*Algebraic & Geometric Topology 12 (2012) 1081–1098* **1081**

# The link concordance invariant from Lee homology

JOHN PARDON

We use the knot homology of Khovanov and Lee to construct link concordance invariants generalizing the Rasmussen $s$–invariant of knots. The relevant invariant for a link is a filtration on a vector space of dimension $2^{|L|}$. The basic properties of the $s$–invariant all extend to the case of links; in particular, any orientable cobordism $\Sigma$ between links induces a map between their corresponding vector spaces which is filtered of degree $\chi(\Sigma)$. A corollary of this construction is that any component-preserving orientable cobordism from a Kh–thin link to a link split into $k$ components must have genus at least $\lfloor k/2 \rfloor$. In particular, no quasi-alternating link is concordant to a split link.

57M25, 57M27, 57Q60

## 1 Introduction

Using Lee's modification [18] of Khovanov homology [14], Rasmussen [26] introduced for every knot $K$ an even integer valued invariant, known as the $s$–invariant. It shares some of the basic properties of the classical knot signature; in particular it is a homomorphism from the group of smooth concordance classes of knots to $2\mathbb{Z}$ and gives a lower bound for twice the smooth slice genus (though the signature also does this in the *topological* category, whereas the $s$–invariant does not). The definition of the $s$–invariant is purely combinatorial, and, like many other knot invariants coming out of quantum algebra, it so far lacks any intrinsic geometric definition. One of the main reasons for interest in this invariant is that it is by definition algorithmically computable (though some cleverness is needed to do large calculations quickly; cf Bar-Natan [3] and Freedman, Gompf, Morrison and Walker [8]) and is one of the few tools known to give useful lower bounds on the smooth slice genus of knots. In particular, Rasmussen [26] showed via direct calculation that $s(T_{p,q}) = (p-1)(q-1)$, thus proving the Milnor conjecture that $g_4(T_{p,q}) = \frac{1}{2}(p-1)(q-1)$, a hard theorem of Kronheimer and Mrowka [15; 17; 16] proved (twice) using gauge theory.

In this paper, we consider the natural generalization of the $s$–invariant to a concordance invariant of links. Everything we do will be in the smooth category. Since the $s$–invariant can detect the deep differences between the smooth and topological

*Published: 7 May 2012* DOI: 10.2140/agt.2012.12.1081

categories in four dimensions, this restriction is in fact necessary for this theory. In particular, knot and link concordance is meant in the smooth sense.

Denote the Khovanov–Lee homology groups of a link by $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$. Lee [18] showed that $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$ is surprisingly simple: there is an isomorphism $\bigoplus_{\text{orientations of } L}\mathbb{Q} \xrightarrow{\sim} \mathrm{Kh}^*_{\mathrm{Lee}}(L)$. We denote the former group by $\mathbb{O}(L)$, and in Section 4, we will define it as a functor (we will specify the maps associated to cobordisms). Kevin Walker [27] informs the author that (with suitable choice of Lee deformation parameter) there is an equivalence of functors between $\mathbb{O}$ and $\mathrm{Kh}^*_{\mathrm{Lee}}$ (Rasmussen [26; 25] has proved a sort of approximate equivalence of functors). The natural generalization of the $s$–invariant is thus the pullback of the $s$–filtration on $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$ to a filtration on $\mathbb{O}(L)$. To get a numerical invariant, we can take the following (which is perhaps slightly coarser).

**Definition 1.1** For an oriented link $L \subseteq \mathbb{R}^3$, we associate a function $d_L\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}_{\geq 0}$ so that the values $d_L(h,s)$ give the dimensions of the associated graded pieces $(\mathrm{Kh}^h_{\mathrm{Lee}}(L))^s/(\mathrm{Kh}^h_{\mathrm{Lee}}(L))^{s+1}$ (where $(\mathrm{Kh}^*_{\mathrm{Lee}}(L))^s$ denotes the subspace of elements of filtration level $\geq s$).

For a knot $K$, it is a theorem of Rasmussen [26] that $d_K(0, s(K)\pm 1) = 1$ and $d_K$ is otherwise zero (this being the defining property of the $s$–invariant). For a link $L$, the vector space $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$ has dimension $2^{|L|}$, and, as one might expect, the support of the function $d_L$ can be much more complicated as we shall see in a few examples.

**Theorem 1.2** *Let $L$ be a link with orientation $\mathfrak{o}$. The invariant $d_L\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}_{\geq 0}$ satisfies the following basic properties:*

(1) $\sum_{s\equiv |L|+k \bmod 4} d_L(h,s)$ *is zero if $k$ is odd, and if $k$ is even it equals one half the number of orientations $\mathfrak{o}_1$ of $L$ such that $\mathrm{lk}(\mathfrak{o}_1)-\mathrm{lk}(\mathfrak{o}) = -h$. Here $\mathrm{lk}(\mathfrak{o}) = \sum_{i<j}\mathrm{lk}(L_i^{\mathfrak{o}}, L_j^{\mathfrak{o}})$ (sum over the components of $L$).*

(2) $d_{L_1\sqcup L_2} = d_{L_1} * d_{L_2}$ *(convolution).*

(3) $d_{\bar{L}}(h,s) = d_L(-h,-s)$.

(4) *If $\Sigma$ is a component-preserving orientable cobordism between $L_1$ and $L_2$ (ie $H_0(L_i) \xrightarrow{\sim} H_0(\Sigma)$), then $\sum_{s\geq a} d_{L_1}(h,s) \leq \sum_{s\geq a+\chi(\Sigma)} d_{L_2}(h,s)$ for all $h\in\mathbb{Z}$.*

(5) $d_L$ *is a link concordance invariant.*

(6) $d_{(L,\mathfrak{o})}(h+\mathrm{lk}(\mathfrak{o}), s+3\,\mathrm{lk}(\mathfrak{o}))$ *is independent of orientation $\mathfrak{o}$.*

For links with a large number of components, it is reasonable to expect that the invariant $d_L$ will be a strong invariant of link concordance. As one sees in the theorem above, the invariant $d_L$ is best suited for studying cobordisms which do not merge components of $L$. In general, if one wants to derive information about a given orientable cobordism, then the relevant object is the $s$–filtration restricted to the subspace of $\mathbb{O}(L)$ generated by those orientations extending to orientations of the cobordism. The larger this subspace, the more likely the invariant is to be useful.

Beliakova and Wehrli have defined an integer $s(L,\mathfrak{o})$ for a link with an orientation [5]. This corresponds to the $s$–filtration restricted to the 2–dimensional subspace of $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$ generated by that orientation and its reverse. Just like for knots, one shows that on this subspace, the filtration is supported in two levels $s \pm 1$, and this defines $s(L,\mathfrak{o})$. This invariant is best suited for studying oriented cobordisms which are allowed to merge components of $L$. Examples show that the function $\mathfrak{o} \mapsto s(L,\mathfrak{o})$ is a weaker invariant that $d_L$. One expects that $d_L$ is a weaker invariant that the filtration on $\mathbb{O}(L)$ but we don't have any examples to prove this at present (mainly because $d_L$ is often easy to derive from $\mathrm{Kh}^*$—which there exist programs to compute—whereas the $s$–filtration on $\mathbb{O}(L)$ is not). We discuss examples in Section 5 and at the end of Section 4.1.

In Section 3, we use the invariants $d_L$ to derive the following corollary, which appears to be new.

**Corollary 1.3** *A component-preserving orientable cobordism between a* Kh*–thin link and a link split into $m$ components must have genus at least $\lfloor m/2 \rfloor$. In particular,* Kh*–thin links (in particular quasi-alternating links; see Definition 3.5) are not concordant to split links.*

It is known (via properties of the Alexander module) that alternating links are not concordant to split links; see Kawauchi [13]. It would be interesting to try to prove Corollary 1.3 (say, restricted to quasi-alternating links) using the Alexander module.[1]

This corollary is interesting because the $s$–invariant for alternating *knots* is equal to the knot signature, and thus gives no new information (the inequality $g_4^{\mathrm{top}}(K) \geq \frac{1}{2}\,|\sigma(K)|$ is classical; see Murasugi [21, page 416, Theorem 9.1]). It is interesting to note that Khovanov homology has a reputation for being easy to compute (at least, compared to gauge theoretic invariants which give results similar to the Milnor conjecture), but hard to use to prove general theorems, since its structure in general is still poorly understood.

[1] We recently learned of a preprint [9] by Stefan Friedl and Mark Powell which apparently presents such a proof.

Thus the above corollary is interesting in that it is a *general* statement which doesn't intrinsically involve Khovanov homology (at least, if one restricts to quasi-alternating links).

There have recently been efforts (see Freedman, Gompf, Morrison and Walker [8]) to prove that some specific proposed counterexamples to the smooth 4–dimensional Poincaré conjecture are in fact exotic by proving some specific links are not slice in the standard $\mathbb{B}^4$ (links which, by virtue of coming from Kirby diagrams for the proposed counterexample, are by definition slice in the proposed exotic $\mathbb{B}^4$). By slice, we mean *strongly slice*, ie bounding a disjoint union of disks in $\mathbb{B}^4$. Since these are usually multicomponent links, it may be helpful to compute the entire filtration on $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$: for a link with many components, this a priori may be a much stronger invariant than the set of $s$–invariant values for some associated knots which are implied to be slice if the link is slice (computing these $s$–invariant values was the strategy employed in [8]). We should, however, also note that, in accordance with the growing relations between Khovanov homology and gauge theory, some would conjecture that the $s$–filtration should be invariant under concordance of links in any *homotopy* $\mathbb{R}^3 \times [0,1]$, and thus would not imply in any straightforward manner that any homotopy $\mathbb{B}^4$ is exotic.

One thinks that an invariant of links similar to $d_L$ could be defined using the Link Floer Homology of Ozsvath–Szabó [22; 24] as an appropriate generalization of the $\tau$–invariant. One would expect this invariant to satisfy similar properties as the $s$–filtration on $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$. It is perhaps interesting to note that the vector space $\mathbb{O}(L)$ appears in the Link Floer Homology theory in the guise of $\wedge^* H_1(\#^{|L|}\mathbb{S}^1 \times \mathbb{S}^2)$ (once we take the union of our link with the unknot).

**Acknowledgements** This paper represents part of the author's Senior Thesis at Princeton University, advised by Zoltán Szabó. I thank him for lots of generous time spent meeting and discussing mathematics. John Baldwin, the second reader for my thesis, also made some useful comments. I thank the referee for a very close reading of this paper and for the many resulting corrections and clarifications, as well as contributions concerning the examples in Section 5.

## 2 Khovanov–Lee homology

In this section, we give a quick review of Lee's deformation [18] of Khovanov homology [14] aimed at our intended application. For a good introduction to Khovanov homology; see Bar-Natan's articles [1; 2]. The maps for cobordisms were first proved consistent by Jacobsson [12].

To be completely explicit, we define Khovanov–Lee homology via Khovanov's chain complex using the following Frobenius algebra $V$:

$$\begin{aligned} V &= \mathbb{Q}\mathbf{v}_- \oplus \mathbb{Q}\mathbf{v}_+, & \iota(1) &= \mathbf{v}_+, \\ \epsilon(\mathbf{v}_+) &= 0, & m(\mathbf{v}_- \otimes \mathbf{v}_-) &= a\mathbf{v}_+, \\ \epsilon(\mathbf{v}_-) &= 1, & m(\mathbf{v}_- \otimes \mathbf{v}_+) &= \mathbf{v}_-, \\ \Delta(\mathbf{v}_+) &= \mathbf{v}_- \otimes \mathbf{v}_+ + \mathbf{v}_- \otimes \mathbf{v}_+, & m(\mathbf{v}_+ \otimes \mathbf{v}_-) &= \mathbf{v}_-, \\ \Delta(\mathbf{v}_-) &= \mathbf{v}_- \otimes \mathbf{v}_- + a\mathbf{v}_+ \otimes \mathbf{v}_+, & m(\mathbf{v}_+ \otimes \mathbf{v}_+) &= \mathbf{v}_+. \end{aligned} \tag{2-1}$$

Setting $a=0$ yields Khovanov homology ($a$ is the Lee deformation parameter). If $a \neq 0$, then $(m, \iota, \Delta, \epsilon)$ admit simple descriptions in terms of the basis $\mathbf{x}_\pm = \mathbf{v}_- \pm \sqrt{a}\mathbf{v}_+$, and this implies that the resulting homology is essentially isomorphic to Lee homology. The only difference between different values of $a \neq 0$ is that the maps associated to a cobordism $\Sigma$ carry a factor of $(2\sqrt{a})^{-\chi(\Sigma)/2}$. This makes Lee's original choice of $a = 1$ slightly inconvenient, so for the remainder of the paper we set $a = \frac{1}{4}$ (as suggested by Walker [27]).

**Theorem 2.1** *For every oriented link $L$, there is an associated $\mathbb{Z}$–graded vector space $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$ over $\mathbb{Q}$ (the grading $*$ is called the* homological *grading). Furthermore, each $\mathrm{Kh}^h_{\mathrm{Lee}}(L)$ carries a descending filtration, called the $s$–filtration. Every oriented cobordism $\Sigma \subseteq \mathbb{R}^3 \times [0,1]$ from $L_1$ to $L_2$ induces a homomorphism $F_\Sigma\colon \mathrm{Kh}^*_{\mathrm{Lee}}(L_1) \to \mathrm{Kh}^*_{\mathrm{Lee}}(L_2)$ (defined up to $\pm 1$) which respects the homological grading, and which is filtered of degree $\chi(\Sigma)$. We have the following additional properties:*

1. $\mathrm{Kh}^h_{\mathrm{Lee}}(L)$ *carries an absolute $\mathbb{Z}/4\mathbb{Z}$ grading which is supported in gradings $\equiv |L| \bmod 2$, and these two pieces have equal dimensions. The $s$–filtration breaks up as a filtration on each of the pieces, and the $s$–filtration on the degree $k \in \mathbb{Z}/4\mathbb{Z}$ piece is supported on integers $s \equiv k \bmod 4$.*
2. $\mathrm{Kh}^*_{\mathrm{Lee}}(L_1 \sqcup L_2) = \mathrm{Kh}^*_{\mathrm{Lee}}(L_1) \otimes \mathrm{Kh}^*_{\mathrm{Lee}}(L_2)$ *(naturally), and this is an isomorphism of the homological grading and the $s$–filtration.*
3. $\mathrm{Kh}^*_{\mathrm{Lee}}(\bar{L})$ *is naturally the dual of* $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$.
4. (Lee [18]) $\dim \mathrm{Kh}^*_{\mathrm{Lee}}(L) = 2^{|L|}$. *In fact,* $\dim \mathrm{Kh}^h_{\mathrm{Lee}}(L)$ *is the number of orientations $\mathfrak{o}$ of $L$ such that $\mathrm{lk}(\mathfrak{o}) - \mathrm{lk}(\mathfrak{o}_1) = -h$, where $\mathfrak{o}_1$ is the given orientation of $L$, and $\mathrm{lk}(\mathfrak{o}) = \sum_{i<j} \mathrm{lk}(L_i^{\mathfrak{o}}, L_j^{\mathfrak{o}})$ (sum over the components of $L$).*
5. $\mathrm{Kh}^*_{\mathrm{Lee}}$ *is a functor from the appropriately defined category of links and cobordisms* (see Clark, Morrison and Walker [7]).
6. $\mathrm{Kh}^{*+\mathrm{lk}(\mathfrak{o})}_{\mathrm{Lee}}(L, \mathfrak{o})_{(-3\,\mathrm{lk}(\mathfrak{o}))}$ *is independent of orientation $\mathfrak{o}$ (where $_{(q_0)}$ means an upwards shift of the $s$–filtration by $q_0$).*

**Remark 2.2** Clark, Morrison, and Walker [7] and Caprau [6] have shown how to define Khovanov–Lee homology (with indeterminate $a$) so that the maps associated to cobordisms no longer have a sign ambiguity. This requires adjoining $i = \sqrt{-1}$ to the coefficient ring.

**Definition 2.3** For an oriented link $L \subseteq \mathbb{R}^3$, we associate a function $d_L\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_{\geq 0}$ so that the values $d_L(h,s)$ give the dimensions of the associated graded pieces $(\mathrm{Kh}^h_{\mathrm{Lee}}(L))^s/(\mathrm{Kh}^h_{\mathrm{Lee}}(L))^{s+1}$ (where $(\mathrm{Kh}^*_{\mathrm{Lee}}(L))^s$ denotes the subspace of elements of filtration level $\geq s$).

Theorem 1.2 (the basic properties of $d_L$) follows directly from the basic properties of $\mathrm{Kh}^*_{\mathrm{Lee}}$ listed in Theorem 2.1.

# 3 Applications to link concordance

**Definition 3.1** A cobordism $\Sigma$ between two links $L_1$ and $L_2$ is said to be *component-preserving* if and only if $H_0(L_1) \xrightarrow{\sim} H_0(\Sigma) \xleftarrow{\sim} H_0(L_2)$. Note that a component-preserving orientable cobordism of genus $0$ is exactly a link concordance.

**Remark 3.2** One is perhaps also interested in relaxing the restrictive notion of *component*-preserving cobordism to *color*-preserving cobordism, where multiple components of the link could have the same color. Now certainly this case is also easily handled using the invariant $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$. The necessary data is a coloring of the link, and a choice of relative orientation on each colored component (by relative orientation, we mean an orientation up to overall reversal). Then the relevant invariant is just the restriction of the $s$–filtration to the subspace of $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$ generated by all orientations agreeing with the given relative orientations on each colored component.

**Lemma 3.3** *The map $F_\Sigma\colon \mathrm{Kh}^*_{\mathrm{Lee}}(L_1) \to \mathrm{Kh}^*_{\mathrm{Lee}}(L_2)$ induced by a component-preserving orientable cobordism is an isomorphism of vector spaces.*

**Proof** This follows from Rasmussen [26, page 434, Proposition 4.1]. □

**Definition 3.4** A link $L$ is said to be Kh–thin if and only if $\mathrm{Kh}^*(L)$ is supported on exactly two diagonals of the form $q = q_0 + 2h \pm 1$ ($h = *$ is the homological grading).

The spectral sequence from $\mathrm{Kh}^*(L)$ to $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$ implies that for a Kh–thin link, the support of $d_L$ is contained in the same two diagonals $s = q_0 + 2h \pm 1$.

**Definition 3.5** Oszváth and Szabó [23] define the set of *quasi-alternating* links to be the set of links generated by the unknot using the following skein operation: if $L_0$ and $L_1$ are quasi-alternating and $\det L_\infty = \det L_0 + \det L_1$, then $L_\infty$ is also quasi-alternating (where $L_0, L_1, L_\infty$ are given as in Figure 1).

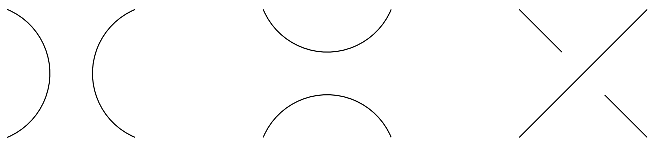

Figure 1. Local pictures of $L_0$, $L_1$, $L_\infty$

It is standard that all nonsplit alternating links are quasi-alternating. Quasi-alternating links are known to be both Kh–thin and $\widehat{\mathrm{HFK}}$–thin by Manolescu and Ozsváth [20], though Greene [10] has shown that there are non-quasi-alternating links that are both Kh–thin and $\widehat{\mathrm{HFK}}$–thin.

**Proposition 3.6** (Corollary 1.3) *Let $L$ be a* Kh*–thin link, and suppose $\Sigma$ is a component-preserving orientable cobordism between $L$ and $M = M_1 \sqcup \cdots \sqcup M_k$. Then $g(\Sigma) \geq \lfloor k/2 \rfloor$.*

**Proof** Fix an orientation on $L$, which thus orients each $M_i$.

We know (from Theorem 1.2(1)) that the support of the $s$–filtration on $\mathrm{Kh}^0_{\mathrm{Lee}}(M_i)$ has diameter at least 2. Thus (by Theorem 1.2(2)) $\mathrm{Kh}^0_{\mathrm{Lee}}(M)$ has $s$–filtration of diameter at least $2k$. Since $L$ is Kh–thin, the $s$–filtration on $\mathrm{Kh}^0_{\mathrm{Lee}}(L)$ has diameter equal to 2 (using the spectral sequence from $\mathrm{Kh}^*$ to $\mathrm{Kh}^*_{\mathrm{Lee}}$).

The cobordism and its reverse induce two maps:

$$\mathrm{Kh}^0_{\mathrm{Lee}}(L) \xrightarrow{F_\Sigma} \mathrm{Kh}^0_{\mathrm{Lee}}(M) \xrightarrow{F_{-\Sigma}} \mathrm{Kh}^0_{\mathrm{Lee}}(L) \tag{3-1}$$

These are both isomorphisms by Lemma 3.3. Also, we know that both maps are filtered of degree $-2g(\Sigma)$.

Without loss of generality, suppose the $s$–filtration on $\mathrm{Kh}^0_{\mathrm{Lee}}(L)$ is supported in degrees $\pm 1$. Then since the isomorphism $F_{-\Sigma}\colon \mathrm{Kh}^0_{\mathrm{Lee}}(M) \to \mathrm{Kh}^0_{\mathrm{Lee}}(L)$ is filtered of degree $-2g(\Sigma)$, the $s$–filtration on $\mathrm{Kh}^0_{\mathrm{Lee}}(M)$ must be supported in degrees $\leq 1 + 2g(\Sigma)$. Similarly, looking at $F_\Sigma\colon \mathrm{Kh}^0_{\mathrm{Lee}}(L) \to \mathrm{Kh}^0_{\mathrm{Lee}}(M)$, we see that the $s$–filtration on $\mathrm{Kh}^0_{\mathrm{Lee}}(M)$ must be supported in degrees $\geq -1 - 2g(\Sigma)$. Thus we have $2 + 4g(\Sigma) \geq 2k$, so $g(\Sigma) \geq \lceil (k-1)/2 \rceil = \lfloor k/2 \rfloor$. □

The following corollary to Proposition 3.6 is already known via properties of the Alexander module [13].

**Corollary 3.7** *No nonsplit alternating link is concordant to a split link.*

# 4 The orientation group

In this section we define a (almost tautological) $(1+1)$–dimensional (projective) TQFT which we call the *orientation group*. It is isomorphic to the TQFT used to define Lee homology (with Lee deformation parameter $a = \frac{1}{4}$). In fact, Walker [27] informs us that the orientation group is isomorphic to $\mathrm{Kh}^*_{\mathrm{Lee}}$ as a functor. The goal of the construction in this section is to give a natural intrinsic description of the maps associated to cobordisms.

For any manifold $X$, we let $|X|$ denote the number of connected components of $X$.

**Definition 4.1** For an orientable manifold $X$, let $O(X)$ denote the set of orientations of $X$. Let $\mathbb{O}(X)$ denote the $\mathbb{Q}$–vector space with basis indexed by $O(X)$. We also define a natural inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{O}(X)$ by declaring that this basis be orthonormal.

**Definition 4.2** Let $\mathfrak{o} \mapsto \overline{\mathfrak{o}}$ denote reversal of orientation; this is an involution of $O(X)$ and of $\mathbb{O}(X)$.

By a *relative orientation* on a manifold $X$, we mean an orientation up to overall reversal of orientation, that is, an element of $O(X)/(\mathfrak{o} \mapsto \overline{\mathfrak{o}})$ (which we often think of as a pair $(\mathfrak{o}, \overline{\mathfrak{o}})$).

**Definition 4.3** We define a mod 4 grading on $\mathbb{O}(X)$ by declaring that the $+1$ eigenspace of $\mathfrak{o} \mapsto \overline{\mathfrak{o}}$ have grading $-|X|$ and that the $-1$ eigenspace of $\mathfrak{o} \mapsto \overline{\mathfrak{o}}$ have grading $2 - |X|$.

**Lemma 4.4** *We have a natural isomorphism* $\mathbb{O}(X_1 \cup X_2) = \mathbb{O}(X_1) \otimes \mathbb{O}(X_2)$ *which respects the involution* $\mathfrak{o} \mapsto \overline{\mathfrak{o}}$ *as well as the mod* 4 *grading.*

**Proof** Clearly $O(X_1 \cup X_2) = O(X_1) \times O(X_2)$, and this gives us the desired isomorphism of vector spaces, which clearly respects reversal of orientation. Now by examining the definition of the mod 4 grading in terms of the map $\mathfrak{o} \mapsto \overline{\mathfrak{o}}$, one easily sees that this implies that the mod 4 grading is preserved as well. □

Henceforth we shall only be interested in $\mathbb{O}(X)$ in the case that $X$ is a 1–manifold.

**Definition 4.5** If $A$ is an orientable cobordism between $X$ and $Y$, then we define a map $F_A\colon \mathbb{O}(X) \to \mathbb{O}(Y)$ (up to overall multiplication by $\pm 1$) as follows. Let $\sigma_A\colon O(A) \to \{\pm 1\}$ satisfy the property that reversing the orientation on some component $A_1 \subseteq A$ multiplies the value of $\sigma_A$ by $(-1)^{(\chi(A_1)-|A_1\cap X|+|A_1\cap Y|)/2}$ (note that since $A$ is orientable, $\chi(A_1)-|A_1\cap X|+|A_1\cap Y| \equiv \chi(\text{closed surface}) \equiv 0 \bmod 2$). Clearly there are two such functions $\sigma_A$, differing by a sign. Then we define (up to $\pm 1$)

$$F_A(\alpha) := \sum_{\mathfrak{o}\in O(A)} \sigma_A(\mathfrak{o})\langle \alpha, \mathfrak{o}|_X\rangle\, \mathfrak{o}|_Y\,. \tag{4-1}$$

By definition, orientations of $X$ which do not extend to $A$ get annihilated by $F_A$. More generally, an orientation is sent to a linear combination of those orientations on $Y$ which are compatible with the cobordism $A$ and the input orientation of $X$. Rasmussen [26, page 434, Proposition 4.1] showed a similar property of $\mathrm{Kh}^*_{\mathrm{Lee}}$ in the process of defining the $s$–invariant.

**Lemma 4.6** *The maps associated to cobordisms are functorial in the sense that if $A$ is a cobordism between $X$ and $Y$ and $B$ is a cobordism between $Y$ and $Z$, then $F_{A\cup_Y B} = F_B\circ F_A$.*

**Proof** We have that

$$\begin{aligned} F_B(F_A(\alpha)) &= \sum_{\mathfrak{o}_B\in O(B)}\sum_{\mathfrak{o}_A\in O(A)} \sigma_B(\mathfrak{o}_B)\sigma_A(\mathfrak{o}_A)\langle \alpha, \mathfrak{o}_A|_X\rangle\langle \mathfrak{o}_A|_Y\,, \mathfrak{o}_B|_Y\rangle\, \mathfrak{o}_B|_Z \\ &= \sum_{\mathfrak{o}\in O(A\cup_Y B)} \sigma_A(\mathfrak{o}|_A)\sigma_B(\mathfrak{o}|_B)\langle \alpha, \mathfrak{o}|_X\rangle\, \mathfrak{o}|_Z\,. \end{aligned} \tag{4-2}$$

Now just observe that the function $O(A\cup_Y B) \to \{\pm 1\}$ given by $\sigma_A(\mathfrak{o}|_A)\sigma_B(\mathfrak{o}|_B)$ satisfies the property which defines $\sigma_{A\cup_Y B}\colon O(A\cup_Y B) \to \{\pm 1\}$ for the construction of $F_{A\cup_Y B}$. □

**Lemma 4.7** *The map $F_A$ on $\mathbb{O}$ is homogeneous of degree $\chi(A)$ with respect to the mod 4 grading.*

**Proof** Note that by definition of the mod 4 grading, we have

$$\begin{aligned} \overline{F_A(\overline{\alpha})} = F_A(\alpha) \quad &\Longleftrightarrow F_A \text{ homogeneous of degree } |X|-|Y|, \\ \overline{F_A(\overline{\alpha})} = -F_A(\alpha) &\Longleftrightarrow F_A \text{ homogeneous of degree } 2+|X|-|Y|. \end{aligned} \tag{4-3}$$

Now we calculate

$$\overline{F_A(\overline{\alpha})} = \sum_{\mathfrak{o}\in O(A)} \sigma_A(\mathfrak{o})\langle\overline{\alpha},\, \mathfrak{o}|_X\rangle\, \overline{\mathfrak{o}}|_Y = \sum_{\mathfrak{o}\in O(A)} \sigma_A(\overline{\mathfrak{o}})\langle\alpha,\, \mathfrak{o}|_X\rangle\, \mathfrak{o}|_Y\,. \tag{4-4}$$

Now by the definition of $\sigma_A$, this equals $(-1)^{(\chi(A)-|X|+|Y|)/2}F_A(\alpha)$. Thus we have

$$\begin{aligned}\chi(A)-|X|+|Y| &\equiv 0 \bmod 4 \Longrightarrow F_A \text{ homogeneous of degree } |X|-|Y|,\\ \chi(A)-|X|+|Y| &\equiv 2 \bmod 4 \Longrightarrow F_A \text{ homogeneous of degree } 2+|X|-|Y|,\end{aligned} \tag{4-5}$$

which exactly says $F_A$ is homogeneous of degree $\chi(A)$. □

The following description shows the isomorphism with Lee's TQFT (with $a=\frac{1}{4}$).

**Lemma 4.8** *The map $F_A$ has the following alternative description. We decompose $A$ into iterated handle additions (eg using a Morse function on $A$), and then to each of the handle additions, we associate maps as follows.*

*For a $0$–handle, we map $\alpha$ to $\alpha\otimes(\mathfrak{o}-\overline{\mathfrak{o}})$, where $\mathfrak{o}$ is an orientation on the new circle.*

*For a $1$–handle which splits a component, the map sends every orientation to its extension to the new manifold.*

*For a $1$–handle which joins two components, the map sends orientations which do not extend to the new manifold to zero, and sends orientations which do extend to their natural extension* multiplied by $\pm 1$ *depending on the orientation of the new merged circle.*

*For a $2$–handle, the map sends $\mathfrak{o}\otimes\alpha$ to $\alpha$ and $\overline{\mathfrak{o}}\otimes\alpha$ to $\alpha$.*

**Proof** That it suffices to splice together the maps for elementary cobordisms follows from Lemma 4.6. We just have to calculate the maps coming from $k$–handle additions, $k\in\{0,1,2\}$. These are given completely explicitly by Definition 4.5, which gives the result. □

It is interesting to note that even with this trivial construction, there is a good reason why if we want to make $\mathbb{O}(L)$ into a functor, we have no choice but to use maps that are only defined up to $\pm 1$. For instance, consider the birth of a circle. Note that the birth of a circle is the same cobordism as the birth of a circle followed by an isotopy from the circle to itself which reverses orientation. Thus they must induce the same map. However, the image of the birth of a circle is $\mathfrak{o} - \overline{\mathfrak{o}}$, and this clearly changes sign under the isotopy.

If we are interested in links *embedded in* $\mathbb{R}^3$, and we want functoriality with respect to orientable cobordisms *embedded in* $\mathbb{R}^3 \times [0,1]$, then it is probably possible to twist by an appropriate homomorphism $\pi_1(\{\text{unoriented loops in } \mathbb{R}^3\}) \to \{\pm 1\}$ to get rid of the sign ambiguity in $\mathbb{O}$. We note that Hatcher [11] has proved the Smale Conjecture, which is equivalent to the fact that the space of unoriented *unknotted* loops in $\mathbb{R}^3$ deformation retracts onto the space of unoriented circles in $\mathbb{R}^3$, and the fundamental group of this space is indeed $\mathbb{Z}/2\mathbb{Z}$. We suspect this type of twisting is morally what fixes the functoriality of Khovanov homology as in [7; 6].

## 4.1 Properties of the $s$–filtration on $\mathbb{O}(L)$

Under the equivalence between $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$ and $\mathbb{O}(L)$, we get a natural definition of the $s$–filtration on $\mathbb{O}(L)$. The space $\mathbb{O}(L)$ carries a number of natural operations, and it is reasonable to ask how they respect the $s$–filtration. We answer a few of these questions in this section, using only the functorial properties of $\mathbb{O}(L)$ under cobordism. Because we use these soft methods, the properties we derive here would also be valid for a hypothetical generalization of the $\tau$–invariant to links.

The following is a rough analogue of Livingston's result [19] that $s(K_-) \le s(K_+) \le s(K_-) + 2$ (here $K_-$ and $K_+$ differ at exactly one crossing, which is positive for $K_+$ and negative for $K_-$).

**Lemma 4.9** *Suppose $L_1$ and $L_2$ differ by a single crossing change. There is of course a natural isomorphism $\phi\colon \mathbb{O}(L_1) \xrightarrow{\sim} \mathbb{O}(L_2)$. Let $\mathbb{O}(L_1)^+$ denote the space generated by orientations in which the given crossing is positive (and similarly define $\mathbb{O}(L_1)^-$, $\mathbb{O}(L_2)^+$ and $\mathbb{O}(L_2)^-$). Pick a strand at the given crossing and an orientation of that strand. Let $\psi\colon \mathbb{O}(L_2) \to \mathbb{O}(L_2)$ be defined by $\psi(\mathfrak{o}) = \sigma(\mathfrak{o})\mathfrak{o}$, where $\sigma(\mathfrak{o}) = 1$ if $\mathfrak{o}$ agrees with the chosen orientation on the chosen strand, and $\sigma(\mathfrak{o}) = -1$ otherwise. Then we have:*

1. *$\psi \circ \phi\colon \mathbb{O}(L_1) \to \mathbb{O}(L_2)$ is filtered of degree $-2$.*
2. *$\phi\colon \mathbb{O}(L_1)^- \to \mathbb{O}(L_2)^+$ is filtered of degree $0$.*

**Proof** Our strategy is to find cobordisms which induce the required maps.

For statement (1), consider the following. Passing the two strands through each other yields an immersed cobordism of Euler characteristic $0$ from $L_1$ to $L_2$. There are two possible resolutions of the double point, giving two maps $\mathbb{O}(L_1) \to \mathbb{O}(L_2)$. Using (4-1), we see that the two maps are

$$\mathbb{O}(L_1) \xrightarrow{\text{projection}} \mathbb{O}(L_1)^+ \xrightarrow{\phi|_{\mathbb{O}(L_1)^+}} \mathbb{O}(L_2) \xrightarrow{\psi} \mathbb{O}(L_2), \tag{4-6}$$

$$\mathbb{O}(L_1) \xrightarrow{\text{projection}} \mathbb{O}(L_1)^- \xrightarrow{\phi|_{\mathbb{O}(L_1)^-}} \mathbb{O}(L_2) \xrightarrow{\psi} \mathbb{O}(L_2). \tag{4-7}$$

Since the Euler characteristic of each resolved cobordism is $-2$, both of these maps are filtered of degree $-2$. The sum of the two projections is the identity map on $\mathbb{O}(L_1)$, so the sum of (4-6) and (4-7) is just $\psi \circ \phi$; hence it is filtered of degree $-2$ as well.

For statement (2), consider the following. By Rasmussen, $\mathbb{O}(T_{2,3})$ is supported in $s$–filtration levels $1$ and $3$. Since the mod 4 grading agrees with the $s$–filtration, we see that $\mathfrak{o} + \overline{\mathfrak{o}}$ lies in filtration level $3$. Thus the map $\mathbb{O}(L_1)^- \to \mathbb{O}(L_1)^- \otimes \mathbb{O}(T_{2,3})$ given by $\alpha \mapsto \alpha \otimes (\mathfrak{o} + \overline{\mathfrak{o}})$ is filtered of degree $3$. Now consider an immersed cobordism starting at $L_1 \sqcup T_{2,3}$ which first passes the strands of the crossing of $L_1$ through each other to get $L_2$, then unknots the $T_{2,3}$ in a similar manner, and then merges the resulting unknot with $L_2$. The two double points are of opposite signs (when the crossing goes from negative in $L_1$ to positive in $L_2$), so they can be tubed together to obtain a cobordism of genus $1$. Thus the resulting map $\mathbb{O}(L_1)^- \otimes \mathbb{O}(T_{2,3}) \to \mathbb{O}(L_2)^+$ is filtered of degree $-3$. The composite is $\phi\colon \mathbb{O}(L_1)^- \to \mathbb{O}(L_2)^+$ (as is clear from (4-1)), so we are done. □

**Definition 4.10** Given a specific orientation $\mathfrak{o}_i$ on a component $L_i$ of $L$, let the map $\mathrm{Res}_{\mathfrak{o}_i}\colon \mathbb{O}(L) \to \mathbb{O}(L)$ be orthogonal projection onto the subspace where $L_i$ is oriented by $\mathfrak{o}_i$. For a relative orientation $\mathfrak{o}_{ij}$ of $L_i \cup L_j$ (two components of $L$), let $\mathrm{Res}_{\mathfrak{o}_{ij},\overline{\mathfrak{o}}_{ij}}\colon \mathbb{O}(L) \to \mathbb{O}(L)$ be projection onto the subspace where $L_i \cup L_j$ has this relative orientation, composed with multiplication by $\sigma\colon O(L) \to \{\pm 1\}$ which flips sign depending on the orientation on $L_i$.

The following should be thought of as a generalization of Rasmussen's theorem that characterizes $d_K$ for knots $K$.

**Lemma 4.11** *The operators* $\mathrm{Res}_{\mathfrak{o}_i}$ *and* $\mathrm{Res}_{\mathfrak{o}_{ij},\overline{\mathfrak{o}}_{ij}}$ *are both filtered of degree* $-2$.

**Proof** For $\mathrm{Res}_{\mathfrak{o}_{ij},\overline{\mathfrak{o}}_{ij}}$, consider the cobordism formed by first adding a $1$–handle connecting $L_i$ and $L_j$ (in such a way that the given relative orientation extends over the cobordism) and then adding a second $1$–handle splitting the resulting component back into $L_i \cup L_j$. Clearly this cobordism induces the map $\mathrm{Res}_{\mathfrak{o}_{ij},\overline{\mathfrak{o}}_{ij}}\colon \mathbb{O}(L) \to \mathbb{O}(L)$, and it has Euler characteristic $-2$, so we are done.

For $\mathrm{Res}_{\mathfrak{o}_i}$, let $U$ be the unknot, and consider the map $\mathbb{O}(L) \to \mathbb{O}(L) \otimes \mathbb{O}(U) = \mathbb{O}(L \sqcup U)$ given by $\alpha \mapsto \alpha \otimes \mathfrak{o}$. This is filtered of degree $-1$. Now compose with the map $\mathbb{O}(L \sqcup U) \to \mathbb{O}(L)$ given by the cobordism obtained by adding a 1–handle to merge the unknot and $L_i$ (such that the orientation $\mathfrak{o}$ and the desired orientation $\mathfrak{o}_i$ extend over the 1–handle). This cobordism has Euler characteristic $-1$, so the composition $\mathbb{O}(L) \to \mathbb{O}(L) \otimes \mathbb{O}(U) \to \mathbb{O}(L)$ is filtered of degree $-2$. This map also clearly equals $\mathrm{Res}_{\mathfrak{o}_i}$. □

**Lemma 4.12** *Let $\psi\colon \mathbb{O}(L) \to \mathbb{O}(L)$ be defined by $\psi(\mathfrak{o}) = \sigma(\mathfrak{o})\mathfrak{o}$, where $\sigma(\mathfrak{o}) = \pm 1$ depending on the orientation of some specific component $L_i \subseteq L$. Then $\psi$ is filtered of degree $-2$.*

**Proof** Such a map is a linear combination of $\mathrm{Res}_{\mathfrak{o}_i}$ and $\mathrm{Res}_{\overline{\mathfrak{o}_i}}$. □

**Lemma 4.13** *For every $\alpha \in \mathbb{O}(L)$, we have $s(\alpha) = \min(s(\alpha + \overline{\alpha}), s(\alpha - \overline{\alpha}))$. In particular, it follows that $s(\alpha) = s(\overline{\alpha})$.*

**Proof** Note that $\alpha + \overline{\alpha}$ and $\alpha - \overline{\alpha}$ are in different mod 4 gradings, so they are sent to different mod 4 gradings in $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$. Thus by Theorem 2.1 property (1), we know that $s(\alpha + \overline{\alpha})$ and $s(\alpha - \overline{\alpha})$ are different mod 4. Thus $s(\alpha + \overline{\alpha}) \neq s(\alpha - \overline{\alpha})$, so $s(\alpha) = \min(s(\alpha + \overline{\alpha}), s(\alpha - \overline{\alpha}))$. □

Suppose we have a relative orientation $(\mathfrak{o}, \overline{\mathfrak{o}})$ of $L$. Let $V_{\mathfrak{o},\overline{\mathfrak{o}}} \subseteq \mathbb{O}(L)$ be the subspace generated by $\mathfrak{o}$ and $\overline{\mathfrak{o}}$. Then let us consider the restriction of the $s$–filtration to $V_{\mathfrak{o},\overline{\mathfrak{o}}} = \mathbb{Q}(\mathfrak{o} + \overline{\mathfrak{o}}) \oplus \mathbb{Q}(\mathfrak{o} - \overline{\mathfrak{o}})$. Note that this direct sum decomposition is into mod 4 graded pieces; thus the elements $\mathfrak{o} + \overline{\mathfrak{o}}$ and $\mathfrak{o} - \overline{\mathfrak{o}}$ are sent to different mod 4 gradings in $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$ which differ by exactly 2. Thus the $s$–filtration on $V_{\mathfrak{o},\overline{\mathfrak{o}}}$ is completely described by the two integers $s(\mathfrak{o} + \overline{\mathfrak{o}})$ and $s(\mathfrak{o} - \overline{\mathfrak{o}})$ (which differ by 2 mod 4). By Lemma 4.12, $s(\mathfrak{o} + \overline{\mathfrak{o}})$ and $s(\mathfrak{o} - \overline{\mathfrak{o}})$ differ by exactly two, and we let the oriented $s(L, \mathfrak{o}) = \frac{1}{2}[s(\mathfrak{o} + \overline{\mathfrak{o}}) + s(\mathfrak{o} - \overline{\mathfrak{o}})]$.

**Definition 4.14** The invariant constructed in the previous paragraph is $s(L, \mathfrak{o})$. It was first defined by Beliakova and Wehrli [5].

For a knot $K$, there is just one relative orientation. This gives Rasmussen's invariant $s(K)$, which determines the $s$–filtration on $\mathrm{Kh}^*_{\mathrm{Lee}}(K)$. For links, however, there is much more to the $s$–filtration on $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$ that is not captured by the function $\mathfrak{o} \mapsto s(L, \mathfrak{o})$. For example, for any alternating link $L$ with zero linking matrix, all $s(L, \mathfrak{o})$ are equal (say, to $s_0$), and $\sum_{i,j} d_L(i,j) t^i q^j = 2^{|L|-1} q^{s_0}(q + q^{-1})$. On the other hand, if $L$ is unlink on $n$ components, then $s(L, \mathfrak{o})$ are all equal (this time to $1-n$), however in this case $\sum_{i,j} d_L(i,j) t^i q^j = (q + q^{-1})^n$. Thus for link concordance, the $s$–filtration on $\mathbb{O}(L)$ is a stronger invariant than the function $\mathfrak{o} \mapsto s(L, \mathfrak{o})$.

# 5 Examples

We now summarize some calculations of the invariant $d_L\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}_{\geq 0}$ for some links $L$. We used the package `KnotTheory`` maintained by Bar-Natan [4], in particular the program to calculate Khovanov homology written by Scott Morrison. This allows us to calculate $\mathrm{Kh}^*(L)$ for the link in question. We use the simple fact that if $\dim \mathrm{Kh}^h(L) = \dim \mathrm{Kh}^h_{\mathrm{Lee}}(L)$, then by virtue of the spectral sequence from $\mathrm{Kh}^*(L)$ to $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$, the support of the $s$–filtration on $\mathrm{Kh}^h_{\mathrm{Lee}}(L)$ is given exactly by the $q$–graded dimension of $\mathrm{Kh}^h(L)$. Many interesting links have lots of crossings, and thus computing the Khovanov homology is time consuming on a computer; we just list the cases that we have been able to compute.

Most of the links in the standard link tables are quasi-alternating, so they do not present a particularly interesting case for the filtration on $\mathrm{Kh}^*_{\mathrm{Lee}}(L)$ (it is just supported in two levels, so only their absolute height is interesting). So instead, we've taken as our examples some links with extra structure.

The function $d_L$ is a link concordance invariant, and thus there are some easy corollaries using Theorem 1.2 distinguishing the link concordance classes of the links we consider below from other links whose $d_L$ one could calculate (eg one can easily see which are concordant to a quasi-alternating link). Theorem 1.2 also implies effective bounds on the genus of component-preserving orientable cobordisms between these links and links with certain splitting numbers.

## 5.1 Cablings of $T_{2,p}$

Let $L_p$ be the $(2,0)$–cabling of $T_{2,p}$. Then the linking matrix of $L_p$ is zero, and we have (for $p$ odd, $1\leq p\leq 11$)

$$\sum_{i,j} d_{L_p}(i,j)\cdot t^i q^j = 1+q^2+q^{2p-4}+q^{2p-2}. \tag{5-1}$$

We conjecture that this is true for all odd $p\geq 1$. We can prove the following:

**Lemma 5.1** *Fix an odd integer $p\geq 5$. Let $(\mathfrak{o}_+,\overline{\mathfrak{o}}_+)$ be the relative orientation of $L_p$ where the two strands are oriented in the same direction, and let $V_+$ be the subspace of $\mathbb{O}(L_p)$ generated by $(\mathfrak{o}_+,\overline{\mathfrak{o}}_+)$. Similarly define $(\mathfrak{o}_-,\overline{\mathfrak{o}}_-)$ and $V_-$ with the two strands oriented in opposite directions.*

*Then the $s$–filtration on $\mathbb{O}(L) = V_+\oplus V_-$ splits up as a filtration on each $V_\pm$. Furthermore, $V_+$ is supported in filtration degrees $(2p-4, 2p-2)$, and $V_-$ is supported* *in degrees* $(0,2)$ *or* $(-2,0)$. *In particular, we have*

$$\sum_{i,j} d_{L_p}(i,j)\cdot t^i q^j = \left\{\begin{matrix} 1+q^2 \\ \text{or} \\ q^{-2}+1 \end{matrix}\right\} + q^{2p-4} + q^{2p-2}. \tag{5-2}$$

Of course, it is not true in general that the $s$–filtration splits up as a direct sum over all relative orientations $(\mathfrak{o},\overline{\mathfrak{o}})$ of filtrations on $V_{\mathfrak{o},\overline{\mathfrak{o}}}$ (for example, this fails for any split link by Theorem 2.1 property (2)).

**Proof** We thank the referee for the argument in this paragraph. The standard diagram for $T_{2,p}$ has $p$ positive crossings, so the 2–cabling in the blackboard framing is the $(2,2p)$–cable, which we call $L'_p$. Orient both $L_p$ and $L'_p$ via $\mathfrak{o}_+$ (we let $\mathfrak{o}_+$ denote the orientation on $L'_p$ corresponding naturally to $\mathfrak{o}_+$ on $L_p$). Now $L'_p$ has a positive diagram coming from the positive diagram of $T_{2,p}$. In this diagram, there are $4p$ crossings and 4 circles in the oriented resolution. Thus by Rasmussen [26, page 439, Section 5.2], we have $s_{L'_p}(\mathfrak{o}_+) = 4p-4$. Now we can transform $L'_p$ into $L_p$ by $p$ crossing changes (they differ by $2p$ half-twists between the two strands). Thus $p$ iterated applications of Lemma 4.9 imply that

$$s_{L_p}(\mathfrak{o}_+) \geq 2p-4. \tag{5-3}$$

On the other hand, $L_p$ bounds two parallel copies of a Seifert surface for $T_{2,p}$, each of genus $\frac{1}{2}(p-1)$. These give a component-preserving cobordism of genus $p-1$ from $L_p$ to the unlink. Every orientation $\mathfrak{o}$ of the unlink has $s(\mathfrak{o}) = -2$. Thus

$$s_{L_p}(\mathfrak{o}_+) \leq s_{\text{unlink}}(\mathfrak{o}) + 2(p-1) = 2p-4. \tag{5-4}$$

Thus $s_{L_p}(\mathfrak{o}_+) = 2p-4$. By the discussion surrounding Definition 4.14, this shows that the restriction of the $s$–filtration to $V_+$ is supported in degrees $(2p-4, 2p-2)$.

Now for $\mathfrak{o}_-$, observe that adding a 1–handle merging the two components of $L_p$ yields the unknot $U$. Thus if we orient $L_p$ by $\mathfrak{o}_-$, we have maps $\mathbb{O}(L_p)\to\mathbb{O}(U)\to\mathbb{O}(L_p)$, and in fact they give *isomorphisms*

$$V_- \xrightarrow{\sim} \mathbb{O}(U) \xrightarrow{\sim} V_- \tag{5-5}$$

which are filtered of degree $-1$. Since $\mathbb{O}(U)$ is supported in degrees $\pm 1$, and $V_-$ is supported in even degrees (which differ by exactly two), we see that the support of $V_-$ is either $(0,2)$ or $(-2,0)$.

Now it remains to show the $s$–filtration on $\mathbb{O}(L)$ decomposes as the direct sum of the filtrations on each $V_\pm$. In other words, we need to show $s(\alpha_+ + \alpha_-) = \min(s(\alpha_+), s(\alpha_-))$

for all pairs $\alpha_\pm \in V_\pm$. Our assumption $p \geq 5$ implies $2p-4 > 2$, so by the results above, we have $s(\alpha_+) \neq s(\alpha_-)$ (unless $\alpha_+ = \alpha_- = 0$). It follows that $s(\alpha_+ + \alpha_-) = \min(s(\alpha_+), s(\alpha_-))$. □

## 5.2 $T_{n,n}$

We now consider the $(n,n)$–torus links for $1 \leq n \leq 6$ (with all components oriented the same direction). We have

$$\sum_{i,j} d_{T_{1,1}}(i,j)\cdot t^i q^j = q + q^{-1},$$

$$\sum_{i,j} d_{T_{2,2}}(i,j)\cdot t^i q^j = [1+q^2] + (tq^3)^2[q^{-2}+1],$$

$$\sum_{i,j} d_{T_{3,3}}(i,j)\cdot t^i q^j = [q^3+q^5] + (tq^3)^4[q^{-3}+3q^{-1}+2q],$$

$$\sum_{i,j} d_{T_{4,4}}(i,j)\cdot t^i q^j = [q^8+q^{10}] + (tq^3)^6[q^{-2}+4+3q^2] + (tq^3)^8[q^{-4}+3q^{-2}+2],$$

$$\begin{aligned}\sum_{i,j} d_{T_{5,5}}(i,j)\cdot t^i q^j &= [q^{15}+q^{17}] + (tq^3)^8[q+5q^3+4q^5] \\ &\quad + (tq^3)^{12}[q^{-5}+5q^{-3}+9q^{-1}+5q],\end{aligned}$$

$$\begin{aligned}\sum_{i,j} d_{T_{6,6}}(i,j)\cdot t^i q^j &= [q^{24}+q^{26}] + (tq^3)^{10}[q^6+6q^8+5q^{10}] \\ &\quad + (tq^3)^{16}[q^{-4}+6q^{-2}+14+9q^2] \\ &\quad + (tq^3)^{18}[q^{-6}+5q^{-4}+9q^{-2}+5].\end{aligned}$$

In accordance with Theorem 1.2 property (6), it is natural to separate out factors of $tq^3$. For $n = 5, 6$, computing the final answer requires the use of Theorem 1.2 property (1), in particular the fact that the dimensions of $\mathrm{Kh}^*_{\mathrm{Lee}}$ supported in $s$–filtration level $|L|$ and $|L|+2$ are equal (this enables us to see which parts of $\mathrm{Kh}^*$ are killed in the spectral sequence).

The referee has noticed the following pattern for the values of $d_{T_{n,n}}$. Define polynomials $P_{n,k} \in \mathbb{Q}[q,q^{-1}]$ for $n \geq 0$ and $0 \leq 2k \leq n$ by the recurrence

$$P_{0,0} = 1, \tag{5-6}$$

$$\text{for } n \geq 1 \quad P_{n,k} = \begin{cases} qP_{n-1,k-1} + q^{-1}P_{n-1,k} & 2k+2 \leq n, \\ qP_{n-1,k-1} + \frac{1}{2}(1+q^{-1})P_{n-1,k} & 2k+1 = n, \\ 2P_{n-1,k-1} & 2k = n, \end{cases} \tag{5-7}$$

where we interpret $P_{n,k}$ as zero if $k < 0$. Certainly $P_{n,k}$ are some sort of $q$–deformed binomial coefficients. Then for $1 \leq p \leq 6$, we have

$$\sum_{i,j} d_{T_{n,n}}(i,j) \cdot t^i q^j = \sum_{k=0}^{n} (tq^3)^{2k(n-k)} q^{(n-2k)^2} P_{n,\min(k,n-k)}(q^2). \tag{5-8}$$

One would naturally conjecture that this equality holds for all larger $p$ as well.

## References


[1] **D Bar-Natan**, *On Khovanov's categorification of the Jones polynomial*, Algebr. Geom. Topol. 2 (2002) 337–370 MR1917056

[2] **D Bar-Natan**, *Khovanov's homology for tangles and cobordisms*, Geom. Topol. 9 (2005) 1443–1499 MR2174270

[3] **D Bar-Natan**, *Fast Khovanov homology computations*, J. Knot Theory Ramifications 16 (2007) 243–255 MR2320156

[4] **D Bar-Natan**, **S Morrison**, `KnotTheory` *Package* (2008) Available at `http://katlas.org/wiki/The_Mathematica_Package_KnotTheory`

[5] **A Beliakova**, **S Wehrli**, *Categorification of the colored Jones polynomial and Rasmussen invariant of links*, Canad. J. Math. 60 (2008) 1240–1266 MR2462446

[6] **C L Caprau**, sl(2) *tangle homology with a parameter and singular cobordisms*, Algebr. Geom. Topol. 8 (2008) 729–756 MR2443094

[7] **D Clark**, **S Morrison**, **K Walker**, *Fixing the functoriality of Khovanov homology*, Geom. Topol. 13 (2009) 1499–1582 MR2496052

[8] **M Freedman**, **R Gompf**, **S Morrison**, **K Walker**, *Man and machine thinking about the smooth 4-dimensional Poincaré conjecture*, Quantum Topol. 1 (2010) 171–208 MR2657647

[9] **S Friedl**, **M Powell**, *Cobordisms to weakly splittable links* `arXiv:1112.3685`

[10] **J Greene**, *Homologically thin, non-quasi-alternating links*, Math. Res. Lett. 17 (2010) 39–49 MR2592726

[11] **A E Hatcher**, *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$, Ann. of Math. (2) 117 (1983) 553–607 MR701256

[12] **M Jacobsson**, *An invariant of link cobordisms from Khovanov homology*, Algebr. Geom. Topol. 4 (2004) 1211–1251 MR2113903

[13] **A Kawauchi**, *On alternation numbers of links*, Topology Appl. 157 (2010) 274–279 MR2556105

[14] **M Khovanov**, *A categorification of the Jones polynomial*, Duke Math. J. 101 (2000) 359–426 MR1740682

[15] **P B Kronheimer**, **T S Mrowka**, *Gauge theory for embedded surfaces, I*, Topology 32 (1993) 773–826 MR1241873

[16] **P B Kronheimer**, **T S Mrowka**, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. 1 (1994) 797–808 MR1306022

[17] **P B Kronheimer**, **T S Mrowka**, *Gauge theory for embedded surfaces, II*, Topology 34 (1995) 37–97 MR1308489

[18] **E S Lee**, *An endomorphism of the Khovanov invariant*, Adv. Math. 197 (2005) 554–586 MR2173845

[19] **C Livingston**, *Computations of the Ozsváth–Szabó knot concordance invariant*, Geom. Topol. 8 (2004) 735–742 MR2057779

[20] **C Manolescu**, **P Ozsváth**, *On the Khovanov and knot Floer homologies of quasi-alternating links*, from: "Proceedings of Gökova Geometry-Topology Conference 2007", (S Akbulut, T Önder, R J Stern, editors), Gökova Geometry/Topology Conference (2008) 60–81 MR2509750

[21] **K Murasugi**, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. 117 (1965) 387–422 MR0171275

[22] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and knot invariants*, Adv. Math. 186 (2004) 58–116 MR2065507

[23] **P Ozsváth**, **Z Szabó**, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. 194 (2005) 1–33 MR2141852

[24] **P Ozsváth**, **Z Szabó**, *Holomorphic disks, link invariants and the multi-variable Alexander polynomial*, Algebr. Geom. Topol. 8 (2008) 615–692 MR2443092

[25] **J Rasmussen**, *Khovanov's invariant for closed surfaces* `arXiv:math.GT/0502527`

[26] **J Rasmussen**, *Khovanov homology and the slice genus*, Invent. Math. 182 (2010) 419–447 MR2729272

[27] **K Walker** (mathoverflownet/users/284), *Is the complete functorial structure for Khovanov–Lee homology known?* (accessed 2011-07-22) Available at `http://mathoverflow.net/questions/70902`

*Department of Mathematics, Stanford University*
*450 Serra Mall, Building 380, Stanford CA 94305, USA*

pardon@math.stanford.edu

http://math.stanford.edu/~pardon/

Received: 25 July 2011 Revised: 9 February 2012