Simplicial volume of $\mathbb{Q}$–rank one locally symmetric spaces covered by the product of $\mathbb{R}$–rank one symmetric spaces

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In this paper, we show that the simplicial volume of $\mathbb{Q}$–rank one locally symmetric spaces covered by the product of $\mathbb{R}$–rank one symmetric spaces is strictly positive.

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1 Introduction

The simplicial volume of a connected, oriented manifold M was introduced by Gromov [12]. This is a topological invariant in $\mathbb{R}_{\geq 0}$ and measures how efficiently the fundamental class of M can be represented by simplices. Gromov conjectured that non-positively curved closed manifolds with negative Ricci curvature have positive simplicial volume.

First, the positivity of the simplicial volume was verified for closed negatively curved manifolds by Thurston [22], Gromov [12] and Inoue and Yano [13]. It was verified for closed locally symmetric spaces covered by $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$ by Savage [21] and Bucher-Karlsson [3]. Indeed, Savage gave a proof of the positivity of the simplicial volume of closed locally symmetric spaces covered by $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ but it turned out to be incomplete. Bucher-Karlsson made a complete proof of the same result in the case of $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$. Then, Lafont and Schmidt [14] showed that the simplicial volume of all closed locally symmetric spaces of non-compact type is positive, which supports the conjecture raised by Gromov.

For open manifolds, the simplicial volume is somewhat odd. Thurston [22] verified that the simplicial volume of complete Riemannian manifolds with pinched negative sectional curvature and finite volume is strictly positive. In contrast, Gromov [12], Löh and Sauer [19] proved that the simplicial volume of open manifolds, which are the Cartesian product of three open manifolds and locally symmetric spaces of $\mathbb{Q}$–rank at least 3, vanishes. Löh and Sauer [20] showed that Hilbert modular varieties have positive simplicial volume, which was the first class of examples of open locally symmetric spaces of $\mathbb{R}$–rank at least 2 for which the positivity of simplicial or minimal

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volume is known. Hilbert modular varieties are special cases of $\mathbb{Q}$–rank one locally symmetric spaces covered by the product of hyperbolic planes. The aim of this paper is to show the positivity of the simplicial volume of $\mathbb{Q}$–rank one locally symmetric spaces covered by the product of $\mathbb{R}$–rank one symmetric spaces.

Theorem 1.1 *Let M be a $\mathbb{Q}$–rank one locally symmetric space covered by the product of $\mathbb{R}$–rank one symmetric spaces. Then, the simplicial volume of M is positive.*

Theorem 1.1 holds for $\mathbb{Q}$–rank one locally symmetric spaces covered by $\mathbb{R}$–rank one symmetric spaces as shown by Thurston. For the proportionality principle of $\mathbb{Q}$–rank one locally symmetric spaces, see the forthcoming paper [5]. Gromov [12] proved a lower bound for the minimal volume of n–dimensional Riemannian manifolds in terms of the simplicial volume:

$$\|M\| \leq (n-1)^n n! \cdot \operatorname{minvol}(M).$$

The theorem implies the positivity of minimal volume of $\mathbb{Q}$–rank one locally symmetric spaces covered by the product of $\mathbb{R}$–rank one symmetric spaces as follows:

Corollary 1.2 *The minimal volume of $\mathbb{Q}$–rank one locally symmetric spaces covered by the product of $\mathbb{R}$–rank one symmetric spaces is positive.*

In fact, Connell and Farb [7; 8; 6] showed that the minimal volume of a locally symmetric space of non-compact type and finite volume with no local direct factors locally isometric to $\mathbb{H}^2$ or $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$ is positive. Nevertheless, the result of Connell and Farb does not imply Corollary 1.2. Note that Corollary 1.2 covers the case that the theorem of Connell and Farb does not cover. Furthermore, we can obtain the degree theorem without any Lipschitz condition on the map from Theorem 1.1.

Theorem 1.3 *Let N be a complete Riemannian n–dimensional manifold of finite volume with Ricci curvature bounded below by $-(n-1)$ and M be a $\mathbb{Q}$–rank one locally symmetric space covered by the product of $\mathbb{R}$–rank one symmetric spaces. For any proper map $f\colon N \to M$ we have*

$$\deg(f) \leq C_n \frac{\operatorname{Vol}(N)}{\operatorname{Vol}(M)},$$

where C_n depends only on n.

The degree theorem for general locally symmetric space of finite volume was proved by Connell and Farb [7; 6] with Lipschitz condition on f. Note that we drop the Lipschitz condition on the map in the degree theorem for $\mathbb{Q}$–rank one locally symmetric spaces covered by the product of $\mathbb{R}$–rank one symmetric spaces. This is one advantage of the ordinary simplicial volume against the Lipschitz simplicial volume. The essential part of our approach is to show that the geodesic straightening map is well defined on the locally finite chain complex of $\mathbb{Q}$–rank one locally symmetric spaces. Indeed, Thurston [22] introduced the geodesic straightening map on the singular chain complex of non-positively curved manifolds, which is homotopic to the identity.

Unfortunately, the geodesic straightening map is generally not defined on the locally finite chain complex of non-positively curved manifolds because the geodesic straightening of a locally finite chain is not necessarily locally finite. However, the situation in the $\mathbb{Q}$–rank one locally symmetric spaces is different. By using Leuzinger's explicit geometric description of $\mathbb{Q}$–rank one locally symmetric spaces [15], one can see that the geodesic straightening of a locally finite chain is locally finite. The presence of the geodesic straightening map on a locally finite chain complex and the uniform upper bound of the volume of geodesic simplices in $\mathbb{Q}$–rank one locally symmetric spaces covered by the product of $\mathbb{R}$–rank one symmetric spaces give rise to the positivity of the simplicial volume.

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2 Preliminaries

In this section, we first collect some definitions and results about the simplicial volume. We begin with the definition of the simplicial volume.

2.1 Simplicial volume

Let M be an n–dimensional, connected, oriented manifold. Denote by $C_*(M)$ the singular chain complex of M with real coefficients. Consider on $C_*(M)$ the ℓ^1–norm with respect to the canonical basis of singular simplices, that is, $\|c\|_1 = \sum_{i=1}^r |a_i|$ for $c = \sum_{i=1}^r a_i \sigma_i$ in $C_*(M)$ where $\sigma_1, \dots, \sigma_r$ are distinct singular simplices. This norm induces a semi-norm on the real coefficient homology $H_*(M)$ of M as follows:

$$\|z\| = \inf_c \|c\|_1,$$

where c runs over all singular cycles representing $z \in H_*(M)$.

The simplicial volume $\|M\|$ of a closed manifold M is defined as the semi-norm of the fundamental class $[M] \in H_n(M)$.

For closed Riemannian manifolds, Gromov proved the remarkable proportionality principle relating the simplicial volume and the volume of the Riemannian manifolds [12, Section 2.3].

Theorem 2.1 (Gromov) *Let M and N be two closed Riemannian manifolds with isometric universal covers. Then,*

$$\frac{\|M\|}{\mathrm{Vol}(M)} = \frac{\|N\|}{\mathrm{Vol}(N)}.$$

There are many ways to prove the proportionality principle. Löh [17] provides a detailed proof of the proportionality principle following the approach of measure homology sketched by Thurston in [22, Chapter 6], while Bucher-Karlsson [4] reformulates a proof of the proportionality principle in the language of continuous bounded cohomology. In addition, Frigerio [11] gives a cohomological proof of the proportionality principle.

If M is open, one cannot choose the fundamental class of M in the singular homology of M because the top dimensional homology of M is trivial. Hence, the simplicial volume of open manifolds is defined in terms of the locally finite chain complex of the manifolds as follows.

Let M be a topological space and let $S_k(M)$ be the set of all continuous maps from the standard k–simplex Δ^k to M. A subset A of $S_k(M)$ is called *locally finite* if any compact subset of M intersects the image of only a finite number of elements of A. Let us denote by $S_k^{\mathrm{lf}}(M)$ the set of all locally finite subsets of $S_k(M)$.

Definition 2.2 Let M be a topological space and let $k \in \mathbb{N}$. The locally finite chain complex of M is the chain complex $C_*^{\mathrm{lf}}(M)$ consisting of the real vector spaces

$$C_k^{\mathrm{lf}}(X) = \left\{ \sum_{\sigma \in A} a_\sigma \cdot \sigma \;\middle|\; A \in S_k^{\mathrm{lf}}(X) \text{ and } (a_\sigma)_{\sigma \in A} \subset \mathbb{R} \right\}$$

equipped with the boundary operator given by the alternating sums of the $(k-1)$–faces. *The locally finite homology $H_*^{\mathrm{lf}}(M)$ of M* is the homology of the locally finite chain complex $C_*^{\mathrm{lf}}(M)$.

The ℓ^1–norm $\|\cdot\|_1$ on the locally finite chain complex of M is defined with respect to the canonical basis of singular simplices. The ℓ^1–norm of a locally finite chain can be infinite. This ℓ^1–norm gives rise to a semi-norm on the locally finite homology of M. Any oriented, connected manifold M possesses a fundamental class, which is a distinguished generator of the top dimensional locally finite homology $H_n^{\mathrm{lf}}(M;\mathbb{Z}) \cong \mathbb{Z}$ with integral coefficients (see Löh [18]). Now, we are ready to define the simplicial volume of open manifolds.

Definition 2.3 Let M be a connected n–dimensional manifold without boundary. Then, *the simplicial volume of M* is defined as

$$\|M\| = \inf\left\{\|c\|_1 \;\middle|\; c \in C_n^{\mathrm{lf}}(M) \text{ is a fundamental cycle of } M\right\}.$$

Note that the simplicial volume of open manifolds may be infinite. The simplicial volume of open manifolds is zero in a large number of cases, including the product of three open manifolds (see Gromov [12, Section 4.2]), locally symmetric manifolds of $\mathbb{Q}$–rank of at least 3 (see Löh and Sauer [19]).

Gromov [12, Section 4.4] introduced another notion of the simplicial volume, so called Lipschitz simplicial volume which is a geometric variant of the ordinary simplicial volume. Let M be an oriented Riemannian manifold. For a locally finite chain $c = \sum_{\sigma \in A} a_\sigma \sigma$, define $\mathrm{Lip}(c)$ as the supremum of the Lipschitz constants of the singular simplices σ with respect to the standard Euclidean metric on the standard simplex.

Definition 2.4 Let M be an n–dimensional, oriented Riemannian manifold. A locally finite chain c is called a *Lipschitz fundamental cycle of M* when c represents the fundamental class of M and $\mathrm{Lip}(c) < \infty$. *The Lipschitz simplicial volume $\|M\|_{\mathrm{Lip}}$ of M* is defined as

$$\|M\|_{\mathrm{Lip}} = \inf\left\{\|c\|_1 \;\middle|\; c \in C_n^{\mathrm{lf}}(M) \text{ is a Lipschitz fundamental cycle of } M\right\}.$$

From the definition of Lipschitz simplicial volume, we have the obvious inequality $\|M\| \le \|M\|_{\mathrm{Lip}}$ for an oriented, Riemannian manifold M. If M is closed, the fundamental cycles in the locally finite chain complex of M involve only a finite number of simplices, and hence, $\|M\| = \|M\|_{\mathrm{Lip}}$. Löh and Sauer [19] prove the proportionality principle for the Lipschitz simplicial volume under the non-positive curvature condition in the non-compact case.

Theorem 2.5 (Löh and Sauer) *Let M and N be complete, non-positively curved Riemannian manifolds of finite volume. Assume that their universal covers are isometric. Then,*

$$\frac{\|M\|_{\mathrm{Lip}}}{\mathrm{Vol}(M)} = \frac{\|N\|_{\mathrm{Lip}}}{\mathrm{Vol}(N)}.$$

By Theorem 2.5, they can show that the Lipschitz simplicial volume of locally symmetric spaces of finite volume and non-compact type is positive and, moreover, they obtain degree theorems for locally symmetric spaces of non-compact type of finite volume, which is originally due to Connell and Farb [6].

Theorem 2.6 (Connell and Farb) *Let M be a locally symmetric n–manifold of non-compact type with finite volume. Assume that M has no local direct factors locally isometric to $\mathbb{H}^2$ or $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$. Then for any complete Riemannian manifold N with finite volume and any proper Lipschitz map $f\colon N \to M$,*

$$\deg(f) \leq C \frac{\mathrm{Vol}(N)}{\mathrm{Vol}(M)}$$

where $C > 0$ depends only on n and the smallest Ricci curvatures of N and M.

Note that the result of Connell and Farb holds with the Lipschitz condition on f. If the positivity of the ordinary simplicial volume of M is verified, one can obtain the degree theorem without the Lipschitz condition on f.

3 $\mathbb{Q}$–rank one locally symmetric spaces

In this section, we recall the definitions of arithmetic lattices, $\mathbb{Q}$–rank, and cusp decomposition in $\mathbb{Q}$–rank one locally symmetric spaces. The cusp decomposition in quotient manifolds by arithmetic lattices is crucial to show the presence of the geodesic straightening map on the locally finite chain complex.

Let X be a connected symmetric space of non-compact type. Let G be the identity component of the isometry group of X. Then, G is a connected semisimple Lie group with trivial center and no compact factors (see Eberlein [10]). We first recall the definition of arithmetic lattice in Zimmer [23].

Definition 3.1 Let G be a connected semisimple Lie group with trivial center and no compact factors. Let $\Gamma \subset G$ be a lattice. Then, Γ is called *arithmetic* if there exist

(i) a semisimple algebraic group $\mathbf{G} \subset \mathrm{GL}(n, \mathbb{C})$ defined over $\mathbb{Q}$ and

(ii) a surjective homomorphism $\rho\colon \mathbf{G}(\mathbb{R})^0 \to G$ with compact kernel

such that $\rho(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^0)$ and Γ are commensurable.

The $\mathbb{Q}$–rank(Γ) is defined as the dimension of any maximal $\mathbb{Q}$–split torus of $\mathbf{G}(\mathbb{Q})$ when Γ is an arithmetic lattice. The structure of the ends of $M = \Gamma\backslash X$ is closely related to the $\mathbb{Q}$–rank(Γ). For instance, a locally symmetric space $M = \Gamma\backslash X$ is compact if and only if the $\mathbb{Q}$–rank(Γ) is zero by the result of Borel and Harish-Chandra. To understand the ends of quotient manifolds by arithmetic lattices, we recall the reduction theory due to Borel and Harish-Chandra [2].

Theorem 3.2 (Borel and Harish-Chandra [2]) *Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$ with associated Riemannian symmetric space X. Let $\mathbf{P}$ be a minimal parabolic $\mathbb{Q}$–subgroup of $\mathbf{G}$ and let Γ be an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. Then*

(i) *the set of double cosets $\mathcal{F} = \Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$ is finite,*

(ii) *there exists a generalized Siegel set $\mathcal{S}_{\omega,\tau}$ such that for a (fixed) set $\{q_i \mid 1 \le i \le m\}$ of representatives of $\mathcal{F}$ the union $\Omega = \bigcup_{i=1}^{m} q_i \cdot \mathcal{S}_{\omega,\tau}$ is a fundamental set for Γ in X.*

By using the reduction theory, Leuzinger [15] gives an explicit differential geometric description of M.

Theorem 3.3 (Leuzinger) *Let X be a Riemannian symmetric space of non-compact type and with $\mathbb{R}$–rank ≥ 2 and let Γ be an irreducible, torsion-free, non-uniform lattice in the isometry group of X. On the locally symmetric space $M = \Gamma\backslash X$ there exists a continuous and piecewise real analytic exhaustion function $h\colon M \to [0,\infty)$ such that, for any $s \ge 0$, the sublevel set $M(s) = \{h \le s\}$ is a compact submanifold with corners of M. Moreover, the boundary of $M(s)$, which is a level set of h, consists of projections of subsets of horospheres in X.*

Note that the exhaustion function $h\colon M \to [0,\infty)$ in Theorem 3.3 is formulated in such a way that each level set of h is the projections of subsets of horospheres in X. More precisely, there exists a union of a countable number of open horoballs $U(s)$ of X such that $X(s) = X - U(s)$ is Γ–invariant and $M(s) = \Gamma\backslash X(s)$. If $s_1 < s_2$, then $U(s_2) \subset U(s_1)$ and the subsets $X(s) = X - U(s)$ exhaust X. These horoballs are in one-to-one correspondence with the vertices of the Tits building of $\mathbf{G}$ over $\mathbb{Q}$. The deleted horoballs are disjoint if and only if Γ is an arithmetic subgroup of a semisimple algebraic group of $\mathbb{Q}$–rank one. In the case of higher $\mathbb{Q}$–rank, the horoballs of $U(s)$ intersect and give rise to corners (see Leuzinger [16]).

Indeed, the theorem of Leuzinger is available for torsion-free arithmetic lattices with $\mathbb{Q}$–rank at least one because the proof in the paper [15] is only based on the reduction theory for arithmetic lattices. He used the arithmeticity theorem of Margulis to show that all irreducible, torsion-free, non-uniform lattices with $\mathbb{R}$–rank≥ 2 are arithmetic lattices.

4 Geodesic straightening map on locally finite chain complex

The geodesic straightening map has played an important role in proving the positivity of the simplicial volume. In this section, we will show the presence of the geodesic straightening map on the locally finite chain complex of $\mathbb{Q}$–rank one locally symmetric spaces.

4.1 Geodesic Straightening

The geodesic straightening map on the level of singular chain complexes was introduced by Thurston [22, Section 6.1]. We recall the definition of the geodesic straightening map.

Let X be a simply connected, complete Riemannian manifold with non-positive sectional curvature. For $x_0, \dots, x_k \in X$, the geodesic simplex $[x_0, \dots, x_k]$ is defined inductively as follows: First, $[x_0]$ is the point $x_0 \in X$, and $[x_0, x_1]$ is the unique geodesic arc from x_1 to x_0. In general, $[x_0, \dots, x_k]$ is the geodesic cone on $[x_0, \dots, x_{k-1}]$ with the top point x_k.

Definition 4.1 Let M be a connected, complete Riemannian manifold with non-positive sectional curvature. Then, *the geodesic straightening map* $st_*\colon C_*(M) \to C_*(M)$ is defined by

$$st_k(\sigma) = \pi_M \circ [\widetilde{\sigma}(e_0), \dots, \widetilde{\sigma}(e_k)] \text{ for a singular } k\text{–simplex } \sigma,$$

where $\pi_M\colon \widetilde{M} \to M$ is the universal covering, $e_0, \dots, e_k$ are the vertices of the standard k–simplex Δ^k, and $\widetilde{\sigma}$ is a lift of σ to the universal cover $\widetilde{M}$.

The following proposition proved by Thurston [22, Chapter 6] makes it possible to obtain the simplicial volume of M by considering only the ℓ^1–norm on the geodesically straight chains of M.

Proposition 4.2 (Thurston [22]) *Let M be a connected, complete Riemannian manifold with non-positive sectional curvature. Then, the geodesic straightening map is chain homotopic to the identity.*

4.2 Straightening locally finite chains

Let M be a $\mathbb{Q}$–rank one locally symmetric space. First, we fix some notations. Let X denote the universal cover of M and Γ the fundamental group of M. Let $h\colon M \to [0, \infty)$ be the exhaustion function of M under Leuzinger's theorem in [15]. For any $s \geq 0$, M admits the following disjoint decomposition

$$M = M(s) \cup \coprod_{i=1}^{l} E_i(s),$$

where the sublevel set $M(s) = \{h \leq s\}$ is a compact submanifold and $E_i(s)$ is a cusp end of $M - M(s)$ for each $i = 1, \dots, l$. Note that for all $i = 1, \dots, l$ and $s \geq 0$, each cusp end $E_i(s)$ is the quotient of an open horoball in X by the formulation of the exhaustion function $h\colon M \to [0,\infty)$ in Theorem 3.3. More precisely, there is a countable number of pairwise disjoint horoballs $U(s)$ in X such that each $E_i(s)$ is obtained by the quotient of an open horoball in $U(s)$ by Γ and $M(s)=\Gamma\backslash(X-U(s))$. Thus, $E_i(s)$ is geodesically convex for each $i=1,\dots,l$.

Lemma 4.3 *Let M be a $\mathbb{Q}$–rank one locally symmetric space. Then, the geodesic straightening of a locally finite chain in $C_*^{\mathrm{lf}}(M)$ is a locally finite chain.*

Proof Let $A \in S_k^{\mathrm{lf}}(M)$. Let us define $st_k(A)=\{st_k(\sigma) \mid \sigma \in A\}$. In order to prove this Lemma, it is sufficient to prove that $st_k(A)$ is locally finite. Let K be a compact subset of M. Then, one can choose a compact submanifold $M(s)$ of M containing K for some $s>0$.

By the local finiteness of A, a compact submanifold $M(s)$ intersects the image of only a finite number of elements of A. Let σ be an element of A whose image does not intersect $M(s)$. Then, we claim that the image of σ has to be contained in only one cusp end of $M-M(s)$. Suppose the image of σ intersects at least two cusp ends of $M-M(s)$, denoted by $E_1(s)$ and $E_2(s)$. Since the image of σ is path-connected, there is a path in the image of σ connecting two different points contained in $E_1(s)$ and $E_2(s)$, respectively. However, any path connecting such two points must passes through $M(s)$. This means that the image of σ intersects $M(s)$, which contradicts the assumption that the image of σ does not intersect $M(s)$.

Now, let's assume that the image of σ is contained in $E_1(s)$. Since $E_1(s)$ is geodesically convex and the image of σ is contained in $E_1(s)$, the image of geodesic straightening $st_k(\sigma)$ of σ is also contained in $E_1(s)$. This implies that the image of $st_k(\sigma)$ does not intersect both $M(s)$ and K. Hence, we conclude that K can intersect the image of $st_k(\tau)$ for only a finite number of elements τ of A intersecting $M(s)$, and so $st_k(A)$ is locally finite. This completes the proof of the lemma. □

By Lemma 4.3, the geodesic straightening map is well defined on the locally finite chain complex of M:

$$st_*^{\mathrm{lf}}\colon C_*^{\mathrm{lf}}(M) \to C_*^{\mathrm{lf}}(M).$$

The map st_*^{lf} is obviously a chain map because it is induced from the geodesic straightening map on the singular chain complex of M. Furthermore, we prove that it is chain homotopic to the identity as follows.

Proposition 4.4 *Let M be a $\mathbb{Q}$–rank one locally symmetric space. Then the geodesic straightening map st_*^{lf} is chain homotopic to the identity.*

Proof First we recall the construction of the chain homotopy $H_*\colon C_*(M)\to C_{*+1}(M)$ from the geodesic straightening map to the identity. The chain homotopy H_k is defined by the straight line homotopy between any k–simplex and its geodesically straight simplex. Moreover, these homotopies, when restricted to lower dimensional faces, agree with the homotopies canonically defined on those faces. For more details, let L_σ be a canonical straight line homotopy

$$L_\sigma\colon \Delta^k\times[0,1]\to M$$

from σ to $st_k(\sigma)$ for any k–simplex σ. Now $\Delta^k\times[0,1]$ has vertices

$$a_0=(e_0,0),\dots,a_k=(e_k,0),b_0=(e_0,1),\dots,b_k=(e_k,1).$$

For each $i=0,\dots,k$, let

$$\alpha_i\colon \Delta^{k+1}\to\Delta^k\times[0,1]$$

be the affine map that maps $e_0,\dots,e_{k+1}$ to $a_0,\dots,a_i,b_i,\dots,b_k$, respectively. Define linear transformation

$$H_k\colon C_k(M;\mathbb{R})\to C_{k+1}(M;\mathbb{R})$$

by the formula

$$H_k(\sigma)=\sum_{i=0}^{k}(-1)^i L_\sigma\circ\alpha_i.$$

This H_* is the chain homotopy from the geodesic straightening map to the identity on the singular chain complex of M.

Let $c=\sum_{\sigma\in A}a_\sigma\sigma$ be a locally finite chain. We claim that $H_*(c)$ is a locally finite chain again. Let K be a compact subset of M. Recall the disjoint decomposition of M by the exhaustion function $h\colon M\to[0,\infty)$ of M in Theorem 3.3 as we describe in the first part of this section. Choose a compact submanifold $M(s)$ containing K for some $s>0$. Suppose that the image of $\sigma\in A$ does not intersect $M(s)$. By a similar argument in Lemma 4.3, the images of both σ and $st_k(\sigma)$ are contained in only one cusp end, denoted by $E(s)$.

Note that $E(s)$ is geodesically convex since $E(s)$ is the quotient of an open horoball. Hence, the image of the straight line homotopy $H_k(\sigma)$ between σ and $st_k(\sigma)$ is contained in $E(s)$. This means that $M(s)$ does not intersect the image of $H_k(\sigma)$ and K does not either. Therefore, K can intersect the image of $H_k(\sigma)$ for only a finite number of elements σ of A intersecting $M(s)$. Since $H_k(\sigma)$ is a finite sum of simplices, K intersects the image of only a finite number of simplices in $H_k(c)$.

In other words, $H_k(c)$ is a locally finite chain. Finally, we obtain the following well-defined map:

$$H_k^{\mathrm{lf}}\colon C_k^{\mathrm{lf}}(M) \to C_{k+1}^{\mathrm{lf}}(M)$$

satisfying

$$\partial_{k+1} H_k^{\mathrm{lf}} + H_{k-1}^{\mathrm{lf}} \partial_k = st_k^{\mathrm{lf}} - id.$$

Therefore, H_*^{lf} is a chain homotopy from st_*^{lf} to the identity. $\square$

As can be seen in the proof of Lemma 4.3 and Proposition 4.4, $\mathbb{Q}$–rank one condition on M is essential to obtain the geodesic straightening map on the locally finite chain complex of M. Since the cusp end of higher $\mathbb{Q}$–rank locally symmetric space has corners, it is not geodesically convex. Hence, Lemma 4.3 and Proposition 4.4 fail in the case of a higher $\mathbb{Q}$–rank locally symmetric space.

5 Positivity of the simplicial volume

Now, we will prove the positivity of the simplicial volume of $\mathbb{Q}$–rank one locally symmetric spaces covered by the product of $\mathbb{R}$–rank one symmetric spaces.

Let X_i be a complete, simply-connected, n_i–dimensional Riemannian manifold with negative sectional curvature bounded away from zero for each $i = 1, \dots, k$. Let X be the n–dimensional product manifold $X_1 \times \cdots \times X_k$ with the product metric. Now, we prove that the volume of geodesic n–simplices in X is uniformly bounded from above.

Lemma 5.1 *The volume of geodesic n–simplices in X is uniformly bounded from above.*

Proof Let $[x_0, \dots, x_n]$ be a geodesic n–simplex for an ordered vertex set

$$\{x_0, \dots, x_n\} \subset X.$$

Let $p_i\colon X_1 \times \cdots \times X_k \to X_i$ denote the projection map from X onto X_i for each $i = 1, \dots, k$. Then, we have

$$\mathrm{Vol}([x_0, \dots, x_n]) \leq \prod_{i=1}^{k} \mathrm{Vol}(p_i([x_0, \dots, x_n])). \tag{5-1}$$

From Equation (5-1), it suffices to show that the volume of $p_i([x_0, \dots, x_n])$ in X_i is uniformly bounded from above for each $i = 1, \dots, k$. First, note that $p_i([x_0, \dots, x_n]) = [p_i(x_0), \dots, p_i(x_n)]$ because a geodesic in X projects to a geodesic in X_i by the projection map $p_i\colon X \to X_i$. In other words, $p_i([x_0, \dots, x_n])$ is a geodesic n–simplex in X_i. When $n \geq n_i$, a geodesic n–simplex in X_i consists of at most $\binom{n}{n_i}$ geodesic n_i–simplices in X_i. More precisely, for a geodesic n–simplex $[y_0, \ldots, y_n]$ in X_i, we have

$$[y_0, \ldots, y_n] = \bigcup_{0 \leq l_0 < \cdots < l_{n_i} \leq n} [y_{l_0}, \ldots, y_{l_{n_i}}].$$

Since X_i has negative sectional curvature bounded away from zero for all $i = 1, \ldots, k$, the volume of geodesic n_i–simplices in X_i is uniformly bounded from above for all $i = 1, \ldots, k$ (see Inoue and Yano [13]). Hence, there exists a uniform bound on the volume of geodesic n–simplices in X_i for all $i = 1, \ldots, k$. By Inequality (5-1), this implies that the volume of geodesic n–simplices in X is uniformly bounded from above, which completes the proof. □

We now prove the main theorems. Let M be an n–dimensional Riemannian manifold. Then, the evaluation map

$$\langle \cdot, \cdot \rangle \colon C^*(M) \otimes C_*(M) \longrightarrow \mathbb{R}$$

is well defined and it induces the Kronecker product on $H^*(M) \otimes H_*(M)$. Let $K \subset M$ be a compact, connected subset with non-empty interior. Let $\Omega^*(M, M-K)$ be the kernel of the restriction homomorphism $\Omega^*(M) \to \Omega^*(M-K)$ on differential forms. The corresponding cohomology groups are denoted by $H^*_{dR}(M, M-K)$. The de Rham map $\Omega^*(M) \to C^*(M)$ restricts to the respective kernels and, thus, induces a homomorphism, called relative de Rham map,

$$\Psi^* \colon H^*_{dR}(M, M-K) \to H^*(M, M-K).$$

The relative de Rham map is an isomorphism. Note that integration gives a homomorphism $\int \colon H^*_{dR}(M, M-K) \to \mathbb{R}$. Moreover, it is well known that

$$\langle \Psi^n[\omega], [M, M-K] \rangle = \int_M \omega$$

holds for all n–forms ω (see Dupont [9]).

Let dvol_M be the Riemannian volume form on M. One can think of dvol_M as a singular cochain in $C^n(M)$ by the de Rham map. Then, for a smooth singular n–simplex $\tau \colon \Delta^n \to M$,

$$\langle \mathrm{dvol}_M, \tau \rangle = \int_{\Delta^n} \tau^* \mathrm{dvol}_M.$$

Note that every geodesic simplex in M is smooth if M is a connected, complete Riemannian manifold with non-positive sectional curvature.

Proposition 5.2 *Let M be an n–dimensional, oriented, $\mathbb{Q}$–rank one locally symmetric space covered by the product of $\mathbb{R}$–rank one symmetric spaces. Let $c=\sum_{k\in\mathbb{N}} a_k\sigma_k$ be a fundamental cycle of M with $\|c\|_1<\infty$. Then,*

$$\sum_{k\in\mathbb{N}} a_k\cdot\langle \mathrm{dvol}_M, st_n(\sigma_k)\rangle = \mathrm{Vol}(M).$$

Proof Every geodesic simplex in M is smooth and the volume of geodesic simplices in M is uniformly bounded from above by Lemma 5.1. Hence, there exists a uniform constant $C>0$ such that we have $|\langle \mathrm{dvol}_M, st_n(\sigma)\rangle|\le C$ for any singular simplex σ in M. From this inequality, one can see that $\sum_{k\in\mathbb{N}} a_k\cdot\langle \mathrm{dvol}_M, st_n(\sigma_k)\rangle$ converges absolutely. Since st_*^{lf} is chain homotopic to the identity, $[st_n^{\mathrm{lf}}(c)]$ is also a fundamental class of M.

Let $K\subset M$ be a connected, compact subset with non-empty interior. For $\delta\in\mathbb{R}_{>0}$, let $g_\delta\colon M\to[0,1]$ be a smooth function supported on the closed δ–neighborhood K_δ of K with $g_\delta|_K=1$. Then, $g_\delta \mathrm{dvol}_M\in\Omega^n(M,M-K_\delta)$ is a cocycle and

$$\mathrm{Vol}(K)=\lim_{\delta\to 0}\int_M g_\delta \mathrm{dvol}_M.$$

The map $H_n(i_\delta)\colon H_n^{\mathrm{lf}}(M)\to H_n(M,M-K_\delta)$ induced by the inclusion $i_\delta:(M,\varnothing)\to(M,M-K_\delta)$ maps the fundamental class of M to the relative fundamental class $[M,M-K_\delta]$ of $(M,M-K_\delta)$, and $H_n(i_\delta)[st_n^{\mathrm{lf}}(c)]$ is represented by

$$\sum_{\mathrm{im}\sigma_k\cap K_\delta\neq\phi} a_k st_n(\sigma_k).$$

Since $H_n(i_\delta)[st_n^{\mathrm{lf}}(c)]$ is also the relative fundamental class of $(M,M-K_\delta)$, we have

$$\begin{aligned}\lim_{\delta\to 0}\sum_{\mathrm{im}\sigma_k\cap K_\delta\neq\phi} a_k\cdot\langle g_\delta \mathrm{dvol}_M, st_n(\sigma_k)\rangle &= \lim_{\delta\to 0}\langle \Psi^n[g_\delta \mathrm{dvol}_M],[M,M-K_\delta]\rangle\\ &=\lim_{\delta\to 0}\int_M g_\delta \mathrm{dvol}_M\\ &=\mathrm{Vol}(K).\end{aligned}$$

For each $k\in\mathbb{N}$ and $\delta\in\mathbb{R}_{>0}$, we also have a uniform upper bound

$$|\langle g_\delta \mathrm{dvol}_M, st_n(\sigma_k)\rangle|\le C,$$

and hence

$$\left|\sum_{k\in\mathbb{N}} a_k\cdot\langle \mathrm{dvol}_M, st_n(\sigma_k)\rangle - \sum_{\mathrm{im}\sigma_k\cap K_\delta\neq\phi} a_k\cdot\langle g_\delta \mathrm{dvol}_M, st_n(\sigma_k)\rangle\right|\le 2C\sum_{\mathrm{im}\sigma_k\subset M-K}|a_k|.$$

Because $\sum_{k\in\mathbb{N}} |a_k| < \infty$, there is an exhausting sequence $(K^m)_{m\in\mathbb{N}}$ of compact, connected subsets of M with non-empty interior satisfying

$$\lim_{m\to\infty} \mathrm{Vol}(K^m) = \mathrm{Vol}(M) \text{ and } \lim_{m\to\infty} \sum_{\mathrm{im}\sigma_k \subset M - K^m} |a_k| = 0.$$

Thus, the estimates of the previous paragraphs yield

$$\begin{aligned}\sum_{k\in\mathbb{N}} a_k \cdot \langle \mathrm{dvol}_M, st_n(\sigma_k)\rangle &= \lim_{m\to\infty} \lim_{\delta\to 0} \sum_{\mathrm{im}\sigma_k \cap K^m_\delta \neq \phi} a_k \cdot \langle g^m_\delta \mathrm{dvol}_M, st_n(\sigma_k)\rangle \\ &= \lim_{m\to\infty} \mathrm{Vol}(K^m) \\ &= \mathrm{Vol}(M),\end{aligned}$$

which establishes the formula. □

From Proposition 5.2, the positivity of the simplicial volume of $\mathbb{Q}$–rank one locally symmetric spaces covered by the product of $\mathbb{R}$–rank one symmetric spaces is directly obtained as follows.

Theorem 5.3 *Let M be a $\mathbb{Q}$–rank one locally symmetric space covered by the product of $\mathbb{R}$–rank one symmetric spaces. Then, the simplicial volume of M is positive.*

Proof To prove this theorem, we can assume that M is orientable since if M is not orientable, its simplicial volume is defined by

$$\|M\| = \tfrac{1}{2}\|\widetilde{M}\|,$$

where $\widetilde{M}$ is the orientable double covering of M. Also, it is enough to consider fundamental cycles with finite ℓ^1–norms.

Let $c = \sum_{k\in\mathbb{N}} a_k\sigma_k \in C_n^{\mathrm{lf}}(M)$ be a fundamental cycle with $\|c\|_1 < \infty$. From Proposition 5.2, we have

$$\mathrm{Vol}(M) = \sum_{k\in\mathbb{N}} a_k \cdot \langle \mathrm{dvol}_M, st_n(\sigma_k)\rangle.$$

A uniform upper bound of the volume of geodesic simplices in M yields the inequality

$$\mathrm{Vol}(M) \leq C \cdot \sum_{k\in\mathbb{N}} |a_k|,$$

where $C > 0$ depends only on the universal cover of M. Dividing and passing to the infimum over all fundamental cycles, it provides the positive lower bound

$$\|M\| \geq \mathrm{Vol}(M)/C > 0.$$

Therefore, we conclude that the simplicial volume of M is positive. □

Gromov [12, Section 0.5] provides a lower bound for the minimal volume minvol(M), which is defined as the infimum of volumes over all complete Riemannian metrics on M with sectional curvatures bounded between -1 and 1, in terms of the simplicial volume of an n–dimensional smooth manifold M:

$$\|M\| \leq (n-1)^n n! \cdot \mathrm{minvol}(M). \tag{5-2}$$

For a compact Riemannian manifold M, Besson, Courtois and Gallot [1] improve the lower bound for the minimal volume of M as follows:

$$\mathrm{minvol}(M) \geq \frac{n^{n/2}}{(n-1)^n (n!)} \|M\|.$$

By Inequality (5-2) and Theorem 5.3, we have an immediate corollary.

Corollary 5.4 *The minimal volume of $\mathbb{Q}$–rank one locally symmetric spaces covered by the product of $\mathbb{R}$–rank one symmetric spaces is positive.*

See the works of Connell and Farb [7; 6] for different approach using Lipschitz maps.

6 Degree theorem

For any proper map

$$f\colon N \to M$$

between complete, finite volume Riemannian manifolds, a locally finite fundamental cycle is mapped to a locally finite cycle. Hence, usual inequality

$$\deg(f) \cdot \|M\| \leq \|N\| \tag{6-1}$$

holds.

Theorem 6.1 *Let N be a complete Riemannian n–dimensional manifold of finite volume with Ricci curvature bounded below by $-(n-1)$ and M be a $\mathbb{Q}$–rank one locally symmetric space covered by the product of $\mathbb{R}$–rank one symmetric spaces. For any proper map $f\colon N \to M$ we have*

$$\deg(f) \leq C_n \frac{\mathrm{Vol}(N)}{\mathrm{Vol}(M)},$$

where C_n depends only on n.

Proof By Gromov,

$$\|N\| \leq (n-1)^n n! \cdot \mathrm{Vol}(N).$$

Since

$$\mathrm{Vol}(M) \leq C \cdot \|M\|,$$

from Inequality (6-1), we get

$$\frac{\deg(f)}{C} \cdot \mathrm{Vol}(M) \leq \deg(f) \cdot \|M\| \leq \|N\| \leq (n-1)^n n! \cdot \mathrm{Vol}(N).$$

Since, for a given dimension n, there are only finitely many symmetric spaces, the constant C can be chosen in such a way that it depends only on n. □

Such a degree theorem is known by Connell and Farb [7; 6] and Löh and Sauer [19] for proper Lipschitz map f with the sectional curvature of N bounded above by 1 and any n–dimensional locally symmetric manifold M of finite volume. Note that they obtained the degree theorem for proper Lipschitz map f by verifying the positivity of the Lipschitz simplicial volume of M. Our result about the positivity of the ordinary simplicial volume of M yields the degree theorem without any Lipschitz condition on map f.

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