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Mutation and SL(2, ℂ)–Reidemeister torsion for hyperbolic knots

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Given a hyperbolic knot, we prove that the Reidemeister torsion of any lift of the holonomy to $\mathrm{SL}(2,\mathbb{C})$ is invariant under mutation along a surface of genus 2, hence also under mutation along a Conway sphere.

57M27; 57M50, 57M25

1 Introduction

Let $K \subset S^3$ be a hyperbolic knot. In this paper we prove that the Reidemeister torsion of the lift of the holonomy in $\mathrm{SL}(2,\mathbb{C})$ is invariant under mutation along a surface of genus 2. Mutation along a Conway sphere is composition of at most two genus 2 mutations (see Dunfield, Garoufalidis, Shumakovitch and Thistlethwaite [4]) so our result implies invariance under Conway mutation.

Let $F \subset S^3 \setminus K$ be an embedded, closed surface of genus 2, and let $\tau\colon F \to F$ denote the hyperelliptic involution, see Figure 1. The knot $K^\tau \subset S^3$ obtained by cutting along F and gluing again after composing with τ is called the mutant knot; the fact that K^τ is indeed a knot in S^3 is proved for instance in [4]. We are interested in comparing torsions of K and K^τ, thus we may assume that F is incompressible in the knot exterior $M = S^3 \setminus \mathcal{N}(K)$ (see [4, Proposition 2.1]).

Ruberman [18] showed that K^τ is also hyperbolic and that $M^\tau = S^3 \setminus \mathcal{N}(K^\tau)$ has the same volume as $M = S^3 \setminus \mathcal{N}(K)$. See [4] and Morton and Ryder [11] for a recent account on invariants that distinguish or not K from K^τ.

Let $\rho\colon \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ be a lift of the holonomy of the hyperbolic structure on the interior of M. If $\mu \in \pi_1(M)$ is a meridian of the knot, then $\mathrm{trace}(\rho(\mu)) = \pm 2$, and there are, up to conjugation, two lifts of the holonomy: one with $\mathrm{trace}(\rho(\mu)) = +2$ and another with $\mathrm{trace}(\rho(\mu)) = -2$. By Menal-Ferrer and Porti [9] ρ is acyclic, namely the homology and cohomology of M with coefficients twisted by ρ vanish; hence the Reidemeister torsion $\mathrm{tor}(M,\rho)$ is well defined. Moreover, as the dimension of $\mathbb{C}^2$ is

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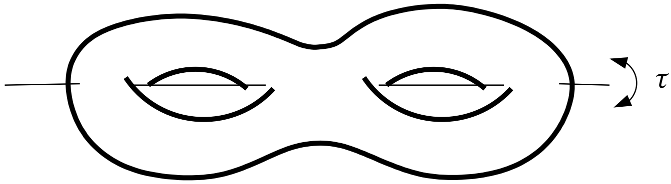

Figure 1: The hyperelliptic involution of a surface of genus 2

even, there is no sign indeterminacy, thus $\operatorname{tor}(M,\rho)$ is a well defined nonzero complex number. Therefore, these torsions are two topological invariants of the hyperbolic knot.

Theorem 1.1 *Let K, M, τ and M^τ be as above. Let $\rho\colon \pi_1(M)\to \mathrm{SL}(2,\mathbb{C})$ and $\rho^\tau\colon \pi_1(M^\tau)\to \mathrm{SL}(2,\mathbb{C})$ be lifts of the holonomy, with $\operatorname{trace}(\rho(\mu))=\operatorname{trace}(\rho^\tau(\mu))$. Then*

$$\operatorname{tor}(M,\rho)=\operatorname{tor}(M^\tau,\rho^\tau).$$

This is not true for any representation of $\pi_1(M)$. Wada proved in [22] that the twisted Alexander polynomials could be used to distinguish mutant knots. N Dunfield, S Friedl, and N Jackson [3] computed the torsion for the representation ρ twisted by the abelianization map (namely, the corresponding twisted Alexander polynomials) and proved that it distinguishes mutant knots up to 15 crossings. The evaluation at ± 1 of these polynomials provides evidence for Theorem 1.1 and has motivated the current paper.

In [9] we proved that when we consider the $2n$–dimensional irreducible representation $\sigma_{2n}=\mathrm{Sym}^{2n-1}\colon \mathrm{SL}(2,\mathbb{C})\to \mathrm{SL}(2n,\mathbb{C})$, then the composition $\sigma_{2n}\circ\rho$ is acyclic, thus its torsion is well defined. We have checked that the torsion of $\sigma_4\circ\rho$ distinguishes the Conway and the Kinoshita–Terasaka mutants, see Section 4.

The paper is organized as follows. In Section 2, we discuss the basic constructions for Reidemeister torsion and representations of mutants, and we give a sufficient criterion in Proposition 2.4 for invariance of the torsion under mutation. The criterion is stated in terms of the action of τ on the cohomology of F with twisted coefficients. This criterion is proved in Section 3, using the rulings of a quadric in $\mathbb{P}^3$ and a deformation argument. In Section 4 we compute an example, the Kinoshita–Terasaka and Conway mutants, and Section 5 is devoted to further discussion.

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2 Mutation

Let $M = S^3 \setminus \mathcal{N}(K)$ be a hyperbolic knot exterior and let $\rho\colon \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ be a lift of the holonomy representation. Let F be an embedded, incompressible, and closed surface of genus 2 in M, and $\tau\colon F \to F$, the hyperelliptic involution. The result of cutting M along F and then gluing back both copies of F using τ is denoted by M^τ. Since M is the exterior of a knot, F separates M into two pieces M_1 and M_2.

Write a commutative diagram for the inclusions:

$$\begin{tikzcd} F \arrow[r, "i_1"] \arrow[d, "i_2"'] & M_1 \arrow[d] \\ M_2 \arrow[r] & M, \end{tikzcd}$$

so that $\pi_1(M)$ is an amalgamated product

$$\pi_1(M) = \pi_1(M_1) *_{\pi_1(F)} \pi_1(M_2).$$

Let ρ_0, ρ_1, and ρ_2 denote the restriction of ρ to $\pi_1(F)$, $\pi_1(M_1)$, and $\pi_1(M_2)$, respectively, so that

$$\rho_1 \circ i_{1*} = \rho_2 \circ i_{2*} = \rho_0.$$

Using the notation

$$\rho_0^a(\gamma) = a\rho_0(\gamma)a^{-1} \quad \text{for all } \gamma \in \pi_1(F),$$

there exists $a \in \mathrm{SL}(2,\mathbb{C})$ which is unique up to sign (see Cooper and Long [1, Lemma 7.4], Ruberman [18, Theorem 2.2] or Tillmann [20, Lemma 2.1.1]), such that

$$\rho_0^a \circ \tau_* = \rho_0.$$

The existence of a is equivalent to the fact that τ is an isometry of $F \times \mathbb{R}$ equipped with the hyperbolic metric of a tubular neighbourhood $\mathcal{N}(F)$. Notice that $a \in \mathrm{SL}(2,\mathbb{C})$ corresponds to a rotation of order two in hyperbolic space, therefore a is conjugate to

$$a \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{1}$$

To construct the representation of $\pi_1(M^\tau)$, we also use the amalgamated product structure with the same inclusion i_1, but with $i_2 \circ \tau$ instead of i_2. The representation $\rho^\tau\colon \pi_1(M^\tau) \to \mathrm{SL}(2,\mathbb{C})$ is then defined by

$$\rho^\tau|_{\pi_1(M_1)} = \rho_1 \qquad \text{and} \qquad \rho^\tau|_{\pi_1(M_2)} = \rho_2^a.$$

This is well defined because $\rho_2^a \circ (i_2 \circ \tau)_* = \rho_0^a \circ \tau_* = \rho_0 = \rho_1 \circ i_{1*}$.

2.1 Cohomology with twisted coefficients

To set notation we recall the basic construction of cohomology with twisted coefficients. Let X be a CW–complex and $\rho\colon \pi_1(X) \to \mathrm{SL}(2,\mathbb{C})$ be a representation. The singular chains of its universal covering $\widetilde{X} \to X$ are denoted by $C_*(\widetilde{X};\mathbb{Z})$, which is a chain complex of left $\mathbb{Z}[\pi_1(X)]$–modules of finite type. We view $\pi_1(X)$ as the group of deck transformations of the universal covering $\widetilde{X} \to X$, so we do not consider any base point. The cochains with twisted coefficients are then

$$C^*(X;\rho) = \hom_{\mathbb{Z}[\pi_1(X)]}(C_*(\widetilde{X};\mathbb{Z}), \mathbb{C}^2_\rho),$$

where $\mathbb{C}^2_\rho \cong \mathbb{C}^2$ is viewed as a left $\mathbb{Z}[\pi_1(X)]$–module through the action induced by ρ. The corresponding cohomology groups are denoted by

$$H^*(|X|;\rho),$$

as they only depend on the underlying topological space $|X|$ of the CW–complex X. We shall mainly work with aspherical spaces, in this case the homology or cohomology of X with twisted coefficients is naturally isomorphic to the group cohomology of $\pi_1(X)$.

We shall also be interested in de Rham cohomology. Assuming that N is a smooth manifold, let $E = \widetilde{N} \times \mathbb{C}^2/\pi_1(N)$ denote the flat bundle with holonomy ρ. The space of p–forms valued on E is $\Omega^p(N;E) = \Gamma(\bigwedge^p T^*N \otimes E)$. The de Rham cohomology of $(\Omega^*(N;E), d_\rho)$ is $H^*(N;E)$ and it is naturally isomorphic to $H^*(N;\rho)$. In this paper N will be for instance the interior of M_1 or M_2, or $F \times \mathbb{R}$.

Many properties of cohomology with constant coefficients hold true when we have twisted coefficients: Mayer–Vietoris, the long exact sequence of the pair, etc. Poincaré duality is discussed in Section 3.1.

2.2 The map induced by an isometry

We discuss the map induced by and isometry, that plays an important role in this paper. The group of orientation preserving isometries of N is denoted by $\mathrm{Isom}^+(N)$ and it is a discrete group. Recall that N may be the interior of M_1 or M_2, or $F \times \mathbb{R}$. Every orientation preserving isometry $\sigma \in \mathrm{Isom}^+(N)$ lifts to an isometry of the universal covering, hence to an element $\pm s \in \mathrm{PSL}(2,\mathbb{C})$. It satisfies the equality

$$\rho \circ \sigma_* = \rho^s,$$

where σ_* denotes the map induced in the fundamental group. Then the induced map in cohomology has to take care of the representations in the coefficients

$$\sigma^*\colon H^1(N,\rho) \to H^1(N,\rho^s).$$

To relate the different coefficients, chose a lift $s \in \mathrm{SL}(2,\mathbb{C})$ and define

$$\begin{aligned} s_*\colon C^*(X;\rho) &\to C^*(X;\rho^s) \\ \theta &\mapsto s \circ \theta, \end{aligned}$$

where $|X| = N$. It is straightforward to check that this defines an isomorphism of complexes. Thus

$$s_*\colon H^*(N;\rho) \to H^*(N;\rho^s) \tag{2}$$

is an isomorphism. Then define

$$\pm\sigma^\natural = \pm s_*^{-1} \circ \sigma^*\colon H^1(N;\rho) \to H^1(N;\rho). \tag{3}$$

Notice that there is a sign indeterminacy, because the lift s is only unique up to sign. The following lemma is straightforward from the definition of $\sigma^\natural$.

Lemma 2.1 *The morphism* $\pm\sigma^\natural$ *is well defined up to sign. In addition, it induces a representation of* $\mathrm{Isom}^+(N)$ *in* $\mathrm{PGL}(H^1(N;\rho))$*, the projective group of linear transformations of* $H^1(N;\rho)$*.*

2.3 Mayer–Vietoris exact sequences with twisted coefficients

We will use Mayer–Vietoris for the pair (M_1, M_2) to compute the torsion of M and of M^τ. Since $H^*(M,\rho) \cong H^*(M^\tau,\rho^\tau) \cong 0$, the Mayer–Vietoris exact sequence gives the isomorphisms

$$i_1^* \oplus i_2^*\colon H^1(M_1;\rho_1) \oplus H^1(M_2;\rho_2) \to H^1(F;\rho_0), \tag{4}$$

$$i_1^* \oplus (i_2 \circ \tau)^*\colon H^1(M_1;\rho_1) \oplus H^1(M_2;\rho_2^a) \to H^1(F;\rho_0). \tag{5}$$

The isomorphism (2) $a_*\colon H^*(M_2;\rho_2) \to H^*(M_2;\rho_2^a)$ relates the cohomology group of conjugate representations. Recall that there are two choices of a up to sign. The action of the involution τ has an induced map defined as in (3):

$$\pm\tau^\natural = \pm a_*^{-1} \circ \tau^* = \pm a_* \circ \tau^*, \tag{6}$$

because $a_*^2 = -\mathrm{Id}$, see (1). Since $\tau^2 = \mathrm{Id}$, we have

$$(\tau^\natural)^2 = -\mathrm{Id}. \tag{7}$$

We also have a commutative diagram

$$\begin{tikzcd} H^1(M_2;\rho_2) \arrow[r, "i_2^*"] \arrow[d, "\pm a_*"'] & H^1(F;\rho_0) \arrow[d, "\pm\tau^\natural"] \\ H^1(M_2;\rho_2^a) \arrow[r, "(i_2\circ\tau)^*"] & H^1(F;\rho_0) \end{tikzcd} \tag{8}$$

because $(i_2\circ\tau)^*\circ(\pm a_*) = \pm a_*\circ(i_2\circ\tau)^* = \pm a_*\circ\tau^*\circ i_2^* = \pm\tau^\natural\circ i_2^*$.

We compute next the cohomology groups of the spaces involved in the Mayer–Vietoris sequence. We start with the surface F of genus 2.

Lemma 2.2 *$H^i(F;\rho_0)=0$ for $i\neq 1$ and $H^1(F;\rho_0)\cong\mathbb{C}^4$.*

Proof Firstly, $H^0(F;\rho_0)\cong H^0(\pi_1(F),\rho_0)$ is isomorphic to the subspace of $\mathbb{C}^2$ of elements that are fixed by $\rho_0(\pi_1(F))$; hence it vanishes because ρ_0 is an irreducible representation. By Poincaré duality $H^2(F;\rho_0)=0$. Finally

$$\dim_{\mathbb{C}} H^1(F;\rho_0) = -\chi(F)\dim(\mathbb{C}^2)=4. \qquad\square$$

Next we compute the cohomology groups of M_1 and M_2.

Lemma 2.3 *For $k=1,2$, $H^i(M_k;\rho_0)=0$ for $i\neq 1$ and $H^1(M_k;\rho_0)\cong\mathbb{C}^2$.*

Proof By Mayer–Vietoris, and using that $H^*(M;\rho)=0$, we get

$$H^i(M_1;\rho_1)\oplus H^i(M_2;\rho_2)\cong H^i(F;\rho_0).$$

The lemma follows from Lemma 2.2, because $\chi(M_k)=\frac{1}{2}\chi(F)=-1$. $\square$

2.4 Reidemeister torsions

Let X be a compact CW–complex equipped with a representation

$$\rho\colon \pi_1(X)\to \mathrm{SL}(2,\mathbb{C}).$$

When $H^*(|X|;\rho)=0$, the Reidemeister torsion can be defined and it is an invariant of X, up to subdivision, and the conjugacy class of ρ. We will not recall the definition, which can be found in Milnor [10] and Turaev [21] for instance. There are two main issues for the torsion we are interested in. Firstly, the torsion is only defined up to sign, but since we consider a two–dimensional vector space, it is sign defined, hence a nonzero complex number. Equivalently, any choice of homology orientation for Turaev's refined torsion [21] gives the same result. Secondly, since we are working with three and two–dimensional manifolds, the PL–structure is not relevant. Thus, for a two and three–dimensional manifold X and an acyclic representation $\rho\colon \pi_1(X) \to \mathrm{SL}(2,\mathbb{C})$, the torsion is denoted by

$$\mathrm{tor}(|X|,\rho) \in \mathbb{C} \setminus \{0\}.$$

When ρ is not acyclic, then we can also use the Reidemeister torsion provided we specify a basis for $H^*(|X|;\rho)$.

Choose b_K a basis for $H^1(M_k;\rho_k)$ as $\mathbb{C}$–vector space. In particular $a_*(b_2)$ is a basis for $H^1(M_2;\rho_2^a)$. By Milnor's formula [10] for the torsion of a long exact sequence applied to (4) and (5),

$$\mathrm{tor}(M,\rho) = \pm \frac{\mathrm{tor}(M_1,\rho_1,b_1)\,\mathrm{tor}(M_2,\rho_2,b_2)}{\mathrm{tor}(F,\rho_0,i_1^*(b_1)\sqcup i_2^*(b_2))},$$
$$\mathrm{tor}(M^\tau,\rho^\tau) = \pm \frac{\mathrm{tor}(M_1,\rho_1,b_1)\ \mathrm{tor}(M_2,\rho_2^a,a_*(b_2))}{\mathrm{tor}(F,\rho_0,i_1^*(b_1)\sqcup (i_2\circ\tau)^*(a_*(b_2)))}.$$

Here $\sqcup$ denotes the disjoint union of basis. Notice that Milnor works with torsions up to sign in [10], but his formalism applies even with sign. The isomorphism of complexes $a_*\colon C^*(M_2;\rho_2) \to C^*(M_2;\rho_2^a)$ can be used to prove that

$$\mathrm{tor}(M_2,\rho_2,b_2) = \mathrm{tor}(M_2,\rho_2^a,a_*(b_2)),$$

see [17, Remarque a2, Section 02]. Since $\pm\tau^\natural = \pm\tau^*\circ a_* = \pm a_*\circ\tau^*$, using (8) we deduce

$$\frac{\mathrm{tor}(M,\rho)}{\mathrm{tor}(M^\tau,\rho^\tau)} = \det(i_1^*(b_1)\sqcup(\pm\tau^\natural(i_2^*(b_2))), i_1^*(b_1)\sqcup i_2^*(b_2)). \tag{9}$$

Namely, the determinant of the matrix whose entries are the coefficients of the basis $i_1^*(b_1)\sqcup(\pm\tau^\natural(i_2^*(b_2)))$ with respect to $i_1^*(b_1)\sqcup i_2^*(b_2)$. Notice that the sign of $\tau^\natural$ is not relevant because $\dim(i_2^*(b_2))) = 2$.

The following is a sufficient criterion for invariance of torsion with respect to mutation.

Proposition 2.4 *If $\tau^\natural\colon H^1(F;\rho_0) \to H^1(F;\rho_0)$ leaves invariant the image of*

$$i_2^*\colon H^1(M_2;\rho_2) \to H^1(F;\rho_0),$$

then $\mathrm{tor}(M,\rho) = \pm\,\mathrm{tor}(M^\tau,\rho^\tau)$. If, in addition, the determinant of the restriction of $\tau^\natural$ to $\mathrm{Im}(i_2^*)$ is equal to 1, then $\mathrm{tor}(M,\rho) = \mathrm{tor}(M^\tau,\rho^\tau)$.

Proof Since $(\tau^\natural)^2 = -\mathrm{Id}$ by (7), $\tau^\natural$ diagonalizes with eigenvalues $\pm i$. Hence, assuming that $\tau^\natural$ leaves invariant the image of i_2^*, the matrix in (9) is conjugate to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & \pm i \end{pmatrix},$$

hence it has determinant ± 1. The other assertion is obvious. □

3 Invariance

In this section we prove Theorem 1.1. In Section 3.1 we recall a natural nondegenerate pairing B on $H^1(F;\rho_0)$. We show that for $k=1,2$ the image of $i_k^*\colon H^1(M_k;\rho_k) \to H^1(F;\rho_0)$ is an isotropic subspace with respect to B. Then in Section 3.2 we analyze properties of isotropic planes of $H^1(F;\rho_0)\cong\mathbb{C}^4$, which are viewed as lines in a ruled quadric in $\mathbb{P}^3$. More precisely, the quadric has two rulings and the hyperelliptic involution acts trivially in one of them, thus the criterion of Proposition 2.4 applies if the image of i_2^* is a projective line in the ruling where $\tau^\natural$ acts trivially. This is checked in Section 3.3, where M_2 is glued to another manifold M_3 that has several symmetries, including one that induces τ on its boundary. Those symmetries suffice to show that the criterion of Proposition 2.4 holds true for this other structure on M_2, the one that matches with M_3, and then a deformation argument is carried out to establish the criterion for the hyperbolic structure we are interested in.

3.1 A nondegenerate pairing

The determinant on $\mathbb{C}^2$ induces a nondegenerate, antisymmetric, bilinear product that is $\mathrm{SL}(2,\mathbb{C})$–invariant:

$$\begin{array}{ccccc} \mathbb{C}^2 & \otimes & \mathbb{C}^2 & \to & \mathbb{C} \\ \begin{pmatrix} a \\ b \end{pmatrix} & \otimes & \begin{pmatrix} c \\ d \end{pmatrix} & \mapsto & \det\begin{pmatrix} a & c \\ b & d \end{pmatrix}. \end{array}$$

Combined with the antisymmetric cup product in cohomology, it yields a *symmetric* bilinear form

$$B\colon H^1(F;\rho_0)\times H^1(F;\rho_0)\to H^2(F;\mathbb{C})\cong\mathbb{C}.$$

This pairing is bilinear, symmetric, nondegenerate (Poincaré duality), and natural. In terms of group cohomology this is described by Goldman in [6], see also Hodgson [8] and Sikora [19]. Here we use also the de Rham cohomology approach, therefore the cup product is represented by the wedge product on the E–valued differential forms.

Lemma 3.1 *The image of* $i_k^*\colon H^1(M_k;\rho_k)\to H^1(F;\rho_0)\cong H^1(F;E)\cong\mathbb{C}^4$ *is an isotropic plane for the product* B*.*

Proof The image $\mathrm{Im}(i_k^*)$ is a plane by Lemma 2.3 and its proof. The fact that the $\mathrm{Im}(i_k^*)$ is an isotropic plane is well known (see Hodgson [8] and Sikora [19]), but we sketch the argument for completeness. Let $[\alpha],[\beta]\in H^1(M_k;E)\cong H^1(M_k;\rho_k)$ with α and β two closed differential 1–forms with values in the flat vector bundle E defined by ρ. Thus $d_\rho\alpha=d_\rho\beta=0$, where d_ρ is the exterior differential on $\Omega^1(M_k;E)$. It is easy to prove that the formula

$$d(\alpha\wedge\beta)=(d_\rho\alpha)\wedge\beta+\alpha\wedge(d_\rho\beta),$$

holds, where $\wedge$ denotes the usual wedge product composed with the determinant. Thus, if $F=\partial M_k$, namely if M_k is disjoint from the knot neighbourhood, then Stokes theorem yields

$$B(i_k^*(\alpha),i_k^*(\beta))=\int_F i_k^*(\alpha)\wedge i_k^*(\beta)=\int_{M_k}d(\alpha\wedge\beta)=0.$$

Otherwise $\partial M_k=F\cup\partial\mathcal{N}(K)$, but since $H^*(\partial\mathcal{N}(K);\rho_k)=0$, the same argument applies. □

3.2 Finding isotropic planes with the ruled quadric

Let $\mathbb{P}^3$ denote the projective space on $H^1(F;\rho_0)\cong\mathbb{C}^4$. Isotropic planes of $H^1(F;\rho_0)$ with respect to B are in bijection with projective lines in the quadric

$$Q=\{x\in\mathbb{P}^3\mid B(x,x)=0\}.$$

Since B is a nondegenerate paring, Q is the standard quadric, which is a ruled surface with two rulings. We recall next its basic properties.

Proposition 3.2 *There are two disjoint families of projective lines* $\mathcal{L}_+$ *and* $\mathcal{L}_-$ *in* Q *such that:*

(i) *Every line in* Q *belongs to either* $\mathcal{L}_+$ or $\mathcal{L}_-$*.*

(ii) *Every point in* Q *belongs to precisely one line in* $\mathcal{L}_+$ *and one in* $\mathcal{L}_-$*.*

(iii) *Two lines in* Q *intersect if, and only if, one is in* $\mathcal{L}_+$ *and the other one is in* $\mathcal{L}_-$*.*

(iv) *Embedding lines in the projective Grassmannian,* $\mathcal{L}_+\cong\mathcal{L}_-\cong\mathbb{P}^1$*.*

This ruling is well known and it is related to the Segre embedding of $\mathbb{P}^1\times\mathbb{P}^1$ in $\mathbb{P}^3$ (see Mumford [12, Section 2B]). We provide a proof for completeness.

Proof We identify $\mathbb{C}^4$ with $\mathrm{M}_{2\times 2}(\mathbb{C})$, the space of 2×2 matrices with complex coefficients, and we may assume that the quadratic form is the determinant. Thus Q is identified to the projectivization of the set of matrices with zero determinant. For a nonzero matrix with vanishing determinant, its rows and its columns satisfy a linear combination. The linear combinations of rows define one of the projective lines containing the matrix, the other being defined by a linear combination of matrix columns. More precisely, for $A\in \mathrm{M}_{2\times 2}(\mathbb{C})$, with $A\neq 0$ but $\det(A)=0$, there exist nonzero $u\in \mathrm{M}_{1\times 2}(\mathbb{C})$ and $v\in \mathrm{M}_{2\times 1}(\mathbb{C})$ such that $u\cdot A=(0\,0)$ and $A\cdot v=\binom{0}{0}$. Then $\mathcal{L}_+$ is the family (of projectivizations) of planes $\{B\in \mathrm{M}_{2\times 2}(\mathbb{C})\mid u\cdot B=(0\,0)\}$ for some nonzero $u\in \mathrm{M}_{1\times 2}(\mathbb{C})$, and $\mathcal{L}_-$ is the corresponding family of lines defined by an equation $B\cdot v=\binom{0}{0}$, for some nonzero $v\in \mathrm{M}_{2\times 1}(\mathbb{C})$. The proposition follows easily from this construction. □

We use again the identification between $\mathbb{C}^4$ and $M_{2\times 2}(\mathbb{C})$ of the previous proof, the quadratic form being the determinant. Consider the action

$$\begin{array}{ccccc} \big(\mathrm{SL}(2,\mathbb{C})\times \mathrm{SL}(2,\mathbb{C})\big) & \times & \mathrm{M}_{2\times 2}(\mathbb{C}) & \to & \mathrm{M}_{2\times 2}(\mathbb{C}) \\ (A,B) & , & C & \mapsto & ACB^{-1}. \end{array}$$

Since this action preserves the determinant, it defines a map $\mathrm{SL}(2,\mathbb{C})\times \mathrm{SL}(2,\mathbb{C})\to \mathrm{SO}(4,\mathbb{C})$. Its kernel is precisely $\{\pm(\mathrm{Id},\mathrm{Id})\}$, and a standard dimensional argument of Lie groups gives the isomorphism

$$\mathrm{SL}(2,\mathbb{C})\times \mathrm{SL}(2,\mathbb{C})/\pm(\mathrm{Id},\mathrm{Id})\cong \mathrm{SO}(4,\mathbb{C}).$$

After projectivizing this induces an isomorphism

$$\mathrm{PSL}(2,\mathbb{C})\times \mathrm{PSL}(2,\mathbb{C})\cong \mathrm{PSO}(4,\mathbb{C}).$$

Using the construction of the ruling in the proof of Proposition 3.2, we immediately obtain the following proposition:

Proposition 3.3 *The action of* $\mathrm{PSL}(2,\mathbb{C})\times \mathrm{PSL}(2,\mathbb{C})\cong \mathrm{PSO}(4,\mathbb{C})$ *is equivalent to the product action on* $\mathcal{L}_+\times\mathcal{L}_-\cong \mathbb{P}^1\times\mathbb{P}^1$ *(by an equivalence that preserves the product).*

See Fulton and Harris [5, Section 18.2] for another description of this action. It follows from this proposition that $\mathrm{PSL}(2,\mathbb{C})\times\{\mathrm{Id}\}$ acts trivially on $\mathcal{L}_-$, and $\{\mathrm{Id}\}\times \mathrm{PSL}(2,\mathbb{C})$ acts trivially on $\mathcal{L}_+$.

Lemma 3.4 *The induced map* $\tau^\natural$ *on* $\mathbb{P}^3$ *lies in one factor* $\mathrm{PSL}(2,\mathbb{C})\times\{\mathrm{Id}\}$ *or* $\{\mathrm{Id}\}\times \mathrm{PSL}(2,\mathbb{C})$. *In particular it acts trivially in one of the rulings.*

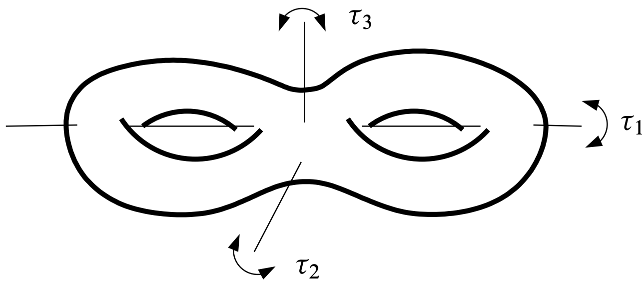

Figure 2: The three involutions of F (that extend to a handlebody)

Proof By looking at the action on $H^1(F;\rho_0)\cong\mathbb{C}^4$, we have $(\tau^\natural)^2=-\mathrm{Id}$, by (7). This implies that $\tau^\natural$ projects to an involution of $\mathbb{P}^3$ preserving B, hence to an involution in $\mathrm{PSL}(2,\mathbb{C})\times\mathrm{PSL}(2,\mathbb{C})$. Notice that if an involution of $\mathrm{PSL}(2,\mathbb{C})\times\mathrm{PSL}(2,\mathbb{C})$ is nontrivial on each factor, then it lifts to an element of $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ whose square is $-(\mathrm{Id},\mathrm{Id})$, hence to an involution in $\mathrm{SO}(4,\mathbb{C})\cong\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})/\pm(\mathrm{Id},\mathrm{Id})$. As $(\tau^\natural)^2=-\mathrm{Id}\in\mathrm{SO}(4,\mathbb{C})$, namely $\tau^\natural$ is not an involution in $\mathrm{SO}(4,\mathbb{C})$, we deduce that $\tau^\natural$ projects to an involution of one of the factors of $\mathrm{PSL}(2,\mathbb{C})\times\mathrm{PSL}(2,\mathbb{C})$ and it is trivial on the other factor. □

Up to permutation, let $\mathcal{L}_-$ denote the ruling on which $\tau^\natural$ acts trivially. Now the goal is to prove that $\mathrm{Im}(i_2^*)\in\mathcal{L}_-$.

3.3 A deformation argument

Let $\tau_1=\tau$ and consider two more involutions of F, τ_2 and τ_3 as in Figure 2. They satisfy $\tau_1\tau_2=\tau_3$, hence they define a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

Lemma 3.5 *There exists an orientable 3–manifold M_3 satisfying:*

(1) *The boundary ∂M_3 is an incompressible surface of genus 2. In addition M_3 has finitely many ends homeomorphic to $T^2\times[0,+\infty)$.*

(2) *The inclusion induces an epimorphism $H^1(M_3,\mathbb{Z}/2\mathbb{Z})\to H^1(\partial M_3,\mathbb{Z}/2\mathbb{Z})$.*

(3) *For $i=1,2,3$, $\tau_i\colon\partial M_3\to\partial M_3$ extends to an involution $\widetilde{\tau}_i$ of M_3 and $\widetilde{\tau}_1\widetilde{\tau}_2=\widetilde{\tau}_3$ is satisfied.*

(4) *The union $M_2\cup_F M_3$ obtained by identifying ∂M_3 with $F\subseteq\partial M_2$ is hyperbolic with finite volume.*

Proof of Lemma 3.5 We start with a handlebody H of genus 2, so that the group $\langle\tau_1,\tau_2\rangle$ that acts on ∂H extends to H. Viewing H as the union of two solid tori along a boundary disc, we consider then the link L_0 with two components that are the core curves of these solid tori. Then $H\setminus L_0$ satisfies (1), (2) and (3). To get (4), we shall remove some more curves in a equivariant way, so that (1), (2) and (3) are still satisfied. For this we apply Myers' theorem [13] in a $\langle\tau_1,\tau_2\rangle$–invariant way as in Paoluzzi and Porti [16] (namely on the orbifold $(H\setminus L_0)/\langle\tau_1,\tau_2\rangle$), so that there is an invariant link $L_1\subset H\setminus L_0$ such that $M_3=H\setminus(L_0\cup L_1)$ is irreducible, atoroidal, anannular, and has incompressible boundary. Then $M_2\cup_F M_3$ is also irreducible and atoroidal; it cannot be Seifert fibered, because the incompressible surface F separates (but it should be horizontal), hence $M_2\cup_F M_3$ is hyperbolic. □

Let $\rho'\colon \pi_1(M_2\cup_F M_3)\to \mathrm{SL}(2,\mathbb{C})$ be a lift of the holonomy of the hyperbolic structure on $M_2\cup_F M_3$, and denote by ρ'_2, ρ'_0, and ρ'_3 the respective restrictions to $\pi_1(M_2)$, $\pi_1(F)$, and $\pi_1(M_3)$. The strategy is to prove the claim for this manifold, namely that the image of $i_2\colon H^1(M_2,\rho'_2)\to H^1(F;\rho'_0)$ is invariant by the hyperelliptic involution, using that M_3 has more symmetries, and then deduce that it holds for the initial manifold by means of a deformation argument.

Consider the induced maps $\pm\tau_i^\natural\colon H^1(F;\rho'_0)\to H^1(F;\rho'_0)$ as in (3). By (7) and Lemma 2.1 we have

$$(\tau_1^\natural)^2=(\tau_2^\natural)^2=(\tau_3^\natural)^2=-\mathrm{Id}\quad\text{and}\quad \pm\,\tau_1^\natural\tau_2^\natural=\pm\tau_3^\natural.$$

Thus $\langle\tau_1^\natural,\tau_2^\natural\rangle$ is a subgroup of $\mathrm{PSO}(4,\mathbb{C})$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

Now consider Q' the quadric on $PH^1(F;\rho'_0)\cong\mathbb{P}^3$ and $\mathcal{L}'_+$ and $\mathcal{L}'_-$ its rulings, so that $Q'=\mathcal{L}'_+\times\mathcal{L}'_-$.

Lemma 3.6 *As a subgroup of* $\mathrm{PSO}(4,\mathbb{C})\cong\mathrm{PSL}(2,\mathbb{C})\times\mathrm{PSL}(2,\mathbb{C})$, $\langle\tau_1^\natural,\tau_2^\natural\rangle$ *lies in either* $\mathrm{PSL}(2,\mathbb{C})\times\{1\}$ *or* $\{1\}\times\mathrm{PSL}(2,\mathbb{C})$. *Hence we may assume that this group fixes pointwise* $\mathcal{L}'_-$.

Proof Lemma 3.4 tells that each $\tau_i^\natural$ lies in either $\mathrm{PSL}(2,\mathbb{C})\times\{1\}$ or $\{1\}\times\mathrm{PSL}(2,\mathbb{C})$. Then we use the relation $\tau_1^\natural\tau_2^\natural=\pm\tau_3^\natural$ to see that the factor is the same. □

Proposition 3.7 *Viewed as a line in* Q', *the image of the map induced by the inclusion* $i_3^*\colon H^1(M_3;\rho'_3)\to H^1(F;\rho'_0)$, *belongs to* $\mathcal{L}'_-$.

Proof Since τ_1 and τ_2 are isometries of M_3, $\mathrm{Im}(i_3^*)\in Q'$ is a line fixed by $\langle\tau_1^\natural,\tau_2^\natural\rangle$. Seeking a contradiction, assume that $\mathrm{Im}(i_3^*)\in\mathcal{L}'_+$. Then there would be a point in

$\mathcal{L}'_+ \cong \mathbb{P}^1$ fixed by a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ isomorphic to $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. This cannot happen, because such a subgroup acts freely on $\mathbb{P}^1$. Hence $\mathrm{Im}(i_3^*)\in\mathcal{L}'_-$. □

Corollary 3.8 *Viewed as a line in $\mathcal{Q}'$, the image of the map induced by the inclusion $i_2^*\colon H^1(M_2;\rho'_2)\to H^1(F;\rho'_0)$ belongs to $\mathcal{L}'_-$.*

Proof This follows from Proposition 3.7 and Proposition 3.2 (iii) applied to the direct sum $\mathrm{Im}(i_3^*)\oplus\mathrm{Im}(i_2^*)=H^1(F;\rho'_0)$. □

Corollary 3.8 is the statement for ρ'_2 and ρ'_0 that we aim for for ρ_2 and ρ_0. To get it for those representations we shall use a deformation argument. In particular we need to consider the variety or representations of $\pi_1(M_2)$ in $\mathrm{SL}(2,\mathbb{C})$, that is denoted by

$$R(M_2)=\hom(\pi_1(M_2),\mathrm{SL}(2,\mathbb{C})),$$

and it is an algebraic subset of the affine space $\mathbb{C}^N$.

Lemma 3.9 $\mathrm{Im}(i_2^*)$ *belongs to* $\mathcal{L}_-$.

Proof We connect $\rho_2\in R(M_2)$ to $\rho'_2\in R(M_2)$, a lift of the holonomy representation of M_2 that matches with M_3. Namely we want to find a path or representations

$$\begin{array}{rcl}[0,1] & \to & R(M_2)\\ t & \mapsto & \varphi_t\end{array}$$

that satisfies:

(i) $\varphi_0=\rho_2$.

(ii) For all $t\in[0,1]$, φ_t is the lift of the holonomy of a hyperbolic structure on M_2.

(iii) For all $t\in[0,1]$, $\dim H_1(M_2;\varphi_t)=2$.

(iv) $\varphi_1=\rho'_2$ is the lift of the holonomy of a hyperbolic structure on M_2 that matches with M_3 in Lemma 3.5.

Assuming we have this path of representations, then

$$\mathrm{Im}\big(i_2^*\colon H^1(M_2;\rho'_2)\to H^1(F;\rho'_0)\big)\in\mathcal{L}'_-,$$

by Corollary 3.8. Now, since there exists the path φ_t, the ruled quadric of $H^1(F;\varphi_t)$ is also deformed continuously (notice that as $\varphi_t|_F$ is irreducible Lemmas 2.2, 2.3, and 3.1 apply to $H^1(F;\varphi_t)$). Hence along the deformation the image of i_2^* is contained in $\mathcal{L}_-$, as $\mathcal{L}_+\cap\mathcal{L}_-=\varnothing$. Therefore

$$\mathrm{Im}\big(i_2^*\colon H^1(M_2;\rho_2)\to H^1(F;\rho_0)\big)\in\mathcal{L}_-,$$

as claimed.

Let us justify the existence of the path ϕ_t between ρ_2 and ρ_2'. If both $\rho_2(\pi_1(M_2))$ and $\rho_2'(\pi_1(M_2))$ are geometrically finite, then they can be connected along the space of geometrically finite structures of the pared manifold, because by Ahlfors–Bers theorem this space is isomorphic to the Teichmüller space of F, see Otal [15]. In addition, this is an open subset of the variety of representations of M_2 to $\mathrm{PSL}(2,\mathbb{C})$, and since the dimension of the cohomology is upper semi–continuous (it can only jump in a Zariski closed subset, see Hartshorne [7]), (iii) can be achieved by avoiding a proper Zariski closed subset (hence of real codimension ≥ 2). If any of $\rho_2(\pi_1(M_2))$ and $\rho_2'(\pi_1(M_2))$ is not geometrically finite, then it lies in the closure of geometrically finite structures (see Otal [14], though this is a particular case of the density theorem), thus there is a path in the space of representations of $\pi_1(M_2)$ in $\mathrm{PSL}(2,\mathbb{C})$ satisfying (ii) and (iii). To lift this path to $\mathrm{SL}(2,\mathbb{C})$, we start with ϕ_0 to be equal to ρ_2, which determines the lift ϕ_t for each $t\in[0,1]$. In particular $\rho_2'=\phi_1$ is determined and we chose ρ_3', the lift of the holonomy of M_3, to satisfy $\rho_3'|_{\pi_1(F)}=\rho_2'|_{\pi_1(F)}$, by using Lemma 3.5 (2). Namely we may replace any lift ρ_3' by $(-1)^\epsilon\rho_3'$ for some $\epsilon\colon \pi_1(M_3)\to\mathbb{Z}/2\mathbb{Z}$, and the fact that $H^1(M_3,\mathbb{Z}/2\mathbb{Z})\to H^1(F,\mathbb{Z}/2\mathbb{Z})$ is an epimorphism suffices to find ϵ so that $(-1)^\epsilon\rho_3'|_{\pi_1(F)}=\rho_2'|_{\pi_1(F)}$. □

By Lemma 3.9 and Proposition 2.4,

$$\mathrm{tor}(M,\rho)=\pm\,\mathrm{tor}(M^\tau,\rho^\tau).$$

We shall prove that there is also equality of signs, concluding the proof of Theorem 1.1:

Proposition 3.10 $\mathrm{tor}(M,\rho)=\mathrm{tor}(M^\tau,\rho^\tau)$.

Proof To remove the sign ambiguity we use again the deformation φ_t of the proof of Lemma 3.9. Since φ_t satisfies the sufficiency criterion of Proposition 2.4 for all $t\in[0,1]$, the eigenvalues of $\tau^\natural$ restricted to the image of i_2^* belong to $\{\pm i\}$, and they do not change as we deform t. Hence the determinant of $\tau^\natural$ restricted to the image of i_2^* is $+1$, because this holds for $\rho_2'=\varphi_1$ (as M_3 is τ–invariant). □

4 Example: Kinoshita–Terasaka and Conway mutants

Let KT and C be the Kinoshita–Terasaka knot and the Conway knot respectively. It is well known that they are hyperbolic and mutant along a Conway sphere. Using the Snap program (see Coulson, Goodman, Hodgson and Neumann [2]), based on J Weeks' SnapPea [23], we have obtained all the necessary information to compute their torsion.

The fundamental groups of these knots have the following presentations:

$$\pi_1(S^3 \setminus C) = \langle abc \mid abACbcbacBCABaBc, aBcBCABacbCbAbacbc\rangle,$$
$$\pi_1(S^3 \setminus KT) = \langle abc \mid aBCbABBCbaBcbbcABcbbaB, abcACaB\rangle.$$

As usual, capital letters denote inverse.

The image of the holonomy representation is contained in $\mathrm{PSL}(2, \mathbb{Q}(\omega))$ where $\mathbb{Q}(\omega)$ is the number field generated by a root ω of the polynomial

$$p(x) = x^{11} - x^{10} + 3x^9 - 4x^8 + 5x^7 - 8x^6 + 8x^5 - 5x^4 + 6x^3 - 5x^2 + 2x - 1.$$

The torsions then are elements of $\mathbb{Q}(\omega)$. In order to express elements in $\mathbb{Q}(\omega)$, we use the $\mathbb{Q}$–basis $(\omega^{10}, \omega^9, \dots, \omega, 1)$. Tables 1 and 2 give the coefficients of the torsions of KT and C with respect to this $\mathbb{Q}$–basis. On each table, the first column gives the element of the basis. We let n denote the dimension of the irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ used to compute the torsion, and the tables show the values for $n = 2$ (that is, the standard representation), but also $n = 4$ and $n = 6$. In order to compare them, the coefficients of the torsion for Kinoshita–Terasaka (KT) and Conway (C) knots are tabulated side by side. We give a table for each lift of the holonomy, one when the trace of the meridian is 2 (Table 1) and another when it is -2 (Table 2).

	$n=2$		$n=4$		$n=6$	
	KT	C	KT	C	KT	C
ω^{10}	356	356	11112880	11112880	676803770859632	662357458754672
ω^9	−620	−620	−38963592	−38963592	−640579476284656	−579216259622896
ω^8	636	636	36107416	36107416	212555254795952	153724448856752
ω^7	−864	−864	−31579196	−31579196	−990061444305088	−943617945204928
ω^6	1228	1228	60889040	60889040	1004678681648016	908722528184976
ω^5	−1080	−1080	−58195768	−58195768	−444238765345264	−349679698188784
ω^4	780	780	36555000	36555000	482101712163904	424247992815424
ω^3	−628	−628	−31740272	−31740272	−371824600930944	−320894530449024
ω^2	428	428	21313180	21313180	51168266257072	15655188602032
ω^1	−188	−188	−8829332	−8829332	−165869512283168	−152117462516768
1	124	124	7476160	7476160	−37602419304496	−50452054740016

Table 1: Torsions for the lift of the holonomy with trace of the meridian 2. The table gives the coefficients of the torsion of n–dimensional representation Sym^{n-1} (with respect to a $\mathbb{Q}$–basis for $\mathbb{Q}(\omega)$).

Of course, for $n = 2$ and for any lift of the holonomy, the torsion of KT and the torsion of C is the same. Notice that for the 4–dimensional representation, they are also the same for one lift but different for the other, and that they differ for both lifts when we use the 6–dimensional representation.

	$n=2$		$n=4$		$n=6$	
	KT	C	KT	C	KT	C
ω^{10}	7352	7352	−106244812	−84923788	−5089618734386048	−5181970358958464
ω^9	12100	12100	−40892392	−98464552	2633363724289740 8	2676752816711398 4
ω^8	−18868	−18868	135740632	176373400	−2613267846488212 8	−2655694343714913 6
ω^7	−16	−16	81031412	30483572	1896152546040371 2	1928233150046387 2
ω^6	−19124	−19124	70025564	154082012	−4126829530431662 4	−4194839392254843 2
ω^5	29448	29448	−188927128	−264857368	4181577625057168 0	4249576690878684 8
ω^4	−14272	−14272	71097428	118825172	−2520799555396448 0	−2562141977708460 8
ω^3	13576	13576	−71628932	−116091140	2231142042715502 4	2267627031570926 4
ω^2	−13352	−13352	98553148	124139068	−1599008323642632 0	−1624828012223854 4
ω^1	2780	2780	−4562444	−18136844	5898804809613840	5996288593045520
1	−5812	−5812	48068144	56560304	−5891958922292320	−5986195442605152

Table 2: Torsions for the lift of the holonomy with trace of the meridian -2, for the n–dimensional representations Sym^{n-1}. Again the table gives the coefficients with respect to a $\mathbb{Q}$–basis for $\mathbb{Q}(\omega)$.

As said in the introduction, when $n=2$, these had been computed by Dunfield, Friedl and Jackson in [3]. They computed numerically a twisted Alexander invariant (which are not mutation invariant) for all knots up to 15 crossings, and the torsions computed here are just the evaluations at ± 1.

5 Mutation for other representations

The proof of Section 3 applies to the following situation.

Proposition 5.1 *Let $\rho\colon \pi_1(M)\to \mathrm{SL}(2,\mathbb{C})$ be a representation satisfying:*

(1) $H^1(M;\rho)=0$*;*

(2) *ρ restricted to $\pi_1(F)$ is irreducible;*

(3) *the representation ρ is in the same irreducible component of $R(M)$ as some representation such that $\mathrm{Im}(i_2^*)$ is $\tau^\natural$–invariant and $\det(\tau^\natural|_{\mathrm{Im}(i_2^*)})=1$.*

Then $\mathrm{tor}(M,\rho)=\mathrm{tor}(M^\tau,\rho)$.

Corollary 5.2 *For a generic representation ρ of the irreducible component of $R(M)$ that contains a lift of the holonomy,* $\mathrm{tor}(M,\rho)=\mathrm{tor}(M^\tau,\rho^\tau)$.

Question The holonomy representation of a hyperbolic knot has two lifts to $\mathrm{SL}(2,\mathbb{C})$, each one with a different sign for the image of the meridian. Do they belong to the same irreducible component of the variety of representations?

This happens to be true for instance if the component of the variety of representations contains a dihedral representation, as this is a ramification point for the map from the variety of representations in $\mathrm{SL}(2,\mathbb{C})$ to those in $\mathrm{PSL}(2,\mathbb{C})$.

The three dimensional representation Sym^2 of $\mathrm{SL}(2,\mathbb{C})$ is conjugate to the adjoint representation in the automorphism group of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. The representation $\mathrm{Ad}\,\rho$ is not acyclic, but a natural choice of basis for homology has been given in Porti [17], hence its torsion is well defined. Moreover, we have:

Proposition 5.3 (Porti [17]) *The torsion* $\mathrm{tor}(M,\mathrm{Ad}\,\rho)$ *is invariant under genus 2 mutation.*

The proof is straightforward, as $H^1(F;\mathrm{Ad}\,\rho)$ is the tangent space to the variety of characters of F, and the action of the hyperelliptic involution is trivial on the variety of characters of F.

We have seen that if we compose the lift of the holonomy with the 6–dimensional representation Sym^5 of $\mathrm{SL}(2,\mathbb{C})$ (or the 4–dimensional one Sym^3 when the trace of the meridian is -2), then the torsion is not invariant under genus 2 mutation, as it is not invariant under Conway mutation, see the example of the previous section.

Question Working with the lift of the holonomy with trace of the meridian $+2$, is the torsion of the 4–dimensional representation Sym^3 invariant under Conway mutation?

To conclude, we notice that our arguments do not apply if we tensorize $\rho\colon \pi_1(M)\to \mathrm{SL}(2,\mathbb{C})$ with the abelianization map $\pi_1(M)\twoheadrightarrow \mathbb{Z}=\langle t\rangle$. This torsion gives the twisted polynomial in $\mathbb{C}[t^{\pm 1}]$ studied by Dunfield, Friedl and Jackson [3], where it is proved not to be mutation invariant. To apply our arguments, in Section 3 we use three involutions of the surface of genus 2, but some of them may be incompatible with the abelianization.

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