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Gromov K–area and jumping curves in $\mathbb{CP}^n$

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We give here some extensions of Gromov's and Polterovich's theorems on k–area of $\mathbb{CP}^n$, particularly in the symplectic and Hamiltonian context. Our main methods involve Gromov–Witten theory, and some connections with Bott periodicity and the theory of loop groups. The argument is closely connected with the study of jumping curves in $\mathbb{CP}^n$, and as an upshot we prove a new symplectic-geometric theorem on these jumping curves.

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1 Introduction

In [3] Gromov proposed an interesting way to probe the macroscopic geometry of Riemannian and symplectic manifolds by means of the geometry of complex vector bundles on the manifold. This is a geometric analogue of K–theory which is partly why it was named k–area. This construction involves minimizing the supremum norm of the curvature over all homologically essential vector bundles and all connections, and Gromov's main theorem in the Riemannian setting for spin, positively curved manifolds (X, g) used somewhat mysteriously the index theorem for the twisted Dirac operator.

To pass to the symplectic world Gromov considered an additional variation over all compatible metrics, ie all compatible almost complex structures with a symplectic form ω on X. At the moment the resulting invariant is still very poorly understood. Here we focus on $X = \mathbb{CP}^n$ and relate this notion to the classical theory of jumping curves and quantum classes originally defined by the author in [10]. Interestingly, this allows us to arrive at a purely algebraic-geometric theorem in the theory of jumping curves, as well as its symplectic generalization.

Some of the symplectic methods of the present paper continue in the spirit of Polterovich [8], and Entov [2].

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1.1

We begin by discussing k–area in the symplectic context. Fix

$$p\colon E \to X,$$

a rank r complex vector bundle with $c_1(E) = 0$ over a closed symplectic manifold (X^{2n}, ω) and let $p\colon P \to X$ denote its projectivization. Let $\mathcal{A}$ be a Hamiltonian connection on P, with curvature 2–form $R^{\mathcal{A}}$, which at $x \in X$ takes values in the Lie algebra of $\mathrm{Ham}(p^{-1}(x))$, ie the space of normalized smooth functions G on $p^{-1}(x)$. Here normalized means that

$$\int_{p^{-1}(x)} G \,\mathrm{Vol}_{\omega_{st}} = 0,$$

and ω_{st} on $p^{-1}(x) \simeq \mathbb{CP}^{r-1}$ is always assumed to be the standard form with $\omega_{st}([\text{line}]) = 1$.

Let j_X be an ω–compatible almost complex structure on X, and g_{j_X} the associated metric. We define the norm of the curvature tensor by

$$\|R^{\mathcal{A}}\|_{g_{j_X}} = \sup_{x \in X,\, \xi,\, \eta \in T_x X} |R^{\mathcal{A}}(\xi, \eta)|^{+}_{H}, \tag{1-1}$$

where $|\cdot|^{+}_{H}$ is "half" the Hofer norm $|G|^{+}_{H} = \max G$, for $G\colon (p^{-1}(x) \simeq \mathbb{CP}^{r-1}) \to \mathbb{R}$ in the Lie algebra of $\mathrm{Ham}(p^{-1}(x))$, and the supremum is over all orthonormal pairs ξ, η. Here is the basic quantity we will study:

$$\text{K–area}^{-1}(X, \omega) = \inf_{E,\, \mathcal{A},\, j_X} \|R^{\mathcal{A}}\|_{g_{j_X}}, \tag{1-2}$$

where the infimum is over all E with some nonvanishing Chern number, and $2r \geq \dim_{\mathbb{R}} X$.

Remark 1.1 The quantity (1-2) is closely related to the one studied by Gromov in [3]. This relationship is discussed in Polterovich [8]. We make a few comments: Gromov does not projectivize and works with unitary connections and consequently with the standard norm on the Lie algebra of $U(r)$, he also works with the inverse of our quantity, which we symbolize by the superscript -1 in K–area^{-1}. The condition that $2r \geq \dim_{\mathbb{R}} X$ is related to stability for homotopy groups of $\mathrm{SU}(n)$ and is vacuous if we restrict to unitary connections, since after stabilizing E we may extend the unitary connection to the stabilization, without affecting the norm (1-1).

Here is our first theorem:

Theorem 1.2

$$\text{K–area}^{-1}(\mathbb{CP}^n, E) \equiv \inf_{\mathcal{A},\, j_X} \|R^{\mathcal{A}}\|_{g_{j_X}} \geq 1, \tag{1-3}$$

where the infimum is over all Hamiltonian connections $\mathcal{A}$ on the projectivization of a fixed complex vector bundle E on $\mathbb{CP}^n$, provided that

$$\operatorname{rank}_{\mathbb{C}} E \geq n,$$

$c_1(E) = 0$ and some other Chern class of E does not vanish. In particular

$$\text{K–area}^{-1}(\mathbb{CP}^n) \geq 1.$$

This answers a question of Polterovich [8] about finding bounds for $\text{K–area}^{-1}(\mathbb{CP}^2, E)$, for rank 2 complex vector bundle over $\mathbb{CP}^2$, with $c_1(E) = 0$, $c_2(E) = 2$. If c_1 does not vanish then the argument is more elementary, and was already worked out in [8], although one can also adapt our discussion to subsume this case. The above theorem extends this:

Theorem 1.3 (Gromov [3])

$$\text{K–area}_U^{-1}(\mathbb{CP}^n, \omega_{st}) \geq 1.$$

Here U in K–area_U^{-1} emphasizes that Gromov worked with unitary connections. Notice we have the same lower bound in the unitary and Hamiltonian case.

Remark/Question 1.4 We can of course define $\text{K–area}^{-1}(\mathbb{CP}^n, g)$ for an arbitrary non almost Kahler metric g on $\mathbb{CP}^n$. And it is easy to find g for which

$$\text{K–area}^{-1}(\mathbb{CP}^n, g) < 1,$$

as it is elementary to check that

$$\text{K–area}^{-1}(\mathbb{CP}^n, E, c \cdot g) = \frac{1}{c}\, \text{K–area}^{-1}(\mathbb{CP}^n, E, g)$$

for $c > 0$. It appears to be much more difficult to construct an example of such a g with the same volume as $g_j = (\omega_{st}, j)$.

1.2 Jumping curves in $\mathbb{CP}^n$

Although the proof of Theorem 1.2 is via transcendental methods of Gromov–Witten theory, it is also closely related to the classical notion of jumping curves in $\mathbb{CP}^n$. In fact, as a corollary, we obtain an interesting phenomenon regarding these jumping curves. Here is a simplified version of the definition, suitable in our context.

Definition 1.5 Let $E \to \mathbb{CP}^n$ be a $\operatorname{rank}_{\mathbb{C}} E = r$ holomorphic vector bundle, with $c_1(E) = 0$. A smooth rational curve C in $\mathbb{CP}^n$ will be called a *jumping curve* if the restriction of E to C (by which we mean pullback) is not trivial as a holomorphic vector bundle. (Actually we will just be concerned with jumping lines.)

Jumping curves C can be further classified by the holomorphic isomorphism type of $E|_C$, which by Birkhoff–Grothendieck theorem is:

$$E|_C \simeq \bigoplus_i \mathcal{O}(\alpha(i)), \quad \text{with } \sum_i \alpha(i) = 0. \tag{1-4}$$

We can give a symplectic generalization of this notion as follows: Let $P_E \to \mathbb{CP}^n$ denote the projectivization of E, which is a Hamiltonian bundle and so its total space has a natural deformation class of symplectic forms Ω, extending the fiber wise symplectic forms ω_{st}; see McDuff and Salamon [5].

For a smooth rational curve C in $\mathbb{CP}^n$, we have a smooth identification of $P_E|_C$ with $X = \mathbb{CP}^{r-1} \times S^2$, and a canonical identification of $H_*(X)$ with $H_*(\mathbb{CP}^{r-1} \times S^2)$, since the group of bundle automorphisms of X acts trivially on homology by Lalonde and McDuff [4, Theorem 1.16].

Definition 1.6 Let $E \to \mathbb{CP}^n$ be a complex vector bundle with $c_1(E) = 0$. Let J be an almost complex structure compatible with Ω on P_E, and such that the projection map $P_E \to \mathbb{CP}^n$ is J–holomorphic. We will call such J *admissible*. A smooth rational curve C in $\mathbb{CP}^n$ is called a *jumping curve* if $P_E|_C \simeq \mathbb{CP}^{r-1} \times S^2$ has a J–holomorphic section in the class $d[\text{line}] + [S^2]$, $d < 0$.

A more natural way of stating this is that $P_E|_C$ has a J–holomorphic section u, with $\langle [\widetilde{\Omega}], [u] \rangle = d$, $d < 0$, where $[\widetilde{\Omega}]$ is the coupling class of P_E; see [5].

When J is induced by a holomorphic structure on E, this notion is equivalent to the classical notion, since in this case $P_E|_C$ is known to be the generalized Hirzebruch bundle

$$P_E|_C \simeq S^3 \times_{S^1} \mathbb{CP}^{r-1},$$

for some circle subgroup $S^1 \in \mathrm{SU}(r)$. If this subgroup is nontrivial and H denotes its generating Hamiltonian, then points $x \in F_{\max}$, (the maximum set of H), are fixed points of the S^1–action on $\mathbb{CP}^{r-1}$, and give holomorphic sections $S^3 \times_{S^1} \{x\}$ of $P_E|_C$ with $d < 0$ above.

If we restrict to lines in $\mathbb{CP}^n$, then the locus of classical jumping lines is a divisor of the variety of all lines in $\mathbb{CP}^n$, and if E is holomorphically nontrivial it is nonempty; see Okonek, Schneider and Spindler [7, Section 3]. We will prove the following partial symplectic generalization of this:

Theorem 1.7 *Let $E \to \mathbb{CP}^n$ be a rank $r \geq n$, complex vector bundle, with $c_1(E) = 0$ and some other Chern class nonzero. Then for any admissible J on P_E it has degree one jumping lines. If we also assume that J is suitably generic then d above can be chosen to be -1.*

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2 Setup

Here are our main conventions. The Hamiltonian vector field generated by $H\colon (M, \omega) \to \mathbb{R}$ is given by

$$\omega(X_H, \cdot\,) = -dH(\cdot).$$

An ω–compatible almost complex structure J is required to satisfy $\omega(v, Jv) > 0$, for $v \neq 0$. Homology is always over $\mathbb{Q}$ unless specified otherwise.

Our main tools are certain characteristic cohomology classes

$$qc_k \in H^{2k}(\Omega \mathrm{SU}(r), QH(\mathbb{CP}^{r-1})), \tag{2-1}$$

where $QH(\mathbb{CP}^{r-1})$ denotes the ungraded vector space $H(\mathbb{CP}^{r-1}, \mathbb{Q})$, with its ungraded quantum product. These classes were originally defined and named *quantum classes* in much more generality by the author in [10]. Note however, that qc_k here is qc_{2k} in [10]. The grading change is for convenience, as in this context all odd quantum classes vanish. We will not need the full definition, just some basic geometric content and the following theorem [9]:

Theorem 2.1 *The classes qc_k on $\Omega\mathrm{SU}(r)$ are algebraically independent and generate cohomology in the stable range $2k \leq 2r - 2$, with coefficients in $QH(\mathbb{CP}^{r-1})$.*

Here is a brief overview of the geometric construction of quantum classes. See [10] for more details of the following discussion. Let $M \hookrightarrow P \to X$ be a Hamiltonian bundle, with a Hamiltonian connection $\mathcal{A}$, with monotone fiber (M, ω), over a smooth manifold X. (Here M is used to denote a symplectic manifold because it serves a different logical purpose to (X, ω) of Section 1, but this M will just be $\mathbb{CP}^{r-1}$ in the rest of the paper.) We have a natural S^1–action on $\Omega^2 X$, induced by the rotation of S^2, along the axis of revolution containing the base point $0 \in S^2$.

Let $\Omega^2 X_{S^1}$ denote the Borel S^1–quotient:

$$\Omega^2 X_{S^1} = \Omega^2 X \times_{S^1} S^\infty.$$

Let B denote a closed, oriented smooth manifold. Given a cycle

$$B \xrightarrow{f} \Omega^2 X_{S^1},$$

there is a naturally induced Hamiltonian bundle $M \hookrightarrow P_f \to Y$, where $Y \to B$ is an oriented S^2–bundle over B, classified by the composition $B \to \Omega^2 X_{S^1} \to \mathbb{CP}^\infty$, with the map to $\mathbb{CP}^\infty$ being the canonical projection. Let us explain this: a map $B \to \Omega^2 X_{S^1}$ induces an S^1–equivariant map $T \to X \times S^\infty$, for T an oriented circle bundle over B. (Just by pulling back the universal circle bundle.) But this is the same thing as an S^1–equivariant map

$$T \times S^2 \to X \times S^\infty,$$

which then induces the S^1–quotient map $Y \to X \times \mathbb{CP}^\infty$, where Y is an oriented S^2–bundle over B. Our M bundle over Y is then just the pull-back by the induced map $Y \to X$. Equivalently we have a bundle

$$F \hookrightarrow P_f \xrightarrow{p} B, \tag{2-2}$$

with p denoting natural projection, where F is a Hamiltonian M bundle over S^2.

We may define classes

$$qc_* \in H^*(B, QH(M))$$

for P_f as in [10], via count of certain p–fiberwise holomorphic curves, with a p–fiberwise family of complex structures on P_f, induced by some Hamiltonian connection $\mathcal{A}$ on $M \hookrightarrow P \to X$. These classes are induced by universal classes

$$qc_* \in H^*(\Omega^2 B\mathrm{Ham}(M, \omega)_{S^1}, QH(M)). \tag{2-3}$$

Here are more details in the case $M = \mathbb{CP}^{r-1}$ and $\mathbb{CP}^{r-1} \hookrightarrow P \to X$ is a projectivization of a rank r complex vector bundle E with $c_1(E) = 0$. For $f\colon B \to \Omega^2 B\mathrm{Ham}(\mathbb{CP}^{r-1}, \omega)_{S^1}$ as above, the fibers F_b of $P_f \to B$ (as in (2-2)) are Hamiltonian bundle diffeomorphic to $F = \mathbb{CP}^{r-1} \times S^2$, although not naturally. The group of $\mathrm{Ham}(\mathbb{CP}^{r-1})$–bundle automorphisms of F_b acts trivially on homology, this follows by [4, Theorem 1.16]. In particular a section class A in $H_2(F_b)$ is uniquely characterized by its "degree" d,

$$A = d[\text{line}] + [\mathbb{CP}^1].$$

Since $Y \to B$ has a pair of canonical sections corresponding to the pair of fixed points of S^1 action on S^2, we have a pair of natural embeddings $I\colon B \times \mathbb{CP}^{r-1} \to P_f$.

The classes we now define "measure" quantum self intersection of $I(B \times \mathbb{CP}^n) \subset P_f$. Let

$$\mathcal{M}(P_f, d, \{J_b\})$$

denote the moduli space of tuples (u, b), where u is a J_b–holomorphic section of X_b in degree d. The virtual dimension of this space is given by the Fredholm index:

$$2n + 2k + 2\langle c_1^{vert}, A\rangle = 2n + 2k + 2d \cdot (n+1).$$

We define $qc_k \in H^{2k}(\Omega \mathrm{Ham}(\mathbb{CP}^n, \omega), QH(\mathbb{CP}^n))$ as follows:

$$\langle qc_k, [f]\rangle = \sum_{d \in \mathbb{Z}} b_d. \tag{2-4}$$

Here $b_d \in H_*(\mathbb{CP}^n)$ is defined by duality:

$$b_d \cdot_{\mathbb{CP}^n} c = ev_d \cdot_{B \times \mathbb{CP}^n} [B] \otimes c,$$

and where

$$\begin{aligned} ev_d\colon \mathcal{M}(P_f, d, \{J_b\}) &\to B \times \mathbb{CP}^n \\ ev_d(b, u) &= (b, u(0)), \end{aligned}$$

and $\cdot_M$, $\cdot_{B\times M}$ denote the intersection pairings in M, respectively $B \times M$. The sum (2-4), is finite and only $d < 0$ contribute for dimensional reasons.

3 Proofs

Proof of Theorem 1.7 Let E be a rank r complex vector bundle over $\mathbb{CP}^n$, with $n \leq r$, and some Chern class nonzero. We may assume without loss of generality that E has a nonvanishing Chern number. (Otherwise, restrict the following discussion to a subspace $\mathbb{CP}^i \subset \mathbb{CP}^n$, corresponding to a nonzero class $c_i(E)$.) Let

$$\mathcal{M}_{0,1}(\mathbb{CP}^n, [\text{line}], j; x_0, [\mathbb{CP}^{n-1}]) \to \mathbb{CP}^n$$

denote the moduli space of curves with 1 free marked point and 2 fixed marked points mapping to x_0, $\mathbb{CP}^{n-1}$, $x_0 \notin \mathbb{CP}^{n-1}$, with

$$ev\colon \mathcal{M}_{0,1}(\mathbb{CP}^n, [\text{line}], j; x_0, [\mathbb{CP}^{n-1}]) \to \mathbb{CP}^n,$$

denoting the evaluation map given by evaluating at the free marked point. It is well known that the standard complex structure on $\mathbb{CP}^n$ is regular and that for this standard j, the evaluation map is a degree one map

$$P \to \mathbb{CP}^n,$$

where P is an S^2–bundle over $\mathbb{CP}^{n-1}$ associated to the Hopf bundle. This is because there is a unique complex line through a pair of points in $\mathbb{CP}^n$. The induced cycle $f\colon \mathbb{CP}^{n-1} \to \Omega^2\mathbb{CP}^n_{S^1}$ represents a class denoted a. Let $e\colon \mathbb{CP}^\infty \to \Omega^2\,\mathrm{BSU}(r)_{S^1} = \Omega^2\,\mathrm{BSU}(r) \times_{S^1} S^\infty$ be the section corresponding to the canonical fixed point of the S^1–action on $\Omega^2\,\mathrm{BSU}(r)$, ie the constant map of S^2 to the based point $x_0 \in X$.

Lemma 3.1

$$0 \neq f_{E_*} a \in H_{2n-2}(\Omega^2\,\mathrm{BSU}(r)_{S^1})/e_* H_*(\mathbb{CP}^\infty),$$

where

$$f_E\colon \Omega^2\mathbb{CP}^n_{S^1} \to \Omega^2\,\mathrm{BSU}(r)_{S^1}$$

is the map induced by $E \to \mathbb{CP}^n$.

Proof Let us suppose otherwise. The composition map

$$ev\colon P \to \mathbb{CP}^n \to \mathrm{BSU}(r),$$

is nonvanishing in homology, since E has a nonvanishing Chern number and

$$ev\colon P \to \mathbb{CP}^n$$

is of degree one by discussion above.

Let

$$H\colon T \to \Omega^2\,\mathrm{BSU}(r)_{S^1}$$

be a bordism of $f_{E_*} a_{i-1}$ to $c \in e_* H_*(\mathbb{CP}^\infty)$. We'll denote the corresponding boundary pieces of T by T_a and T_c. Consequently, the bordism H induces an S^2–bundle P_T over T, it is the pull-back of the tautological S^2–bundle

$$(\Omega^2_{x_0}\,\mathrm{BSU}(r) \times E^\infty) \times_{S^1} S^2 \to \Omega^2_{x_0}\,\mathrm{BSU}(r)_{S^1},$$

and of course P_T restricts over T_a to P. We have a natural "evaluation" map

$$ev_T\colon P_T \to \mathrm{BSU}(r),$$

restricting to evaluation maps ev, ev_c over the boundary, and so a homology of $[ev]$ to $[ev_c]$, but ev_c is the constant map to the based point $x_0 \in BU$; a contradiction. □

Lemma 3.2 *For some integer $\{\alpha_i, \beta_i\}$ with at least one α_i, β_i nonzero:*

$$\left\langle \prod_{i,j} q c_{\beta_i}^{\alpha_i} \wedge c_1^j, f_{E_*} a_{n-1} \right\rangle \neq 0, \tag{3-1}$$

where c_1 is the pullback to $\Omega^2\,\mathrm{BSU}(r)_{S^1}$ of the canonical generator of $H^2(\mathbb{CP}^\infty)$ by the natural projection $\Omega^2\,\mathrm{BSU}(r)_{S^1} \to \mathbb{CP}^\infty$.

Proof Note that all of the rational cohomology of $\Omega^2\,\mathrm{BSU}(r) \simeq \Omega\mathrm{SU}(r)$, is in even degree, since by Milnor–Morre [6] and Cartan–Serre [1], the rational homology algebra of $\Omega\mathrm{SU}(r)$ is generated as a ring with Pontryagin product by the rational homotopy groups, (via the Hurewicz homomorphism) which are all in even degrees since the rational homotopy groups of $\mathrm{SU}(r)$ are well known to be all in odd degrees. (In fact $\mathrm{SU}(r)$ has the rational homotopy type of the product of odd spheres $S^3 \times S^5 \times \cdots$) Consequently, the Serre spectral sequence for the fibration

$$\Omega^2\,\mathrm{BSU}(r) \hookrightarrow \Omega^2\,\mathrm{BSU}(r)_{S^1} \to \mathbb{CP}^\infty$$

degenerates at the second page and so:

$$H^*(\Omega^2\,\mathrm{BSU}(r)_{S^1}) \simeq H^*(\Omega^2\,\mathrm{BSU}(r)) \otimes H^*(\mathbb{CP}^\infty) \simeq H^*(\Omega\mathrm{SU}(r)) \otimes H^*(\mathbb{CP}^\infty).$$

Our lemma then follows by Lemma 3.1 and Theorem 2.1. □

The theorem then readily follows. Since by construction of quantum classes and Lemma 3.2, for any fixed (not necessarily regular) complex structure J on P compatible with Ω, and with projection to $\mathbb{CP}^n$ for some complex line l in $\mathbb{CP}^n$, the restriction of P to l, which is diffeomorphic to $\mathbb{CP}^{r-1} \times S^2$ has a J holomorphic stable section u in the total class $S = -[\text{line}] + S^2$, as otherwise the relevant Gromov–Witten invariants in class S all vanish and (3-1) is impossible. Of course the stable section u may be in the form of a holomorphic section u_p in a class $d[\text{line}] + S^2$, with $d < -1$ together with some vertical holomorphic bubbles, but this still implies our claim. □

Proof of Theorem 1.2 We just need the following lemma:

Lemma 3.3 *The norm of the curvature $\|R^{\mathcal{A}}\|$ of the projectivization $\mathbb{CP}^{r-1} \hookrightarrow P \to \mathbb{CP}^n$ is at least 1.*

Proof Let $\widetilde{\Omega}$ denote the coupling form of the Hamiltonian fibration P associated to $\mathcal{A}$, (see for example [5] for discussion on coupling forms). This is a certain closed form associated to the curvature form of $\mathcal{A}$, with the following properties.

The restriction of $\widetilde{\Omega}$ to fibers $M \simeq \mathbb{CP}^{r-1}$ of $P \to \mathbb{CP}^n$ coincides with ω_{st}. The $\widetilde{\Omega}$–orthogonal subspaces in TP to the fibers are horizontal subspaces, whose value on horizontal lifts

$$\widetilde{v}, \widetilde{w} \in T_{m,z}P$$

of $v, w \in T_z\mathbb{CP}^n$ are given by

$$\widetilde{\Omega}(\widetilde{v}, \widetilde{w}) = -R^{\mathcal{A}}(v, w)(m),$$

for $m \in M_z$; in other words we evaluate the Lie algebra element of $\mathrm{Ham}(M_z, \omega)$, $R^{\mathcal{A}}(v, w)$ (ie, a function on M_z), at m.

Consider the symplectic form $\Omega = \widetilde{\Omega} + (\|R^{\mathcal{A}}\| + \epsilon)\omega_{st}$, where ω_{st} is the standard Fubini–Study symplectic form on the base normalized by the condition that the area of a complex line is 1, and $\epsilon > 0$. Pick any compatible complex structure $J_{\mathcal{A}}$. By Theorem 1.7, for some complex line l in $\mathbb{CP}^n$, the restrictions of P to l, which is diffeomorphic to $\mathbb{CP}^{r-1} \times S^2$ (by the assumption that $c_1(E) = 0$) has a $J_{\mathcal{A}}$–holomorphic section u in class $S = d \cdot [\text{line}] + S^2$, with $d \leq -1$. Here [line] is the class of the complex line in $\mathbb{CP}^{r-1}$.

Since $J_{\mathcal{A}}$ is Ω compatible, for the class S, $J_{\mathcal{A}}$–holomorphic section u of $P|_l$ we get

$$0 \leq [\Omega]([u]) = [\widetilde{\Omega}]([u]) + \|R^{\mathcal{A}}\| + \epsilon.$$

On the other hand $[\widetilde{\Omega}] = [\omega_{st}]$ on $P|_l$ since the cohomology class of the coupling form is independent of the choice of connection, and the form ω_{st} on $\mathbb{CP}^{r-1} \times S^2$, is another coupling form associated to the trivial connection on this bundle. Since $[\omega_{st}]([\text{line}]) = 1$ by our normalization, it follows that $[\widetilde{\Omega}]([u]) = d$, since $[u] = d[\text{line}] + S^2$. So we get

$$-d \leq \|R^{\mathcal{A}}\| + \epsilon,$$

for every $\epsilon > 0$. □

This finishes the proof of the theorem. □

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