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On the augmentation quotients of the IA-automorphism group of a free group

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We study the augmentation quotients of the IA-automorphism group of a free group and a free metabelian group. First, for any group G, we construct a lift of the k–th Johnson homomorphism of the automorphism group of G to the k–th augmentation quotient of the IA-automorphism group of G. Then we study the images of these homomorphisms for the case where G is a free group and a free metabelian group. As a corollary, we detect a $\mathbb{Z}$–free part in each of the augmentation quotients, which can not be detected by the abelianization of the IA-automorphism group.

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1 Introduction

Let F_n be a free group of rank $n \geq 2$, and $\mathrm{Aut}\, F_n$ the automorphism group of F_n. Let $\rho\colon \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ denote the natural homomorphism induced from the abelianization $F_n \to H$. The kernel of ρ is called the IA-automorphism group of F_n, denoted by IA_n. The subgroup IA_n reflects much of the richness and complexity of the structure of $\mathrm{Aut}\, F_n$ and plays important roles in various studies of $\mathrm{Aut}\, F_n$. Although the study of the IA-automorphism group has a long history since its finitely many generators were obtained by Magnus [13] in 1935, the combinatorial group structure of IA_n is still quite complicated. For instance, no presentation for IA_n is known in general.

We have studied IA_n mainly using the Johnson filtration of $\mathrm{Aut}\, F_n$ so far. The Johnson filtration is one of a descending central series

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

consisting of normal subgroups of $\mathrm{Aut}\, F_n$, whose first term is IA_n. (For details, see Section 2.3.) Each graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ naturally has a $\mathrm{GL}(n, \mathbf{Z})$–module structure, and from it we can extract some valuable information about IA_n. For example, $\mathrm{gr}^1(\mathcal{A}_n)$ is just the abelianization of IA_n due to Cohen and Pakianathan [6; 7], Farb [8] and Kawazumi [12]. Pettet [18] determined the image of the cup product $\cup_{\mathbf{Q}}\colon \Lambda^2 H^1(\mathrm{IA}_n, \mathbf{Q}) \to H^2(\mathrm{IA}_n, \mathbf{Q})$ by using the $\mathrm{GL}(n, \mathbf{Q})$–module

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structure of $\mathrm{gr}^2(\mathcal{A}_n) \otimes_{\mathbf{Z}} \mathbf{Q}s$. At the present stage, however, the structures of the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ are far from well-known.

On the other hand, compared with the Johnson filtration, the lower central series $\Gamma_{\mathrm{IA}_n}(k)$ of IA_n and its graded quotients $\mathcal{L}_{\mathrm{IA}_n}(k) := \Gamma_{\mathrm{IA}_n}(k)/\Gamma_{\mathrm{IA}_n}(k+1)$ are somewhat easier to handle since we can obtain finitely many generators of $\mathcal{L}_{\mathrm{IA}_n}(k)$ using the Magnus generators of IA_n. Since the Johnson filtration is central, $\Gamma_{\mathrm{IA}_n}(k) \subset \mathcal{A}_n(k)$ for any $k \geq 1$. Andreadakis conjectured that $\Gamma_{\mathrm{IA}_n}(k) = \mathcal{A}_n(k)$ for each $k \geq 1$ and showed $\Gamma_{\mathrm{IA}_2}(k) = \mathcal{A}_2(k)$ for each $k \geq 1$. It is currently known that $\Gamma_{\mathrm{IA}_n}(2) = \mathcal{A}_n(2)$ due to Bachmuth [2], and that $\Gamma_{\mathrm{IA}_n}(3)$ has at most finite index in $\mathcal{A}_n(3)$ due to Pettet [18].

In this paper, we consider the augmentation quotients of IA_n. Let $\mathbf{Z}[G]$ be the integral group ring of a group G, and $\Delta(G)$ the augmentation ideal of $\mathbf{Z}[G]$. We denote by $Q^k(G) := \Delta^k(G)/\Delta^{k+1}(G)$ the k–th augmentation quotient of G. The augmentation quotients $Q^k(\mathrm{IA}_n)$ of IA_n seem to be closely related to the lower central series $\Gamma_{\mathrm{IA}_n}(k)$ as follows. If Andreadakis' conjecture is true, then each of the graded quotients $\mathcal{L}_{\mathrm{IA}_n}(k)$ is free abelian. Using a work of Sandling and Tahara [20] (for details, see Section 4.1), we obtain a conjecture for the $\mathbf{Z}$–module structure of $Q^k(\mathrm{IA}_n)$:

Conjecture 1 *For any $k \geq 1$,*

$$Q^k(\mathrm{IA}_n) \cong \sum \bigotimes_{i=1}^{k} S^{a_i}(\mathcal{L}_{\mathrm{IA}_n}(i))$$

as a $\mathbf{Z}$–module. Here the sum runs over all nonnegative integers $a_1, \ldots, a_k$ such that $\sum_{i=1}^k i a_i = k$, and $S^a(M)$ means the symmetric tensor product of a $\mathbf{Z}$–module M such that $S^0(M) = \mathbf{Z}$.

We see that this is true for $k = 1$ and 2 from a general argument in group ring theory. (For $k = 2$, see (1) below.) For $k \geq 3$, however, it is still an open problem. In general, one of the most standard methods to study the augmentation quotients $Q^k(\mathrm{IA}_n)$ is to consider a natural surjective homomorphism $\pi_k \colon Q^k(\mathrm{IA}_n) \to Q^k(\mathrm{IA}_n^{\mathrm{ab}})$ induced from the abelianization $\mathrm{IA}_n \to \mathrm{IA}_n^{\mathrm{ab}}$ of IA_n. Furthermore, since $\mathrm{IA}_n^{\mathrm{ab}}$ is free abelian, we have a natural isomorphism $Q^k(\mathrm{IA}_n^{\mathrm{ab}}) \cong S^k(\mathcal{L}_{\mathrm{IA}_n}(1))$. Hence, in the conjecture above, we can detect $S^k(\mathcal{L}_{\mathrm{IA}_n}(1))$ in $Q^k(\mathrm{IA}_n)$ by the abelianization of IA_n.

Then we have a natural problem to consider: Determine the structure of the kernel of π_k. More precisely, clarify the $\mathrm{GL}(n, \mathbf{Z})$–module structure of $\mathrm{Ker}(\pi_k)$. In order to attack this problem, in this paper we construct and study a certain homomorphism defined on $Q^k(\mathrm{IA}_n)$ whose restriction to $\mathrm{Ker}(\pi_k)$ is nontrivial. For a group G, let

$\alpha_k = \alpha_{k,G}\colon \mathcal{L}_G(k) \to Q^k(G)$ be a homomorphism defined by $\sigma \mapsto \sigma - 1$. One of the main purposes of the paper is to construct a $\mathrm{GL}(n, \mathbf{Z})$–equivariant homomorphism

$$\mu_k\colon Q^k(\mathrm{IA}_n) \to \mathrm{Hom}_{\mathbf{Z}}(H, \alpha_{k+1}(\mathcal{L}_n(k+1))),$$

where $\mathcal{L}_n(k)$ is the k–th graded quotient of the lower central series of F_n. Furthermore, for the k–th Johnson homomorphism

$$\tau'_k\colon \mathcal{L}_{\mathrm{IA}_n}(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$$

defined by $\sigma \mapsto (x \mapsto x^{-1}x^{\sigma})$ (see Section 2.3 for details), we show that $\mu_k \circ \alpha_k = \alpha^*_{k+1} \circ \tau'_k$ where α^*_{k+1} is a natural homomorphism induced from α_{k+1}. Since α_{k,F_n} is a $\mathrm{GL}(n, \mathbf{Z})$–equivariant injective homomorphism for each $k \geq 1$, if we identify $\mathcal{L}_n(k)$ with its image $\alpha_k(\mathcal{L}_n(k))$, we obtain $\mu_k \circ \alpha_k = \tau'_k$. Hence, the homomorphism μ_k can be considered as a lift of the Johnson homomorphism τ'_k. In the following, we naturally identify $\mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ with $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ for $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$.

Historically, the study of the Johnson homomorphisms was originally begun in 1980 by D Johnson [10] who determined the abelianization of the Torelli subgroup of the mapping class group of a surface in [11]. Now, there is a broad range of remarkable results for the Johnson homomorphisms of the mapping class group. (For example, see Hain [9] and Morita [14; 15; 16].) These works also inspired the study of the Johnson homomorphisms of $\mathrm{Aut}\, F_n$. Using it, we can investigate the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ and $\mathcal{L}_{\mathrm{IA}_n}(k)$. Recently, good progress has been achieved by many authors, for example, Cohen and Pakianathan [6; 7], Farb [8], Kawazumi [12], Morita [14; 15; 16] and Pettet [18]. In particular, in our previous work [23], we determined the cokernel of the rational Johnson homomorphism $\tau'_{k,\mathbf{Q}} := \tau'_k \otimes \mathrm{id}_{\mathbf{Q}}$ for $2 \leq k \leq n-2$.

The main theorem of the paper is:

Theorem 1 (See Theorem 4.4.) *For $3 \leq k \leq n-2$, the $\mathrm{GL}(n, \mathbf{Z})$–equivariant homomorphism*

$$\mu_k \oplus \pi_k\colon Q^k(\mathrm{IA}_n) \to (H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n(k+1))) \oplus Q^k(\mathrm{IA}_n^{\mathrm{ab}})$$

defined by $\sigma \mapsto (\mu_k(\sigma), \pi_k(\sigma))$ is surjective.

Next, we consider the framework above for a free metabelian group. Let $F_n^M := F_n/[[F_n, F_n], [F_n, F_n]]$ be a free metabelian group of rank n. By the same argument as the free group case, we can consider the IA-automorphism group IA_n^M and the Johnson homomorphism

$$\tau'_k\colon \mathcal{L}_{\mathrm{IA}_n^M}(k) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^M(k+1)$$

of $\mathrm{Aut}\,F_n^M$ where $\mathcal{L}_{\mathrm{IA}_n^M}(k)$ is the k–th graded quotient of the lower central series of IA_n^M, and $\mathcal{L}_n^M(k)$ is that of F_n^M. In our previous work [22], we studied the Johnson homomorphism of $\mathrm{Aut}\,F_n^M$ and determined its cokernel. In particular, we showed that there appears only the Morita obstruction $S^k H$ in $\mathrm{Coker}(\tau_k')$ for any $k \geq 2$ and $n \geq 4$. We remark that in [22], we determined the cokernel of the Johnson homomorphism τ_k which is defined on the graded quotient of the Johnson filtration of $\mathrm{Aut}\,F_n^M$. Observing our proof, we verify that $\mathrm{Coker}(\tau_k') = \mathrm{Coker}(\tau_k)$.

Now, similarly to the free group case, we can also construct a $\mathrm{GL}(n,\mathbf{Z})$–equivariant homomorphism

$$\mu_k\colon Q^k(\mathrm{IA}_n^M) \to \mathrm{Hom}_{\mathbf{Z}}(H, \alpha_{k+1}(\mathcal{L}_n^M(k+1)))$$

such that $\mu_k \circ \alpha_k = \alpha_{k+1}^* \circ \tau_k'$. The second purpose of the paper is to show:

Theorem 2 (See Theorem 5.3.) *For $k \geq 2$ and $n \geq 4$, the $\mathrm{GL}(n,\mathbf{Z})$–equivariant homomorphism*

$$\mu_k \oplus \pi_k\colon Q^k(\mathrm{IA}_n^M) \to (H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))) \oplus S^k((\mathrm{IA}_n^M)^{\mathrm{ab}})$$

defined by $\sigma \mapsto (\mu_k(\sigma), \pi_k(\sigma))$ is surjective.

In this paper, for an arbitrary group G, we construct a lift of the Johnson homomorphism of the automorphism group of G to the augmentation quotients of G. In order to do this, in Section 2, after fixing notation and conventions, we recall the associated graded Lie algebra of a group G, the Johnson homomorphism of the automorphism group of G, and the associated graded ring of the integral group ring $\mathbf{Z}[G]$ of G. In Section 3, we construct an $\mathrm{Aut}\,G/\mathrm{IA}(G)$–equivariant homomorphism μ_k which is considered as a lift of the Johnson homomorphism. In Sections 4 and 5, we consider the case where G is a free group and a free metabelian group respectively.

2 Preliminaries

2.1 Notation and conventions

Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G.

- The abelianization of G is denoted by G^{ab}.
- The group $\mathrm{Aut}\,G$ of G acts on G from the right. For any $\sigma \in \mathrm{Aut}\,G$ and $x \in G$, the action of σ on x is denoted by x^σ.

- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion.
- For elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2.2 Associated graded Lie algebra of a group

For a group G, we define the lower central series of G by the rule

$$\Gamma_G(1) := G, \quad \Gamma_G(k) := [\Gamma_G(k-1), G], \quad k \geq 2.$$

We denote by $\mathcal{L}_G(k) := \Gamma_G(k)/\Gamma_G(k+1)$ the graded quotient of the lower central series of G, and by $\mathcal{L}_G := \bigoplus_{k \geq 1} \mathcal{L}_G(k)$ the associated graded sum. The graded sum $\mathcal{L}_G$ naturally has a graded Lie algebra structure induced from the commutator bracket on G, and called the associated graded Lie algebra of G.

For any $g_1, \ldots, g_t \in G$, a commutator of weight k type of

$$[[\cdots[[g_{i_1}, g_{i_2}], g_{i_3}], \ldots], g_{i_k}], \quad i_j \in \{1, \ldots, t\},$$

with all of its left brackets to the left of all the elements occurring is called a simple k–fold commutator among the components $g_1, \ldots, g_t$, and we denote it by

$$[g_{i_1}, g_{i_2}, \ldots, g_{i_k}]$$

for simplicity. In general, if G is generated by $g_1, \ldots, g_t$, then the graded quotient $\mathcal{L}_G(k)$ is generated by the simple k–fold commutators

$$[g_{i_1}, g_{i_2}, \ldots, g_{i_k}], \quad 1 \leq i_j \leq t,$$

as a $\mathbf{Z}$–module.

Let $\rho_G\colon \operatorname{Aut} G \to \operatorname{Aut} G^{\mathrm{ab}}$ be the natural homomorphism induced from the abelianization of G. The kernel $\mathrm{IA}(G)$ of ρ_G is called the IA-automorphism group of G. Then the automorphism group $\operatorname{Aut} G$ naturally acts on $\mathcal{L}_G(k)$ for each $k \geq 1$, and $\mathrm{IA}(G)$ acts on it trivially. Hence the action of $\operatorname{Aut} G/\mathrm{IA}(G)$ on $\mathcal{L}_G(k)$ is well-defined.

2.3 Johnson homomorphisms

For $k \geq 1$, the action of $\operatorname{Aut} G$ on each nilpotent quotient $G/\Gamma_G(k+1)$ induces a homomorphism

$$\operatorname{Aut} G \to \operatorname{Aut}(G/\Gamma_G(k+1)).$$

For $k=1$, this homomorphism is just ρ_G. We denote the kernel of the homomorphism above by $\mathcal{A}_G(k)$. Then the groups $\mathcal{A}_G(k)$ define a descending central filtration

$$\mathrm{IA}_G = \mathcal{A}_G(1) \supset \mathcal{A}_G(2) \supset \mathcal{A}_G(3) \supset \cdots .$$

(See Andreadakis [1] for details.) We call it the Johnson filtration of $\mathrm{Aut}\, G$. For each $k \geq 1$, the group $\mathrm{Aut}\, G$ acts on $\mathcal{A}_G(k)$ by conjugation, and it naturally induces an action of $\mathrm{Aut}\, G/\mathrm{IA}(G)$ on $\mathrm{gr}^k(\mathcal{A}_G)$. The graded sum $\mathrm{gr}(\mathcal{A}_G) := \bigoplus_{k\geq 1} \mathrm{gr}^k(\mathcal{A}_G)$ has a graded Lie algebra structure induced from the commutator bracket on $\mathrm{IA}(G)$.

To study the $\mathrm{Aut}\, G/\mathrm{IA}(G)$–module structure of each graded quotient $\mathrm{gr}^k(\mathcal{A}_G)$, we define the Johnson homomorphisms of $\mathrm{Aut}\, G$ in the following way. For each $k \geq 1$, we consider a homomorphism $\mathcal{A}_G(k) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, \mathcal{L}_G(k+1))$ defined by

$$\sigma \mapsto (g \mapsto g^{-1} g^{\sigma}), \quad x \in G.$$

Then the kernel of this homomorphism is just $\mathcal{A}_G(k+1)$. Hence it induces an injective homomorphism

$$\tau_k = \tau_{G,k} \colon \mathrm{gr}^k(\mathcal{A}_G) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, \mathcal{L}_G(k+1)).$$

The homomorphism τ_k is called the k–th Johnson homomorphism of $\mathrm{Aut}\, G$. It is easily seen that each τ_k is an $\mathrm{Aut}\, G/\mathrm{IA}(G)$–equivariant homomorphism. Since each Johnson homomorphism τ_k is injective, it is natural question to determine the image, or equivalently, the cokernel of τ_k in the study of the $\mathrm{Aut}\, G/\mathrm{IA}(G)$–module $\mathrm{gr}^k(\mathcal{A}_G)$.

Here, we consider another descending filtration of $\mathrm{IA}(G)$. Let $\Gamma_{\mathrm{IA}(G)}(k)$ be the k–th subgroup of the lower central series of $\mathrm{IA}(G)$. Then for each $k \geq 1$, $\Gamma_{\mathrm{IA}(G)}(k)$ is a subgroup of $\mathcal{A}_G(k)$ since the Johnson filtration is a central filtration of $\mathrm{IA}(G)$. In general, it is a natural question to ask whether $\Gamma_{\mathrm{IA}(G)}(k)$ coincides with $\mathcal{A}_G(k)$ or not. For the case where G is a free group F_n of rank n, it is conjectured that $\Gamma_{\mathrm{IA}(F_n)}(k)$ coincides with $\mathcal{A}_{F_n}(k)$ by Andreadakis.

Consider $\mathcal{L}_{\mathrm{IA}(G)}(k) := \Gamma_{\mathrm{IA}(G)}(k)/\,\Gamma_{\mathrm{IA}(G)}(k+1)$ for each $k \geq 1$. Similarly to $\mathrm{gr}(\mathcal{A}_G)$, the graded sum $\mathcal{L}_{\mathrm{IA}(G)} := \bigoplus_{k\geq 1} \mathcal{L}_{\mathrm{IA}(G)}(k)$ has a graded Lie algebra structure induced from the commutator bracket on $\mathrm{IA}(G)$. The restriction of the homomorphism $\mathcal{A}_G(k) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, \mathcal{L}_G(k+1))$ to $\Gamma_{\mathrm{IA}(G)}(k)$ also induces an $\mathrm{Aut}\, G/\mathrm{IA}(G)$–equivariant homomorphism

$$\tau'_k = \tau'_{G,k} \colon \mathcal{L}_{\mathrm{IA}(G)}(k) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, \mathcal{L}_G(k+1)).$$

In this paper, we also call τ'_k the k–th Johnson homomorphism of $\mathrm{Aut}\, G$.

2.4 Associated graded ring of a group ring

For a group G, let $\mathbf{Z}[G]$ be a group ring of G over $\mathbf{Z}$, and $\Delta(G)$ the augmentation ideal of $\mathbf{Z}[G]$. Namely, $\Delta(G)$ is the kernel of the augmentation map $\varepsilon : \mathbf{Z}[G] \to \mathbf{Z}$ defined by

$$\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g, \quad a_g \in \mathbf{Z}.$$

We denote by $\Delta^k(G) := (\Delta(G))^k$ the k–times product of the augmentation ideal $\Delta(G)$ in $\mathbf{Z}[G]$. For each $k \geq 1$, set

$$Q^k(G) := \Delta^k(G)/\Delta^{k+1}(G),$$
$$\mathrm{gr}(\mathbf{Z}[G]) := \bigoplus_{k\geq 1} Q^k(G).$$

The quotients $Q^k(G)$ are called the augmentation quotients of G. The graded sum $\mathrm{gr}(\mathbf{Z}[G])$ naturally has an associative graded ring structure induced from the product in $\mathbf{Z}[G]$. The ring $\mathrm{gr}(\mathbf{Z}[G])$ is called the associated graded ring of the group ring $\mathbf{Z}[G]$.

In general, one of the most standard methods to study $Q^k(G)$ is to consider a natural surjective homomorphism $\pi_k = \pi_{k,G}\colon Q^k(G) \to Q^k(G^{\mathrm{ab}})$ induced from the abelianization $G \to G^{\mathrm{ab}}$. Furthermore, if G^{ab} is free abelian, we have an natural isomorphism $Q^k(G^{\mathrm{ab}}) \cong S^k(G^{\mathrm{ab}}) = S^k(\mathcal{L}_G(1))$ where S^k means the k–th symmetric power. (See Passi [17, Corollary 8.2].) In Section 4.2, we study the kernel of π_k for $G = F_n$. We remark that for a group G and $k \geq 1$, $\mathrm{Ker}(\pi_k)$ is generated by elements

$$(g_1 - 1)\cdots(g_k - 1) - (g_{\sigma(1)} - 1)\cdots(g_{\sigma(k)} - 1)$$

as a $\mathbf{Z}$–module for any $g_1, \ldots, g_k \in G$ and $\sigma \in \mathfrak{S}_k$. Here $\mathfrak{S}_k$ denotes the symmetric group of degree k.

Here we consider a relation between $\mathrm{gr}(\mathbf{Z}[G])$ and $\mathcal{L}_G$. For any $g \in \Gamma_G(k)$, it is well known that an element $g - 1 \in \mathbf{Z}[G]$ belongs to $\Delta^k(G)$. Then a map $\Gamma_G(k) \to \Delta^k(G)$ defined by $g \mapsto g - 1$ induces a $\mathbf{Z}$–linear map

$$\alpha_k = \alpha_{k,G}\colon \mathcal{L}_G(k) \to Q^k(G)$$

and a Lie algebra homomorphism

$$\alpha_G := \bigoplus_{k\geq 1} \alpha_k\colon \mathcal{L}_G \to \mathrm{gr}(\mathbf{Z}[G]),$$

where we consider $\mathrm{gr}(\mathbf{Z}[G])$ as a Lie algebra with a Lie bracket $[x, y] := xy - yx$ for any $x, y \in \mathbf{Z}[G]$. We remark that for any group G, $\alpha_{1,G}\colon G^{\mathrm{ab}} \to Q^1(G)$ is an isomorphism. Hence, so is π_1. For $k \geq 2$, however, π_k is not injective in general. For $k = 2$, if G is a finitely generated, then we have a split exact sequence of $\mathbf{Z}$–modules:

$$0 \to \mathcal{L}_G(2) \xrightarrow{\alpha_{2,G}} Q^2(G) \xrightarrow{\pi_{2,G}} Q^2(G^{\mathrm{ab}}) \to 0. \tag{1}$$

(For a proof, see [17, Corollary 8.13, Chapter VIII].) We denote by

$$\alpha^*_{k+1} = \alpha^*_{k+1,G}\colon \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, \mathcal{L}_G(k+1)) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, Q^{k+1}(G))$$

the natural homomorphism induced from α_{k+1}.

3 A lift of the Johnson homomorphisms to the augmentation quotients

In this section, for a group G, we construct an $\mathrm{Aut}\, G/\mathrm{IA}(G)$–equivariant homomorphism $\mu_k\colon Q^k(\mathrm{IA}(G)) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, Q^{k+1}(G))$ such that

$$\mu_k \circ \alpha_{k,\mathrm{IA}(G)} = \alpha^*_{k+1,G} \circ \tau'_k. \tag{2}$$

3.1 Construction of μ_k

For any $\sigma \in \mathrm{Aut}\, G$ and $x \in G$, set $s_\sigma(x) := x^{-1}x^\sigma \in G$. First, we recall an important and useful lemma due to Andreadakis [1]:

Lemma 3.1 *For any k, $l \geq 1$, $\sigma \in \mathcal{A}_G(k)$ and $x \in \Gamma_G(l)$, we have $s_\sigma(x) \in \Gamma_G(k+l)$.*

For the proof of Lemma 3.1, see [1]. From this lemma, we see that $s_\sigma(x) - 1 \in \Delta^{k+l}(G)$ for any $\sigma \in \mathcal{A}_G(k)$ and $x \in \Gamma_G(l)$. We often use these facts without any quotation. In order to define a lift of the Johnson homomorphism, we prepare some lemmas.

Lemma 3.2 *For any σ, $\tau \in \mathrm{IA}(G)$ and x, $y \in G$, we have*

(1) $s_{\sigma\tau}(x) = s_\tau(x) \cdot s_\sigma(x)^\tau = s_\tau(x) s_\sigma(x) s_\tau(s_\sigma(x))$.

(2) $s_\sigma(xy) = y^{-1}s_\sigma(x)y \cdot s_\sigma(y) = [y^{-1}, s_\sigma(x)]s_\sigma(x)s_\sigma(y)$.

Proof The equations follow from

$$s_{\sigma\tau}(x) = x^{-1}x^{\sigma\tau} = x^{-1}x^\tau \cdot (x^{-1}x^\sigma)^\tau = x^{-1}x^\tau \cdot x^{-1}x^\sigma \cdot (x^{-1}x^\sigma)^{-1} \cdot (x^{-1}x^\sigma)^\tau,$$
$$s_\sigma(xy) = y^{-1}x^{-1}x^\sigma y^\sigma = y^{-1}x^{-1}x^\sigma y \cdot y^{-1}y^\sigma. \qquad \square$$

Lemma 3.3 *For any $x \in \Gamma_G(k)$ and $\sigma \in \mathrm{IA}(G)$, we have*

$$x^\sigma - x \equiv s_\sigma(x) - 1 \pmod{\Delta^{k+2}(G)}.$$

Proof This is clear from

$$\begin{aligned} x^\sigma - x &= (x^\sigma - 1) - (x-1) \\ &= (x(x^{-1}x^\sigma) - 1) - (x-1) \\ &= (x-1)(s_\sigma(x) - 1) + (s_\sigma(x) - 1) \end{aligned}$$

as $s_\sigma(x) - 1 \in \Delta^{k+1}(G)$, and hence $(x-1)(s_\sigma(x)-1) \in \Delta^{k+2}(G)$. □

Lemma 3.4 *For any $a \in \Delta^k(G)$ and $\sigma \in \mathrm{IA}(G)$, we have $a^\sigma - a \in \Delta^{k+1}(G)$.*

Proof Any element of $\Delta^k(G)$ can be written as a $\mathbf{Z}$–linear combination of elements types of

$$(x_1 - 1) \cdots (x_k - 1)$$

for $x_i \in G$. Hence it suffices to show the lemma for $a = (x_1 - 1)\cdots(x_k - 1)$. Then we have

$$\begin{aligned} a^\sigma - a &= (x_1(x_1^{-1}x_1^\sigma) - 1)\cdots(x_k(x_k^{-1}x_k^\sigma) - 1) - (x_1 - 1)\cdots(x_k - 1), \\ &= \{(x_1 - 1)(x_1^{-1}x_1^\sigma - 1) + (x_1 - 1) + (x_1^{-1}x_1^\sigma - 1)\} \\ &\qquad \cdots \{(x_k - 1)(x_k^{-1}x_k^\sigma - 1) + (x_k - 1) + (x_k^{-1}x_k^\sigma - 1)\} \\ &\qquad - (x_1 - 1)\cdots(x_k - 1) \\ &\equiv (x_1 - 1)\cdots(x_k - 1) - (x_1 - 1)\cdots(x_k - 1) = 0 \pmod{\Delta^{k+1}(G)}. \end{aligned}$$ □

For any $x \in G$, consider a $\mathbf{Z}$–linear homomorphism $\varphi_x \colon \mathbf{Z}[\mathrm{IA}(G)] \to \Delta(G)$ defined by $\sigma \mapsto s_\sigma(x) - 1$ for any $\sigma \in \mathrm{IA}(G)$.

Lemma 3.5 *For any $k, l \geq 1$, $x \in \Gamma_G(l)$, and $\sigma_1, \ldots, \sigma_k \in \mathrm{IA}(G)$, we have*

$$\varphi_x((\sigma_1 - 1)\cdots(\sigma_k - 1)) \equiv s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots)) - 1 \pmod{\Delta^{k+l+1}(G)}.$$

Proof We prove this lemma by the induction on $k \geq 1$. For $k = 1$, it is obvious by the definition. Assume that $k \geq 2$. Write

$$(\sigma_1 - 1)\cdots(\sigma_{k-1} - 1) = \sum_{\sigma \in \mathrm{IA}(G)} a_\sigma \sigma \in \mathbf{Z}[\mathrm{IA}(G)]$$

for $a_\sigma \in \mathbf{Z}$. Then we have

$$\begin{aligned}
\varphi_x&((\sigma_1-1)\cdots(\sigma_{k-1}-1)(\sigma_k-1)),\\
&=\varphi_x((\sigma_1-1)\cdots(\sigma_{k-1}-1)\sigma_k-(\sigma_1-1)\cdots(\sigma_{k-1}-1)),\\
&=\varphi_x\Bigl(\sum_{\sigma\in \mathrm{IA}(G)} a_\sigma\,\sigma\sigma_k-\sum_{\sigma\in \mathrm{IA}(G)} a_\sigma\sigma\Bigr),\\
&=\sum_{\sigma\in \mathrm{IA}(G)} a_\sigma\{(s_{\sigma\sigma_k}(x)-1)-(s_\sigma(x)-1)\},\\
&=\sum_{\sigma\in \mathrm{IA}(G)} a_\sigma\{(s_{\sigma_k}(x)s_\sigma(x)^{\sigma_k}-1)-(s_\sigma(x)-1)\},\\
&=\sum_{\sigma\in \mathrm{IA}(G)} a_\sigma\{(s_{\sigma_k}(x)-1)(s_\sigma(x)^{\sigma_k}-1)+(s_{\sigma_k}(x)-1)\\
&\qquad\qquad+(s_\sigma(x)^{\sigma_k}-1)-(s_\sigma(x)-1)\}.
\end{aligned}$$

Here we see

$$\begin{aligned}
\sum_{\sigma\in \mathrm{IA}(G)} a_\sigma(s_{\sigma_k}(x)-1)(s_\sigma(x)^{\sigma_k}-1)&=(s_{\sigma_k}(x)-1)\Bigl(\sum_{\sigma\in \mathrm{IA}(G)} a_\sigma(s_\sigma(x)-1)\Bigr)^{\sigma_k}\\
&\equiv 0 \pmod{\Delta^{k+l+1}(G)}
\end{aligned}$$

since $s_{\sigma_k}(x)-1\in\Delta^2(G)$ and $\sum_{\sigma\in \mathrm{IA}(G)} a_\sigma(s_\sigma(x)-1)\in\Delta^{k+l-1}(G)$ by the inductive hypothesis, and see

$$\sum_{\sigma\in \mathrm{IA}(G)} a_\sigma(s_{\sigma_k}(x)-1)=(s_{\sigma_k}(x)-1)\sum_{\sigma\in \mathrm{IA}(G)} a_\sigma=0.$$

On the other hand, by the inductive hypothesis, we have

$$\begin{aligned}
\sum_{\sigma\in \mathrm{IA}(G)} & a_\sigma\{(s_\sigma(x)^{\sigma_k}-1)-(s_\sigma(x)-1)\}\\
&=\Bigl(\sum_{\sigma\in \mathrm{IA}(G)} a_\sigma(s_\sigma(x)-1)\Bigr)^{\sigma_k}-\sum_{\sigma\in \mathrm{IA}(G)} a_\sigma(s_\sigma(x)-1)\\
&=(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots)-1)^{\sigma_k}-(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots)-1)+a^{\sigma_k}-a
\end{aligned}$$

for some $a\in\Delta^{k+l}(G)$. Then, by Lemmas 3.3 and 3.4, we see this is congruent to

$$s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots))-1 \pmod{\Delta^{k+l+1}(G)}.$$

This completes the proof of Lemma 3.5. □

For each $k \geq 1$, since $\Delta^k(\mathrm{IA}(G))$ is generated by elements types of

$$(\sigma_1 - 1)\cdots(\sigma_k - 1)$$

for $\sigma_i \in \mathrm{IA}(G)$ as a $\mathbf{Z}$–module, by Lemma 3.5 we obtain:

Corollary 3.6 *For any $k, l \geq 1$ and $x \in \Gamma_G(l)$, we have*

$$\varphi_x(\Delta^k(\mathrm{IA}(G))) \subset \Delta^{k+l}(\mathrm{IA}(G)).$$

Remark 3.7 For any $x \in \Gamma_G(l)$ a homomorphism $\mathbf{Z}[\mathrm{IA}(G)] \to Q^{k+l}(\mathrm{IA}(G))$ defined by $a \mapsto \varphi_x(a)$ is a polynomial map of degree $\leq k$. (For details for polynomial maps, see Passi [17], for example.)

Lemma 3.8 *For any k, $l \geq 1$ and $x, y \in \Gamma_G(l)$, we have*

$$s_{\sigma_k}(\cdots(s_{\sigma_1}(xy))\cdots) \equiv s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)\cdot s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots) \pmod{\Gamma_G(k+2l)}$$

for any $\sigma_1, \ldots, \sigma_k \in \mathrm{IA}(G)$.

Proof We prove this lemma by the induction on $k \geq 1$. If $k = 1$, it is trivial from the part (2) of Lemma 3.2. Assume $k \geq 2$. By the inductive hypothesis, we see

$$s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(xy))) = c\, s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x)))\cdot s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y)))$$

for some $c \in \Gamma_G(k+2l-1)$. Then, using the part (2) of Lemma 3.2 we have

$$\begin{aligned}
&s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(xy))))\\
&\quad= s_{\sigma_k}(c\, s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x)))\cdot s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y))))\\
&\quad= [\{s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x)))s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y)))\}^{-1}, s_{\sigma_k}(c)]\\
&\qquad\qquad\cdot s_{\sigma_k}(c)\cdot s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x)))\cdot s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y))))\\
&\quad\equiv s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x)))\cdot s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y)))),\\
&\quad= [s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y))))^{-1}, s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))))]\\
&\qquad\qquad\cdot s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))))\cdot s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y))))\\
&\quad\equiv s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))))\cdot s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(y)))).
\end{aligned}$$

modulo $\Gamma_G(k+2l)$. □

Lemma 3.9 *For any k, $l \geq 1$, x, $y \in \Gamma_G(l)$, and $a \in \Delta^k(\mathrm{IA}(G))$, we have*

$$\varphi_{xy}(a) \equiv \varphi_x(a) + \varphi_y(a) \pmod{\Delta^{k+l+1}(G)}.$$

Proof First, we consider the case where $a=(\sigma_1-1)\cdots(\sigma_k-1)$ for some $\sigma_i\in\mathrm{IA}(G)$. From Lemma 3.5 and Lemma 3.8, we see

$$\begin{aligned}\varphi_{xy}(a) &\equiv s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(xy))\cdots))-1\\ &= cs_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)\cdot s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots)-1\end{aligned}$$

for some $c\in\Gamma_G(k+2l)$. Hence we have

$$\begin{aligned}\varphi_{xy}(a) &= (c-1)(s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)\cdot s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots)-1),\\ &\qquad +(c-1)+(s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)\cdot s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots)-1)\\ &\equiv s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)\cdot s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots)-1\\ &= (s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)-1)(s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots)-1)\\ &\qquad +(s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)-1)+(s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots)-1)\\ &\equiv (s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots)-1)+(s_{\sigma_k}(\cdots(s_{\sigma_1}(y))\cdots)-1)\\ &= \varphi_x(a)+\varphi_y(a)\end{aligned}$$

modulo $\Delta^{k+l+1}(G)$.

For a general case, $a\in\Delta^k(\mathrm{IA}(G))$ is written as a $\mathbf{Z}$–linear combination of elements types of

$$(\sigma_1-1)\cdots(\sigma_k-1).$$

Therefore, using the argument above, we obtain the lemma for any $a\in\Delta^k(\mathrm{IA}(G))$. □

Lemma 3.10 *For any $a\in\Delta^k(\mathrm{IA}(G))$, a map $\mu_k(a)\colon G^{\mathrm{ab}}\to Q^{k+1}(G)$ defined by $x\mapsto\varphi_x(a)$ is a homomorphism.*

Proof To begin with, we check that $\mu_k(a)$ is well-defined. Consider elements $x,y\in G$ such that $y=xc$ for some $c\in\Gamma_G(2)$. Then by Lemma 3.9,

$$\varphi_y(a)=\varphi_{xc}(a)\equiv\varphi_x(a)+\varphi_c(a)\quad(\mathrm{mod}\ \Delta^{k+2}(G)).$$

On the other hand, by Corollary 3.6, we see $\varphi_c(a)\in\Delta^{k+2}(G)$. Hence $\varphi_y(a)=\varphi_x(a)\in Q^{k+1}(G)$.

To show $\mu_k(a)$ is a homomorphism, take any x and $y\in G$. Then by Lemma 3.9,

$$\mu_k(a)(xy)=\varphi_{xy}(a)\equiv\varphi_x(a)+\varphi_y(a)=\mu_k(a)(x)+\mu_k(a)(y)$$

modulo $\Delta^{k+2}(G)$. □

Now, we are ready to define a lift of the Johnson homomorphism τ'_k. For any $k \geq 1$, define a map

$$\mu_k\colon \Delta^k(\mathrm{IA}(G)) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, Q^{k+1}(G))$$

by

$$a \mapsto (x \mapsto \varphi_x(a)).$$

Using Lemma 3.3, it is easy to check that the map μ_k is a homomorphism. Furthermore $\Delta^{k+1}(\mathrm{IA}(G))$ is contained in $\mathrm{Ker}(\mu_k)$. Hence μ_k induces a homomorphism

$$Q^k(\mathrm{IA}(G)) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, Q^{k+1}(G)).$$

We also denote by μ_k this induced homomorphism, and call it the k–th Johnson homomorphism of $\mathbf{Z}[\mathrm{IA}(G)]$. We see that the compatibility (2) follows by the definition of τ'_k and μ_k.

3.2 Actions of Aut G

Next we consider actions of $\mathrm{Aut}\, G$. Since $\mathrm{IA}(G)$ is a normal subgroup of $\mathrm{Aut}\, G$, the group $\mathrm{Aut}\, G$ acts on $\mathbf{Z}[\mathrm{IA}(G)]$ from the right by

$$\Bigg(\sum_{\sigma \in \mathrm{IA}(G)} a_\sigma \sigma\Bigg) \cdot \tau := \sum_{\sigma \in \mathrm{IA}(G)} a_\sigma (\tau^{-1}\sigma\tau)$$

for any $\tau \in \mathrm{Aut}\, G$. For each $k \geq 1$, since $\Delta^k(\mathrm{IA}(G))$ is preserved by the action of $\mathrm{Aut}\, G$, the group $\mathrm{Aut}\, G$ also acts on each of the graded quotient $Q^k(\mathrm{IA}(G))$. Then $\mathrm{IA}(G)$ acts on $Q^k(\mathrm{IA}(G))$ trivially. In fact, for any $\tau \in \mathrm{IA}(G)$, we have

$$\begin{aligned}
(\sigma_1 - 1)\cdots(\sigma_k - 1)\cdot\tau &= (\tau^{-1}\sigma_1\tau - 1)\cdots(\tau^{-1}\sigma_k\tau - 1)\\
&= ([\tau^{-1},\sigma_1]\sigma_1 - 1)\cdots([\tau^{-1},\sigma_k]\sigma_k\tau - 1)\\
&= \{([\tau^{-1},\sigma_1]-1)(\sigma_1 - 1) + ([\tau^{-1},\sigma_1]-1) + (\sigma_1 - 1)\}\\
&\qquad \cdots\{([\tau^{-1},\sigma_k]-1)(\sigma_k - 1) + ([\tau^{-1},\sigma_k]-1) + (\sigma_k - 1)\}\\
&\equiv (\sigma_1 - 1)\cdots(\sigma_k - 1)
\end{aligned}$$

modulo $\Delta^{k+1}(\mathrm{IA}(G))$ since $[\tau^{-1},\sigma_i] \in \Gamma_{\mathrm{IA}(G)}(2)$ and $[\tau^{-1},\sigma_i] - 1 \in \Delta^2(\mathrm{IA}(G))$. Since $Q^k(\mathrm{IA}(G))$ is generated by elements $(\sigma_1 - 1)\cdots(\sigma_k - 1)$ for $\sigma_i \in \mathrm{IA}(G)$ as a $\mathbf{Z}$–module, we verify that the action of $\mathrm{IA}(G)$ on $Q^k(\mathrm{IA}(G))$ is trivial. Hence the quotient group $\mathrm{Aut}\, G/\mathrm{IA}(G)$ naturally acts on each of $Q^k(\mathrm{IA}(G))$ from the right.

Now, $\mathrm{Aut}\, G$ naturally acts on $\mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, Q^{k+1}(G))$. Then it is easily seen that the action of $\mathrm{IA}(G)$ on $\mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, Q^{k+1}(G))$ is trivial. Hence the quotient group $\mathrm{Aut}\, G/\mathrm{IA}(G)$ also acts on it. To show that μ_k is $\mathrm{Aut}\, G/\mathrm{IA}(G)$–equivariant, we prepare:

Lemma 3.11 *For any $k \geq 1$, and $\sigma, \sigma_1, \ldots, \sigma_k \in \operatorname{Aut} G$, we have*

$$(s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots))^{\sigma} = s_{\sigma^{-1}\sigma_k\sigma}(\cdots(s_{\sigma^{-1}\sigma_1\sigma}(x^{\sigma}))\cdots).$$

Proof We prove this lemma by the induction on $k \geq 1$. For $k = 1$, it is clear by

$$s_{\sigma_1}(x)^{\sigma} = (x^{-1}x^{\sigma_1})^{\sigma} = (x^{\sigma})^{-1}x^{\sigma_1\sigma} = (x^{\sigma})^{-1}(x^{\sigma})^{\sigma^{-1}\sigma_1\sigma} = s_{\sigma^{-1}\sigma_1\sigma}(x^{\sigma}).$$

Assume $k \geq 2$. Using the inductive hypothesis, we obtain

$$\begin{aligned}
&(s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots))^{\sigma}\\
&= \big((s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots))^{-1}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots))^{\sigma_k}\big)^{\sigma}\\
&= \{(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots))^{\sigma}\}^{-1}\{(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots))^{\sigma}\}^{\sigma^{-1}\sigma_k\sigma}\\
&= \{s_{\sigma^{-1}\sigma_{k-1}\sigma}(\cdots(s_{\sigma^{-1}\sigma_1\sigma}(x^{\sigma}))\cdots)\}^{-1}\{s_{\sigma^{-1}\sigma_{k-1}\sigma}(\cdots(s_{\sigma^{-1}\sigma_1\sigma}(x^{\sigma}))\cdots)\}^{\sigma^{-1}\sigma_k\sigma}\\
&= s_{\sigma^{-1}\sigma_k\sigma}(\cdots(s_{\sigma^{-1}\sigma_1\sigma}(x^{\sigma}))\cdots).
\end{aligned}$$

This completes the proof of Lemma 3.11. □

Proposition 3.12 *For any $k \geq 1$, the Johnson homomorphism μ_k is an $\operatorname{Aut} G/\mathrm{IA}(G)$–equivariant homomorphism.*

Proof It is enough to show that $\mu_k(a^{\sigma}) = (\mu_k(a))^{\sigma}$ for $\sigma \in \mathrm{IA}(G)$ and $a = (\sigma_1 - 1)\cdots(\sigma_k - 1) \in Q^k(\mathrm{IA}(G))$. Then, for any $x \in G^{\mathrm{ab}}$ we have

$$\begin{aligned}
\mu_k(a^{\sigma})(x) &= \mu_k((\sigma^{-1}\sigma_1\sigma - 1)\cdots(\sigma^{-1}\sigma_k\sigma - 1))(x)\\
&= s_{\sigma^{-1}\sigma_k\sigma}(\cdots(s_{\sigma^{-1}\sigma_1\sigma}(x))\cdots) - 1.
\end{aligned}$$

On the other hand, by Lemma 3.11,

$$\begin{aligned}
(\mu_k(a))^{\sigma}(x) = (\mu_k(a)(x^{\sigma^{-1}}))^{\sigma} &= (s_{\sigma_k}(\cdots(s_{\sigma_1}(x^{\sigma^{-1}}))\cdots) - 1)^{\sigma}\\
&= s_{\sigma^{-1}\sigma_k\sigma}(\cdots(s_{\sigma^{-1}\sigma_1\sigma}(x))\cdots) - 1.
\end{aligned}$$

for any $x \in G^{\mathrm{ab}}$. □

3.3 Some properties of μ_k

Here we observe some properties of μ_k. First, we consider the image of μ_k. In general, μ_k is not surjective.

Lemma 3.13 *For each $k \geq 1$, the image of μ_k is contained in that of $\alpha^*_{k+1,G}$.*

Proof Since $Q^k(\mathrm{IA}(G))$ is generated by $(\sigma_1-1)\cdots(\sigma_k-1)$ for $\sigma_i \in \mathrm{IA}(G)$ as a $\mathbf{Z}$–module, it suffices to show $\mu_k(a) \in \mathrm{Im}(\alpha^*_{k+1,G})$ for $a=(\sigma_1-1)\cdots(\sigma_k-1)$. On the other hand, using Lemma 3.1 recursively, we see that $s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots))$ belongs to $\Gamma_G(k+1)$ for any $x \in G$. Hence

$$s_{\sigma_k}(s_{\sigma_{k-1}}(\cdots(s_{\sigma_1}(x))\cdots))-1 \in \alpha_{k+1,G}(\mathcal{L}_G(k+1)). \qquad \square$$

By this lemma, in the following, we write the k–th Johnson homomorphism as

$$\mu_k\colon Q^k(\mathrm{IA}(G)) \to \mathrm{Hom}_{\mathbf{Z}}(G^{\mathrm{ab}}, \alpha_{k+1,G}(\mathcal{L}_G(k+1))).$$

Next, we consider a calculation of $\mu_{k+1}(a(\tau-1))$ for a given $a \in Q^k(\mathrm{IA}(G))$ and $\tau \in \mathrm{IA}(G)$. Let

$$a = \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}(\sigma_1-1)\cdots(\sigma_k-1)$$

for $m_{\sigma_1,\ldots,\sigma_k} \in \mathbf{Z}$. Then for any $x \in G$, we have

$$\begin{aligned}\mu_{k+1}(a(\tau-1))(x) &= \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}\mu_{k+1}((\sigma_1-1)\cdots(\sigma_k-1)(\tau-1))(x)\\ &\equiv \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}\{s_\tau(s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots))-1\}\end{aligned}$$

modulo $\Delta^{k+3}(G)$. If we set $X := s_{\sigma_k}(\cdots(s_{\sigma_1}(x))\cdots) \in \Gamma_G(k+1)$, then

$$\begin{aligned}&\mu_{k+1}(a(\tau-1))(x)\\ &\quad= \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}\{X^{-1}X^\tau-1\}\\ &\quad= \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}\{(X^{-1}-1)(X^\tau-1)+(X^{-1}-1)+(X^\tau-1)\}\\ &\quad\equiv \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}\{(X^\tau-1)-(X-1)\}\\ &\quad= \left\{\sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}(X-1)\right\}^\tau - \sum_{\sigma_1,\ldots,\sigma_k \in \mathrm{IA}(G)} m_{\sigma_1,\ldots,\sigma_k}(X-1)\\ &\quad\equiv \{\mu_k(a)(x)\}^\tau - \mu_k(a)(x)\end{aligned}$$

modulo $\Delta^{k+3}(G)$. Hence we have

$$\mu_{k+1}(a(\tau-1))(x) = \{\mu_k(a)(x)\}^\tau - \mu_k(a)(x) \in Q^{k+2}(\mathrm{IA}(G)).$$

This formula is sometimes convenient for a calculation of the image of μ_k.

4 Free group case

In this section, we mainly consider the case where $G = F_n$. For simplicity, we often omit the capital F from the subscript F_n if there is no confusion. For example, we write $\mathcal{L}_n$, $\mathcal{L}_n(k)$, $\mathrm{IA}_n, \ldots$ for $\mathcal{L}_{F_n}$, $\mathcal{L}_{F_n}(k)$, $\mathrm{IA}(F_n), \ldots$ respectively. Here, we study the structure of graded quotients $Q^k(\mathrm{IA}_n)$ as a $\mathrm{GL}(n, \mathbf{Z})$–module.

4.1 Preliminary results for $G = F_n$

In this subsection, we recall some well-known properties of the IA-automorphism group IA_n, the graded Lie algebra $\mathcal{L}_n$ and the graded ring $\mathrm{gr}(\mathbf{Z}[F_n])$. Let $H := F_n^{\mathrm{ab}}$ be the abelianization of F_n. The natural homomorphism $\rho = \rho_{F_n} \colon \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ induced from the abelianization of $F_n \to H$ is surjective. Throughout the paper, we identify $\mathrm{Aut}\, H$ with the general linear group $\mathrm{GL}(n, \mathbf{Z})$ by fixing a basis of H induced from the basis $x_1, \ldots, x_n$ of F_n. Namely, we have $\mathrm{GL}(n, \mathbf{Z}) \cong \mathrm{Aut}\, F_n / \mathrm{IA}_n$.

Magnus [13] showed that for any $n \geq 3$, IA_n is finitely generated by automorphisms

$$K_{ij} \colon x_t \mapsto \begin{cases} {x_j}^{-1} x_i x_j & t = i, \\ x_t & t \neq i \end{cases}$$

for distinct $1 \leq i,\ j \leq n$, and

$$K_{ijl} \colon x_t \mapsto \begin{cases} x_i [x_j, x_l] & t = i, \\ x_t & t \neq i \end{cases}$$

for distinct $1 \leq i,\ j,\ l \leq n$ and $j < l$. Recently, Cohen and Pakianathan [6; 7], Farb [8] and Kawazumi [12] independently showed

$$\mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H \tag{3}$$

as a $\mathrm{GL}(n, \mathbf{Z})$–module. In particular, from their result, we see that $\mathrm{IA}_n^{\mathrm{ab}}$ is a free abelian group of rank $n^2(n-1)/2$ with basis the coset classes of the Magnus generators K_{ij} and K_{ijl}.

It is classically known due to Magnus that the graded Lie algebra $\mathcal{L}_n$ is isomorphic to the free Lie algebra generated by H over $\mathbf{Z}$. (See Reutenauer [19], for example, for basic material concerning the free Lie algebra.) Each of the degree k part $\mathcal{L}_n(k)$ of $\mathcal{L}_n$ is a free abelian group, which rank is given by Witt's formula

$$\mathrm{rank}_{\mathbf{Z}}(\mathcal{L}_n(k)) = \frac{1}{k} \sum_{d|k} \mu(d) n^{k/d} \tag{4}$$

where μ is the Möbius function.

We next consider an embedding of the free Lie algebra $\mathcal{L}_n$ into the graded sum $\mathrm{gr}(\mathbf{Z}[F_n])$. In general, it is known that the graded Lie algebra homomorphism $\alpha_{F_n}\colon \mathcal{L}_n \to \mathrm{gr}(\mathbf{Z}[F_n])$ induced from $x \mapsto x-1$ for any $x \in F_n$ is a $\mathrm{GL}(n,\mathbf{Z})$–equivariant injective homomorphism, and that $\mathrm{gr}(\mathbf{Z}[F_n])$ is naturally isomorphic to the universal enveloping algebra $\mathcal{U}(\mathcal{L}_n)$ of $\mathcal{L}_n$. (See [17, Theorem 6.2, Chapter VIII].) For simplicity, in the following, we identify $\mathcal{L}_n(k)$ with its image $\alpha_k(\mathcal{L}_n(k))$ in $Q^k(F_n)$.

Here we observe a conjecture for the $\mathbf{Z}$–module structure of $Q^k(\mathrm{IA}_n)$. For a group G such that each of the graded quotients $\mathcal{L}_G(k)$ is a free abelian group for $k \geq 1$, Sandling and Tahara [20] showed that as a $\mathbf{Z}$–module,

$$Q^k(G) \cong \sum \bigotimes_{i=1}^{k} S^{a_i}(\mathcal{L}_G(i))$$

for each $k \geq 1$. Here $\sum$ runs over all nonnegative integers $a_1, \dots, a_k$ such that

$$\sum_{i=1}^{k} i a_i = k,$$

and $S^a(\mathcal{L}_G(i))$ means the symmetric tensor product of $\mathcal{L}_G(i)$ of degree a such that $S^0(\mathcal{L}_G(i)) = \mathbf{Z}$.

On the other hand, it is conjectured by Andreadakis that the lower central series $\Gamma_{\mathrm{IA}_n}(k)$ coincides with the Johnson filtration $\mathcal{A}_n(k)$. He [1] showed that this is true for $n=2$. Since each of the graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ of the Johnson filtration $\mathcal{A}_n(k)$ is free abelian, the Andreadakis's conjecture let us conjecture:

Conjecture 4.1 For any $k \geq 1$,

$$Q^k(\mathrm{IA}_n) \cong \sum \bigotimes_{i=1}^{k} S^{a_i}(\mathcal{L}_{\mathrm{IA}_n}(i))$$

as a $\mathbf{Z}$–module. Here the sum runs over all nonnegative integers $a_1, \dots, a_k$ such that $\sum_{i=1}^{k} i a_i = k$.

To study $Q^k(\mathrm{IA}_n)$, first, we consider the surjective homomorphism $\pi_k\colon Q^k(\mathrm{IA}_n) \to Q^k(\mathrm{IA}_n^{\mathrm{ab}})$ induced from the abelianization of IA_n for $k \geq 1$. We remark that each of π_k is an $\mathrm{GL}(n,\mathbf{Z})$–equivariant surjective homomorphism, and that $Q^k(\mathrm{IA}_n^{\mathrm{ab}}) \cong S^k(\mathrm{IA}_n^{\mathrm{ab}})$ since $\mathrm{IA}_n^{\mathrm{ab}}$ is free abelian as mentioned before. For $k=1$, $\pi_k\colon Q^1(\mathrm{IA}_n) \to Q^1(\mathrm{IA}_n^{\mathrm{ab}})$ is an isomorphism, and $Q^1(\mathrm{IA}_n) \cong \mathrm{IA}_n^{\mathrm{ab}} = H^* \otimes_{\mathbf{Z}} \Lambda^2 H$. In general, however, π_k is not injective for $k \geq 2$, and seems to have a large kernel from the conjecture above. In this paper, to investigate the $\mathrm{GL}(n,\mathbf{Z})$–module structure of $\mathrm{Ker}(\pi_k)$, we use the Johnson homomorphism μ_k.

4.2 The image of $\mu_k|_{\mathrm{Ker}(\pi_k)}$

Here we study the image of the Johnson homomorphism

$$\mu_k\colon Q^k(\mathrm{IA}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \subset H^* \otimes_{\mathbf{Z}} Q^{k+1}(F_n)$$

restricted to the kernel of π_k for a sufficiently large n. Note that $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) = H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n(k+1))$ is generated by elements

$$x_i^* \otimes ([x_{i_1}, \dots, x_{i_{k+1}}] - 1), \quad 1 \le i,\ i_j \le n$$

as a $\mathbf{Z}$–module. First we consider the case where $k \ge 3$.

Proposition 4.2 *For any $k \ge 3$ and $n \ge k+2$, the homomorphism*

$$\mu_k|_{\mathrm{Ker}(\pi_k)}\colon \mathrm{Ker}(\pi_k) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

is surjective.

Proof For any $x_i^* \otimes ([x_{i_1}, \dots, x_{i_{k+1}}] - 1)$, since $n \ge k+2$, there exists some $1 \le j \le n$ such that $j \ne i_1, \dots, i_{k+1}$.

Case 1 The case where $i_{k+1} \ne i$. Set

$$a := \begin{cases} (K_{iji_{k+1}} - 1)(K_{ji_k} - 1) \cdots (K_{ji_3} - 1)(K_{ji_1i_2} - 1) & \text{if } j \ne i, \\ (K_{ji_{k+1}} - 1)(K_{ji_k} - 1) \cdots (K_{ji_3} - 1)(K_{ji_1i_2} - 1) & \text{if } j = i. \end{cases}$$

Then we have $\mu_k(a) = x_i^* \otimes ([x_{i_1}, \dots, x_{i_{k+1}}] - 1)$. On the other hand, if we set

$$b := \begin{cases} (K_{ji_1i_2} - 1)(K_{ji_3} - 1) \cdots (K_{ji_k} - 1)(K_{iji_{k+1}} - 1) & \text{if } j \ne i, \\ (K_{ji_1i_2} - 1)(K_{ji_3} - 1) \cdots (K_{ji_k} - 1)(K_{ji_{k+1}} - 1) & \text{if } j = i, \end{cases}$$

then $\mu_k(b) = 0$. Hence we obtain $\mu_k(a-b) = x_i^* \otimes ([x_{i_1}, \dots, x_{i_{k+1}}] - 1)$ for $a - b \in \mathrm{Ker}(\pi_k)$.

Case 2 The case where $i_{k+1} = i$. Set

$$a' := (K_{ij}^{-1} - 1)(K_{ji_k} - 1) \cdots (K_{ji_3} - 1)(K_{ji_1i_2} - 1).$$

Then $\mu_k(a') = x_i^* \otimes ([x_{i_1}, \dots, x_{i_{k+1}}] - 1)$. On the other hand, if we set

$$b' := (K_{ji_1i_2} - 1)(K_{ji_3} - 1) \cdots (K_{ji_k} - 1)(K_{ij}^{-1} - 1),$$

$\mu_k(b') = 0$. Hence we obtain $\mu_k(a' - b') = x_i^* \otimes ([x_{i_1}, \dots, x_{i_{k+1}}] - 1)$ for $a' - b' \in \mathrm{Ker}(\pi_k)$. This completes the proof of Proposition 4.2. □

It seems to difficult to show above for $2 \leq n \leq k+2$ since we can not take $1 \leq j \leq n$ such that $j \neq i_1, \dots, i_{k+1}$ in general.

As a corollary to Proposition 4.2, we see the surjectivity of μ_k of $\mathbf{Z}[\mathrm{IA}(G)]$ for the case where G is a certain quotient group of F_n. Let C be a characteristic subgroup of F_n such that $C \subset \Gamma_n(2)$, and set $G := F_n/C$. Then we have a natural isomorphism $G^{\mathrm{ab}} \cong H$. The natural projection $\phi\colon F_n \to G$ induces homomorphisms $Q^k(F_n) \to Q^k(G)$, also denoted by ϕ. Since C is characteristic, $\phi\colon F_n \to G$ induces a homomorphism $\overline{\phi}\colon \mathrm{Aut}\, F_n \to \mathrm{Aut}(G)$. Clearly, $\overline{\phi}(\mathrm{IA}_n) \subset \mathrm{IA}(G)$. Furthermore, $\overline{\phi}$ naturally induces homomorphisms $Q^k(\mathrm{IA}_n) \to Q^k(\mathrm{IA}(G))$ which is also denoted by $\overline{\phi}$.

Corollary 4.3 *With the notation above, for any $k \geq 3$ and $n \geq k+2$, the homomorphism $\mu_k\colon \mathrm{Ker}(\pi_{k,\mathrm{IA}(G)}) \to H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_G(k+1))$ is surjective.*

Proof It is clear from a commutative diagram

$$\begin{array}{ccc} \mathrm{Ker}(\pi_{k,\mathrm{IA}_n}) & \xrightarrow{\mu_k} & H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n(k+1)) \\ {\scriptstyle\overline{\phi}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\phi} \\ \mathrm{Ker}(\pi_{k,\mathrm{IA}(G)}) & \xrightarrow{\mu_k} & H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_G(k+1)), \end{array}$$

where the first row and $\mathrm{id} \otimes \phi$ are surjective. □

For example, if G is a free metabelian group $G = F_n/[\Gamma_n(2), \Gamma_n(2)]$, then the Johnson homomorphism $\mu_k\colon \mathrm{Ker}(\pi_{k,\mathrm{IA}(G)}) \to H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_G(k+1))$ is surjective for any $k \geq 3$ and $n \geq k+2$. In Section 5, we show that we can improve the condition $k \geq 3$ and $n \geq k+2$ above for $G = F_n/[\Gamma_n(2), \Gamma_n(2)]$.

By Proposition 4.2 and Corollary 4.3, we have:

Theorem 4.4 *Let C and G be as above. For $k \geq 3$ and $n \geq k+2$, an $\mathrm{Aut}(G)/\mathrm{IA}(G)$–equivariant homomorphism*

$$\mu_k \oplus \pi_k\colon Q^k(\mathrm{IA}(G)) \to (H^* \otimes_{\mathbf{Z}} \alpha_{k+1,G}(\mathcal{L}_G(k+1))) \bigoplus Q^k(\mathrm{IA}(G)^{\mathrm{ab}})$$

defined by $\sigma \mapsto (\mu_k(\sigma), \pi_k(\sigma))$ is surjective.

In particular, for $C = \{1\}$, and hence $G = F_n$, we have a $\mathrm{GL}(n, \mathbf{Z})$–equivariant surjective homomorphism

$$\mu_k \oplus \pi_k\colon Q^k(\mathrm{IA}_n) \to (H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)) \oplus S^k(\mathrm{IA}_n^{\mathrm{ab}})$$

for $k \geq 3$ and $n \geq k+2$.

Finally, we consider the case where $k=2$. Observing a split exact sequence (1), we see that $\mathrm{Ker}(\pi_2)=\alpha_{2,\mathrm{IA}(G)}(\mathcal{L}_{\mathrm{IA}(G)}(2))$. Hence, from the compatibility (2), we see that $\mathrm{Im}(\mu_2|_{\mathrm{Ker}(\pi_2)})=\alpha^*_{3,F_n}(\mathrm{Im}(\tau_2'))$. In [21], we showed that for any $n\geq 2$, $\mathrm{Im}(\tau_2')$, which is equal to $\mathrm{Im}(\tau_2)$, satisfies an exact sequence

$$0\to \mathrm{Im}(\tau_2')\xrightarrow{\tau_2'} H^*\otimes_{\mathbf{Z}}\mathcal{L}_n(3)\to S^2H\to 0$$

of $\mathrm{GL}(n,\mathbf{Z})$–modules. Hence we see that:

Proposition 4.5 *For $n\geq 2$, $\mathrm{Im}(\mu_2|_{\mathrm{Ker}(\pi_2)})$ is a $\mathrm{GL}(n,\mathbf{Z})$–equivariant proper submodule of $H^*\otimes_{\mathbf{Z}}\alpha_3(\mathcal{L}_n(3))$, which rank is given by*

$$\frac{1}{6}n(n+1)(2n^2-2n-3).$$

Here we remark that μ_2 is surjective.

Lemma 4.6 *For any $n\geq 2$, the map $\mu_2\colon Q^2(\mathrm{IA}_n)\to H^*\otimes_{\mathbf{Z}}\mathcal{L}_n(3)$ is surjective.*

Proof Take an element $x_i^*\otimes([x_{i_1},x_{i_2},x_{i_3}]-1)$. We may assume $i_1\neq i_2$. If $i_j\neq i$ for $1\leq j\leq 3$, we see that

$$\mu_2((K_{ii_3}-1)(K_{ii_1i_2}-1))=x_i^*\otimes([x_{i_1},x_{i_2},x_{i_3}]-1).$$

If $i_3=i$ and $i_1,i_2\neq i$, then

$$\mu_2((K_{ii_1}^{-1}-1)(K_{i_1i_2}-1))=x_i^*\otimes([x_{i_1},x_{i_2},x_i]-1).$$

If $i_1=i$ and $i_2,i_3\neq i$, then

$$\mu_2((K_{ii_3}-1)(K_{ii_2}-1))=x_i^*\otimes([x_i,x_{i_2},x_{i_3}]-1).$$

If $i_2=i$ and $i_1,i_3\neq i$, then

$$\mu_2((K_{ii_3}-1)(K_{ii_1}^{-1}-1))=x_i^*\otimes([x_{i_1},x_i,x_{i_3}]-1).$$

If $i_1=i_3=i$, then

$$\mu_2((K_{ii_2}^{-1}-1)(K_{i_2i}^{-1}-1))=x_i^*\otimes([x_i,x_{i_2},x_i]-1).$$

If $i_2=i_3=i$, then

$$\mu_2((K_{ii_1}^{-1}-1)(K_{i_1i}^{-1}-1))=x_i^*\otimes([x_{i_1},x_i,x_i]-1).$$

Hence the generators of $H^*\otimes_{\mathbf{Z}}\mathcal{L}_n(3)$ are contained in the image of μ_2. □

5 Free metabelian case

In this section, we mainly consider the case where $G = F_n^M := F_n/[\Gamma_n(2), \Gamma_n(2)]$. For simplicity, we often omit the capital F from the subscript F_n^M if there is no confusion. For example, we write $\mathcal{L}_n^M$, $\mathcal{L}_n^M(k)$, $\mathrm{IA}_n^M, \dots$, for $\mathcal{L}_{F_n^M}$, $\mathcal{L}_{F_n^M}(k)$, $\mathrm{IA}(F_n^M), \dots$, respectively. Here, we study the structure of graded quotients $Q^k(\mathrm{IA}_n^M)$ as a $\mathrm{GL}(n, \mathbf{Z})$–module.

5.1 Preliminary results for $G = F_n^M$

In this subsection, we recall some properties of the IA-automorphism group IA_n^M and the graded Lie algebras $\mathcal{L}_n^M$.

To begin with, we have $(F_n^M)^{\mathrm{ab}} = H$, and hence $\mathrm{Aut}\,(F_n^M)^{\mathrm{ab}} = \mathrm{Aut}(H) = \mathrm{GL}(n, \mathbf{Z})$. Since the surjective map $\rho_{F_n}\colon \mathrm{Aut}\, F_n \to \mathrm{GL}(n, \mathbf{Z})$ factors through $\mathrm{Aut}\, F_n^M$, a map $\rho_{F_n^M}\colon \mathrm{Aut}\, F_n^M \to \mathrm{GL}(n, \mathbf{Z})$ is also surjective. So we can identify $\mathrm{Aut}\, F_n^M/\mathrm{IA}(F_n^M)$ with $\mathrm{GL}(n, \mathbf{Z})$.

Let $\nu_n\colon \mathrm{Aut}\, F_n \to \mathrm{Aut}\, F_n^M$ be the natural homomorphism induced from the action of $\mathrm{Aut}\, F_n$ on F_n^M. Restricting ν_n to IA_n gives a homomorphism $\nu_n|_{\mathrm{IA}_n}\colon \mathrm{IA}_n \to \mathrm{IA}_n^M$. Bachmuth and Mochizuki [4] showed that $\nu_n|_{\mathrm{IA}_n}$ is surjective for $n \geq 4$. They also showed that $\nu_3|_{\mathrm{IA}_3}$ is not surjective and IA_3^M is not finitely generated [3]. Hence IA_n^M is finitely generated for $n \geq 4$ by the (coset classes of) Magnus generators K_{ij} and K_{ijl}. We remark that since $\mathrm{Ker}(\nu_n|_{\mathrm{IA}_n})$ is contained in $\mathcal{A}_n(3)$, we have isomorphisms

$$(\mathrm{IA}_n^M)^{\mathrm{ab}} \cong \mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H$$

as a $\mathrm{GL}(n, \mathbf{Z})$–module.

The associated Lie algebra $\mathcal{L}_n^M = \bigoplus_{k\geq 1} \mathcal{L}_n^M(k)$ is called the free metabelian Lie algebra generated by H or the Chen Lie algebra. It is also classically known due to Chen [5] that each $\mathcal{L}_n^M(k)$ is a $\mathrm{GL}(n, \mathbf{Z})$–equivariant free abelian group of rank

$$\mathrm{rank}_{\mathbf{Z}}(\mathcal{L}_n^M(k)) := (k-1)\binom{n+k-2}{k}.$$

We remark that $\mathcal{L}_n(k) = \mathcal{L}_n^M(k)$ for $1 \leq k \leq 3$.

By the same argument as in Section 4.1, for each $k \geq 2$, we can detect $S^k((\mathrm{IA}_n^M)^{\mathrm{ab}})$ in $Q^k(\mathrm{IA}_n^M)$ by the $\mathrm{GL}(n, \mathbf{Z})$–equivariant surjective homomorphism $\pi_k^M\colon Q^k(\mathrm{IA}_n^M) \to Q^k((\mathrm{IA}_n^M)^{\mathrm{ab}})$ induced from the abelianization of IA_n^M. In order to investigate the $\mathrm{GL}(n, \mathbf{Z})$–module structure of $\mathrm{Ker}(\pi_k^M)$, we use the Johnson homomorphism μ_k.

5.2 The image of $\mu_k|_{\mathrm{Ker}(\pi_k^M)}$

Here we study the image of the Johnson homomorphism

$$\mu_k\colon Q^k(\mathrm{IA}_n^M) \to H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))$$

restricted to the kernel of π_k^M for $n \geq 4$. First, in order to get a reasonable generators of $\mathcal{L}_n^M(k+1)$, we consider some lemmas. Let $\mathfrak{S}_l$ be the symmetric group of degree l. Then we have:

Lemma 5.1 *Let $l \geq 2$ and $n \geq 2$. For any element $[x_{i_1}, x_{i_2}, x_{j_1}, \dots, x_{j_l}] \in \mathcal{L}_n^M(l+2)$ and any $\lambda \in \mathfrak{S}_l$,*

$$[x_{i_1}, x_{i_2}, x_{j_1}, \dots, x_{j_l}] = [x_{i_1}, x_{i_2}, x_{j_{\lambda(1)}} \dots, x_{j_{\lambda(l)}}].$$

Proof Since $\mathfrak{S}_l$ is generated by transpositions $(m\ m+1)$ for $1 \leq m \leq l-1$, it suffices to prove the lemma for each $\lambda = (m\ m+1)$. Now we have

$$\begin{aligned}
&[[[x_{i_1}, x_{i_2}, x_{j_1}, \dots, x_{j_{m-1}}], x_{j_m}], x_{j_{m+1}}]] \\
&\quad = -[[x_{j_m}, x_{j_{m+1}}], [x_{i_1}, x_{i_2}, x_{j_1}, \dots, x_{j_{m-1}}]] \\
&\qquad\qquad - [[x_{j_{m+1}}, [x_{i_1}, x_{i_2}, x_{j_1}, \dots, x_{j_{m-1}}]], x_{j_m}] \\
&\quad = [[[x_{i_1}, x_{i_2}, x_{j_1}, \dots, x_{j_{m-1}}], x_{j_{m+1}}], x_{j_m}]
\end{aligned}$$

in $\mathcal{L}_n^M(m+3)$ by the Jacobi identity. Hence,

$$\begin{aligned}
[x_{i_1}, x_{i_2}, x_{j_1}, \dots, x_{j_l}] &= [x_{i_1}, x_{i_2}, x_{j_1}, \dots, x_{j_{m-1}}, x_{j_{m+1}}, x_{j_m}, \dots, x_{j_l}] \\
&= [x_{i_1}, x_{i_2}, x_{j_{\lambda(1)}} \dots, x_{j_{\lambda(l)}}].
\end{aligned}$$

in $\mathcal{L}_n^M(l+2)$. □

Similarly to $H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n(k+1))$, the $\mathbf{Z}$–module $H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))$ is generated by elements

$$x_i^* \otimes ([x_{i_1}, \dots, x_{i_{k+1}}] - 1), \quad 1 \leq i,\ i_j \leq n.$$

On the other hand, using Lemma 5.1, elements $[x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}] \in \mathcal{L}_n^M(k+1)$ is rewritten as

$$[x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_{l-1}}, x_i, x_i, \dots, x_i]$$

in $\mathcal{L}_n^M(k+1)$ for some l, $3 \leq l \leq k+2$ such that $i_3, i_4, \dots, i_{l-1} \neq i$. Hence $H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))$ is generated by elements

$$x_i^* \otimes ([x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_{l-1}}, x_i, x_i, \dots, x_i] - 1)$$

for some l, $3 \leq l \leq k+2$ such that $i_3, \dots, i_{l-1} \neq i$. Furthermore, without loss of generality, we may assume $i_2 \neq i$ in the generators above.

Proposition 5.2 *For any $k \geq 2$ and $n \geq 4$, the homomorphism*

$$\mu_k|_{\mathrm{Ker}(\pi_k^M)} \colon \mathrm{Ker}(\pi_k^M) \to H^* \otimes_{\mathbf{Z}} \oslash_{k+1}(\mathcal{L}_n^M(k+1))$$

is surjective.

Proof Take a generator

$$x_i^* \otimes ([x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_{l-1}}, x_i, x_i, \dots, x_i] - 1)$$

of $H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))$ for some l, $3 \leq l \leq k+2$ such that $i_2, \dots, i_{l-1} \neq i$ as mentioned above. Since $n \geq 4$, there exists some $1 \leq j \leq n$ such that $j \neq i, i_1, i_2$. First, consider an element

$$a := (K_{ij}^{-1} - 1)(K_{ji} - 1) \cdots (K_{ji} - 1) \in \Delta^{k-l+2}(\mathrm{IA}_n^M),$$

where $(K_{ji} - 1)$ appears $k-l+1$ times in the product. Then we see

$$\mu_{k-l+3}(a) = x_i^* \otimes ([x_j, x_i, \dots, x_i] - 1),$$

where x_i appears $k-l+2$ times among the component.

Next, set

$$b := \begin{cases} K_{jii_{l-1}} - 1 & \text{if } j \neq i_{l-1}, \\ K_{ji}^{-1} - 1 & \text{if } j = i_{l-1}, \end{cases}$$

$$c := (K_{ii_{l-2}} - 1)(K_{ii_{l-3}} - 1) \cdots (K_{ii_3} - 1) \in \Delta^{l-4}(\mathrm{IA}_n^M),$$

$$d := \begin{cases} K_{ii_1i_2} - 1 & \text{if } i \neq i_1, \\ K_{ii_2} - 1 & \text{if } i = i_1. \end{cases}$$

Then we have

$$\mu_k(abcd) = x_i^* \otimes ([x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_{l-1}}, x_i, x_i, \dots, x_i] - 1).$$

On the other hand, $\mu_k(dbac) = 0$. Hence we have

$$\mu_k(abcd - dbac) = x_i^* \otimes ([x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_{l-1}}, x_i, x_i, \dots, x_i] - 1).$$

Therefore since $abcd - dbac \in \mathrm{Ker}(\pi_k^M)$, we conclude that $\mu_k|_{\mathrm{Ker}(\pi_k^M)}$ is surjective. This completes the proof of Proposition 5.2. □

Then we have:

Theorem 5.3 *For $k \geq 2$ and $n \geq 4$, a $\mathrm{GL}(n, \mathbf{Z})$–equivariant homomorphism*

$$\mu_k \oplus \pi_k \colon Q^k(\mathrm{IA}_n^M) \to (H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))) \bigoplus S^k((\mathrm{IA}_n^M)^{\mathrm{ab}})$$

defined by $\sigma \mapsto (\mu_k(\sigma), \pi_k(\sigma))$ is surjective.

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