Algebraic & Geometric Topology 13 (2013) 237–275

Cascades and perturbed Morse–Bott functions

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Let $f\colon M \to \mathbb{R}$ be a Morse–Bott function on a finite-dimensional closed smooth manifold M. Choosing an appropriate Riemannian metric on M and Morse–Smale functions $f_j\colon C_j \to \mathbb{R}$ on the critical submanifolds C_j, one can construct a Morse chain complex whose boundary operator is defined by counting cascades [16]. Similar data, which also includes a parameter $\varepsilon > 0$ that scales the Morse–Smale functions f_j, can be used to define an explicit perturbation of the Morse–Bott function f to a Morse–Smale function $h_\varepsilon\colon M \to \mathbb{R}$ [3; 6]. In this paper we show that the Morse–Smale–Witten chain complex of h_ε is the same as the Morse chain complex defined using cascades for any $\varepsilon > 0$ sufficiently small. That is, the two chain complexes have the same generators, and their boundary operators are the same (up to a choice of sign). Thus, the Morse Homology Theorem implies that the homology of the cascade chain complex of $f\colon M \to \mathbb{R}$ is isomorphic to the singular homology $H_*(M;\mathbb{Z})$.

57R70; 37D05, 37D15, 58E05

1 Introduction

Let $f\colon M \to \mathbb{R}$ be a Morse–Bott function on a finite-dimensional closed smooth Riemannian manifold (M, g) with connected critical submanifolds C_j for $j = 1, \ldots, l$. There are at least three approaches to computing the homology of M using moduli spaces of gradient flow lines:

(1) Perturb $f\colon M \to \mathbb{R}$ to a Morse–Smale function and use the Morse–Smale–Witten chain complex, whose boundary operator is defined using moduli spaces of gradient flow lines of the perturbed function (see for instance the authors' [4], Schwarz [29], and the references therein).

(2) Introduce Morse functions $f_j\colon C_j \to \mathbb{R}$ on the critical submanifolds $C_1, \ldots, C_l$ and use a Morse chain complex whose boundary operator is defined using moduli spaces of cascades (see Frauenfelder [16]).

Published: 16 February 2013 DOI: 10.2140/agt.2013.13.237

(3) Use the Morse–Bott–Smale multicomplex, where the homomorphisms in the multicomplex are defined using fibered products of moduli spaces of gradient flow lines of the Morse–Bott function $f\colon M \to \mathbb{R}$ (see the authors' [7]).

A fourth approach might involve using the filtration determined by the Morse–Bott function $f\colon M \to \mathbb{R}$ to define a spectral sequence, but the differentials in the spectral sequence determined by the filtration are not defined using moduli spaces of gradient flow lines (see the authors' [7] and the second author's [19]). In addition, there are approaches to computing the cohomology/homology of M from a Morse–Bott function using differential forms and/or currents (see Austin and Braam [3], Cho and Hong [12] and Latschev [22]), but we will not discuss differential forms or currents in this paper.

The main goal of this paper is to show that for a finite-dimensional closed smooth manifold M the first two approaches are essentially the same. That is, the auxiliary Morse functions $f_j\colon C_j \to \mathbb{R}$ on the critical submanifolds C_j for $j = 1, \ldots, l$ required to define the cascade chain complex and a parameter $\varepsilon > 0$ determine an explicit perturbation of the Morse–Bott function $f\colon M \to \mathbb{R}$ to a Morse function $h_\varepsilon\colon M \to \mathbb{R}$ (see Austin and Braam [3] and the authors' [6]). Moreover, under certain transversality assumptions the Morse–Smale–Witten chain complex of $h_\varepsilon\colon M \to \mathbb{R}$ has the same generators and the same boundary operator as the cascade chain complex (up to a choice of sign).

We now describe the cascade chain complex for a Morse–Bott function. To the best of our knowledge, moduli spaces of cascades were first introduced within the context of symplectic Floer homology by Frauenfelder [16], and cascade-like objects were simultaneously introduced within the context of contact homology by Bourgeois [9]. Moduli spaces of cascades have since been used in the contexts of contact homology and gauge theory by several authors (see Bourgeois and Oancea [10; 11] Cieliebak and Frauenfelder [13] and Swoboda [30]). Our approach to constructing moduli spaces of cascades and their compactifications is given in Sections 3 and 4 for a function $f\colon M \to \mathbb{R}$ on a finite-dimensional closed smooth Riemannian manifold (M, g) that satisfies the Morse–Bott–Smale transversality condition. The moduli spaces of cascades are constructed using finite-dimensional fibered products similar to those found in the authors' [7], and the compactifications of the moduli spaces are described in terms of the Hausdorff topology.

Cascades

Let $f\colon M \to \mathbb{R}$ be a Morse–Bott function on a finite-dimensional closed smooth Riemannian manifold (M, g) with connected critical submanifolds $C_1, \ldots, C_l$. Choose

Morse–Smale functions $f_j\colon C_j \to \mathbb{R}$ on the critical submanifolds for all $j = 1, \dots, l$, and define the *total index* of a critical point of f_j to be its Morse index on C_j plus the Morse–Bott index of the critical submanifold C_j. Roughly speaking, a *cascade* between two critical points is a concatenation of some gradient flow lines of the function f and pieces of the gradient flow lines of the functions f_j on the critical submanifolds. Choosing appropriate Riemannian metrics on M and the critical submanifolds C_j, it is shown in the appendix to Frauenfelder [16] that the moduli space of cascades $\mathcal{M}^c(q, p)$ between two critical points q and p is a smooth manifold of dimension $\lambda_q - \lambda_p - 1$, where λ_q and λ_p denote the total indices of q and p respectively. Moreover, $\mathcal{M}^c(q, p)$ has a compactification consisting of broken flow lines with cascades between q and p.

Since the moduli space of cascades $\mathcal{M}^c(q, p)$ has properties similar to those of a moduli space of gradient flow lines of a Morse–Smale function, it is natural to define a chain complex analogous to the Morse–Smale–Witten chain complex but using moduli spaces of cascades in place of moduli spaces of gradient flow lines. Thus, we define the k–th chain group $C_k^c(f)$ to be the free abelian group generated by the critical points of total index k of the Morse–Smale functions f_j for all $j = 1, \dots, l$. In the appendix to [16] a boundary operator ∂_*^c is defined by counting the number of cascades between critical points of relative index one mod 2, and a continuation theorem is stated that implies that the homology of the chain complex $(C_*^c(f) \otimes \mathbb{Z}_2, \partial_*^c)$ is isomorphic to the singular homology $H_*(M; \mathbb{Z}_2)$. In Section 5 of this paper we show that it is possible to define the boundary operator ∂_*^c over $\mathbb{Z}$ by counting the elements of $\mathcal{M}^c(q, p)$ with sign when $\lambda_q - \lambda_p = 1$, and we prove that the homology of the resulting chain complex $(C_*^c(f), \partial_*^c)$ is isomorphic to the singular homology $H_*(M; \mathbb{Z})$.

Perturbing the Morse–Bott function

The particular Morse–Smale functions $f_j\colon C_j \to \mathbb{R}$ chosen to define the chain complex $(C_*^c(f), \partial_*^c)$ can also be used to define an explicit perturbation of the Morse–Bott function $f\colon M \to \mathbb{R}$ to a Morse–Smale function $h_\varepsilon\colon M \to \mathbb{R}$. This perturbation technique was used by Austin and Braam [3] in relation to a de Rham version of Morse–Bott cohomology. It was also used by the authors [6] to give a dynamical systems approach to the proof of the Morse–Bott inequalities with somewhat different orientation assumptions than the classical "half-space" method using the Thom Isomorphism Theorem (see Bott [8], Farber [15, Appendix C], and Nicolaescu [25, Section 2.6]).

To define the Morse–Smale function $h_\varepsilon\colon M \to \mathbb{R}$ near f choose "small" tubular neighborhoods T_j of each of the critical submanifolds C_j for all $j = 1, \dots, l$ and extend the Morse–Smale functions f_j to the tubular neighborhoods T_j by making them constant in the direction normal to C_j. Choose bump functions ρ_j on the tubular neighborhoods T_j for all $j = 1, \dots, l$ that are equal to one in an open neighborhood of C_j, constant in the direction parallel to C_j, and equal to zero outside of T_j. The function

$$h_\varepsilon = f + \varepsilon \left(\sum_{k=1}^{l} \rho_k f_k \right)$$

is a Morse function near f for any sufficiently small $\varepsilon > 0$, and the critical set of h_ε is the union of the critical points of the functions $f_j\colon C_j \to \mathbb{R}$ for $j = 1, \dots, l$. In fact, the total index λ_q of a critical point q is the same as the Morse index of q viewed as a critical point of $h_\varepsilon\colon M \to \mathbb{R}$.

Correspondence

If we choose the Riemannian metric g on M so that $h_\varepsilon\colon M \to \mathbb{R}$ satisfies the Morse–Smale transversality condition with respect to g, then the moduli space $\mathcal{M}_{h_\varepsilon}(q, p)$ of gradient flow lines of h_ε between two critical points q and p is a smooth manifold with $\dim \mathcal{M}_{h_\varepsilon}(q, p) = \lambda_q - \lambda_p - 1$. We show in Section 3 that if $f\colon M \to \mathbb{R}$ satisfies the Morse–Bott–Smale transversality condition and we choose the Morse functions f_j on the critical submanifolds so that some additional transversality conditions are satisfied, then the moduli space of cascades $\mathcal{M}^c(q, p)$ is also a smooth manifold of dimension $\lambda_q - \lambda_p - 1$.

In Section 5 we prove that when the dimension of these moduli spaces is zero they have the same number of elements.

Theorem 1.1 (Correspondence of moduli spaces) *Let $p, q \in \mathrm{Cr}(h_\varepsilon)$ with $\lambda_q - \lambda_p = 1$. For any sufficiently small $\varepsilon > 0$ there is a bijection between unparameterized cascades and unparameterized gradient flow lines of the Morse–Smale function $h_\varepsilon\colon M \to \mathbb{R}$ between q and p,*

$$\mathcal{M}^c(q, p) \longleftrightarrow \mathcal{M}_{h_\varepsilon}(q, p).$$

Choosing orientations on the unstable manifolds of the Morse–Smale function $h_\varepsilon\colon M \to \mathbb{R}$ associates a sign ± 1 to each component of $\mathcal{M}_{h_\varepsilon}(q, p)$ when $\lambda_q - \lambda_p = 1$, and thus we can use the correspondence theorem for moduli spaces to transport the signs to the components of $\mathcal{M}^c(q, p)$. This allows us to define the boundary operator in the cascade chain complex over $\mathbb{Z}$, and we have the following as an immediate corollary.

Corollary 1.2 (Correspondence of chain complexes) *For $\varepsilon > 0$ sufficiently small, the Morse–Smale–Witten chain complex $(C_*(h_\varepsilon), \partial_*)$ associated to the perturbation*

$$h_\varepsilon = f + \varepsilon \left(\sum_{k=1}^{l} \rho_k f_k \right)$$

of a Morse–Bott function $f\colon M \to \mathbb{R}$ is the same as the cascade chain complex $(C^c_*(f), \partial^c_*)$. That is, the chain groups of both complexes have the same generators and their boundary operators are the same (up to a choice of sign).

This corollary, together with the Morse Homology Theorem, implies immediately that the homology of the chain complex $(C^c_*(f), \partial^c_*)$ is isomorphic to the singular homology $H_*(M;\mathbb{Z})$.

Outline of the paper

In Section 2 we recall some basic definitions and facts about the Morse–Smale–Witten chain complex. In Section 3 we give a detailed construction of the smooth moduli space of cascades $\mathcal{M}^c(q,p)$ under the assumption that $f\colon M \to \mathbb{R}$ satisfies the Morse–Bott–Smale transversality condition with respect to the metric g on M. Our construction requires that the Morse functions $f_j\colon C_j \to \mathbb{R}$ satisfy the Morse–Smale transversality condition with respect to the restriction of the Riemannian metric g to the critical submanifolds for all $j = 1, \dots, l$ and that all the unstable and stable manifolds on the critical submanifolds are transverse to certain beginning and endpoint maps (Definition 3.8). Lemma 3.9 shows that it is always possible to choose the auxiliary Morse functions $f_j\colon M \to \mathbb{R}$ so that these transversality conditions are satisfied. Theorem 3.10 shows that under the above assumptions $\mathcal{M}^c(q,p)$ is a smooth manifold of dimension $\lambda_q - \lambda_p - 1$ that is stratified by smooth manifolds with corners.

In Section 4 we study the compactness properties of $\mathcal{M}^c(q,p)$. We show using the Hausdorff metric that $\mathcal{M}^c(q,p)$ can be compactified using broken flow lines with cascades, which implies that $\mathcal{M}^c(q,p)$ is compact when $\lambda_q - \lambda_p = 1$. In Section 5 we give a detailed construction of the perturbation $h_\varepsilon\colon M \to \mathbb{R}$, and we prove that it is possible to choose a single Riemannian metric g so that $h_\varepsilon\colon M \to \mathbb{R}$ satisfies the Morse–Smale transversality condition with respect to g for all $\varepsilon > 0$ sufficiently small (Lemma 5.1). We also prove that as $\varepsilon \to 0$ a sequence of gradient flow lines of h_ε between two critical points q and p must have a subsequence that converges to a broken flow line with cascades from q to p (Lemma 5.3).

The correspondence theorem for moduli spaces (Theorem 5.4) is proved in Section 5 using recent results from geometric singular perturbation theory. In particular, our proof uses the exchange lemma for fast-slow systems (see Jones [21] and Schecter [27; 28]), which says (roughly) that a manifold M_0 that is transverse to the stable manifold of a normally hyperbolic locally invariant submanifold C will have subsets that flow forward in time under the full fast-slow system to be near subsets of the unstable manifold of C. The correspondence theorem for the Morse–Smale–Witten chain complex of $h_\varepsilon\colon M \to \mathbb{R}$ and the cascade chain complex (Corollary 5.7) follows as an immediate corollary to the correspondence theorem for moduli spaces.

2 The Morse–Smale–Witten chain complex

In this section we briefly recall the construction of the Morse–Smale–Witten chain complex and the Morse Homology Theorem. For more details see [4].

Let $\mathrm{Cr}(f) = \{p \in M \mid df_p = 0\}$ denote the set of critical points of a smooth function $f\colon M \to \mathbb{R}$ on a smooth m–dimensional manifold M. A critical point $p \in \mathrm{Cr}(f)$ is said to be *nondegenerate* if the Hessian $H_p(f)$ is nondegenerate. The *index* λ_p of a nondegenerate critical point p is the dimension of the subspace of T_pM where $H_p(f)$ is negative definite. If all the critical points of f are nondegenerate, then f is called a *Morse function*. If $f\colon M \to \mathbb{R}$ is a Morse function on a finite-dimensional compact smooth Riemannian manifold (M, g), then the *stable manifold* $W_f^s(p)$ and the *unstable manifold* $W_f^u(p)$ of a critical point $p \in \mathrm{Cr}(f)$ are defined to be

$$W_f^s(p) = \bigl\{x \in M \mid \lim_{t\to\infty} \varphi_t(x) = p\bigr\},$$
$$W_f^u(p) = \bigl\{x \in M \mid \lim_{t\to-\infty} \varphi_t(x) = p\bigr\},$$

where φ_t is the 1–parameter group of diffeomorphisms generated by minus the gradient vector field, ie $-\nabla f$. The index of p coincides with the dimension of $W_f^u(p)$. The stable/unstable manifold theorem for a Morse function says that the tangent space at p splits as

$$T_pM = T_p^sM \oplus T_p^uM,$$

where the Hessian is positive definite on $T_p^sM \overset{\text{def}}{=} T_pW_f^s(p)$ and negative definite on $T_p^uM \overset{\text{def}}{=} T_pW_f^u(p)$. Moreover, the stable and unstable manifolds of p are surjective images of smooth embeddings

$$E^s\colon T_p^sM \longrightarrow W_f^s(p) \subseteq M,$$
$$E^u\colon T_p^uM \longrightarrow W_f^u(p) \subseteq M.$$

Hence, $W_f^s(p)$ is a smoothly embedded open disk of dimension $m - \lambda_p$, and $W_f^u(p)$ is a smoothly embedded open disk of dimension λ_p.

If the stable and unstable manifolds of a Morse function $f\colon M \to \mathbb{R}$ all intersect transversally, then the function f is called *Morse–Smale*. For any metric g on M the set of smooth Morse–Smale functions is dense by the Kupka–Smale Theorem [4, Theorem 6.6 and Remark 6.7], and for a given Morse function $f\colon M \to \mathbb{R}$ one can choose a Riemannian metric on M so that f is Morse–Smale with respect to the chosen metric [1, Theorem 2.20]. Moreover, if f is Morse–Smale and $p, q \in \mathrm{Cr}(f)$ then $W_f(q,p) = W_f^u(q) \cap W_f^s(p)$ is an embedded submanifold of M of dimension $\lambda_q - \lambda_p$, and when $\lambda_q - \lambda_p = 1$ the number of gradient flow lines from q to p is finite [4, Corollary 6.29].

If we choose an orientation for each of the unstable manifolds of f, then there is an induced orientation on the normal bundles of the stable manifolds. Thus, we can define an integer associated to any two critical points p and q of relative index one by counting the number of gradient flow lines from q to p with signs determined by the orientations. This integer is denoted by $n_f(q,p) = \#\mathcal{M}_f(q,p)$, where $\mathcal{M}_f(q,p) = W_f(q,p)/\mathbb{R}$ is the moduli space of gradient flow lines of f from q to p. The *Morse–Smale–Witten chain complex* is defined to be the chain complex $(C_*(f), \partial_*)$ where $C_k(f)$ is the free abelian group generated by the critical points q of index k and the boundary operator $\partial_k\colon C_k(f) \to C_{k-1}(f)$ is given by

$$\partial_k(q) \;=\; \sum_{p \in \mathrm{Cr}_{k-1}(f)} n_f(q,p)\,p,$$

where $\mathrm{Cr}_{k-1}(f)$ denotes the set of critical points with index $k-1$.

Theorem 2.1 (Morse Homology Theorem) *The pair $(C_*(f), \partial_*)$ is a chain complex, and the homology of $(C_*(f), \partial_*)$ is isomorphic to the singular homology $H_*(M;\mathbb{Z})$.*

Note that the Morse Homology Theorem implies that the homology of $(C_*(f), \partial_*)$ is independent of the Morse–Smale function $f\colon M \to \mathbb{R}$, the Riemannian metric, and the chosen orientations.

3 Morse–Bott functions and cascades

Let $f\colon M \to \mathbb{R}$ be a smooth function whose critical set $\mathrm{Cr}(f)$ contains a submanifold C of positive dimension. Pick a Riemannian metric on M and use it to split $T_*M|_C$ as

$$T_*M|_C = T_*C \oplus \nu_*C,$$

where T_*C is the tangent space of C and ν_*C is the normal bundle of C. Let $p \in C$, $V \in T_pC$, $W \in T_pM$, and let $H_p(f)$ be the Hessian of f at p. We have

$$H_p(f)(V,W) = V_p \cdot (\widetilde{W} \cdot f) = 0$$

since $V_p \in T_pC$ and any extension of W to a vector field $\widetilde{W}$ satisfies $df(\widetilde{W})|_C = 0$. Therefore, the Hessian $H_p(f)$ induces a symmetric bilinear form $H_p^\nu(f)$ on ν_pC.

Definition 3.1 A smooth function $f\colon M \to \mathbb{R}$ on a smooth manifold M is called a *Morse–Bott function* if the set of critical points $\mathrm{Cr}(f)$ is a disjoint union of connected submanifolds and for each connected submanifold $C \subseteq \mathrm{Cr}(f)$ the bilinear form $H_p^{\nu}(f)$ is nondegenerate for all $p \in C$.

Often one says that the Hessian of a Morse–Bott function f is nondegenerate in the direction normal to the critical submanifolds.

For a proof of the following lemma see [4, Section 3.5] or [5].

Lemma 3.2 (Morse–Bott Lemma) *Let $f\colon M \to \mathbb{R}$ be a Morse–Bott function and $C \subseteq \mathrm{Cr}(f)$ a connected component. For any $p \in C$ there is a local chart of M around p and a local splitting $\nu_* C = \nu_*^- C \oplus \nu_*^+ C$, identifying a point $x \in M$ in its domain to (u, v, w) where $u \in C$, $v \in \nu_*^- C$, $w \in \nu_*^+ C$, such that within this chart f assumes the form*

$$f(x) = f(u, v, w) = f(C) - |v|^2 + |w|^2.$$

Definition 3.3 Let $f\colon M \to \mathbb{R}$ be a Morse–Bott function on a finite-dimensional smooth manifold M, and let C be a critical submanifold of f. For any $p \in C$ let λ_p denote the index of $H_p^{\nu}(f)$. This integer is the dimension of $\nu_p^- C$ and is locally constant by the preceding lemma. If C is connected, then λ_p is constant throughout C and we call $\lambda_p = \lambda_C$ the *Morse–Bott index* of C.

Cascades

Let $f\colon M \to \mathbb{R}$ be a Morse–Bott function on a finite-dimensional compact smooth manifold, and let

$$\mathrm{Cr}(f) = \coprod_{j=1}^{l} C_j,$$

where $C_1, \dots, C_l$ are disjoint connected critical submanifolds of Morse–Bott index $\lambda_1, \dots, \lambda_l$ respectively. Let $f_j\colon C_j \to \mathbb{R}$ be a Morse function on the critical submanifold C_j for all $j = 1, \dots, l$. If $q \in C_j$ is a critical point of $f_j\colon C_j \to \mathbb{R}$, then we will denote the Morse index of q relative to f_j by λ_q^j, the stable manifold of q relative to f_j by $W_{f_j}^s(q) \subseteq C_j$, and the unstable manifold of q relative to f_j by $W_{f_j}^u(q) \subseteq C_j$.

Definition 3.4 If $q \in C_j$ is a critical point of the Morse function $f_j\colon C_j \to \mathbb{R}$ for some $j = 1, \dots, l$, then the *total index* of q, denoted λ_q, is defined to be the sum of the Morse–Bott index of C_j and the Morse index of q relative to f_j, ie

$$\lambda_q = \lambda_j + \lambda_q^j.$$

The following is a restatement of [16, Definition A.5].

Definition 3.5 For $q \in \mathrm{Cr}(f_j)$, $p \in \mathrm{Cr}(f_i)$, and $n \in \mathbb{N}$, a *flow line with n cascades from q to p* is a $(2n-1)$–tuple:

$$\big((x_k)_{1\le k\le n}, (t_k)_{1\le k\le n-1}\big),$$

where $x_k \in C^\infty(\mathbb{R}, M)$ and $t_k \in \mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$ satisfy the following for all k:

(1) Each x_k is a nonconstant gradient flow line of f, ie

$$\frac{d}{dt}x_k(t) = -(\nabla f)(x_k(t)).$$

(2) For the first cascade $x_1(t)$ we have

$$\lim_{t\to-\infty} x_1(t) \in W^u_{f_j}(q) \subseteq C_j$$

and for the last cascade $x_n(t)$ we have

$$\lim_{t\to\infty} x_n(t) \in W^s_{f_i}(p) \subseteq C_i.$$

(3) For $1 \le k \le n-1$ there are critical submanifolds C_{j_k} and gradient flow lines $y_k \in C^\infty(\mathbb{R}, C_{j_k})$ of f_{j_k}, ie

$$\frac{d}{dt}y_k(t) = -(\nabla f_{j_k})(y_k(t)),$$

such that $\lim_{t\to\infty} x_k(t) = y_k(0)$ and $\lim_{t\to-\infty} x_{k+1}(t) = y_k(t_k)$.

When $j = i$ a *flow line with zero cascades from q to p* is a gradient flow line of f_j from q to p.

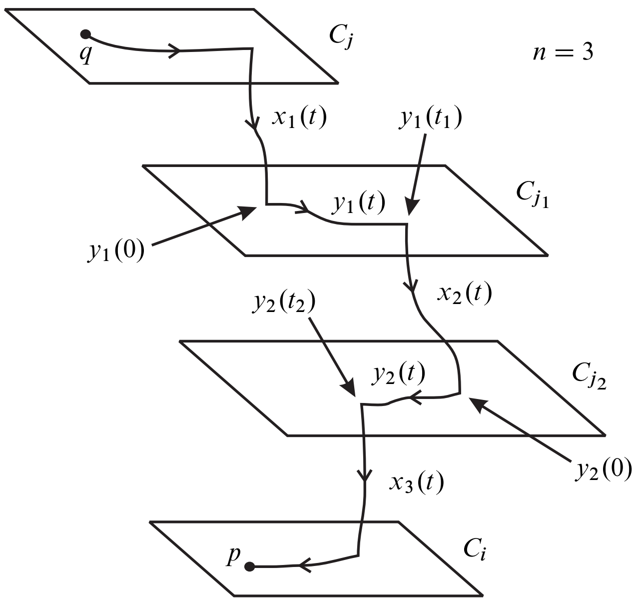
Note When $j \neq i$ a flow line with cascades from q to p must have at least one cascade.

Note With respect to the notation in the preceding definition, we will say that the flow line with n cascades $((x_k)_{1\le k\le n}, (t_k)_{1\le k\le n-1})$ *begins* at q and *ends* at p if the conditions listed in (2) hold, ie

$$\lim_{t\to-\infty} x_1(t) \in W^u_{f_j}(q) \subseteq C_j$$

and

$$\lim_{t\to\infty} x_n(t) \in W^s_{f_i}(p) \subseteq C_i.$$



Note In the preceding definition the parameterizations of the gradient flow lines $y_k(t)$ of the Morse functions $f_{j_k}\colon C_{j_k} \to \mathbb{R}$ are fixed in (3) by $\lim_{t\to\infty} x_k(t) = y_k(0)$, and the entry t_k records the time spent flowing along the critical submanifold C_{j_k} (or resting at a critical point). However, the parameterizations of the *cascades* $x_1(t), \ldots, x_n(t)$ are not fixed. Hence, there is an action of $\mathbb{R}^n$ on a flow line with n cascades given by

$$\big((x_k(t))_{1\leq k\leq n}, (t_k)_{1\leq k\leq n-1}\big) \longmapsto \big((x_k(t+s_k))_{1\leq k\leq n}, (t_k)_{1\leq k\leq n-1}\big)$$

for $(s_1, \ldots, s_n) \in \mathbb{R}^n$.

Definition 3.6 For $q \in \mathrm{Cr}(f_j)$, $p \in \mathrm{Cr}(f_i)$, and $n \in \mathbb{N}$ we denote the space of flow lines from q to p with n cascades by $W_n^c(q,p)$, and we denote the quotient of $W_n^c(q,p)$ by the action of $\mathbb{R}^n$ by

$$\mathcal{M}_n^c(q,p) = W_n^c(q,p)/\mathbb{R}^n.$$

The *set of unparameterized flow lines with cascades from q to p* is defined to be

$$\mathcal{M}^c(q,p) = \bigcup_{n\in\mathbb{Z}_+} \mathcal{M}_n^c(q,p),$$

where $\mathcal{M}_0^c(q,p) = W_0^c(q,p)/\mathbb{R}$. We will say that an element of $\mathcal{M}^c(q,p)$ *begins* at q and *ends* at p.

We now prove that $\mathcal{M}^c(q,p)$ is a smooth manifold of dimension $\lambda_q - \lambda_p - 1$ when $f\colon M \to \mathbb{R}$ satisfies the Morse–Bott–Smale transversality condition with respect to the metric g, the Morse functions $f_k\colon C_k \to \mathbb{R}$ satisfy the Morse–Smale transversality condition with respect to the restriction of g to C_k for all $k = 1, \dots, l$, and the stable and unstable manifolds of the Morse–Smale functions $f_i\colon C_i \to \mathbb{R}$ and $f_j\colon C_j \to \mathbb{R}$ are transverse to certain beginning and endpoint maps. Our proof uses fibered product constructions on smooth manifolds with corners similar to those found in [7].

Definition 3.7 (Morse–Bott–Smale transversality) A Morse–Bott function $f\colon M \to \mathbb{R}$ is said to satisfy the *Morse–Bott–Smale transversality* condition with respect to a given Riemannian metric g on M if for any two connected critical submanifolds C and C', $W_f^u(q)$ intersects $W_f^s(C')$ transversely in M, ie $W_f^u(q) \pitchfork W_f^s(C') \subseteq M$, for all $q \in C$.

Note Given a Morse function on a Riemannian manifold it is always possible to perturb the Riemannian metric to make the Morse–Smale transversality condition hold with respect to the perturbed metric (see [1, Theorem 2.20]). However, there are examples of Morse–Bott functions which do not satisfy the Morse–Bott–Smale transversality condition with respect to any Riemannian metric on the manifold (see [22, Section 2]).

Let C_k and $C_{k'}$ be two connected critical submanifolds of f, and let $W_f^u(C_k)$ and $W_f^s(C_{k'})$ denote the unstable and stable manifolds of C_k and $C_{k'}$ with respect to the flow of $-\nabla f$. The Morse–Bott–Smale transversality assumption implies that the moduli space of gradient flow lines of f,

$$\mathcal{M}_f(C_k, C_{k'}) = (W_f^u(C_k) \cap W_f^s(C_{k'}))/\mathbb{R},$$

is either empty or a smooth manifold of dimension $\lambda_k - \lambda_{k'} + \dim C_k - 1$. Moreover, the beginning and endpoint maps $\partial_-\colon \mathcal{M}_f(C_k, C_{k'}) \to C_k$ and $\partial_+\colon \mathcal{M}_f(C_k, C_{k'}) \to C_{k'}$ are smooth, and the beginning point map ∂_- is a submersion (see [7, Lemma 5.19]).

Now assume that the following moduli spaces and fibered products are nonempty. Then for distinct $k, k', k'' \in \{1, 2, \dots, l\}$ and $t \in \mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$ we can consider the fibered product

$$\begin{tikzcd}
(\mathbb{R}_+ \times \mathcal{M}_f(C_k, C_{k'})) \times_{C_{k'}} \mathcal{M}_f(C_{k'}, C_{k''}) \arrow[r, dashed] \arrow[d, dashed] & \mathcal{M}_f(C_{k'}, C_{k''}) \arrow[d, "\partial_-"] \\
\mathbb{R}_+ \times \mathcal{M}_f(C_k, C_{k'}) \arrow[r, "\varphi_t \circ \partial_+ \circ \pi_2"] & C_{k'},
\end{tikzcd}$$

where π_2 denotes projection onto the second component and φ_t denotes the gradient flow of $f_{k'}$ along the critical submanifold $C_{k'}$ for time $t \in \mathbb{R}_+$. This fibered product is a smooth manifold with boundary because $\partial_-\colon \mathcal{M}_f(C_{k'}, C_{k''}) \to C_{k'}$ is a submersion, and its dimension is

$$(\lambda_k - \lambda_{k'} + \dim C_k) + (\lambda_{k'} - \lambda_{k''} + \dim C_{k'} - 1) - \dim C_{k'} = \lambda_k - \lambda_{k''} + \dim C_k - 1$$

(see [7, Lemma 4.5 and Lemma 5.21]). Similarly, for any set of distinct integers $\{j_1, j_2, \dots, j_{n-1}\} \subseteq \{1, 2, \dots, l\}$ such that the following moduli spaces are nonempty, the iterated fibered product

$$\begin{aligned}\big(\mathbb{R}_+ \times \mathcal{M}_f(C_j, C_{j_1})\big) \times_{C_{j_1}} \big(\mathbb{R}_+ \times \mathcal{M}_f(C_{j_1}, C_{j_2})\big) \times_{C_{j_2}} \cdots \\ \times_{C_{j_{n-2}}} \big(\mathbb{R}_+ \times \mathcal{M}_f(C_{j_{n-2}}, C_{j_{n-1}})\big) \times_{C_{j_{n-1}}} \mathcal{M}_f(C_{j_{n-1}}, C_i)\end{aligned}$$

is a smooth manifold with corners because $\partial_- \circ \pi_2\colon \mathbb{R}_+ \times \mathcal{M}_f(C_k, C_{k'}) \to C_k$ is a submersion and a stratum submersion for all $k, k' = 1, \dots, l$. We will denote this smooth manifold with corners by $\mathcal{M}_n^c(C_j, C_{j_1}, \dots, C_{j_{n-1}}, C_i)$. Its dimension is

$$\begin{aligned}&(\lambda_j - \lambda_{j_1} + \dim C_j) \\ &\quad + (\lambda_{j_1} - \lambda_{j_2} + \dim C_{j_1}) - \dim C_{j_1} + \cdots \\ &\quad + (\lambda_{j_{n-2}} - \lambda_{j_{n-1}} + \dim C_{j_{n-2}}) - \dim C_{j_{n-2}} \\ &\quad + (\lambda_{j_{n-1}} - \lambda_i + \dim C_{j_{n-1}} - 1) - \dim C_{j_{n-1}} = \lambda_j - \lambda_i + \dim C_j - 1,\end{aligned}$$

which is independent of $j_1, j_2, \dots, j_{n-1}$. Note that we have smooth beginning and endpoint maps

$$\begin{aligned}\partial_-\colon \mathcal{M}_n^c(C_j, C_{j_1}, \dots, C_{j_{n-1}}, C_i) &\longrightarrow C_j, \\ \partial_+\colon \mathcal{M}_n^c(C_j, C_{j_1}, \dots, C_{j_{n-1}}, C_i) &\longrightarrow C_i.\end{aligned}$$

We can now state our transversality assumptions for the stable and unstable manifolds $W_{f_i}^s(p)$ and $W_{f_j}^u(q)$ of the Morse–Smale functions $f_i\colon C_i \to \mathbb{R}$ and $f_j\colon C_j \to \mathbb{R}$ with respect to these beginning and endpoint maps.

Definition 3.8 The stable and unstable manifolds $W_{f_i}^s(p)$ and $W_{f_j}^u(q)$ are *transverse to the beginning and endpoint maps* if for any set (possibly empty) of distinct integers $\{j_1, j_2, \dots, j_{n-1}\} \subseteq \{1, 2, \dots, l\}$ such that the space $\mathcal{M}_n^c(C_j, C_{j_1}, \dots, C_{j_{n-1}}, C_i)$ is not empty the map

$$\mathcal{M}_n^c(C_j, C_{j_1}, \dots, C_{j_{n-1}}, C_i) \xrightarrow{(\partial_-, \partial_+)} C_j \times C_i$$

is transverse and stratum transverse to $W_{f_j}^u(q) \times W_{f_i}^s(p)$.

Note When $\{j_1, j_2, \dots, j_{n-1}\} = \varnothing$ we have $\mathcal{M}_1^c(C_j, C_i) = \mathcal{M}_f(C_j, C_i)$.

Lemma 3.9 *There exist arbitrarily small perturbations of $f_i\colon C_i \to \mathbb{R}$ and $f_j\colon C_j \to \mathbb{R}$ to smooth Morse–Smale functions $\tilde{f}_i$ and $\tilde{f}_j$ such that all the stable and unstable manifolds of $\tilde{f}_i$ and $\tilde{f}_j$ are transverse to the beginning and endpoint maps. Moreover, there exist open neighborhoods of $\tilde{f}_i$ and $\tilde{f}_j$ consisting of smooth Morse–Smale functions whose stable and unstable manifolds are all transverse to the beginning and endpoint maps.*

Proof Let $\{j_1, j_2, \ldots, j_{n-1}\} \subseteq \{1, 2, \ldots, l\}$ be a (possibly empty) set of distinct integers such that the moduli space $\mathcal{M}^c_n(C_j, C_{j_1}, \ldots, C_{j_{n-1}}, C_i)$ is not empty, and let X be a stratum of $\mathcal{M}^c_n(C_j, C_{j_1}, \ldots, C_{j_{n-1}}, C_i)$. Let

$$E^s_{f_i}\colon \mathbb{R}^{\dim C_i - \lambda^i_p} \longrightarrow W^s_{f_i}(p) \subseteq C_i \quad \text{and} \quad E^u_{f_j}\colon \mathbb{R}^{\lambda^j_q} \longrightarrow W^u_{f_j}(q) \subseteq C_j$$

be the surjective smooth embeddings from Section 2, where $p \in \mathrm{Cr}(f_i)$, $q \in \mathrm{Cr}(f_j)$, and we have identified $T^s_p C_i = \mathbb{R}^{\dim C_i - \lambda^i_p}$ and $T^u_q C_j = \mathbb{R}^{\lambda^j_q}$. The stable and unstable manifolds $W^s_{f_j}(p)$ and $W^u_{f_i}(q)$ are transverse to $(\partial_-, \partial_+)\colon X \to C_j \times C_i$ if and only if the map

$$\big(E^u_{f_j}, E^s_{f_i}\big) \times (\partial_-, \partial_+)\colon \big(\mathbb{R}^{\lambda^j_q} \times \mathbb{R}^{\dim C_i - \lambda^i_p}\big) \times X \longrightarrow (C_j \times C_i) \times (C_j \times C_i)$$

is transverse to the diagonal $\Delta \subset (C_j \times C_i) \times (C_j \times C_i)$.

For any $r \geq 2$ the set of C^r Morse–Smale functions on a smooth Riemannian manifold (M, g) is an open and dense subset of the set of all C^r functions on M, and the phase diagram of a Morse–Smale function is stable under small C^r perturbations [26]. Thus there exists a neighborhood $\mathcal{N}_{f_i} \subset C^r(M, \mathbb{R})$ of f_i such that $\tilde{f}_i \in \mathcal{N}_{f_i}$ implies that $\tilde{f}_i$ is a Morse–Smale function with critical points of the same index and near the critical points of f_i. Similarly, there exists a neighborhood $\mathcal{N}_{f_j} \subset C^r(M, \mathbb{R})$ of f_j such that $\tilde{f}_j \in \mathcal{N}_{f_j}$ implies that $\tilde{f}_j$ is a Morse–Smale function with critical points of the same index and near the critical points of f_i. Moreover, we can choose these neighborhoods small enough so that the maps

$$E^s\colon \mathcal{N}_{f_i} \longrightarrow C^r\big(\mathbb{R}^{\dim C_i - \lambda^i_p}, C_i\big) \quad \text{and} \quad E^u\colon \mathcal{N}_{f_j} \longrightarrow C^r\big(\mathbb{R}^{\lambda^j_q}, C_j\big)$$

defined by sending $\tilde{f}_i \in \mathcal{N}_{f_i}$ to the embedding ${E^s}_{\tilde{f}_i}$ (with respect to the critical point $\tilde{p}$ near p) and $\tilde{f}_j \in \mathcal{N}_{f_j}$ to the embedding ${E^u}_{\tilde{f}_j}$ (with respect to the critical point $\tilde{q}$ near q) are well defined and of class C^r. In particular, we can choose the neighborhoods small enough so that we can identify $T^s_{\tilde{p}} C_i = T^s_p C_i = \mathbb{R}^{\dim C_i - \lambda^i_p}$ and $T^u_{\tilde{q}} C_j = T^u_q C_j = \mathbb{R}^{\lambda^j_q}$.

The map

$$(E^u \times E^s) \times (\partial_-, \partial_+)\colon \big(\mathcal{N}_{f_j} \times \mathcal{N}_{f_i}\big) \times \big(\mathbb{R}^{\lambda^j_q} \times \mathbb{R}^{\dim C_i - \lambda^i_p} \times X\big) \longrightarrow (C_j \times C_i) \times (C_j \times C_i)$$

defined by

$$\big((E^u \times E^s) \times (\partial_-, \partial_+)\big)\big((\tilde{f}_j, \tilde{f}_i) \times (x, y, \gamma)\big) = \big(E^u{}_{\tilde{f}_j}(x), E^s{}_{\tilde{f}_i}(y)\big) \times (\partial_-(\gamma), \partial_+(\gamma))$$

is of class C^r (see [2, Theorem 12.3]) and transverse to $\Delta \subset (C_j \times C_i) \times (C_j \times C_i)$. Hence, by the transversality density theorem [2, Theorem 19.1] the set of Morse–Smale functions $(\tilde{f}_j, \tilde{f}_i) \in \mathcal{N}_{f_j} \times \mathcal{N}_{f_i}$ such that

$$\big(E^u{}_{\tilde{f}_j}, E^s{}_{\tilde{f}_i}\big) \times (\partial_-, \partial_+)\colon \big(\mathbb{R}^{\lambda_q^j} \times \mathbb{R}^{\dim C_i - \lambda_p^i}\big) \times X \longrightarrow (C_j \times C_i) \times (C_j \times C_i)$$

is transverse to Δ is residual (and hence dense) in $\mathcal{N}_{f_j} \times \mathcal{N}_{f_i}$ for $r \geq 2$ large enough, eg $r > 3 \dim M$.

Since there are only finitely many subsets $\{j_1, j_2, \ldots, j_{n-1}\} \subseteq \{1, 2, \ldots, l\}$, finitely many critical points of f_i and f_j, and finitely many strata X, we can intersect finitely many such residual sets to obtain a residual (and hence dense) subset $\mathcal{R} \subseteq \mathcal{N}_{f_j} \times \mathcal{N}_{f_i}$ such that

$$\mathcal{M}_n^c(C_j, C_{j_1}, \ldots, C_{j_{n-1}}, C_i) \xrightarrow{(\partial_-, \partial_+)} C_j \times C_i$$

is transverse and stratum transverse to $W^u_{\tilde{f}_j}(\tilde{q}) \times W^s_{\tilde{f}_i}(\tilde{p})$ for all $\tilde{q}$ in $\mathrm{Cr}(\tilde{f}_j)$ and $\tilde{p}$ in $\mathrm{Cr}(\tilde{f}_i)$ whenever $(\tilde{f}_j, \tilde{f}_i) \in \mathcal{R}$. Also, since the space of smooth Morse–Smale functions on M is dense in the space of C^r Morse–Smale functions on M, the openness of transversal intersection theorem [2, Theorem 18.2] implies that we can find open neighborhoods of smooth functions arbitrarily close to f_j and f_i consisting of Morse–Smale functions $\tilde{f}_j$ and $\tilde{f}_i$ with $(\tilde{f}_j, \tilde{f}_i) \in \mathcal{R}$. □

Note The critical points of f_i and f_j may not be preserved by the perturbations in the preceding lemma. However, it is possible to choose the perturbations so that the phase diagrams of f_i and f_j do not change [26]. In particular, the number of critical points of index k remains the same for all $k = 1, \ldots, m$, which also follows from [14, Rigidity Theorem 1.19].

The next theorem should be compared with [16, Theorem A.12], whose proof uses the modern infinite-dimensional techniques of Floer homology. [16, Theorem A.12] is proved under the assumption that the Riemannian metric g on M is generic, which is necessary to ensure that a certain Fredholm operator used in the proof of the theorem is surjective.

Theorem 3.10 *Assume that f satisfies the Morse–Bott–Smale transversality condition with respect to the Riemannian metric g on M, $f_k\colon C_k \to \mathbb{R}$ satisfies the Morse–Smale transversality condition with respect to the restriction of g to C_k for all $k = 1, \ldots, l$, and the unstable and stable manifolds $W^u_{f_j}(q)$ and $W^s_{f_i}(p)$ are transverse to the beginning and endpoint maps.*

(1) *When $n = 0, 1$ the set $\mathcal{M}^c_n(q, p)$ is either empty or a smooth manifold without boundary.*

(2) *For $n > 1$ the set $\mathcal{M}^c_n(q, p)$ is either empty or a smooth manifold with corners.*

(3) *The set $\mathcal{M}^c(q, p)$ is either empty or a smooth manifold without boundary.*

In each case the dimension of the manifold is $\lambda_q - \lambda_p - 1$. When M is orientable and C_k is orientable for all $k = 1, \ldots, l$, the above manifolds are orientable.

Proof For more details concerning the notation and dimension formulas used in the following we refer the reader to [7, Sections 3 and 4]. We first prove statements (1) and (2) using pullback constructions. A gluing theorem is then used to show that the space $\mathcal{M}^c_{\leq n}(C_j, C_i)$ consisting of flow lines with at most n cascades beginning at any point in C_j and ending at any point in C_i is a manifold without boundary. Pulling back $W^u_{f_j}(q) \times W^s_{f_i}(p)$ via the beginning and endpoint maps on $\mathcal{M}^c_{\leq l}(C_j, C_i)$ then shows that $\mathcal{M}^c(q, p)$ is a smooth manifold without boundary of dimension $\lambda_q - \lambda_p - 1$.

The space $\mathcal{M}^c_0(q, p)$ is empty unless $i = j$, and when $i = j$ the theorem follows from the fact that f_j satisfies the Morse–Smale transversality condition. For the case $n = 1$ note that the assumption that

$$\mathcal{M}^c_1(C_j, C_i) \xrightarrow{(\partial_-,\partial_+)} C_j \times C_i$$

is transverse to $W^u_{f_j}(q) \times W^s_{f_i}(p)$ implies that

$$\mathcal{M}^c_1\big(W^u_{f_j}(q), W^s_{f_i}(p)\big) \stackrel{\text{def}}{=} (\partial_-, \partial_+)^{-1}\big(W^u_{f_j}(q) \times W^s_{f_i}(p)\big)$$

is either empty or a smooth manifold. In the second case, the codimension of the manifold $\mathcal{M}^c_1(W^u_{f_j}(q), W^s_{f_i}(p))$ is $\dim C_j - \lambda^j_q + \lambda^i_p$, and hence

$$\dim \mathcal{M}^c_1(W^u_{f_j}(q), W^s_{f_i}(p)) = \lambda_j + \lambda^j_q - (\lambda_i + \lambda^i_p) - 1$$

since

$$\dim \mathcal{M}^c_1(C_j, C_i) = \lambda_j - \lambda_i + \dim C_j - 1$$

(see [4, Theorem 5.11]). This shows that $\mathcal{M}^c_1(q, p) = \mathcal{M}^c_1(W^u_{f_j}(q), W^s_{f_i}(p))$ is a smooth manifold without boundary of dimension $\lambda_q - \lambda_p - 1$.

Now assume that $n > 1$ and the following moduli spaces and fibered products are nonempty. Then for distinct $j_1, j_2, \ldots, j_{n-1} \in \{1, 2, \ldots, l\}$ the assumption that

$$\mathcal{M}^c_n(C_j, C_{j_1}, \ldots, C_{j_{n-1}}, C_i) \xrightarrow{(\partial_-,\partial_+)} C_j \times C_i$$

is transverse and stratum transverse to $W^u_{f_j}(q) \times W^s_{f_i}(p)$ implies that

$$\mathcal{M}^c_n\big(W^u_{f_j}(q), C_{j_1}, \ldots, C_{j_{n-1}}, W^s_{f_i}(p)\big) \overset{\text{def}}{=} (\partial_-, \partial_+)^{-1}\big(W^u_{f_j}(q) \times W^s_{f_i}(p)\big)$$

is a smooth manifold with corners of dimension $\lambda_q - \lambda_p - 1$. This shows that

$$\mathcal{M}^c_n(q, p) \;=\; \bigcup_{\{j_1,\ldots,j_{n-1}\}} \mathcal{M}^c_n\big(W^u_{f_j}(q), C_{j_1}, \ldots, C_{j_{n-1}}, W^s_{f_i}(p)\big)$$

is a smooth manifold with corners of dimension $\lambda_q - \lambda_p - 1$, where the union is taken over all sets of distinct integers $\{j_1, \ldots, j_{n-1}\} \subseteq \{1, 2, \ldots, l\}$. This completes the proof of statements (1) and (2).

We now use a gluing theorem to define smooth charts on

$$\mathcal{M}^c_{\leq n}(C_j, C_i) \overset{\text{def}}{=} \bigcup_{k=0}^{n} \mathcal{M}^c_k(C_j, C_i),$$

where $\mathcal{M}^c_k(C_j, C_i)$ denotes the union of $\mathcal{M}^c_k(C_j, C_{j_1}, \ldots, C_{j_{k-1}}, C_i)$ over all sets of distinct integers $\{j_1, \ldots, j_{k-1}\} \subseteq \{1, 2, \ldots, l\}$ when $k > 1$. For distinct k, k', k'' in $\{1, 2, \ldots, l\}$ there exists an $\varepsilon > 0$ and a smooth injective local diffeomorphism

$$G\colon \mathcal{M}_f(C_k, C_{k'}) \times_{C_{k'}} \mathcal{M}_f(C_{k'}, C_{k''}) \times (-\varepsilon, 0) \longrightarrow \mathcal{M}_f(C_k, C_{k''})$$

onto an end of $\mathcal{M}_f(C_k, C_{k''})$, where the fibered product is taken with respect to the beginning and endpoint maps ∂_- and ∂_+. (See for instance [3, Appendix A.3] or [7, Theorem 4.8].) Let $\rho\colon (-\varepsilon, \infty) \to (-\varepsilon, \infty)$ be a smooth map that is smoothly homotopic to

$$\chi(t) = \begin{cases} t & t \geq 0, \\ 0 & t \leq 0, \end{cases}$$

and satisfies

$$\rho(t) = \begin{cases} t & t \geq \varepsilon/2, \\ 0 & t \leq 0. \end{cases}$$

For $\varepsilon > 0$ sufficiently small we can replace the maps $\varphi_t \circ \partial_+ \circ \pi_2$ in the iterated fibered product that defines $\mathcal{M}^c_n(C_j, C_{j_1}, \ldots, C_{j_{n-1}}, C_i)$ with the maps $\varphi_{\rho(t)} \circ \partial_+ \circ \pi_2$ and obtain a smooth manifold with corners that is smoothly diffeomorphic to the original manifold. Moreover, if we choose $\varepsilon > 0$ small enough, then $W^u_{f_j}(q)$ and $W^s_{f_i}(p)$ will still be transverse to the beginning and endpoint maps from the modified fibered product space.

Using the maps $\varphi_{\rho(t)} \circ \partial_+ \circ \pi_2$ and ∂_- we consider the fibered product

$$((-\varepsilon, \infty) \times \mathcal{M}_f(C_j, C_k)) \times_{C_k} \mathcal{M}_f(C_k, C_i),$$

where $k \in \{1, 2, \dots, l\}$. The part of this smooth manifold where $-\varepsilon < t < 0$ is diffeomorphic to an end of $\mathcal{M}_1^c(C_j, C_i)$ by the above gluing theorem, and the part of the space where $t \geq 0$ is diffeomorphic to $\mathcal{M}_2^c(C_j, C_k, C_i)$. Therefore, there are smooth charts on the above manifold around the points where $t = 0$ which are compatible with the smooth charts on $\mathcal{M}_1^c(C_j, C_i)$ and the smooth charts on $\mathcal{M}_2^c(C_j, C_k, C_i)$. This shows that the space $\mathcal{M}_{\leq 2}^c(C_j, C_i)$ of unparameterized flow lines with at most 2 cascades from C_j to C_i is a smooth manifold without boundary of dimension $\lambda_j - \lambda_i + \dim C_j - 1$.

Continuing by induction, for distinct $j_1, j_2, \dots, j_{n-1} \in \{1, 2, \dots, l\}$ the fibered product

$$\big((-\varepsilon, \infty) \times \mathcal{M}_f(C_j, C_{j_1})\big) \times_{C_{j_1}} \big((-\varepsilon, \infty) \times \mathcal{M}_f(C_{j_1}, C_{j_2})\big) \times_{C_{j_2}} \cdots \\ \times_{C_{j_{n-2}}} \big((-\varepsilon, \infty) \times \mathcal{M}_f(C_{j_{n-2}}, C_{j_{n-1}})\big) \times_{C_{j_{n-1}}} \mathcal{M}_f(C_{j_{n-1}}, C_i)$$

with respect to the maps $\varphi_{\rho(t)} \circ \partial_+ \circ \pi_2$ and $\partial_- \circ \pi_2$ is a smooth manifold. The part of the space where $-\varepsilon < t_k < 0$ for some k is diffeomorphic to an end of $\mathcal{M}_{\leq n-1}^c(C_j, C_i)$, and the part of the space where $t_k \geq 0$ for all k is diffeomorphic to $\mathcal{M}_n^c(C_j, C_{j_1}, \dots, C_{j_{n-1}}, C_i)$. Thus, the space $\mathcal{M}_{\leq n}^c(C_j, C_i)$ of unparameterized flow lines with at most n cascades from C_j to C_i is a smooth manifold without boundary of dimension $\lambda_j - \lambda_i + \dim C_j - 1$. Moreover,

$$\mathcal{M}_{\leq n}^c(C_j, C_i) \xrightarrow{(\partial_-, \partial_+)} C_j \times C_i$$

is transverse to $W_{f_j}^u(q) \times W_{f_i}^s(p)$. The pullback of $W_{f_j}^u(q) \times W_{f_i}^s(p)$ under this map is the space of unparameterized flow lines with at most n cascades from q to p:

$$\mathcal{M}_{\leq n}^c(q, p) = \bigcup_{k=0}^{n} \mathcal{M}_k^c(q, p).$$

Hence, for any $0 \leq n \leq l$ the space $\mathcal{M}_{\leq n}^c(q, p)$ is either empty or a smooth manifold without boundary of dimension $\lambda_q - \lambda_p - 1$. Taking $n = l$ we see that $\mathcal{M}^c(q, p)$ is either empty or a smooth manifold without boundary of dimension $\lambda_q - \lambda_p - 1$.

Now, an orientation on M and orientations on C_j for all $j = 1, \dots, l$ determine orientations on the above fibered products by the results in [7, Section 5.2]. If we choose the gluing diffeomorphisms to be compatible with these orientations, then we obtain an orientation on $\mathcal{M}^c(q, p)$. □

4 Broken flow lines with cascades

We will now consider the compactness properties of $\mathcal{M}^c(q, p)$. In general, $\mathcal{M}^c(q, p)$ will be a noncompact manifold because a sequence of unparameterized flow lines with cascades from q to p may converge to a broken flow line with cascades from q to p. Throughout this section we will assume that f satisfies the Morse–Bott–Smale transversality condition with respect to the Riemannian metric g on a compact smooth manifold M, $f_k\colon C_k \to \mathbb{R}$ satisfies the Morse–Smale transversality condition with respect to the restriction of g to C_k for all $k = 1, \ldots, l$, and the unstable and stable manifolds $W^u_{f_j}(q)$ and $W^s_{f_i}(p)$ are transverse to the beginning and endpoint maps. It is well known that any sequence of unparameterized gradient flow lines between two critical points of a Morse–Smale function must have a subsequence that converges to a broken flow line. However, making this statement precise requires a discussion of the topology on the space of broken flow lines. The topology on the space of broken flow lines can be defined in several ways, including as the compact open topology (after picking specific parameterizations for the flow lines), in terms of Floer–Gromov convergence, and using the Hausdorff metric (after identifying a broken flow line with its image). For a detailed discussion concerning different ways to define the topology on the space of broken flow lines of a Morse–Smale function and proofs that the resulting spaces are homeomorphic see [23].

To prove a similar result for cascades we first need to explain what we mean by a broken flow line with cascades. Roughly speaking, a broken flow line with cascades is a concatenation of unparameterized flow line with cascades that either flows along an intermediate critical submanifold for infinite time or rests at an intermediate critical point of one of the Morse functions $f_k\colon C_k \to \mathbb{R}$ for some $k = 1, \ldots, l$ for infinite time. To make this more precise, recall that a flow line with cascades is of the form $((x_k)_{1 \leq k \leq n}, (t_k)_{1 \leq k \leq n-1})$, where $t_k \in \mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$. In particular, $t_k < \infty$, but we might have $t_k = 0$ for some k. If $t_k = 0$ for some k, then the flow line with cascades "looks like" it contains a broken flow line. That is, if $t_k = 0$, then $\lim_{t\to\infty} x_k(t) = \lim_{t\to-\infty} x_{k+1}(t)$ and (x_k, x_{k+1}) is a broken flow line of the Morse–Bott function $f\colon M \to \mathbb{R}$. However, $(x_k, x_{k+1}, 0)$ is an unbroken flow line with 2 cascades.

Since a flow line with cascades must begin and end at critical points of the Morse functions chosen on the critical submanifolds, it's clear that (x_k, x_{k+1}) should not be called a broken flow line with cascades when $\lim_{t\to\infty} x_k(t) = \lim_{t\to-\infty} x_{k+1}(t)$ is not a critical point of $f_{j_k}\colon C_{j_k} \to \mathbb{R}$. In order to be consistent, we will *not* call (x_k, x_{k+1}) a broken flow line with cascades even if $\lim_{t\to\infty} x_k(t) = \lim_{t\to-\infty} x_{k+1}(t) = r$ is a critical point of $f_{j_k}\colon C_{j_k} \to \mathbb{R}$. Instead, we will always assume that the time spent resting at the intermediate critical point is zero, unless the time is otherwise specified. That is, we will identify (x_k, x_{k+1}) with the flow line with 2 cascades $(x_k, x_{k+1}, 0)$.

In general, suppose that we have an n–tuple of unparameterized flow lines with cascades $(v_1, \ldots, v_n)$ such that v_1 begins at $q \in \mathrm{Cr}(f_j)$, v_n ends at $p \in \mathrm{Cr}(f_i)$, and v_ν begins where $v_{\nu-1}$ ends for $2 \le \nu \le n$. Suppose that v_ν is represented by

$$\big((x_k^\nu)_{1\le k\le n_\nu}, (t_k^\nu)_{1\le k\le n_\nu-1}\big)$$

and $v_{\nu-1}$ is represented by

$$\big((x_k^{\nu-1})_{1\le k\le n_{\nu-1}}, (t_k^{\nu-1})_{1\le k\le n_{\nu-1}-1}\big).$$

The statement that v_ν begins where $v_{\nu-1}$ ends means that there is a critical point r of one of the Morse functions $f_k\colon C_k \to \mathbb{R}$ for some $k = 1,\dots,l$ such that $\lim_{t\to\infty} x_{n_{\nu-1}}^{\nu-1}(t) \in W_{f_k}{}^s(r)$ and $\lim_{t\to-\infty} x_{n_\nu}^\nu(t) \in W_{f_k}{}^u(r)$. So, it appears that $(v_{\nu-1}, v_\nu)$ differs from an unparameterized flow line with cascades in that $(v_{\nu-1}, v_\nu)$ flows along the intermediate critical submanifold C_k for infinite time. However, if $\lim_{t\to\infty} x_{n_{\nu-1}}^{\nu-1}(t) = \lim_{t\to-\infty} x_{n_\nu}^\nu(t) = r$, then $(v_{\nu-1}, v_\nu)$ determines an unparameterized flow line with $n_{\nu-1}+n_\nu$ cascades where the time spent resting at the intermediate critical point q is 0, ie the unparameterized flow line with cascades represented by

$$\big((x_k^{\nu-1})_{1\le k\le n_{\nu-1}}, (x_k^\nu)_{1\le k\le n_\nu}, (t_k^{\nu-1})_{1\le k\le n_{\nu-1}-1}, 0, (t_k^\nu)_{1\le k\le n_\nu-1}\big).$$

In this case, we will identify $(v_{\nu-1}, v_\nu)$ with the unparameterized flow line with cascades represented by the above tuple.

It is interesting to consider what this convention means for a Morse–Smale function $f\colon M \to \mathbb{R}$. Suppose that $p, q, r \in \mathrm{Cr}(f)$, γ_1 is a gradient flow line from q to r and γ_2 is a gradient flow line from r to p. Then with this convention we are identifying the broken gradient flow line represented by (γ_1, γ_2) with the flow line with 2 cascades $(\gamma_1, \gamma_2, 0)$. In fact, for a Morse–Smale function this convention means that the only truly broken flow lines with cascades have representations of the form $((x_k)_{1\le k\le n}, (t_k)_{1\le k\le n-1})$, where $t_k = \infty$ for some k.

Definition 4.1 A *broken flow line with cascades* from $q \in \mathrm{Cr}(f_j)$ to $p \in \mathrm{Cr}(f_i)$ is an n–tuple of unparameterized flow lines with cascades $(v_1, \dots, v_n)$ such that v_1 begins at q, v_n ends at p, and v_ν begins where $v_{\nu-1}$ ends for $2 \le \nu \le n$, subject to the following restriction: If the last cascade of $v_{\nu-1}$ and the first cascade of v_ν meet at a critical point of one of the Morse functions $f_k\colon C_k \to \mathbb{R}$ for some $k = 1,\dots,l$, then the time spent resting at the critical point is infinity.

A sequence of unparameterized flow lines with cascades from $q \in \mathrm{Cr}(f_j)$ to $p \in \mathrm{Cr}(f_i)$ must have a subsequence that converges to a broken flow line with cascades from q to p. This is proved in [16, Theorem A.10] with respect to Floer–Gromov convergence [16, Definition A.9]. Our approach to this theorem will be in terms of the Hausdorff metric.

Definition 4.2 Let (X,d) be a compact metric space and let K_1 and K_2 be nonempty closed subsets of X. The *Hausdorff distance* between K_1 and K_2 is defined to be

$$\begin{aligned} d_H(K_1,K_2) &= \max\left\{ \sup_{x_1\in K_1} \inf_{x_2\in K_2} d(x_1,x_2), \sup_{x_2\in K_2} \inf_{x_1\in K_1} d(x_1,x_2) \right\} \\ &= \inf\bigl\{\varepsilon > 0 \mid K_1 \subseteq N_\varepsilon(K_2) \text{ and } K_2 \subseteq N_\varepsilon(K_1)\bigr\}, \end{aligned}$$

where $N_\varepsilon(K) = \bigcup_{y\in K}\{x \in X \mid d(x,y) \leq \varepsilon\}$.

Note The Hausdorff distance on the set of all nonempty closed subsets $\mathcal{P}^c(X)$ of a compact metric space (X,d) is a metric, and the two definitions of the Hausdorff metric given above are equivalent. Moreover, the space $\mathcal{P}^c(X)$ is itself compact in the topology determined by the Hausdorff metric. (See for instance [24, Section 7.3].)

We would now like to identify a broken flow line with cascades with a closed subset of some compact metric space. For broken flow lines without cascades this is done by identifying a broken flow line of a Morse–Bott–Smale function with its image in the compact manifold M (see [18, Section 2]). However, a flow line with cascades may have a cascade x_k that ends at a critical point. In this case the parameter t_k records the time spent resting at the critical point instead of time spent flowing along the critical submanifold. Hence, the map that sends a broken flow line with cascades to its image in M is not injective. To make this map injective we should keep track of the times t_k, in addition to the image of the broken flow line.

Following [29] we make the following definition.

Definition 4.3 Define the compactification of $\mathbb{R}$ to be $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ equipped with the structure of a bounded manifold by the requirement that $\psi\colon \overline{\mathbb{R}} \to [-1,1]$ given by

$$\psi(t) = \frac{t}{\sqrt{1+t^2}}$$

be a diffeomorphism.

We also make the following definition regarding the different gradient flows.

Definition 4.4 Let $f\colon M \to \mathbb{R}$ be a Morse–Bott function on a Riemannian manifold (M,g) with critical set $\mathrm{Cr}(f) = \coprod_{j=1}^{l} C_j$, and let $f_j\colon C_j \to \mathbb{R}$ be a Morse function on the critical submanifold C_j for $j = 1,\dots,l$. We define the *flow* of $\{f, f_1, \dots, f_l\}$ on M to be the action $\phi\colon \mathbb{R}\times M \to M$ given on a point $x \in M$ for time $t \in \mathbb{R}$ by

$$\phi_t(x) = \begin{cases} \varphi_t^f(x) & \text{if } x \notin \mathrm{Cr}(f) = C_1 \cup \cdots \cup C_l, \\ \varphi_t^{f_j}(x) & \text{if } x \in C_j \text{ for some } j = 1,\dots,l, \end{cases}$$

where φ_t^f denotes the 1–parameter group of diffeomorphisms generated by $-\nabla f$ and $\varphi_t^{f_j}$ denotes the 1–parameter group of diffeomorphisms generated by $-\nabla f_j$ (with respect to the restriction of g to C_j) for all $j = 1, \ldots, l$. We extend this action to $\overline{\mathbb{R}}$ by taking limits as t approaches $\pm\infty$.

Note The flow of $\{f, f_1, \ldots, f_l\}$ defines a map $\phi\colon \overline{\mathbb{R}} \times M \to M$ that is smooth when restricted to $\mathbb{R} \times (M - \mathrm{Cr}(f))$ or to $\mathbb{R} \times \mathrm{Cr}(f)$.

We now explain how to identify a broken flow line with cascades with an element of the compact metric space $\mathcal{P}^c(M) \times \overline{\mathbb{R}}^l$, where l is the number of components of $\mathrm{Cr}(f) = \coprod_{j=1}^{l} C_j$. Recall that the space of all nonempty closed subsets of M, $\mathcal{P}^c(M)$, is a compact metric space with respect to the Hausdorff metric. For the metric on $\overline{\mathbb{R}}$ we will use the totally bounded metric determined by the diffeomorphism $\psi\colon \overline{\mathbb{R}} \to [-1, 1]$. That is, for $x, y \in \mathbb{R}$ we define

$$d(x, y) = \left| \frac{x}{\sqrt{1+x^2}} - \frac{y}{\sqrt{1+y^2}} \right| \in [0, 2]$$

and note that d has a unique continuous extension to a metric on $\overline{\mathbb{R}}$. The space $\mathcal{P}^c(M) \times \overline{\mathbb{R}}^l$ is then a compact metric space with respect to the product metric.

We will map a broken flow line with cascades $(v_1, \ldots, v_n)$ to its image $\mathrm{Im}(v_1, \ldots, v_n)$ in M and the time t_j spent flowing along or resting on each critical submanifold C_j for all $j = 1, \ldots, l$. This gives a nonempty closed subset of M and an l–tuple in $\overline{\mathbb{R}}^l$, ie an element $\mathrm{Im}(v_1, \ldots, v_n) \times (t_1, \ldots, t_l) \in \mathcal{P}^c(M) \times \overline{\mathbb{R}}^l$.

More explicitly, we define $\mathrm{Im}(v_1, \ldots, v_n) \subset M$ for a broken flow line with cascades $(v_1, \ldots, v_n)$ as follows. Let $\nu \in \{1, \ldots, n\}$ and suppose that the unparameterized flow line with n_ν cascades v_ν has a parameterization

$$\big((x_k^\nu)_{1 \le k \le n_\nu}, (t_k^\nu)_{1 \le k \le n_\nu - 1}\big),$$

where $x_k^\nu \in C^\infty(\mathbb{R}, M)$ and $t_k^\nu \in \mathbb{R}_+$. Then the image of v_ν in M is defined to be

$$\mathrm{Im}(v_\nu) = \bigcup_{k=1}^{n_\nu} x_k^\nu(\overline{\mathbb{R}}) \;\cup\; \bigcup_{k=1}^{n_\nu - 1} \phi_{[0,t_k^\nu]}(x_k^\nu(\infty)) \subset M,$$

where $\phi_{[0,t_k^\nu]}(x_k^\nu(\infty)) = \bigcup_{0 \le t \le t_k^\nu} \phi_t(x_k^\nu(\infty))$ and $x_k^\nu(\infty) = \lim_{t\to\infty} x_k^\nu(t)$. This definition is clearly independent of the parameterization, and we define $\mathrm{Im}(v_1, \ldots, v_n) \subset M$ to be the union of the images of v_ν for all $\nu = 1, \ldots, n$. Note that $\mathrm{Im}(v_1, \ldots, v_n)$ is the image of a continuous injective path between two critical points which is $\overline{\mathbb{R}}$–equivariant with respect to the flow ϕ of $\{f, f_1, \ldots, f_l\}$.

For the other components we map $(v_1, \ldots, v_n)$ to an l–tuple of elements $(t_1, \ldots, t_l)$ in $\overline{\mathbb{R}}^l$ that records the time spent flowing along or resting on each critical submanifold. Explicitly, the j–th component of this map is defined to be:

$$\begin{cases} 0 & \text{if the image of } (v_1, \ldots, v_n) \text{ does not intersect } C_j, \\ t_j & \text{if for some } \nu = 1, \ldots, n \text{ the cascade } v_\nu \text{ flows along or rests on} \\ & \text{the critical submanifold } C_j \text{ for finite time } t_j, \\ \infty & \text{otherwise.} \end{cases}$$

Altogether, this defines an injective map

$$(v_1, \ldots, v_n) \mapsto \operatorname{Im}(v_1, \ldots, v_n) \times (t_1, \ldots, t_l) \in \mathcal{P}^c(M) \times \overline{\mathbb{R}}^l.$$

Definition 4.5 The topology on the space of broken flow lines with cascades is defined by the requirement that the above injection be a homeomorphism onto its image.

For $q \in \operatorname{Cr}(f_j)$ and $p \in \operatorname{Cr}(f_i)$ we will identify the space of broken flow lines with cascades from q to p with its image under the above injection and denote this space by $\overline{\mathcal{M}}^c(q, p) \subset \mathcal{P}^c(M) \times \overline{\mathbb{R}}^l$.

Theorem 4.6 *The space $\overline{\mathcal{M}}^c(q, p)$ is compact, and the injection defined above restricts to a continuous embedding*

$$\mathcal{M}^c(q, p) \hookrightarrow \overline{\mathcal{M}}^c(q, p) \subset \mathcal{P}^c(M) \times \overline{\mathbb{R}}^l.$$

Hence, every sequence of unparameterized flow lines with cascades from q to p has a subsequence that converges to a broken flow line with cascades from q to p.

Proof Since the space $\mathcal{P}^c(M) \times \overline{\mathbb{R}}^l$ is compact, any sequence of broken flow lines $\{(v_1^k, \ldots, v_{n_k}^k)\}$ in $\overline{\mathcal{M}}^c(q, p) \subset \mathcal{P}^c(M) \times \overline{\mathbb{R}}^l$ must have a subsequence that converges to some element $C_M \times (t_1, \ldots, t_l) \in \mathcal{P}^c(M) \times \overline{\mathbb{R}}^l$. We need to show that there exists a subsequence of $\{(v_1^k, \ldots, v_{n_k}^k)\}$ (which we still denote by $\{(v_1^k, \ldots, v_{n_k}^k)\}$) such that the limit of this subsequence (which we still denote by $C_M \times (t_1, \ldots, t_l)$) is in $\overline{\mathcal{M}}^c(q, p) \subset \mathcal{P}^c(M) \times \overline{\mathbb{R}}^l$.

We will first show that there exists a subsequence of $\{(v_1^k, \ldots, v_{n_k}^k)\}$ such that $C_M = \operatorname{Im}(v_1, \ldots, v_n)$ for some broken flow line with cascades $(v_1, \ldots, v_n)$ from q to p. To see this, note that since $\operatorname{Im}(v_1^k, \ldots, v_{n_k}^k) \subset M$ is $\overline{\mathbb{R}}$–equivariant with respect to the flow ϕ of $\{f, f_1, \ldots, f_l\}$ and $\lim_{k\to\infty} \operatorname{Im}(v_1^k, \ldots, v_{n_k}^k) = C_M$ in the Hausdorff metric, C_M is also $\overline{\mathbb{R}}$–equivariant with respect to the flow ϕ. Moreover for every k, $\operatorname{Im}(v_1^k, \ldots, v_{n_k}^k)$ is the image of a continuous injective path from q to p with at most one point on each level set $f^{-1}(y)$ for every regular value y of f and at most one point on each level set $f_j^{-1}(y)$ for every value $y \in \mathbb{R}$ for all $j = 1, \dots, l$. Thus, we can pass to a subsequence of $\{(v_1^k, \dots, v_{n_k}^k)\}$ such that the same holds for the limit. This shows that after passing to an appropriate subsequence we have $C_M = \mathrm{Im}(v_1, \dots, v_n)$ for some broken flow line with cascades $(v_1, \dots, v_n)$ from q to p.

Now let $j \in \{1, \dots, l\}$. For $(t_1, \dots, t_l)$ there are two cases to consider: (1) Either the sequence $\{\mathrm{Im}(v_1^k, \dots, v_{n_k}^k)\}$ does not intersect the critical submanifold C_j for any k, or (2) the sequence $\{\mathrm{Im}(v_1^k, \dots, v_{n_k}^k)\}$ intersects the critical submanifold C_j for all k sufficiently large. Otherwise we can pass to a subsequence that fits one of these two cases. For the first case, note that the limit C_M, which is the image of a broken flow line with cascades, can intersect C_j in at most one point since f decreases along its gradient flow lines. Thus, for $\mathrm{Im}(v_1^k, \dots, v_{n_k}^k) \times (t_1^k, \dots, t_l^k) \in \mathcal{P}^c(M) \times \overline{\mathbb{R}}^l$ we have $t_j^k = 0$ for all k, and $t_j = 0$. For the second case, note that since $\overline{\mathbb{R}}$ is a compact metric space, we can pass to a subsequence such that $t_j^k \to t_j$ for some $t_j \in \overline{\mathbb{R}}$. By passing to a subsequence for each $j = 1, \dots, l$ we obtain an element $(t_1, \dots, t_l) \in \overline{\mathbb{R}}^l$ such that

$$\mathrm{Im}(v_1^k, \dots, v_{n_k}^k) \times (t_1^k, \dots, t_l^k) \to \mathrm{Im}(v_1, \dots, v_n) \times (t_1, \dots, t_l) \in \mathcal{P}^c(M) \times \overline{\mathbb{R}}^l$$

as $k \to \infty$ and t_j records the time $(v_1, \dots, v_n)$ spends flowing along or resting on each critical submanifold C_j for all $j = 1, \dots, l$. Therefore, every sequence of broken flow lines with cascades from q to p has a subsequence that converges to a broken flow line with cascades from q to p in $\overline{\mathcal{M}}^c(q, p) \subset \mathcal{P}^c(M) \times \overline{\mathbb{R}}^l$.

To see that the injection defined above restricts to a continuous embedding

$$\mathcal{M}^c(q, p) \hookrightarrow \overline{\mathcal{M}}^c(q, p) \subset \mathcal{P}^c(M) \times \overline{\mathbb{R}}^l$$

note that the fibered product and gluing constructions used in the proof of Theorem 3.10 are compatible with the Hausdorff metric. That is, if a sequence of points v^k contained in a smooth chart of $\mathcal{M}^c(q, p)$ converges to a point v in the chart, then

$$\mathrm{Im}(v^k) \times (t_1^k, \dots, t_l^k) \to \mathrm{Im}(v) \times (t_1, \dots, t_l)$$

as $k \to \infty$. □

Corollary 4.7 *If $\lambda_q - \lambda_p = 1$, then $\mathcal{M}^c(q, p)$ is compact and hence a finite set.*

Proof Let v^k be a sequence of unparameterized flow lines with cascades from q to p. By the preceding theorem v^k has a subsequence that converges to a broken flow line with cascades $(v_1, \dots, v_n)$ from q to p. Suppose that v_1 ends at a critical point p' with $p' \neq p$. Then Theorem 3.10 implies that $\lambda_q > \lambda_{p'} > \lambda_p$, which contradicts the assumption that $\lambda_q - \lambda_p = 1$. Thus, $p' = p$, $n = 1$, and every sequence in $\mathcal{M}^c(q, p)$ has a subsequence that converges to an element of $\mathcal{M}^c(q, p)$. Therefore, $\mathcal{M}^c(q, p)$ is a compact zero-dimensional manifold, ie a finite set of points. □

The preceding corollary allows us to make the following definition under the following assumptions:

(1) f satisfies the Morse–Bott–Smale transversality condition with respect to the Riemannian metric g on M.

(2) $f_k\colon C_k \to \mathbb{R}$ satisfies the Morse–Smale transversality condition with respect to the restriction of g to C_k for all $k = 1, \dots, l$.

(3) For all (i, j) and for each pair of critical points $(q, p) \in \mathrm{Cr}(f_j) \times \mathrm{Cr}(f_i)$ the unstable and stable manifolds $W^u_{f_j}(q)$ and $W^s_{f_i}(p)$ are transverse to the beginning and endpoint maps.

Recall that the total index of a critical point of f_j was defined in Definition 3.4 as the Morse index relative to f_j plus the Morse–Bott index of the critical submanifold C_j. Let $\mathrm{Cr} = \bigcup_{j=1}^{l} \mathrm{Cr}(f_j)$ be the set of critical points of the Morse functions $f_j\colon C_j \to \mathbb{R}$, and let $\mathrm{Cr}_k \subseteq \mathrm{Cr}$ be the subset of critical points whose total index is k for all $k = 0, \dots, m$.

Definition 4.8 Define the k–th chain group $C^c_k(f)$ to be the free abelian group generated by the critical points of total index k of the Morse–Smale functions f_j for all $j = 1, \dots, l$, and define $n^c(q, p; \mathbb{Z}_2)$ to be the number of flow lines with cascades between a critical point q of total index k and a critical point p of total index $k-1$ counted mod 2. Let

$$C^c_*(f) \otimes \mathbb{Z}_2 = \bigoplus_{k=0}^{m} C^c_k(f) \otimes \mathbb{Z}_2$$

and define a homomorphism $\partial^c_k\colon C^c_k(f) \otimes \mathbb{Z}_2 \to C^c_{k-1}(f) \otimes \mathbb{Z}_2$ by

$$\partial^c_k(q) = \sum_{p \in \mathrm{Cr}_{k-1}} n^c(q, p; \mathbb{Z}_2)\, p.$$

The pair $(C^c_*(f) \otimes \mathbb{Z}_2, \partial^c_*)$ is called the *cascade chain complex* with $\mathbb{Z}_2$ coefficients.

In the appendix to [16] there is a continuation theorem that implies that the cascade chain complex with $\mathbb{Z}_2$ coefficients is, in fact, a chain complex whose homology is isomorphic to the singular homology $H_*(M; \mathbb{Z}_2)$. We will not prove this here. Instead, we will use the Morse–Smale functions $f_j\colon C_j \to \mathbb{R}$ for $j = 1, \dots, l$ to define an explicit perturbation of $f\colon M \to \mathbb{R}$ to a Morse–Smale function $h_\varepsilon\colon M \to \mathbb{R}$ such that for every $k = 0, \dots, m$

$$\mathrm{Cr}_k(h_\varepsilon) = \bigcup_{\lambda_j + n = k} \mathrm{Cr}_n(f_j),$$

where λ_j is the Morse–Bott index of the critical submanifold C_j.

By proving a correspondence theorem, we will show that for any q in $\mathrm{Cr}(f_j)$ and p in $\mathrm{Cr}(f_i)$ with $\lambda_q - \lambda_p = 1$ there is a one-dimensional trivial cobordism between $\mathcal{M}^c(q,p)$ and $\mathcal{M}_{h_\varepsilon}(q,p)$. This cobordism induces an orientation on $\mathcal{M}^c(q,p)$, which allows us to define the above homomorphism ∂_*^c over $\mathbb{Z}$. Moreover, the cobordism shows that ∂_*^c is a boundary operator that agrees with the Morse–Smale–Witten boundary operator of h_ε up to sign.

5 The correspondence theorem

In this section we define a 1–parameter family of Morse–Smale functions $h_\varepsilon\colon M \to \mathbb{R}$ in terms of an explicit perturbation of the Morse–Bott–Smale function $f\colon M \to \mathbb{R}$. For any $\varepsilon > 0$ the critical set of h_ε is given by $\mathrm{Cr}(h_\varepsilon) = \bigcup_{k=1}^{l} \mathrm{Cr}(f_k)$, and the index of a critical point $p \in \mathrm{Cr}(h_\varepsilon)$ agrees with the total index of p.

We prove a correspondence theorem which says that for any $\varepsilon > 0$ sufficiently small there is a bijection between unparameterized flow lines with cascades and unparameterized gradient flow lines of $h_\varepsilon\colon M \to \mathbb{R}$ between any two critical points $p, q \in \mathrm{Cr}(h_\varepsilon)$ with $\lambda_q - \lambda_p = 1$. The correspondence theorem allows us to count the number of unparameterized flow lines with cascades between $q \in \mathrm{Cr}_k(h_\varepsilon)$ and $p \in \mathrm{Cr}_{k-1}(h_\varepsilon)$ with sign, which defines an integer $n^c(q,p) \in \mathbb{Z}$.

The integers $n^c(q,p)$ define a homomorphism ∂_k^c analogous to the Morse–Smale–Witten boundary operator such that $\partial_k^c = -\partial_k$ (where ∂_k denotes the Morse–Smale–Witten boundary operator of h_ε). This shows directly that $\partial_{k-1}^c \circ \partial_k^c = 0$ and the homology of the cascade chain complex $(C_*^c(f), \partial_*^c)$ is isomorphic to the homology of the Morse–Smale–Witten chain complex $(C_*(h_\varepsilon), \partial_*)$. The Morse Homology Theorem then implies that the homology of the cascade chain complex with integer coefficients is isomorphic to the singular homology $H_*(M;\mathbb{Z})$.

5.1 An explicit perturbation

The following perturbation technique, based on [3], the Morse–Bott Lemma, and a folk theorem proved in [1], produces an explicit Morse–Smale function $h_\varepsilon\colon M \to \mathbb{R}$ arbitrarily close to a given Morse–Bott–Smale function $f\colon M \to \mathbb{R}$ such that $h_\varepsilon = f$ outside of a neighborhood of the critical set $\mathrm{Cr}(f)$. A similar technique was used in [6] to give a proof of the Morse–Bott inequalities with somewhat different orientation assumptions than the classical "half-space" method using the Thom Isomorphism Theorem (see [8], [15, Appendix C], and [25, Section 2.6]).

Let $f\colon M \to \mathbb{R}$ be a Morse–Bott–Smale function on a finite-dimensional smooth closed Riemannian manifold (M,g). Let T_j be a small open tubular neighborhood around each connected component $C_j \subseteq \mathrm{Cr}(f)$ for every $j = 1, \ldots, l$ with local coordinates (u, v, w) consistent with those from the Morse–Bott Lemma 3.2. By "small" we mean that the following conditions hold:

(1) Each T_j is contained in the union of the domains of the charts from the Morse–Bott Lemma.

(2) For $i \neq j$ we have $T_i \cap T_j = \varnothing$ and f decreases by at least three times $\max\{\mathrm{var}(f, T_j) \mid j = 1, \ldots, l\}$ along any gradient flow line from T_i to T_j, where $\mathrm{var}(f, T_j) = \sup\{f(x) \mid x \in T_j\} - \inf\{f(x) \mid x \in T_j\}$.

(3) If $f(C_i) \neq f(C_j)$, then $\mathrm{var}(f, T_i) + \mathrm{var}(f, T_j) < \frac{1}{3}|f(C_i) - f(C_j)|$.

(4) For every flow line with n cascades between critical points of relative index one $((x_k)_{1 \leq k \leq n}, (t_k)_{1 \leq k \leq n-1})$, the image of x_k for $k = 1, \ldots, n$ intersects the closure of exactly two of the tubular neighborhoods $\{T_j\}_{j=1}^{l}$ (see Definition 3.5 and Corollary 4.7).

In addition, we will assume that the tubular neighborhoods are small enough so that $f\colon M \to \mathbb{R}$ still satisfies the Morse–Bott–Smale transversality condition after modifying the Riemannian metric on the tubular neighborhoods to make the charts from the Morse–Bott Lemma isometries on T_j with respect to the standard Euclidean metric on $\mathbb{R}^m$ for all $j = 1, \ldots, l$. From now on we will assume that the Riemannian metric g has been so modified, ie the charts from the Morse–Bott Lemma are isometries on the tubular neighborhoods with respect to g and the standard Euclidean metric on $\mathbb{R}^n$.

Pick positive Morse functions $f_k\colon C_k \to \mathbb{R}$ satisfying the Morse–Smale transversality condition with respect to the restriction of g to C_k for all $k = 1, \ldots, l$ such that for all $i, j = 1, \ldots, l$ and for every pair of critical points $(q, p) \in \mathrm{Cr}(f_j) \times \mathrm{Cr}(f_i)$ the unstable and stable manifolds $W^u_{f_j}(q)$ and $W^s_{f_i}(p)$ are transverse to the beginning and endpoint maps (see Lemma 3.9). For every $k = 1, \ldots, l$ extend $f_k\colon C_k \to \mathbb{R}$ to a function on T_k by making $f_k\colon T_k \to \mathbb{R}$ constant in the directions normal to C_k, ie f_k is constant in the v and w coordinates coming from the Morse–Bott Lemma. Let $\widetilde{T}_k \subset T_k$ be a smaller open tubular neighborhood of C_k with the same coordinates as T_k, and let ρ_k be a smooth bump function which is constant in the u coordinates, equal to 1 on $\widetilde{T}_k$, equal to 0 outside of T_k, and strictly decreasing on $T_k - \widetilde{T}_k$ with respect to $|v|$ and $|w|$.

Finally, choose $\varepsilon > 0$ small enough so that

$$\sup_{T_k - \widetilde{T}_k} \varepsilon \|\nabla \rho_k f_k\| < \inf_{T_k - \widetilde{T}_k} \|\nabla f\|$$

for all $k = 1, \dots, l$, and define

$$h_\varepsilon = f + \varepsilon \left(\sum_{k=1}^{l} \rho_k f_k \right).$$

The function $h_\varepsilon \colon M \to \mathbb{R}$ is a Morse function close to the Morse–Bott–Smale function f, and the critical points of h_ε are exactly the critical points of the Morse–Smale functions f_j for $j = 1, \dots, l$. Moreover, if $q \in C_j$ is a critical point of $f_j \colon C_j \to \mathbb{R}$ of index λ_q^j, then q is a critical point of h_ε of index $\lambda_q^{h_\varepsilon} = \lambda_j + \lambda_q^j$, where λ_j is the Morse–Bott index of C_j.

Lemma 5.1 *There exists an arbitrarily small perturbation of the Riemannian metric g such that $h_{\varepsilon'} \colon M \to \mathbb{R}$ is Morse–Smale for all $0 < \varepsilon' \leq \varepsilon$ with respect to the perturbed metric. The perturbed metric can be chosen so that it agrees with g on the union of the tubular neighborhoods $\{T_j\}_{j=1}^{l}$.*

Proof Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a countable dense subset of $(0, \varepsilon)$. For every $1 \leq i < \infty$ we can apply [1, Theorem 2.20] to conclude that there is a residual subspace $\mathcal{R}_i$ of the open unit ball $\mathcal{K}_1$ in a Banach space $\mathcal{K}$ such that the function $h_{\varepsilon_i} \colon M \to \mathbb{R}$ is Morse–Smale with respect to the Riemannian metric $g + k_i$ for all $k_i \in \mathcal{R}_i$. Moreover, we can choose the function $\theta \colon M \to [0, \infty)$ in the statement of [1, Theorem 2.20] to be zero on $\bigcup_{j=1}^{l} T_j$ so that $k_i = 0$ on $\bigcup_{j=1}^{l} T_j$ for all $1 \leq i < \infty$.

For any $k \in \bigcap_{i=1}^{\infty} \mathcal{R}_i$ the Riemannian metric $g + k$ is a metric that agrees with g on $\bigcup_{j=1}^{l} T_j$ such that $h_{\varepsilon_i} \colon M \to \mathbb{R}$ is Morse–Smale with respect to $g + k$ for all $1 \leq i < \infty$. Moreover, since $\bigcap_{i=1}^{\infty} \mathcal{R}_i$ is dense in $\mathcal{K}_1$ we can choose $k \in \bigcap_{i=1}^{\infty} \mathcal{R}_i$ arbitrarily close to zero. This completes the proof of the lemma since the set of Morse–Smale gradient vector fields is an open and dense subset of the space of all gradient vector fields on a Riemannian manifold [26]. □

Note that we can choose the perturbation of the Riemannian metric small enough so that $f \colon M \to \mathbb{R}$ still satisfies the Morse–Bott–Smale transversality condition with respect to the perturbed metric and for all (i, j) and for every pair of critical points $(q, p) \in \mathrm{Cr}(f_j) \times \mathrm{Cr}(f_i)$ the unstable and stable manifolds $W_{f_j}^u(q)$ and $W_{f_i}^s(p)$ are still transverse to the beginning and endpoint maps.

Lemma 5.2 *Let $p, q \in \mathrm{Cr}(h_\varepsilon)$ with $\lambda_q - \lambda_p = 1$, and let $0 < \varepsilon' \leq \varepsilon$. If $h_{\varepsilon'} \colon M \to \mathbb{R}$ and $h_\varepsilon \colon M \to \mathbb{R}$ are Morse–Smale with respect to the same Riemannian metric, then the number of gradient flow lines of $h_{\varepsilon'}$ from q to p is equal to the number of gradient flow lines of h_ε from q to p.*

Proof The lemma will be proved by constructing a one-dimensional compact smooth manifold with boundary $\overline{\mathcal{M}}_{F_{21}}(q,p)$ that is a trivial cobordism between $\mathcal{M}_{h_\varepsilon}(q,p)$ and $\mathcal{M}_{h_{\varepsilon'}}(q,p)$.

Using the notation in [7, Section 6], we take $f_1 = h_\varepsilon$, $f_2 = h_{\varepsilon'}$, and a smooth homotopy $F_{21}\colon M \times \mathbb{R} \to \mathbb{R}$ that is strictly decreasing in its second component such that for some large $T \gg 0$ we have

$$F_{21}(x,t) = \begin{cases} h_\varepsilon(x) - \rho(t) & \text{if } t < -T, \\ \hat{h}_t(x) & \text{if } -T \le t \le T, \\ h_{\varepsilon'}(x) - \rho(t) & \text{if } t > T, \end{cases}$$

where $\hat{h}_t(x)$ is an approximation to $\frac{1}{2}(T-t)(h_\varepsilon(x)-\rho(t)) + \frac{1}{2}(T+t)(h_{\varepsilon'}(x)-\rho(t))$ that makes F_{21} smooth and $\rho\colon \mathbb{R} \to (-1,1)$ is a smooth strictly increasing function such that $\lim_{t\to-\infty}\rho(t) = -1$ and $\lim_{t\to+\infty}\rho(t) = 1$. The moduli space of gradient flow lines of $F_{21}\colon M \times \mathbb{R} \to \mathbb{R}$ has a component

$$\mathcal{M}_{F_{21}}(q,p) = \left(W_{F_{21}}{}^u(q) \cap W_{F_{21}}{}^s(p)\right)/\mathbb{R}$$

of dimension 1 (see [7, Lemma 6.2]) that can be compactified to a smooth manifold with boundary $\overline{\mathcal{M}}_{F_{21}}(q,p)$ using piecewise gradient flow lines (see [7, Theorem 6.4]).

Moreover, the boundary of the compactified space consists of the fibered products

$$\partial\overline{\mathcal{M}}_{F_{21}}(q,p) = \overline{\mathcal{M}}_{h_\varepsilon}(q,p) \times_p \overline{\mathcal{M}}_{F_{21}}(p,p) \coprod \overline{\mathcal{M}}_{F_{21}}(q,q) \times_q \overline{\mathcal{M}}_{h'_\varepsilon}(q,p).$$

Since

$$\begin{aligned} \overline{\mathcal{M}}_{h_\varepsilon}(q,p) \times_p \overline{\mathcal{M}}_{F_{21}}(p,p) &\approx \mathcal{M}_{h_\varepsilon}(q,p), \\ \overline{\mathcal{M}}_{F_{21}}(q,q) \times_q \overline{\mathcal{M}}_{h'_\varepsilon}(q,p) &\approx \mathcal{M}_{h'_\varepsilon}(q,p), \end{aligned}$$

and $F_{21}\colon M \times \mathbb{R} \to \mathbb{R}$ is strictly decreasing in its second component, $\overline{\mathcal{M}}_{F_{21}}(q,p)$ is a one-dimensional trivial cobordism between $\mathcal{M}_{h_\varepsilon}(q,p)$ and $\mathcal{M}_{h'_\varepsilon}(q,p)$. Thus, $\mathcal{M}_{h_\varepsilon}(q,p)$ and $\mathcal{M}_{h'_\varepsilon}(q,p)$ have the same number of elements. □

Remark The moduli space $\mathcal{M}_{F_{21}}(q,p)$ used in the preceding proof is, in the language of [29], a space of λ–parameterized trajectories between the trivial regular homotopies h_ε and $h_{\varepsilon'}$ (see [29, Definition 2.29]). A general moduli space of λ–parameterized trajectories is constructed in [29, Theorem 2, Section 2.3.2], and its compactification is discussed in [29, Section 2.4.4].

In summary, we have a Riemannian metric g on M and a 1–parameter family of Morse functions $h_\varepsilon\colon M \to \mathbb{R}$ such that the following conditions hold for all $\varepsilon > 0$ sufficiently small and for all $k = 1, \ldots, l$:

(1) The function $h_0 = f\colon M \to \mathbb{R}$ satisfies the Morse–Bott–Smale transversality condition with respect to the metric g.

(2) The functions $h_\varepsilon\colon M \to \mathbb{R}$ and $f_k\colon C_k \to \mathbb{R}$ satisfy the Morse–Smale transversality condition with respect to g.

(3) For all $i, j = 1, \dots, l$ and for each pair of critical points $(q, p) \in \mathrm{Cr}(f_j) \times \mathrm{Cr}(f_i)$ the unstable and stable manifolds $W^u_{f_j}(q)$ and $W^s_{f_i}(p)$ are transverse to the beginning and endpoint maps.

(4) The function $h_\varepsilon = f$ outside of the union of the tubular neighborhoods T_k.

(5) The function $h_\varepsilon = f + \varepsilon f_k$ on the smaller tubular neighborhoods $\widetilde{T}_k$.

(6) The charts from the Morse–Bott Lemma within the tubular neighborhoods T_k are isometries with respect to the metric on M and the standard Euclidean metric on $\mathbb{R}^m$.

(7) In the local coordinates (u, v, w) of a tubular neighborhood T_k we have $f = f(C) - |v|^2 + |w|^2$, ρ_k depends only on the v and w coordinates, and f_k depends only on the u coordinates. In particular, $\nabla f \perp \nabla f_k$ on T_k by the previous condition.

(8) The gradient ∇f dominates $\varepsilon \nabla \rho_k f_k$ on $T_k - \widetilde{T}_k$.

(9) For $q, p \in \mathrm{Cr}(h_\varepsilon)$ with $\lambda_q - \lambda_p = 1$, the number of gradient flow lines of h_ε from q to p is independent of $\varepsilon > 0$.

Lemma 5.3 *Let $\varepsilon > 0$ be small enough so that the above conditions hold, and let $\{\varepsilon_\nu\}_{\nu=1}^\infty$ be a decreasing sequence such that $0 < \varepsilon_\nu \le \varepsilon$ for all ν and $\lim_{\nu\to\infty} \varepsilon_\nu = 0$. Let $q, p \in \mathrm{Cr}(h_\varepsilon)$, and suppose that $\gamma_{\varepsilon_\nu} \in \mathcal{M}_{h_{\varepsilon_\nu}}(q, p)$ for all ν. Then there exists a broken flow line with cascades $\gamma \in \overline{\mathcal{M}}^c(q, p)$ and a subsequence of $\{\mathrm{Im}(\gamma_{\varepsilon_\nu})\}_{\nu=1}^\infty$ that converges to $\mathrm{Im}(\gamma)$ in the Hausdorff topology.*

Proof Let $q \in C_j$, $p \in C_i$, and $\gamma_{\varepsilon_\nu} \in \mathcal{M}_{h_{\varepsilon_\nu}}(q, p)$ where $\lim_{\nu\to\infty} \varepsilon_\nu = 0$. Recall that outside of the open tubular neighborhoods $\{T_k\}_{k=1}^l$ we have $h_{\varepsilon_\nu} = f$, and inside T_k we have

$$h_{\varepsilon_\nu} = f + \varepsilon_\nu \rho_k f_k,$$

where $\nabla f \perp \nabla f_k$, $0 \le \rho_k \le 1$, and $f_k > 0$. Moreover, $\nabla h_{\varepsilon_\nu} = \nabla f + \varepsilon_\nu \nabla f_k$ on the smaller open tubular neighborhood $\widetilde{T}_k \subset T_k$, and ∇f dominates $\varepsilon_\nu \nabla \rho_k f_k$ on $T_k - \widetilde{T}_k$. By passing to a subsequence of $\{\gamma_{\varepsilon_\nu}\}_{\nu=1}^\infty$ we may assume that there exists a set of distinct integers $\{j_1, j_2, \dots, j_{n-1}\} \subseteq \{1, 2, \dots, l\}$ such that for all ν we have $\mathrm{Im}(\gamma_\nu) \cap T_{j_k} \ne \varnothing$ for all $k = 1, \dots, n-1$ and $\mathrm{Im}(\gamma_\nu) \cap T_k = \varnothing$ if $k \in \{1, 2, \dots, l\} - \{i, j_1, j_2, \dots, j_{n-1}, j\}$. Since $\mathcal{P}^c(M)$ is compact in the Hausdorff topology, there exists a subsequence of $\{\gamma_{\varepsilon_\nu}\}_{\nu=1}^\infty$, which we still denote by $\{\gamma_{\varepsilon_\nu}\}_{\nu=1}^\infty$, such that the compact sets

$$C_{\varepsilon_\nu} = \mathrm{Im}(\gamma_{\varepsilon_\nu}) - \left(T_i \cup \bigcup_{k=1}^{n-1} T_{j_k} \cup T_j \right)$$

converge to some compact set $C \in \mathcal{P}^c(M)$ as $\nu \to \infty$. The interior of each C_{ε_ν} is locally invariant under the flow of $-\nabla f$, and hence the interior of the limit C is also locally invariant with respect to the flow of $-\nabla f$. Moreover, for every regular value y of f the level set $f^{-1}(y)$ contains at most one element of C_{ε_ν} for each ν, and hence we can pass to a subsequence of $\{\gamma_{\varepsilon_\nu}\}_{\nu=1}^\infty$ such that the same holds for C. Therefore, there exists a subsequence of $\{\gamma_{\varepsilon_\nu}\}_{\nu=1}^\infty$, which we still denote by $\{\gamma_{\varepsilon_\nu}\}_{\nu=1}^\infty$, and gradient flow lines $x_1, \ldots, x_n$ of $-\nabla f$ (not necessarily distinct) such that

$$\mathrm{Im}(\gamma_{\varepsilon_\nu}) - \left(T_i \cup \bigcup_{k=1}^{n-1} T_{j_k} \cup T_j \right) \longrightarrow \bigcup_{k=1}^{n} \mathrm{Im}(x_k) - \left(T_i \cup \bigcup_{k=1}^{n-1} T_{j_k} \cup T_j \right)$$

in the Hausdorff topology as $\nu \to \infty$. Moreover, since $\nabla h_{\varepsilon_\nu} = \nabla f + \varepsilon_\nu \nabla \rho_k f_k$ and there is a positive lower bound for $\|\nabla f\|$ on $T_k - \widetilde{T}_k$ for all $k = 1, \ldots, l$ we have

$$\mathrm{Im}(\gamma_{\varepsilon_\nu}) - \left(\widetilde{T}_i \cup \bigcup_{k=1}^{n-1} \widetilde{T}_{j_k} \cup \widetilde{T}_j \right) \longrightarrow \bigcup_{k=1}^{n} \mathrm{Im}(x_k) - \left(\widetilde{T}_i \cup \bigcup_{k=1}^{n-1} \widetilde{T}_{j_k} \cup \widetilde{T}_j \right)$$

in the Hausdorff topology as $\nu \to \infty$. We will order the gradient flow lines $x_1, \ldots, x_n$ as in Definition 3.5, ie $x_k(t)$ flows into T_{j_k} as t increases for all $k = 1, \ldots, n-1$.

On the tubular neighborhood $\widetilde{T}_j$ we have $\nabla h_{\varepsilon_\nu} = \nabla f + \varepsilon_\nu \nabla f_j$, where $\nabla f \perp \nabla f_j$, and hence there is a subsequence of $\{\gamma_{\varepsilon_\nu}\}_{\nu=1}^\infty$ such that $\mathrm{Im}(\gamma_{\varepsilon_\nu}) \cap \widetilde{T}_j$ converges to a curve consisting of the union of $\mathrm{Im}(x_1) \cap \widetilde{T}_j$ and a (possibly broken) gradient flow line of f_j from q to $\lim_{t \to -\infty} x_1(t)$. Similar statements apply to the tubular neighborhood $\widetilde{T}_i$.

For each tubular neighborhood $\widetilde{T}_{j_1}, \ldots, \widetilde{T}_{j_{n-1}}$ there are two cases to consider:

(1) There exists a neighborhood $U \subseteq \widetilde{T}_{j_k}$ of C_{j_k} such that $\mathrm{Im}(\gamma_{\varepsilon_\nu}) \cap U = \varnothing$ for all ν.

(2) For every neighborhood $U \subseteq \widetilde{T}_{j_k}$ of C_{j_k} we have $\mathrm{Im}(\gamma_{\varepsilon_\nu}) \cap U \neq \varnothing$ for all ν sufficiently large.

Otherwise we can pass to a subsequence of $\{\gamma_{\varepsilon_\nu}\}_{\nu=1}^\infty$ such that one of these cases applies. In the first case, there is a positive lower bound for $\|\nabla f\|$ on $\mathrm{Im}(\gamma_{\varepsilon_\nu}) \cap \widetilde{T}_{j_k}$ independent of ν, and hence $\nabla h_{\varepsilon_\nu}$ converges to ∇f on $\mathrm{Im}(\gamma_{\varepsilon_\nu}) \cap \widetilde{T}_{j_k}$ as $\nu \to \infty$.

Thus, $x_k(t)$ and $x_{k+1}(t)$ are the same gradient flow line of f, and $\mathrm{Im}(\gamma_{\varepsilon_\nu}) \cap \widetilde{T}_{j_k}$ converges to $\mathrm{Im}(x_k) \cap \widetilde{T}_{j_k}$ as $\nu \to \infty$.

In the second case, $\lim_{t\to\infty} x_k(t) \in C_{j_k}$ since $\mathrm{Im}(\gamma_{\varepsilon_\nu}) \cap f^{-1}(y)$ converges to $\mathrm{Im}(x_k) \cap f^{-1}(y)$ for any $y > f(C_{j_k})$ with $\mathrm{Im}(\gamma_{\varepsilon_\nu}) \cap f^{-1}(y) \in \widetilde{T}_{j_k}$. Similarly, $\lim_{t\to-\infty} x_{k+1}(t) \in C_{j_k}$. Also, $\mathrm{Im}(\gamma_{\varepsilon_\nu}) \cap \widetilde{T}_{j_k}$ converges to the union of $\mathrm{Im}(x_k) \cap \widetilde{T}_{j_k}$, $\mathrm{Im}(x_{k+1}) \cap \widetilde{T}_{j_k}$ and a curve in C_{j_k} from $\lim_{t\to\infty} x_k(t)$ to $\lim_{t\to-\infty} x_{k+1}(t)$. Since $\nabla h_{\varepsilon_\nu} = \nabla f + \varepsilon_\nu \nabla f_{j_k}$ in $\widetilde{T}_{j_k}$ where $\nabla f \perp \nabla f_{j_k}$, the curve in C_{j_k} must be a subset of the image of a (possibly broken) gradient flow line of f_{j_k}. Therefore, there exists a subsequence of $\{\gamma_{\varepsilon_\nu}\}_{\nu=1}^{\infty}$ and a broken flow line with cascades $\gamma \in \overline{\mathcal{M}}^c(q,p)$ such that $\{\mathrm{Im}(\gamma_{\varepsilon_\nu})\}_{\nu=1}^{\infty}$ converges to $\mathrm{Im}(\gamma)$ in the Hausdorff topology. □

5.2 Correspondence theorem

Throughout this subsection we will assume that the function

$$h_\varepsilon = f + \varepsilon\left(\sum_{k=1}^{l} \rho_k f_k\right)$$

and the Riemannian metric g on M satisfy all the conditions listed above. The main goal of this subsection is to prove the following.

Theorem 5.4 (Correspondence of moduli spaces) *Let $p, q \in \mathrm{Cr}(h_\varepsilon)$ with $\lambda_q - \lambda_p = 1$. For any sufficiently small $\varepsilon > 0$ there is a bijection between unparameterized cascades and unparameterized gradient flow lines of the Morse–Smale function $h_\varepsilon\colon M \to \mathbb{R}$ between q and p,*

$$\mathcal{M}^c(q,p) \longleftrightarrow \mathcal{M}_{h_\varepsilon}(q,p).$$

We will prove this theorem using results from geometric singular perturbation theory [20]. In particular, we will use the exchange lemma for fast-slow systems (see Jones [21] and Schecter [27; 28]). Roughly speaking, the exchange lemma says that a manifold M_0 that is transverse to the stable manifold of a normally hyperbolic locally invariant submanifold C will have subsets that flow forward in time under the full fast-slow system to be near subsets of the unstable manifold of C. The exchange lemma can be viewed as a generalization of the λ–lemma, which applies to hyperbolic fixed points (see for instance [4, Theorem 6.17 and Corollary 6.20]).

In our setup, we have tubular neighborhoods T_j of the critical submanifolds C_j for all $j = 1, \ldots, l$ and local coordinate charts on T_j that are isometries with respect to the standard Euclidean metric on $\mathbb{R}^m$. We also have smaller tubular neighborhoods $\widetilde{T}_j \subset T_j$ such that within the smaller tubular neighborhoods the negative gradient flow of $h_\varepsilon\colon T \to \mathbb{R}$ constitutes a fast-slow system because $\nabla h_\varepsilon = \nabla f + \varepsilon \nabla f_j$ and $\nabla f \perp \nabla f_j$. Moreover, we have coordinates (u, v, w) where the function $f|_{\tilde{T}_j}$ depends only on the (v, w) coordinates, which are the fast variables, and the function $f_j|_{\tilde{T}_j}$ depends only on the u variables, which are the slow variables.

Proof of Theorem 5.4 Let $q \in \mathrm{Cr}(f_j)$ and $p \in \mathrm{Cr}(f_i)$. An unparameterized cascade $\gamma \in \mathcal{M}^c(q, p)$ can be represented by a flow line with n cascades from q to p: $((x_k)_{1\le k\le n}, (t_k)_{1\le k\le n-1})$, where t_k is the time spent flowing along (or resting on) the intermediate critical submanifold C_{j_k}. For $1 \le k \le n-1$, let $y_k\colon \mathbb{R} \to C_{j_k}$ be the parameterized gradient flow line of $f_{j_k}\colon C_{j_k} \to \mathbb{R}$ satisfying $y_k(0) = \lim_{t\to\infty} x_k(t)$ and $y_k(t_k) = \lim_{t\to-\infty} x_{k+1}(t)$ (as in Definition 3.5). Assume that $y_k(0) \neq y_k(t_k)$ for any $1 \le k \le n-1$. This last condition is required in order to apply the exchange lemma, and it holds whenever $\lambda_q - \lambda_p = 1$. To see this, note that if $y_k(0) = y_k(t_k)$ then there is a piecewise gradient flow line of f from the beginning of x_k to the end of x_{k+1}. Hence, there is a 1–parameter family of gradient flow lines of f from the beginning of x_k to the end of x_{k+1} by the gluing theorem for Morse–Bott moduli spaces (see the proof of Theorem 3.10). Each of these gradient flow lines determines a unique flow line with cascade from q to p, and hence $\dim \mathcal{M}^c(q, p) \ge 1$.

For every $1 \le k \le n-1$, let $S_k \subset C_{j_k}$ be a tubular neighborhood of the image $y_k([0, t_k])$ that is diffeomorphic to some contractible open subset $U_k \subset \mathbb{R}^{\dim C_{j_k}}$. The tubular neighborhood S_k exists because $y_k([0, t_k])$ is contractible and hence has a trivial normal bundle in C_{j_k}. Similarly, the normal bundle of $S_k \subset M$ is trivial, and hence S_k has a contractible tubular neighborhood in $\tilde{T}_{j_k}$. This establishes Fenichel coordinates (u, v, w) near S_k. (See [21, Proposition 1 and Section 6], but note that we do not need S_k to vary with ε.)

Let B^k_{Δ,U_k} be a small "box" in the phase space $\mathbb{R}^m$ with respect to the Fenichel coordinates near S_k, eg

$$B^k_{\Delta,U_k} = \left\{(u, v, w) \in \mathbb{R}^m \;\middle|\; |v| < \Delta,\ |w| < \Delta,\ u \in U_k\right\}$$

for some small $\Delta > 0$, and let B_k denote the image of B^k_{Δ,U_k} in M. We will show that for $\Delta > 0$ and $\varepsilon > 0$ sufficiently small there exist submanifolds $M_k \subset W^u_{h_\varepsilon}(q)$ that satisfy the following conditions for every $1 \le k \le n-1$:

(D1) $\lambda_{j_k} \le \dim M_k \le \lambda_{j_k} + \dim C_{j_k} - 1$.

(T1) There exists a point $q_k \in M_k \cap \overline{B}_k$ such that $M_k \pitchfork_{q_k} W^s_f(S_k)$.

(T2) The omega limit set $J_k = \omega(M_k \cap W^s_f(S_k) \cap V_k) \subset S_k$ with respect to the flow of $-\nabla f$ is a manifold of dimension $\dim M_k - \lambda_{j_k}$, where V_k is a small enough open neighborhood of q_k to ensure that $M_k \cap W^s_f(S_k) \cap V_k$ is a manifold, and ∇f_{j_k} is not tangent to J_k.

(T3) The tangent space to M_k at q_k intersects the tangent space of $W_f^s(\omega(q_k))$ in a zero-dimensional space.

(I1) If $\mathrm{Im}(\gamma_\varepsilon) \cap M_k \neq \varnothing$ for some $\gamma_\varepsilon \in \mathcal{M}_{h_\varepsilon}(q,p)$, then $\mathrm{Im}(\gamma'_\varepsilon) \cap M_k \neq \varnothing$ for every $\gamma'_\varepsilon \in \mathcal{M}_{h_\varepsilon}(q,p)$ with $d_H(\mathrm{Im}(\gamma'_\varepsilon), \mathrm{Im}(\gamma)) \leq d_H(\mathrm{Im}(\gamma_\varepsilon), \mathrm{Im}(\gamma))$.

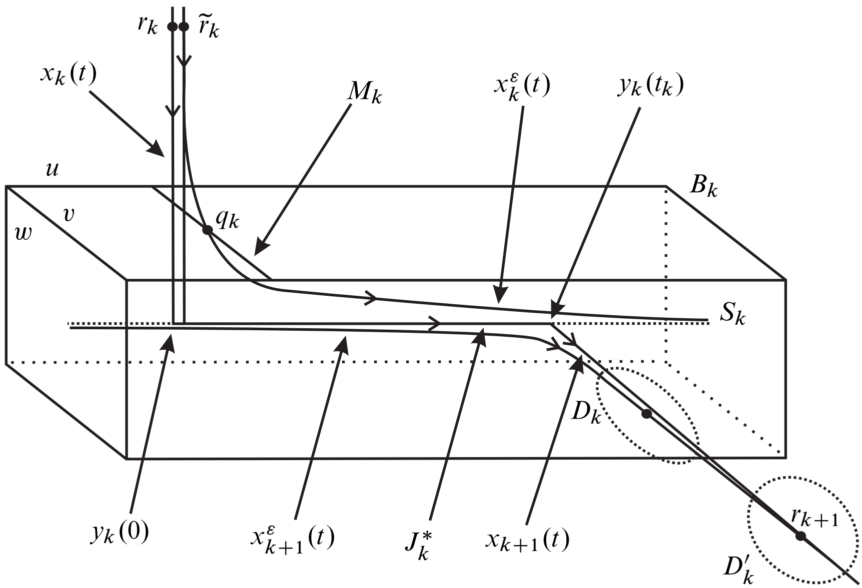
The manifold M_1 exists as long as $\varepsilon > 0$ is small enough so that the conditions listed in the previous subsection hold. That is, the conditions in the previous subsection imply that $\lim_{t\to-\infty} x_1(t) \in W^u_{h_\varepsilon}(q)$ and $W_f^u(\lim_{t\to-\infty} x_1(t)) \pitchfork W_f^s(S_1)$. Thus, we can find a small open neighborhood in $W^u_{h_\varepsilon}(q)$ around the point r_1 where the image of x_1 intersects the boundary of T_j with a cross section that intersects $W_f^s(S_1)$ transversally. This cross section flows forward under the flow of $-\nabla h_\varepsilon$ to a submanifold $\widetilde{M}_1$ of dimension $\lambda_q - 1$ that intersects $\overline{B}_1 \cap W_f^s(S_1)$ at some point q_1. The Morse–Bott–Smale transversality condition implies that $\lambda_{j_1} < \lambda_j$ (see [7, Lemma 3.6]), and hence $\lambda_{j_1} \leq \lambda_j + \lambda_q^j - 1 = \lambda_q - 1 = \dim \widetilde{M}_1$. If $\dim \widetilde{M}_1 \leq \lambda_{j_1} + \dim C_{j_1} - 1$, then we can take $M_1 = \widetilde{M}_1$. Otherwise, we can find a small open ball $M_1 \subset \widetilde{M}_1$ of dimension $\lambda_{j_1} + \dim C_{j_1} - 1$ that satisfies the above conditions. Thus, M_1 exists and $\dim M_1 = \min\{\lambda_q - 1, \dim C_{j_1} + \lambda_{j_1} - 1\}$.

We will see by induction using the exchange lemma that for $\Delta > 0$ and $\varepsilon > 0$ sufficiently small $M_k \subseteq W^u_{h_\varepsilon}(q)$ exists for $k = 2, \ldots, n-1$. For this purpose, assume that $\Delta > 0$ and $\varepsilon > 0$ are small enough so that the conditions listed in the previous subsection hold, the exchange lemma applies around S_k for all $k = 1, \ldots, n-1$, and M_1 exists. Assume that for some k there exists a submanifold $M_k \subseteq W^u_{h_\varepsilon}(q)$ that satisfies the above conditions, and let M_k^* and J_k^* denote the manifolds obtained by flowing M_k and J_k forward in time with respect to $-\nabla h_\varepsilon$ on the time interval $[0, \infty)$. The dimension of M_k^* is $\dim M_k + 1$, and $\dim J_k^* = \dim M_k^* - \lambda_{j_k}$.

Let $x^\varepsilon_{k+1}(t)$ be the gradient flow line of h_ε through the point r_{k+1} where the image of $x_{k+1}(t)$ intersects the boundary of T_{j_k}. We have

$$\lim_{t\to-\infty} x^\varepsilon_{k+1}(t) = \lim_{t\to-\infty} y_k(t).$$

Hence, as long as $\varepsilon > 0$ is sufficiently small, the point where $x^\varepsilon_{k+1}(t)$ exits the box B_k will be in $W_f^u(J_k^*)$. Choose a small open disk D_k in $W_f^u(J_k^*)$ of dimension $\dim M_k^*$ around this point. The exchange lemma implies that by decreasing $\varepsilon > 0$ we can find an open disk $\widetilde{D}_k$ in M_k^* as close as we like to D_k. (See for instance [21, Theorem 6.5], [20, Lemma 6], or [28, Theorem 2.3].) In this context "close" can be in the sense of [4, Definition 6.13] or "close" in the sense that $\widetilde{D}_k$ can be expressed as the graph of a vector valued function over D_k that goes to zero exponentially along with its derivatives up to finite order as $\varepsilon \to 0$ [28].



The open disk D_k flows forward in finite time under the flow of $-\nabla h_\varepsilon$ to a neighborhood D_k' of r_{k+1}, and the open disk $\widetilde{D}_k$ flows forward under the same flow to an open set $\widetilde{D}_k{}'$ in M_k^* close to D_k'. In fact, inside T_{j_k} the disks $\widetilde{D}_k$ and D_k get closer under the forward time flow of $-\nabla h_\varepsilon$. The Morse–Bott–Smale transversality condition implies that $D_k' \pitchfork W_f^s(S_{k+1})$, and hence $\widetilde{D}_k{}' \pitchfork W_f^s(S_{k+1})$ if $\widetilde{D}_k{}'$ is close enough to D_k' since the collection of maps transverse to a given submanifold is locally stable (see for instance [4, Theorem 5.16] or [17, Theorem 3.2.1]). Thus, we can decrease $\varepsilon > 0$, if necessary, to obtain an open set $\widetilde{D}_k{}' \subset M_k^*$ such that $\widetilde{D}_k{}' \pitchfork W_f^s(S_{k+1})$. Moreover, $r_{k+1} \in D_k' \cap W_f^s(S_{k+1}) \neq \varnothing$, and hence there exists a point $\widetilde{r}_{k+1} \in \widetilde{D}_k{}' \cap W_f^s(S_{k+1})$ such that $\widetilde{D}_k{}' \pitchfork_{\widetilde{r}_{k+1}} W_f^s(S_{k+1})$.

For $\varepsilon > 0$ sufficiently small, the point $\widetilde{r}_{k+1} \in W_f^s(S_{k+1})$ flows forward in time under the flow of $-\nabla h_\varepsilon$ to a point $q_{k+1} \in \partial \overline{B}_{k+1}$ since the tubular neighborhoods $\{T_j\}_{j=1}^l$ were chosen small enough so that the image of x_{k+1} does not intersect the closure of any of the tubular neighborhoods other than $\overline{T}_{j_k}$ and $\overline{T}_{j_{k+1}}$. Moreover, $\widetilde{D}_k{}' \subseteq M_k^*$ flows forward in time under the flow of $-\nabla h_\varepsilon$ to a submanifold of $W_{h_\varepsilon}^u(q)$ that is transverse to $W_f^s(S_{k+1})$ at q_{k+1}. Thus, we can find a manifold $M_{k+1} \subset M_k^* \subset W_{h_\varepsilon}^u(q)$ of dimension $\min\{\dim M_k, \dim C_{j_{k+1}} + \lambda_{j_{k+1}} - 1\}$ that satisfies the above conditions. This completes the induction. Note that if we have to decrease $\varepsilon > 0$ during the induction, then we also have to modify $M_k \subset W_{h_\varepsilon}^u(q)$. However, $\varepsilon > 0$ will only need to be decreased a finite number of times. Hence, we can find a sufficiently small $\varepsilon > 0$ so that $M_k \subset W_{h_\varepsilon}^u(q)$ exists for all $k = 1, \ldots, n-1$.

To summarize, we have shown that for $\Delta > 0$ and $\varepsilon > 0$ sufficiently small there exist submanifolds $M_k \subseteq W^u_{h_\varepsilon}(q)$ and points q_k such that $M_k \pitchfork_{q_k} W^s_f(S_k)$ for all $k = 1, \dots, n-1$. Moreover, the point q_k is the image under the forward time flow of $-\nabla h_\varepsilon$ of a point $\tilde{r}_k \in W^s_f(S_k) \cap W^u_{h_\varepsilon}(q)$ close to the point r_k where the image of $x_k(t)$ intersects the boundary of $T_{j_{k-1}}$,

$$M^*_{n-1} \subset M^*_{n-2} \subset \cdots \subset M^*_2 \subset M^*_1 \subset W^u_{h_\varepsilon}(q),$$

and every gradient flow line in $\mathcal{M}_{h_\varepsilon}(q, p)$ whose image is sufficiently close to the image of the cascade $\gamma \in \mathcal{M}^c(q, p)$ intersects M^*_{n-1} (and hence M_k for all $k = 1, \dots, n-1$). We can now repeat the above argument involving the exchange lemma for M_{n-1} to see that for $\Delta > 0$ and $\varepsilon > 0$ sufficiently small we can find an open neighborhood $\widetilde{D}'_{n-1} \subset M^*_{n-1}$ as close as we like to a small open neighborhood $D'_{n-1} \subset W^u_f(J^*_{n-1})$ around the point r_n where the image of $x_n(t)$ intersects the boundary of $T_{j_{n-1}}$.

Now recall the assumption that

$$\mathcal{M}^c_n(C_j, C_{j_1}, \dots, C_{j_{n-1}}, C_i) \xrightarrow{(\partial_-, \partial_+)} C_j \times C_i$$

is transverse and stratum transverse to $W^u_{f_j}(q) \times W^s_{f_i}(p)$ (Definition 3.8). This implies that

$$\mathcal{M}_f(J^*_{n-1}, C_i) \xrightarrow{\partial_+} C_i$$

is transverse to $W^s_{f_i}(p)$ at $\lim_{t\to\infty} x_n(t) \in W^s_{f_i}(p)$, since the endpoint map

$$\partial_+ \colon \mathcal{M}^c_n(W^u_{f_j}(q), C_{j_1}, \dots, C_{j_{n-1}}, C_i) \longrightarrow C_i$$

factors through $\partial_+ \colon \mathcal{M}_f(J^*_{n-1}, C_i) \to C_i$ and is transverse to $W^s_{f_i}(p)$ at $\lim_{t\to\infty} x_n(t)$. Therefore, $D'_{n-1} \pitchfork_{r_n} W^s_f(W^s_{f_i}(p))$ as long as D'_{n-1} is sufficiently small. Thus if $\varepsilon > 0$ is sufficiently small, there exists a point

$$\tilde{r}_n \in \widetilde{D}'_{n-1} \cap W^s_f(W^s_{f_i}(p))$$

close to r_n such that $\widetilde{D}'_{n-1} \pitchfork_{\tilde{r}_n} W^s_f(W^s_{f_i}(p))$. The unparameterized gradient flow line of h_ε that passes through $\tilde{r}_n$ is an element $\gamma_{\tilde{r}_n} \in \mathcal{M}_{h_\varepsilon}(q, p)$ whose image is close to the image of the cascade in $\mathcal{M}^c(q, p)$ represented by $((x_k)_{1\le k\le n}, (t_k)_{1\le k\le n-1})$.

Also, if $\lambda_q - \lambda_p = 1$ then we can choose the D'_k small enough so that $\gamma \in \mathcal{M}^c(q, p)$ is the unique element whose image intersects D'_k for all $k = 1, \dots, n-1$. Then if $\widetilde{D}'_{n-1}$ is sufficiently close to D'_{n-1}, the gradient flow line of h_ε through $\tilde{r}_n$ will be the unique element of $\mathcal{M}_{h_\varepsilon}(q, p)$ whose image intersects $\widetilde{D}'_{n-1} \subset M^*_{n-1}$. Thus for $\lambda_q - \lambda_p = 1$ and $\varepsilon > 0$ sufficiently small we have defined an injective map

$$\mathcal{M}^c(q, p) \longrightarrow \mathcal{M}_{h_\varepsilon}(q, p)$$

that sends a cascade $\gamma \in \mathcal{M}^c(q,p)$ to a gradient flow line $\gamma_\varepsilon \in \mathcal{M}_{h_\varepsilon}(q,p)$ such that $\mathrm{Im}(\gamma)$ is close to $\mathrm{Im}(\gamma_\varepsilon)$ in the Hausdorff topology. To see that this map is surjective, first recall that Lemma 5.2 says that if $\varepsilon > 0$ is sufficiently small, then the (finite) number of elements in $\mathcal{M}_{h_\varepsilon}(q,p)$ does not depend on $\varepsilon > 0$. So, if the above map were not surjective, we could pick a decreasing sequence $\{\varepsilon_\nu\}_{\nu=1}^\infty$ with $\lim_{\nu\to\infty} \varepsilon_\nu = 0$ and a sequence of elements $\gamma_{\varepsilon_\nu} \in \mathcal{M}_{h_{\varepsilon_\nu}}(q,p)$ such that γ_{ε_ν} is not in the image of the map

$$\mathcal{M}^c(q,p) \longrightarrow \mathcal{M}_{h_\varepsilon}(q,p) \longleftrightarrow \mathcal{M}_{h_{\varepsilon_\nu}}(q,p)$$

for all ν. Lemma 5.3 would then imply that there exists a subsequence of $\{\mathrm{Im}(\gamma_{\varepsilon_\nu})\}_{\nu=1}^\infty$ (which we still denote by $\{\mathrm{Im}(\gamma_{\varepsilon_\nu})\}_{\nu=1}^\infty$) that converges to the image of some element $\gamma \in \overline{\mathcal{M}}^c(q,p) = \mathcal{M}^c(q,p)$ in the Hausdorff topology. But if we were to apply the above construction to γ, then for ν sufficiently large we would get an element $\gamma_{\tilde{r}_n}$ in $\mathcal{M}_{h_{\varepsilon_\nu}}(q,p)$ that intersects an open neighborhood $\widetilde{D}'_{n-1} \subset M^*_{n-1}$ near $W_f^u(C_{j_{n-1}})$. Since the sequence $\{\mathrm{Im}(\gamma_{\varepsilon_\nu})\}_{\nu=1}^\infty$ is converging to $\mathrm{Im}(\gamma)$ we must have

$$\mathrm{Im}(\gamma_{\varepsilon_\nu}) \cap \widetilde{D}'_{n-1} \neq \varnothing$$

for ν sufficiently large by condition (I1), and since $\gamma_{\tilde{r}_n}$ is the unique gradient flow line in $\mathcal{M}_{h_{\varepsilon_\nu}}(q,p)$ whose image intersects $\widetilde{D}'_{n-1}$, we see that $\gamma_{\varepsilon_\nu} = \gamma_{\tilde{r}_n}$ is in the image of the above map for ν sufficiently large. This implies that the above map is surjective and hence bijective. $\square$

5.3 Correspondence of chain complexes

Fix $\varepsilon > 0$ small enough so that the conclusion of Theorem 5.4 holds. If we identify $\mathcal{M}^c(q,p)$ with $\mathcal{M}_{h_\varepsilon}(q,p) \times \{0\}$ using Theorem 5.4, then

$$\mathcal{M}_{h_\varepsilon}(q,p) \times [0,\varepsilon]$$

determines a trivial smooth cobordism between

$$\mathcal{M}^c(q,p) \quad \text{and} \quad \mathcal{M}_{h_\varepsilon}(q,p) \approx \mathcal{M}_{h_\varepsilon}(q,p) \times \{\varepsilon\}.$$

If we choose orientations for the unstable manifolds of h_ε, then $\mathcal{M}_{h_\varepsilon}(q,p)$ becomes an oriented zero–dimensional manifold and there is an induced orientation on $\mathcal{M}_{h_\varepsilon}(q,p) \times [0,\varepsilon]$.

Definition 5.5 Let $p,q \in \mathrm{Cr}(h_\varepsilon)$ with $\lambda_q - \lambda_p = 1$, define an orientation on the zero-dimensional manifold $\mathcal{M}^c(q,p)$ by identifying it with the left hand boundary of $\mathcal{M}_{h_\varepsilon}(q,p) \times [0,\varepsilon]$.

An orientation on $\mathcal{M}^c(q,p)$ assigns a $+1$ or -1 to each point in $\mathcal{M}^c(q,p)$. This determines an integer $n^c(q,p) = \#\mathcal{M}^c(q,p) \in \mathbb{Z}$. Moreover, the one-dimensional manifold $\mathcal{M}_{h_\varepsilon}(q,p) \times [0,\varepsilon]$ consists of finitely many closed intervals where the right hand boundary is identified with $\mathcal{M}_{h_\varepsilon}(q,p)$. Thus,

$$n^c(q,p) = -n_{h_\varepsilon}(q,p).$$

Definition 5.6 Define the k–th chain group $C_k^c(f)$ to be the free abelian group generated by the critical points of total index k of the Morse–Smale functions f_j for all $j = 1,\dots,l$, and define $n^c(q,p)$ to be the number of flow lines with cascades between a critical point q of total index k and a critical point p of total index $k-1$ counted with signs determined by the orientations. Let

$$C_*^c(f) = \bigoplus_{k=0}^{m} C_k^c(f)$$

and define a homomorphism $\partial_k^c\colon C_k^c(f) \to C_{k-1}^c(f)$ by

$$\partial_k^c(q) = \sum_{p \in \mathrm{Cr}_{k-1}} n^c(q,p)\,p.$$

Corollary 5.7 (Correspondence of chain complexes) *For $\varepsilon > 0$ sufficiently small we have $C_k^c(f) = C_k(h_\varepsilon)$ and $\partial_k^c = -\partial_k$ for all $k = 0,\dots,m$, where ∂_k denotes the Morse–Smale–Witten boundary operator determined by the Morse–Smale function h_ε. In particular, $(C_*^c(f), \partial_*^c)$ is a chain complex whose homology is isomorphic to the singular homology $H_*(M;\mathbb{Z})$.*

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Received: 22 March 2012