ u &\mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot u \end{aligned}$$

gives a linear projection of V onto the subspace fixed by G . Thus, to prove that $H^k(P\Sigma_n; \mathbb{Q})$ contains no copies of the trivial representation, it suffices to show that $\varphi(v) = 0$ for every basis element v .

Suppose that, for some fixed basis element v , there were an involution $\omega \in W_n$ such that $\omega \cdot v = -v$. Then by partitioning the sum by left cosets of $H = \{\omega, 0\}$, we find that $\varphi(v)$ vanishes:

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in W_n} g \cdot v = \frac{1}{|G|} \sum_{gH \subseteq W_n} (g \cdot v + g\omega \cdot v) = \frac{1}{|G|} \sum_{gH \subseteq W_n} (g \cdot (v - v)) = 0$$

For any basis element $v = \alpha_{i_1, j_1}^* \wedge \alpha_{i_2, j_2}^* \wedge \dots \wedge \alpha_{i_k, j_k}^*$ of $H^k(P\Sigma_n; \mathbb{Q})$, we will find such an involution. Suppose that there is some number $j \in [n]$ that appears as a second index for an odd number of exterior factors of v . Then $\rho_j \cdot v = -v$, which implies that $\varphi(v) = 0$. Next, suppose that there is no such number j . This means that, if we take the rooted forest associated to v as described in Section 4, every vertex has even outdegree. Choosing any tree in the forest with at least one edge, we can find two terminal vertices p and q that have the same direct predecessor r . This implies v includes the factors $\alpha_{p,r}^*$ and $\alpha_{q,r}^*$, but that the numbers p and q do not appear as any other index. Thus, the transposition $\omega = (p\ q) \in S_n$ induces an odd permutation on the factors of the exterior product v , and $\varphi(v) = 0$. This completes the proof that $H^k(\Sigma_n; \mathbb{Q}) = 0$ for $k \geq 1$.

The work of Collins [7] (later Jensen and Wahl [18]) implies that $P\Sigma_n$ is of type FP_∞ . Since $[\Sigma_n : P\Sigma_n] = 2^n n! < \infty$, it follows that Σ_n is of type FP_∞ ; a standard reference is Brown [4, Chapter 8]. The integral homology and cohomology groups are finitely generated and torsion; they are finite groups. \square

An approach to computing these groups is developed in Griffin [14].

8 The cohomology of the braid-permutation group

This section originated in conversations with Allen Hatcher. It also owes to insights from Tom Church.

Consider the group $\Sigma_n^+ = \langle \tau_i, \alpha_{i,j} \rangle$ of string motions that preserve the orientations of the circles. This group has been called the *braid-permutation group*, since it is generated by the classical braid group $B_n = \langle \alpha_{i,i+1} \tau_i \rangle$ and the symmetric group $S_n = \langle \tau_i \rangle$. Fenn, Rimanyi and Rourke [9; 10] describe the group as a generalized braid group in which some crossings are “welded”. They give a presentation for the group, and also show that it is isomorphic to the automorphism group of the so-called *free quandle*. In [22], Vershinin studies the algebraic topology of the braid-permutation group. Applications of the group to mathematical physics are described by Baez, Crans and Wise in [1]. Brendle and Hatcher [3] give a geometric description of Σ_n^+ as the *untwisted ring group*, the fundamental group of the $4n$ -manifold of configurations of n Euclidean circles in \mathbb{R}^3 lying parallel to a fixed plane.

The symmetries of the groups $H^k(P\Sigma_n; \mathbb{Q})$ encode the structure of the cohomology of Σ_n^+ . We see from the short exact sequence

$$1 \longrightarrow P\Sigma_n \longrightarrow \Sigma_n^+ \longrightarrow S_n \longrightarrow 1$$

that the cohomology group $H^k(\Sigma_n^+; \mathbb{Q})$ is the group of invariants $H^k(P\Sigma_n; \mathbb{Q})^{S_n}$. The representation stability of the trivial S_n -representation $V(0)_n$ in Theorem 6.4 therefore implies classical rational stability for $H^k(\Sigma_n^+; \mathbb{Q})$.

Corollary 8.1 *The braid-permutation group Σ_n^+ of orientation-preserving string motions is rationally cohomologically stable in the classical sense. For each $k \geq 0$, $H^k(\Sigma_n^+; \mathbb{Q}) \cong H^k(\Sigma_{n+1}^+; \mathbb{Q})$ once $n \geq 4k$.*

Remark In contrast to the case with the $P\Sigma_n$, we cannot construct a map $\Sigma_{n+1}^+ \rightarrow \Sigma_n^+$ by “forgetting” a circle, as there is no consistent way to chose among $(n + 1)$ unlabeled circles. Since \mathbb{R}^3 is open, however, there are maps $\Sigma_n^+ \rightarrow \Sigma_{n+1}^+$ that may be constructed by ‘introducing’ a circle far from the origin of \mathbb{R}^3 . These maps induce the isomorphisms in Corollary 8.1.

For low degree k , we can use the symmetries of $H^k(P\Sigma_n; \mathbb{Q})$ to directly compute the S_n -invariant subspace:

For $n \geq 2$, the first cohomology $H^1(\Sigma_n^+; \mathbb{Q}) = \mathbb{Q}$ is one-dimensional, corresponding to the trivial representation spanned by

$$\sum_{i \neq j \in [n]} \alpha_{i,j}^* \in H^1(P\Sigma_n; \mathbb{Q}).$$

For $n \geq 3$, the second cohomology $H^2(\Sigma_n^+; \mathbb{Q}) = \mathbb{Q}$ is also one-dimensional, corresponding to the trivial representation spanned by

$$\sum_{i,j,k \in [n] \text{ distinct}} \alpha_{i,j}^* \wedge \alpha_{j,k}^* \in H^2(P\Sigma_n; \mathbb{Q}).$$

For $n \geq 5$, the cohomology $H^3(\Sigma_n^+; \mathbb{Q}) = \mathbb{Q}^3$ corresponding to the trivial representations spanned by

$$\begin{aligned} &\sum_{i,j,k,l \in [n] \text{ distinct}} \alpha_{i,j}^* \wedge \alpha_{j,k}^* \wedge \alpha_{k,l}^* \in H^3(P\Sigma_n; \mathbb{Q}), \\ &\sum_{i,j,k,l \in [n] \text{ distinct}} \alpha_{i,j}^* \wedge \alpha_{j,k}^* \wedge \alpha_{l,k}^* \in H^3(P\Sigma_n; \mathbb{Q}), \text{ and} \\ &\sum_{i,j,k,l,m \in [n] \text{ distinct}} \alpha_{i,j}^* \wedge \alpha_{j,k}^* \wedge \alpha_{l,m}^* \in H^3(P\Sigma_n; \mathbb{Q}). \end{aligned}$$

These three vectors are obtained by summing over the S_n -orbits of each of the three basis elements shown (respectively) in Figure 3.

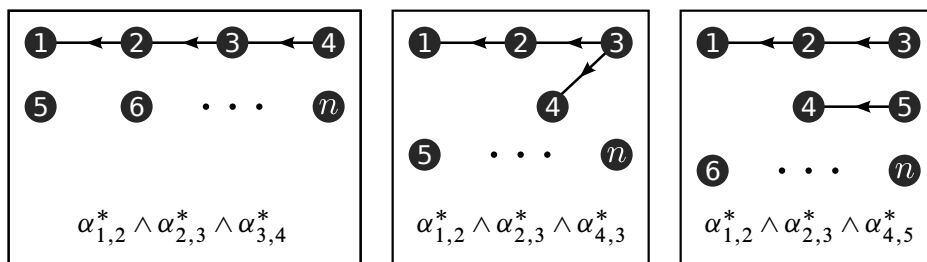


Figure 3. Elements of $H^3(P\Sigma_n; \mathbb{Q})$ corresponding to a basis for $H^3(\Sigma_n^+; \mathbb{Q})$

In general, each of these trivial representations is spanned by the symmetrization of some basis element $v \in H^k(P\Sigma_n; \mathbb{Q})$. To ensure that this symmetrization does not vanish, we need $v \in H^k(P\Sigma_n; \mathbb{Q})$ with the property that no element of S_n negates v . Taking the basis for $H^k(P\Sigma_n; \mathbb{Q})$ described in Lemma 4.2 and Lemma 4.3, the symmetries of a basis element may be seen from the associated rooted forests. Specifically, the trivial representations will correspond to forests such that any permutation of the vertices preserving the labelled, rooted graph structure will induce an even permutation of the factors of the associated basis element. Such forests exist for all $1 \leq k \leq (n-1)$; consider, for example, the simple path $1 \leftarrow 2 \leftarrow \dots \leftarrow k \leftarrow (k+1)$ along with isolated vertices $(k+2)$ through n . Thus, for all n , $H^k(\Sigma_n^+; \mathbb{Q})$ is nontrivial in all dimensions $0 \leq k \leq (n-1)$.

In [14, Example 6.2], Griffin gives a method of computing the integral homology of Σ_n^+ from the structure of these rooted forests.

In [22], Vershinin proves that the plus construction of the classifying space of the infinite braid-permutation group is homotopic to a component of $\Omega^\infty S^\infty \times S^1 \times Y$, where Y is an infinite loop space. A better understanding of the combinatorics of the rooted forests could illuminate the cohomological structure of this space.

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