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Highly transitive actions of free products

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We characterize free products admitting a faithful and highly transitive action. In particular, we show that the group $\mathrm{PSL}_2(\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ admits a faithful and highly transitive action on a countable set.

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Introduction

Let X be a countable[1] set and let G be a countable group acting on X. The action is called *highly transitive* if, for all $k \in \mathbb{N}^*$, it is transitive on ordered k–tuples of distinct elements[2].

Dixon [2] proved that for any integer $k \geq 2$, generically in Baire's sense, k permutations $x_1, \dots, x_k \in \mathrm{Sym}(\mathbb{N})$ such that the subgroup $\langle x_1, \dots, x_k \rangle$ acts without finite orbits generate a free group of rank k which acts highly transitively on $\mathbb{N}$. Adapting this approach, Kitroser [7] showed that the fundamental groups of surfaces of genus at least 2 admit a faithful and highly transitive action.

Garion and Glasner [3] proved that for $n \geq 4$ the group of outer automorphisms of the free group on n generators $\mathrm{Out}(\mathbb{F}_n) = \mathrm{Aut}(\mathbb{F}_n)/\,\mathrm{Inn}(\mathbb{F}_n)$ admits a faithful and highly transitive action. They asked whether $\mathrm{Out}(\mathbb{F}_2) \simeq \mathrm{GL}_2(\mathbb{Z})$ and $\mathrm{Out}(\mathbb{F}_3)$ admit a highly transitive action. In this paper, with methods in Dixon's spirit, we obtain the following result.

Theorem 1 *Let G, H be nontrivial finite or countable groups. Then, the following statements are equivalent:*

(1) *the free product $G * H$ admits a faithful and highly transitive action;*

(2) *at least one of the factors G, H is not isomorphic to the cyclic group $\mathbb{Z}/2\mathbb{Z}$.*

[1] In this paper, "countable" means "infinite countable".
[2] We denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{N}^*$ the set of positive integers.

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In particular, the group $\mathrm{PSL}_2(\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ admits a faithful and highly transitive action. As a consequence, the group $\mathrm{SL}_2(\mathbb{Z})$ admits a highly transitive action on a countable set. On the other hand, this group cannot admit faithful and highly transitive actions since it has nontrivial center (see Corollary 1.5).

The paper is organized as follows. Section 1 contains preliminaries about highly transitive actions and Baire's theory. Sections 2 and 3 are devoted to the proof of Theorem 1.

Note added in proof

Pierre Fima showed us recently papers by Steven G Gunhouse [5] and K K Hickin [6] where Theorem 1 of the present article was proven with different methods than ours. In fact, Gunhouse used a former partial result of Glass and McCleary [4].

What we prove beyond the existence of highly transitive actions (when they exist), is that if G is a group acting on X, then a generic choice of an action of another group H defines a highly transitive and faithful action on X of free product $G * H$ (except when G and H both have two elements). As far as we are aware, this method of genericity is new.

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1 Preliminaries

1.1 Generalities about group actions

Let us begin with a general fact concerning actions with infinite orbits.

Lemma 1.1 (B H Neumann, P Neumann) *Let G be a group acting on some set X and F be a finite subset of X. If every orbit of the points in F is infinite, then there exists $g \in G$ such that $g \cdot F \cap F = \varnothing$.*

Proof This lemma follows from B H Neumann [9, Lemma 4.1] and P Neumann [10, Lemma 2.3]. Indeed, let us suppose that for every $g \in G$, $gF \cap F \neq \varnothing$. If we denote $K_{xy} := \{g \in G \mid gx = y\}$, for all $x,\ y \in F$, then by hypothesis we have $G=\bigcup_{x,y\in F}K_{xy}$. When $K_{xy}\neq\varnothing$, we have $K_{xy}=\mathrm{Stab}(y)g_{xy}$ with some $g_{xy}\in K_{xy}$. Then

$$G=\bigcup_{x,\,y\in F\text{ such that }K_{xy}\neq\varnothing}\mathrm{Stab}(y)g_{xy}.$$

Then by [9, Lemma 4.1], there exists $y\in F$ such that the index of $\mathrm{Stab}(y)$ is finite. Therefore the orbit Gy is finite. $\square$

From the above lemma, we immediately get the following.

Remark 1.2 Let X be a G–set and F_1, F_2 be finite subsets of X. If every orbit of the points in F_1 and F_2 are infinite, then there exists $g\in G$ such that $g\cdot F_1\cap F_2=\varnothing$.

1.2 Highly transitive actions

Let G be a group acting on some set X. Let us recall that the action is called *faithful* if the corresponding homomorphism $G\to\mathrm{Sym}(X)$ is injective and *transitive* if for any $x,y\in X$, there exists $g\in G$ such that $g\cdot x=y$. Given a positive integer k, we set

$$X^{(k)}=\{(x_1,\dots,x_k)\in X^k\mid x_i\neq x_j\text{ for all }i\neq j\},$$

and the action $G\curvearrowright X$ is called k*–transitive* if the diagonal G–action on $X^{(k)}$ is transitive.

Definition 1.3 Assume that G and X are countable. The action $G\curvearrowright X$ is called highly transitive if it is k–transitive for any positive integer k.

Defining highly transitive actions on a finite set Y would not be interesting, since $Y^{(k)}$ is empty for all $k>|Y|$.

We are interested to determine which groups admit highly transitive actions respectively faithful and highly transitive actions. Here are some general facts, which are probably well-known by experts; see eg [3, Section 5.1] for item (2).

Proposition 1.4 *Let $G\curvearrowright X$ be a highly transitive action. Then:*

(1) *any central element of G acts trivially;*

(2) *for any normal subgroup $K\lhd G$, the action $K\curvearrowright X$ is either trivial, or highly transitive;*

(3) *for any finite index subgroup $H<G$, the action $H\curvearrowright X$ is highly transitive.*

Proof (1) Let g be an element of G which acts nontrivially and let $x_1 \in X$ such that x_1 and $x_2 := gx_1$ are distinct. Let $y_1, y_2 \in X$ such that y_2 is distinct from y_1 and gy_1 (this is possible since X is infinite). Then, by high transitivity, there is an element $h \in G$ such that $hx_1 = y_1$ and $hx_2 = y_2$. We have

$$hgx_1 = hx_2 = y_2 \quad \text{and} \quad ghx_1 = gy_1 \neq y_2 ,$$

which proves that g is not a central element.

(2) Suppose that the action is not trivial, ie that there exists $x \in X$ and $k \in K$ such that $x \neq kx$. For any $y \in X$ different from x, there exists $g \in G$ such that $gx = x$ and $gy = kx$. Then $g^{-1}kgx = y$ and therefore y is in $K \cdot x$ by normality of K in G. This proves that the action $K \curvearrowright X$ is transitive.

Let $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ in $X^{(k)}$. By Lemma 1.1, there exists $h \in K$ such that

$$\{hy_1, \dots, hy_k\} \cap (\{y_1, \dots, y_k\} \cup \{x_1, \dots, x_k\}) = \varnothing.$$

Then we have $(x_1, \dots, x_k, hy_1, \dots, hy_k)$ is in $X^{(2k)}$. Take $(z_1, \dots, z_k) \in X^{(k)}$. By Lemma 1.1 again, there exists $h' \in K$ such that $\{h'z_1, \dots, h'z_k\} \cap \{z_1, \dots, z_k\} = \varnothing$. Consequently, $(z_1, \dots, z_k, h'z_1, \dots, h'z_k)$ is in $X^{(2k)}$. Since the G–action on X is highly transitive, there exists $g \in G$ such that

$$g(x_1, \dots, x_k, hy_1, \dots, hy_k) = (z_1, \dots, z_k, h'z_1, \dots, h'z_k).$$

Then $z_i = gx_i$ and $ghy_i = h'z_i = h'gx_i$, so

$$y_i = h^{-1}g^{-1}h'gx_i,$$

for every $i = 1, \dots, k$. Since K is normal in G, the element $h^{-1}g^{-1}h'g$ is in K and therefore $K \curvearrowright X$ is highly transitive.

(3) There exists a normal subgroup $K \lhd G$, contained in H, which has finite index in G. It cannot act trivially since $[G : K]$ is finite and the unique G–orbit is infinite. Thus the assertion follows from (2). □

For faithful and highly transitive actions, we have the following straightforward corollary.

Corollary 1.5 *Assume that $G \curvearrowright X$ is a faithful and highly transitive action. Then:*

(1) *the center $Z(G)$ is trivial;*

(2) *for any nontrivial normal subgroup $K \lhd G$, the action $K \curvearrowright X$ is faithful and highly transitive.*

Corollary 1.6 *If $G \curvearrowright X$ is a faithful and highly transitive action, then G is not solvable.*

Proof For any $n \in \mathbb{N}$, the n–th derived subgroup $G^{(n)}$ is a normal subgroup of G. If $G^{(k)}$ is nontrivial, then it acts highly transitively on X by Corollary 1.5(2), so that it is nonabelian, by Corollary 1.5(1). Hence $G^{(k+1)}$ is nontrivial. This proves (by induction) that G is not solvable. □

Notice that if G contains a finite index subgroup which admits a faithful and highly transitive action, this does *not* imply that G itself admits a faithful and highly transitive action. For example, $\mathrm{SL}_2(\mathbb{Z})$ has a free subgroup of index 12, but does not admit a faithful and highly transitive action since its center is nontrivial.

1.3 Baire spaces

Let X be a countable set. For any enumeration $X = \{x_0, x_1, x_2, \dots\}$, one can consider the distance on the group $\mathrm{Sym}(X)$ defined by

$$d(\sigma, \tau) = 2^{-\inf\{k \in \mathbb{N} \mid \sigma(x_k) \neq \tau(x_k) \text{ or } \sigma^{-1}(x_k) \neq \tau^{-1}(x_k)\}}.$$

Then, $\mathrm{Sym}(X)$ becomes a complete ultrametric space and a topological group. Note that a sequence (σ_n) in $\mathrm{Sym}(X)$ converges to a permutation σ if and only if, given any finite subset $F \subset X$, the permutations σ and σ_n, respectively σ^{-1} and σ_n^{-1}, coincide on F for n large enough. Hence the topology on $\mathrm{Sym}(X)$ is independent of the chosen enumeration. One can notice that a subgroup Γ of $\mathrm{Sym}(X)$ is dense if and only if the Γ–action on X is highly transitive.

As a complete metrizable space, $\mathrm{Sym}(X)$ is a *Baire space*, that is a topological space in which every countable intersection of dense open subsets is still dense. In such a space, a countable intersection of dense open subsets is called *generic subset*, or *co-meager subset*, while its complement (that is a countable union of closed sets with empty interior) is called *meager subset*. In particular, generic subsets are dense, thus nonempty.

The case of free products $G * H$ with two infinite factors (see Section 2) will be treated by genericity arguments in $\mathrm{Sym}(X)$. For the case of free products $G * H$ with a finite factor, we need to consider a clever Baire space that we introduce now. Let us consider two nontrivial finite or countable groups G, H and assume that X is endowed with some G–action such that it is isomorphic (in the category of G–sets) to $G \times \mathbb{N}$, where G acts by left multiplication on the first factor. The product $\mathrm{Sym}(X)^H$ admits the complete metric

$$d((\sigma_h)_{h \in H}, (\tau_h)_{h \in H}) = \max\{d(\sigma_h, \tau_h) \mid h \in H\},$$

where $\mathrm{Sym}(X)$ is endowed with the metric defined as above. One can again see that the topology on $\mathrm{Sym}(X)^H$ does not depend on the choice of an enumeration of X. Moreover, when H is finite, this topology coincides with the product topology. The set of H–actions on X identifies with the subset $\mathrm{Hom}(H, \mathrm{Sym}(X)) \subset \mathrm{Sym}(X)^H$. It is easy to check that this subset is closed in $\mathrm{Sym}(X)^H$, hence is a complete metrizable space.

Definition 1.7 Let X be a G–set. We call an action $\sigma\colon H \to \mathrm{Sym}(X)$ *admissible* if all orbits of $\langle G, \sigma(H)\rangle$ in X are infinite.

The set of admissible actions will be denoted by $\mathcal{A}(G, H, X)$.

Notice that $\mathcal{A}(G, H, X)$ is nonempty. Indeed, if we identify $\mathbb{N}$ to $G\backslash(G * H)$ (which is indeed countable), X is identified (as a G–set) to $G * H$. Then the H–action by left multiplication on $G * H$ corresponds to a H–action on X which is admissible.

Lemma 1.8 *The space $\mathcal{A}(G, H, X)$ is a complete metrizable space.*

In particular, the space $\mathcal{A}(G, H, X)$ is a Baire space.

Proof It suffices to check $\mathcal{A}(G, H, X)$ is closed in $\mathrm{Hom}(H, \mathrm{Sym}(X))$. To do so, let us consider a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathcal{A}(G, H, X)$ converging to $\sigma \in \mathrm{Hom}(H, \mathrm{Sym}(X))$ and prove that σ is an admissible action. If we assume that F is a finite orbit of the subgroup $\langle G, \sigma(H)\rangle$, then for n large enough, the components of σ_n (and their inverses) would coincide with the components of σ (and their inverses) on F and F would be a finite orbit of the subgroup $\langle G, \sigma_n(H)\rangle$, which is impossible since σ_n is an admissible action. □

2 Case with two infinite factors

The aim of this section is to prove the following result.

Theorem 2.1 *If G and H are countable groups, then the free product $G * H$ admits a faithful and highly transitive action.*

It will be a direct consequence of two propositions in the following setting. Let X be a countable set and let G, H be two subgroups of $\mathrm{Sym}(X)$. For any $\sigma \in \mathrm{Sym}(X)$, let us consider the action $\phi_\sigma\colon G * H \to \mathrm{Sym}(X)$ defined by

$$\phi_\sigma(w) = w^\sigma := g_1\sigma^{-1}h_1\sigma \cdots g_k\sigma^{-1}h_k\sigma,$$

where $w = g_1h_1 \cdots g_kh_k$ with $g_1, \dots, g_k \in G$ and $h_1, \dots, h_k \in H$.

Proposition 2.2 *Suppose that every orbit of G and H on X is infinite. Then*

$$\mathcal{H} := \{\sigma \in \mathrm{Sym}(X) \mid \phi_\sigma \text{ is highly transitive}\}$$

is generic in $\mathrm{Sym}(X)$.

Proposition 2.3 *Suppose that every nontrivial element of G and H has infinite support. Then the set*

$$\mathcal{F} = \{\sigma \in \mathrm{Sym}(X) \mid \phi_\sigma \text{ is faithful}\}$$

is generic in $\mathrm{Sym}(X)$.

Proof of Theorem 2.1 based on the propositions Let G, H be countable groups; let X be the countable set considered above. One can endow X with a G–action and a H–action which are both transitive and free. Then, G and H can be identified with their images in $\mathrm{Sym}(X)$. Moreover, by Propositions 2.2 and 2.3, we can take a permutation $\sigma \in \mathcal{H} \cap \mathcal{F}$ (in fact, $\mathcal{H} \cap \mathcal{F}$ is generic in $\mathrm{Sym}(X)$); the $G * H$–action ϕ_σ is then highly transitive and faithful. □

Proof of Proposition 2.2 Let

$$U_{k,x,y} = \{\sigma \in \mathrm{Sym}(X) \mid \exists w \in G * H \text{ such that } w^\sigma(x_i) = y_i, \forall i = 1, \dots, k\},$$

for every $k \in \mathbb{N}^*$ and $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_k) \in X^{(k)}$. Since we have $\mathcal{H} = \bigcap_{k \in \mathbb{N}^*} \bigcap_{x,y \in X^{(k)}} U_{k,x,y}$, it is enough to prove that the set $U_{k,x,y}$ is open and dense.

Let $\sigma \in U_{k,x,y}$ and let w such that $w^\sigma(x_i) = y_i$ for every $i = 1, \dots, k$. The map $\sigma \mapsto w^\sigma$ is continuous and the inverse image of the open set

$$\{\alpha \in \mathrm{Sym}(X) \mid \alpha(x_i) = y_i, \forall i = 1, \dots, k\}$$

contains σ and is contained in $U_{k,x,y}$. Thus the set $U_{k,x,y}$ is a neighborhood of σ and this shows that $U_{k,x,y}$ is open.

Let us show that $U_{k,x,y}$ is dense. Let $F \subset X$ be a finite subset of X and $\tau \in \mathrm{Sym}(X)$. Given a subset $Y \subseteq X$, we denote by $\tau^{\pm 1}(Y)$ the union $\tau(Y) \cup \tau^{-1}(Y)$. Let $I = \{x_1, \dots, x_k\}$ and $J = \{y_1, \dots, y_k\}$. We start by a variation of Remark 1.2.

Claim 2.4 *For any finite subsets A, B of X, there exists $g \in G$ such that*

$$(gA \cup \tau^{\pm 1}(gA)) \cap (B \cup \tau^{\pm 1}(B)) = \varnothing.$$

Similarly, there exists $h \in H$ such that $(hA \cup \tau^{\pm 1}(hA)) \cap (B \cup \tau^{\pm 1}(B)) = \varnothing$.

Proof Indeed, set $B' = B \cup \tau^{\pm 1}(B)$. By Remark 1.2, there exists $g \in G$ (respectively $h \in H$) such that $gA \cap B' = \varnothing$ and $gA \cap \tau^{\pm 1}(B') = \varnothing$. This implies $gA \cap B' = \varnothing$ and $\tau^{\pm 1}(gA) \cap B' = \varnothing$, hence $(gA \cup \tau^{\pm 1}(gA)) \cap (B \cup \tau^{\pm 1}(B)) = \varnothing$. The claim is proved. □

Hence, there exists $g_1 \in G$ such that $(F \cup \tau^{\pm 1} F) \cap (g_1 I \cup \tau^{\pm 1} g_1 I) = \varnothing$. Then, taking $A = J$ and $B = F \cup g_1 I$, Claim 2.4 shows that there exists $g_2 \in G$ such that the sets $F \cup \tau^{\pm 1}(F)$, $g_1 I \cup \tau^{\pm 1}(g_1 I)$ and $g_2 J \cup \tau^{\pm 1}(g_2 J)$ are pairwise disjoint. We then choose a finite subset $M = \{z_1, \dots, z_k\} \subset X$ such that the set $M \cup \tau^{\pm 1} M$ is disjoint from the finite sets considered so far. Again by Claim 2.4 (with $A = M$ and $B = F \cup g_1 I \cup g_2 J \cup M$), there exists $h \in H$ such that the sets

$$F \cup \tau^{\pm 1}(F), \quad g_1 I \cup \tau^{\pm 1}(g_1 I), \quad g_2 J \cup \tau^{\pm 1}(g_2 J),$$
$$M \cup \tau^{\pm 1} M, \quad h(M \cup \tau^{\pm 1} M),$$

are pairwise disjoint.

We then define a permutation σ of X by

$$\sigma(g_1 x_j) = z_j, \qquad \sigma(\tau^{-1}(z_j)) = \tau(g_1 x_j),$$
$$\sigma(g_2(y_j)) = h(z_j), \qquad \sigma(\tau^{-1}(h(z_j))) = \tau(g_2(y_j)),$$

for every $j = 1, \dots, k$, and $\sigma(x) := \tau(x)$ for every other points of X. In particular, $\sigma|_F = \tau|_F$ and $(g_2^{-1} h g_1)^\sigma(x_i) = y_i$ for all $i = 1, \dots, k$. This shows that $\sigma \in U_{k,x,y}$ and the set $U_{k,x,y}$ is dense. □

Proof of Proposition 2.3 This follows from the genericity of $\mathcal{O}_1$ by the first author in [8]; here we give a self-contained proof in the case of free products.

For every $w \in G * H$, let $U_w = \{\sigma \in \mathrm{Sym}(X) \mid w^\sigma \neq \mathrm{id}_X\}$. We have

$$\mathcal{F} = \bigcap_{w \in G * H \setminus \{1\}} U_w \,.$$

So it is enough to show that for every $w \in G * H \setminus \{1\}$, the set U_w is open and dense.

It is clear that U_w is open. Let us show that U_w is dense. If w is a nontrivial element of G or H, then $U_w = \mathrm{Sym}(X)$ since G and H act faithfully on X. If $w \notin G \cup H$ and $w \neq gh$ (with $g \in G \setminus \{1\}$ and $h \in H \setminus \{1\}$), then we can write

$$w = g_k h_k \cdots g_1 h_1,$$

with $k \geq 2$, $g_k \in G$, $g_{k-1}, \dots, g_1 \in G \setminus \{1\}$, $h_k, \dots, h_2 \in H \setminus \{1\}$ and $h_1 \in H$.

Let $\sigma' \in \mathrm{Sym}(X)$ and let F be a finite subset of X. Since the elements $g_1, \dots, g_{k-1}$, $h_2, \dots, h_k$ have infinite supports, there exist $x_0, \dots, x_{2k-1}, y_1, \dots, y_{2k} \in X$ so that:

- none of these points are in $F \cup \sigma'^{\pm 1}(F)$;
- these points are pairwise disjoint, except possibly $x_0 = x_1$ and $y_{2k} = y_{2k-1}$;
- for every $j = 0, \dots, k-1$, we have $h_{j+1}(x_{2j}) = x_{2j+1}$;
- for every $j = 1, \dots, k$, we have $g_j(y_{2j-1}) = y_{2j}$.

If $x_0 = x_1$, put $y_0 = y_1$; if not, put $y_0 = x_0$. Then put $\sigma(y_i) = x_i$ for every $i = 0, \dots, 2k-1$ and $\sigma(x) = \sigma'(x)$ for all $x \in F$. This defines a bijection between $F \cup \{y_0, \dots, y_{2k-1}\}$ and $\sigma'(F) \cup \{x_0, \dots, x_{2k-1}\}$. By extending the definition of σ to the other points, we thus obtain a permutation $\sigma \in \mathrm{Sym}(X)$ such that $\sigma|_F = \sigma'|_F$ and $w^\sigma(y_0) = y_{2k} \neq y_0$. In case where $w = gh$ with $g \in G \setminus \{1\}$ and $h \in H \setminus \{1\}$, there exist pairwise disjoint points y_0, x_0, x_1, y_1, y_2 outside of $F \cup \sigma'^{\pm 1}(F)$ such that $hx_0 = x_1$ and $gy_1 = y_2$. Then we define a permutation $\sigma \in \mathrm{Sym}(X)$ such that $\sigma(y_0) = x_0$, $\sigma(y_1) = x_1$ and $\sigma|_F = \sigma'|_F$ so that $w^\sigma(y_0) = y_2 \neq y_0$. This proves that $\sigma \in U_w$ and therefore U_w is dense in $\mathrm{Sym}(X)$. □

3 Case with one finite factor

3.1 Definitions and notation

Let G, H be two nontrivial finite or countable groups[3]. In this section, the set X will be identified with the disjoint union of a countable collection of copies of G:

$$X = \bigsqcup_{j \in \mathbb{N}} G_j, \quad \text{where } G_j = G \text{ for every } j.$$

Moreover, G will always act on X by left multiplications on each copy G_j (note that X is isomorphic to $G \times \mathbb{N}$, as G–sets).

First of all, we give some definitions and fix the notation. Any action $\sigma\colon H \to \mathrm{Sym}(X)$ induces an action of $G * H$ on X. We denote by X_σ the Schreier graph of this action with respect to the generating set $G \cup H$ and by d_σ the distance on X_σ. Given $u \in G * H$, we denote by u^σ the image of u in the subgroup $\langle G, \sigma(H) \rangle$ of $\mathrm{Sym}(X)$.

[3]The reader can think of H as a finite group from now on: this will be an essential assumption in Theorem 3.3.

Definition 3.1 Let $w \in G * H$ and $x \in X$. We call *σ–trajectory of w from x* the sequence

$$(x, s_1(w)^\sigma(x), \dots, s_{|w|-1}(w)^\sigma(x), w^\sigma(x)),$$

where $s_j(w)$ is the suffix of w of length j (that is, if $w = w_{|w|} w_{|w|-1} \cdots w_2 w_1$ is written as a normal form, then $s_j(w) = w_j w_{j-1} \cdots w_2 w_1$).

Consider now the graph where the vertices are the right cosets Gw and Hw, with $w \in G * H$, and the edges are the elements of $G * H$, such that the edge w links two vertices Gw and Hw. Recall that by Serre [11] this is a tree, called *Bass–Serre tree* of $G * H$, and denote by T its geometric realization (which is a real tree)[4]. If $G * H$ is endowed with the right invariant word metric with respect to the generating set $G \cup H$, the map of $G * H$ in T which sends an element w to the middle point between the vertices Gw and Hw is an isometric injection. From now on, we will identify $G * H$ with the image (see Figure 1).

Definition 3.2 Let Z be a real tree and p, $q \in Z$. We call *shadow of q at p* the set of the points $z \in Z$ such that the geodesic from p to z passes the point q (see Figure 2). We will denote it by $\mathrm{Shadow}(q)_p$.

Note that $\mathrm{Shadow}(q)_p$ is a subtree of Z and that q is the closest point to p in this subtree. In addition, it is easy to see the following properties:

- if r is in $\mathrm{Shadow}(q)_p$, then $\mathrm{Shadow}(r)_p$ is contained in $\mathrm{Shadow}(q)_p$;
- two shadows $\mathrm{Shadow}(q)_p$ and $\mathrm{Shadow}(q')_p$ are either disjoint or nested.

Let $T_+ := \mathrm{Shadow}(H)_1$ be the shadow (of the image) of the vertex H at 1 in T, and let

$$Y = T_+ \cap (G * H).$$

Then $Y = \bigsqcup_w Gw$ where w runs in the set of nontrivial elements of $G * H$ such that the normal form starts and terminates with an element of H. Let

$$\overline{Y} = Y \cup \{1\}.$$

Then $\overline{Y} = \bigsqcup_w Hw$ where w runs in the set of elements of $G * H$ such that the normal form of w is either 1, or starts with an element of G and terminates with an element of H. Therefore, Y is invariant under G–action (by left multiplication) and $\overline{Y}$ is invariant under H–action[5].

[4] We recall that the Bass–Serre tree is locally finite if and only if G and H are finite.

[5] Notice that these actions do not preserve the tree structure. In fact, right multiplications are tree automorphisms, but left multiplications are not.

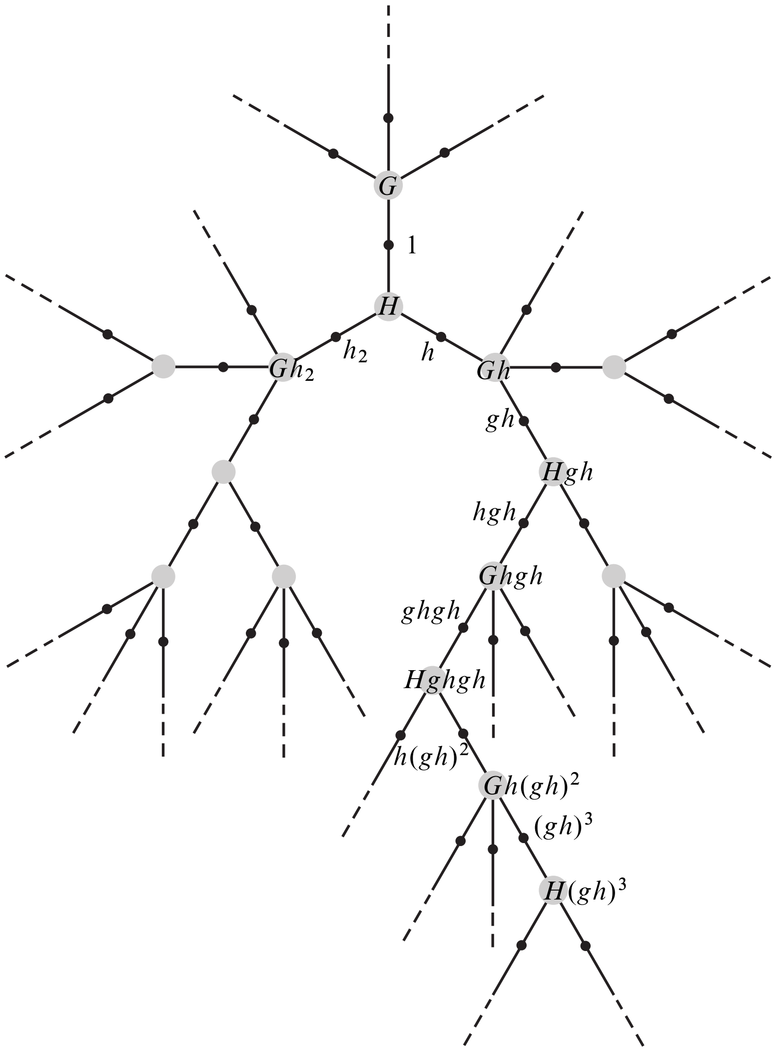

Figure 1: The image of $G * H$ in the Bass–Serre tree

3.2 Main result of this section

Let us consider the Baire space $\mathcal{A} = \mathcal{A}(G, H, X)$ of admissible actions of H on X (see Section 1.3), and

- $\mathcal{H} = \{\sigma\colon H \to \mathrm{Sym}(X) \mid \langle G, \sigma(H)\rangle \curvearrowright X \text{ is highly transitive}\}$,
- $\mathcal{F} = \{\sigma\colon H \to \mathrm{Sym}(X) \mid G * H \to \langle G, \sigma(H)\rangle \text{ is an isomorphism}\}$.

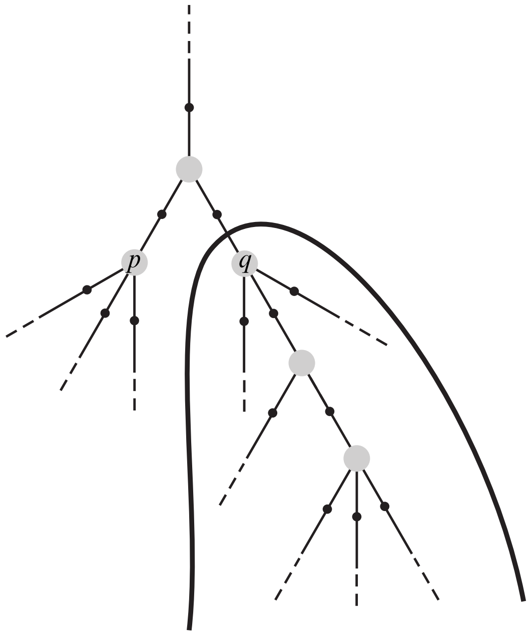

Figure 2: The shadow of q at p, $\mathrm{Shadow}(q)_p$

In other words, an action $\sigma\colon H \to \mathrm{Sym}(X)$ is in the set $\mathcal{H}$ if and only if the induced $G*H$–action is highly transitive and it is in $\mathcal{F}$ if and only if the induced $G*H$–action is faithful.

Theorem 3.3 *If H is finite and $|G| \geq 3$, then $\mathcal{A} \cap \mathcal{H} \cap \mathcal{F}$ is generic in $\mathcal{A}$.*

Note that G can be either finite or countable in this theorem. We now turn to the proof.

For $w \in G*H$, $k \in \mathbb{N}^*$ and $\overline{x}, \overline{y} \in X^{(k)}$, where $\overline{x} = (x_1, \dots, x_k)$ and $\overline{y} = (y_1, \dots, y_k)$, we put

$$\begin{aligned} \mathcal{U}_{k,\overline{x},\overline{y}} &= \{\sigma \in \mathcal{A} \mid \exists \tau \in \langle G, \sigma(H)\rangle \text{ such that } \tau(x_j) = y_j, \forall j = 1, \dots, k\}, \\ \mathcal{U}'_w &= \{\sigma \in \mathcal{A} \mid w^\sigma \neq 1 \text{ in } \mathrm{Sym}(X)\}. \end{aligned}$$

Then we have

$$\begin{aligned} \mathcal{A} \cap \mathcal{H} &= \bigcap_{k \in \mathbb{N}^*} \bigcap_{\overline{x},\overline{y} \in X^{(k)}} \mathcal{U}_{k,\overline{x},\overline{y}}, \\ \mathcal{A} \cap \mathcal{H} \cap \mathcal{F} &= \Big(\bigcap_{k \in \mathbb{N}^*} \bigcap_{\overline{x},\overline{y} \in X^{(k)}} \mathcal{U}_{k,\overline{x},\overline{y}} \Big) \cap \Big(\bigcap_{w \in (G*H)\setminus\{1\}} \mathcal{U}'_w \Big). \end{aligned}$$

So it is enough to prove that the sets $\mathcal{U}_{k,\bar{x},\bar{y}}$ and $\mathcal{U}'_w$ are open and dense in $\mathcal{A}$.

Since $\mathcal{U}_{k,\bar{x},\bar{y}} = \cup_{w\in G*H}\mathcal{O}_{k,\bar{x},\bar{y},w}$ where

$$\mathcal{O}_{k,\bar{x},\bar{y},w} = \{\sigma \in \mathcal{A} \mid w^{\sigma}(x_j) = y_j, \forall j = 1,\dots,k\},$$

which is open, the set $\mathcal{U}_{k,\bar{x},\bar{y}}$ is open. Furthermore the set $\mathcal{U}'_w$ is clearly open.

We shall now prove that $\mathcal{U}_{k,\bar{x},\bar{y}}$ and $\mathcal{U}'_w$ are dense in $\mathcal{A}$. We fix from now on $k \in \mathbb{N}^*$, $\bar{x}, \bar{y} \in X^{(k)}$ and F a finite subset of X. Let $\sigma \in \mathcal{A}$. To see that the set $\mathcal{U}_{k,\bar{x},\bar{y}}$ is dense, we need to show that there exists $\alpha \in \mathcal{U}_{k,\bar{x},\bar{y}}$ such that $\alpha|_F = \sigma|_F$. By taking a bigger finite set containing F if necessary, we can suppose that $x_1,\dots,x_k,y_1,\dots,y_k$ are contained in F. Let

$$K = \bigcup_{z\in F} \sigma(H)\cdot z.$$

Since F and H are finite, K is also finite. Additionally let

$$\bar{K} = \bigcup_{z\in K} G\cdot z.$$

Notice that $\bar{K}$ is infinite if G is infinite, but it has finitely many G–orbits. Note that $\bar{K}\setminus K$ is not empty since otherwise K would be formed with finite $\langle G,\sigma(H)\rangle$–orbits which contradicts the assumption that σ is in $\mathcal{A}$.

Recall that T_+ is the shadow of H at 1 in T and $Y = T_+ \cap (G*H)$. Since $X\setminus\bar{K}$ is formed by infinitely many G–orbits (ie infinitely many copies G_j), there exists a G–equivariant bijection between $Y\times(\bar{K}\setminus K)$, where G acts trivially on the second factor, and $X\setminus\bar{K}$. We can then extend this to a bijection ϕ between $\bar{Y}\times(\bar{K}\setminus K)$ and $X\setminus K$ by sending $(1,z)$ on z for every $z\in\bar{K}\setminus K$. Henceforth, we denote by Y_z (resp. $\bar{Y}_z$), the image of $Y\times\{z\}$ (resp. $\bar{Y}\times\{z\}$) in $X\setminus K$.

Since K is $\sigma(H)$–invariant, we can define an action $\beta\colon H\to \mathrm{Sym}(X)$ as follows (see Figure 3):

- $\beta|_K = \sigma|_K$;
- for every $z\in\bar{K}\setminus K$, the restriction of β to $\bar{Y}_z$ corresponds to the action of H on $\bar{Y}\times\{z\}$ by left multiplication on the first factor.

Claim 3.4 *The action β is in $\mathcal{A}$.*

Proof The $\langle G,\beta(H)\rangle$–orbits are infinite since for the points in $\bar{Y}_z$, it follows from the construction, and for the points in K, it is because the $\langle G,\sigma(H)\rangle$–orbits are infinite and thus $\beta\in\mathcal{A}$. □

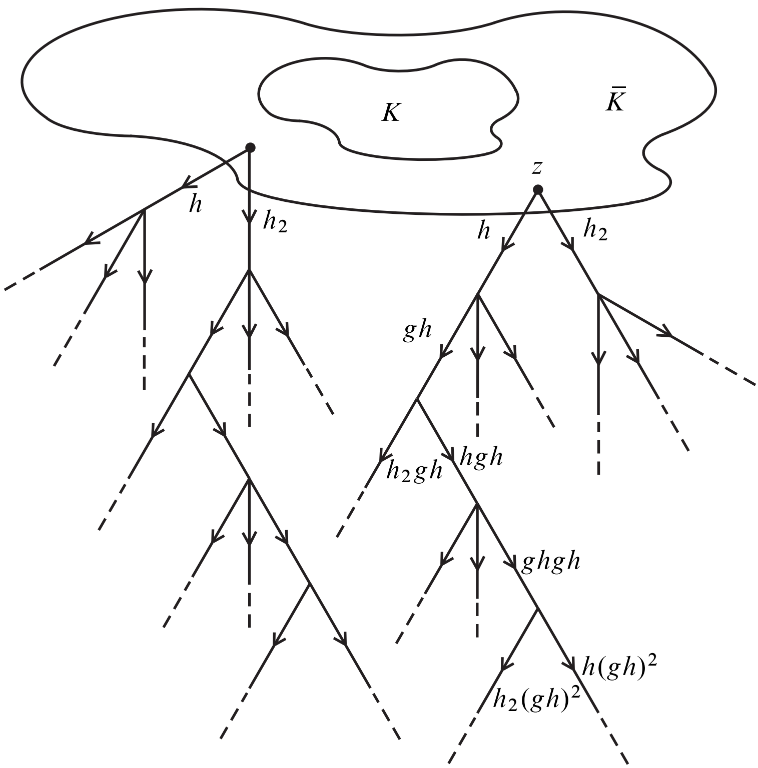

Figure 3: Schreier graph of the action associated to β

Recall that X_β and d_β denote the Schreier graph of the induced $G * H$–action on X with respect to the generating set $G \cup H$, and its distance respectively. Note that

- for every $z \neq z'$ in $\bar{K} \setminus K$, there is no edge of X_β that links an element of Y_z and an element of $Y_{z'}$;
- the edges of X_β that link $\bar{K}$ to a subset Y_z are labeled by elements of H, and they link $z = \phi(1, z)$ to an element of the form $\phi(h, z)$ with $h \in H \setminus \{1\}$;
- the restriction of the distance d_β to $\bar{Y}_z$ corresponds via ϕ^{-1} to the right invariant word metric on $\bar{Y}$.

Since $\bar{Y}$ embeds isometrically in the real tree T, each $\bar{Y}_z$ can be embedded isometrically into a real tree T_z, and we can moreover require that no subtree of T_z contains the image of $\bar{Y}_z$. This real tree T_z is essentially unique (see for example Bestvina [1, Lemma 2.13]). Notice that G and H do not act on the union of X and the trees T_z.

Claim 3.5 *Let $w \in G * H$ and $x \in K$. Suppose that the β–trajectory of w from x is not contained in K and let $z = s_j(w)^\beta(x)$ be the first point of this trajectory that is outside of K. Then z is contained in $\bar{K} \setminus K$ and the end of this trajectory is a geodesic sequence in $\bar{Y}_z$. Therefore, we have*

$$d_\beta(z, s_n(w)^\beta(x)) < d_\beta(z, s_m(w)^\beta(x)),$$

for every $j \le n < m \le |w|$.

Proof Let us write $w = a_{|w|} \cdots a_1$ as the normal form. By hypothesis, we have

$$y := (a_{j-1} \cdots a_1)^\beta(x) \in K, \quad z = a_j^\beta(y) \notin K.$$

Since K is $\beta(H)$–invariant, a_j is in G, a_{j+1} is in H and $a_{j+2}, \dots, a_{|w|}$ are alternatively in G and H. The end of the β–trajectory of the word $a_{|w|} \cdots a_{j+1}$ from z satisfies

$$(a_\ell \cdots a_{j+1})^\beta(z) = \phi(a_\ell \cdots a_{j+1}, z)$$

for every $\ell = j+1, \dots, |w|$. Thus this trajectory is a geodesic sequence in $\bar{Y}_z$ and this proves the claim. □

Claim 3.6 *There exist v_1, $v_2 \in G * H$ such that:*

1. *their normal forms start with an element of G;*
2. *the sets K, $v_1^\beta(K)$ and $v_2^\beta(K)$ are pairwise disjoint.*

Proof Since β is in $\mathcal{A}$, then by Lemma 1.1, there exists $u_1 \in G * H$ such that $u_1^\beta(K) \cap K = \varnothing$. Let $g \in G \setminus \{1\}$ and $h \in H \setminus \{1\}$. If the normal form of u_1 starts with an element of G, we put $v_1 := u_1$; otherwise, we put $v_1 := g u_1$. In both cases, the normal form of v_1 starts with an element of G. In addition, for every $x \in K$, the β–trajectory of v_1 from x passes the point $u_1^\beta(x)$, which is not in K. Thus by Claim 3.5, we have that $v_1^\beta(K) \cap K = \varnothing$. Let

$$d := \mathrm{diam}_\beta(K \cup v_1^\beta(K)), \quad v_2 := (gh)^{2d} v_1.$$

The normal form of v_2 starts with an element of G. Furthermore, for every $x \in K$, the β–trajectory of v_2 from x passes the point $v_1^\beta(x)$, which is not in K. So by Claim 3.5, we have

$$d_\beta(v_2(K), K) \ge 2d,$$

thus the sets K, $v_1^\beta(K)$ and $v_2^\beta(K)$ are pairwise disjoint. This concludes the claim. □

Given a point $x \in X \setminus K$, there exists a unique point $z = z_x \in \bar{K} \setminus K$ such that x is in $\bar{Y}_z$. For the rest of the proof, we denote by $\mathrm{Shadow}(x) := \mathrm{Shadow}(x)_z$ the shadow of x at z in T_z.

Claim 3.7 *Let M be a finite subset of $X \setminus K$ such that every element $y \in M$ can be written as $y = v_y^\beta(x_y)$, where $x_y \in K$ and the normal form of $v_y \in G * H$ starts with an element of $G \setminus \{1\}$. Then there exists $w \in G * H$ such that:*

- *the normal form of w starts with an element of G and terminates with an element of H;*
- *$w^\beta(M) \cap K = \varnothing$;*
- *$\mathrm{Shadow}(p) \cap \mathrm{Shadow}(p') = \varnothing$, for every $p \neq p'$ in $w^\beta(M)$.*

Proof Let $y \neq y' \in M$. If $\mathrm{Shadow}(y) \cap \mathrm{Shadow}(y') = \varnothing$, then for all $g \in G \setminus \{1\}$ and $h \in H \setminus \{1\}$, the intersection $\mathrm{Shadow}((gh)^\beta(y)) \cap \mathrm{Shadow}((gh)^\beta(y'))$ is also empty since one has

$$\mathrm{Shadow}((gh)^\beta(y)) \subseteq \mathrm{Shadow}(y), \quad \mathrm{Shadow}((gh)^\beta(y')) \subseteq \mathrm{Shadow}(y').$$

Now let us suppose that $\mathrm{Shadow}(y) \cap \mathrm{Shadow}(y') \neq \varnothing$. Without loss of generality, we suppose that $\mathrm{Shadow}(y')$ is contained in $\mathrm{Shadow}(y)$. Notice that $d_\beta(y, y') \geq 2$ since $y, y' \in M$ and $y \neq y'$. Let $h' \in H$ and $g' \in G$ be the labels of the first two edges of the geodesic from y to y' in X_β. There exists $g \in G$ different from 1 and g', since G has at least 3 elements. Thus $\mathrm{Shadow}((gh')^\beta(y))$ is disjoint to $\mathrm{Shadow}(y')$ and $\mathrm{Shadow}((gh')^\beta(y'))$.

Given a finite subset $S \subset X \setminus K$, we denote by $n_s(S)$ the number of pairs $(q, q') \in S \times S$ such that $q \neq q'$ and $\mathrm{Shadow}(q) \cap \mathrm{Shadow}(q') \neq \varnothing$. If $n_s(M) > 0$, we have proven the existence of elements $g \in G \setminus \{1\}$ and $h \in H \setminus \{1\}$ such that

$$n_s((gh)^\beta(M)) < n_s(M).$$

In addition, Claim 3.5 guaranties that $(gh)^\beta(M)$ does not intersect with K. By repeating this operation at most $|M|^2$ times, we obtain an element w as we wished. □

End of the proof of Theorem 3.3 Take elements $g \neq g' \in G \setminus \{1\}$ and $h \in H \setminus \{1\}$ and let

$$M := v_1^\beta(K) \sqcup v_2^\beta(K),$$

where v_1, v_2 are the elements as in Claim 3.6. Then there is w as in Claim 3.7. We thus have four elements $w_j = ghwv_j$ and $w'_j = g'hwv_j$ in $G*H$ (for $j=1,2$) such that:

- the normal form of w_j and w'_j (for $j=1,2$) starts with an element of G;
- the shadows of the elements of $w_1^\beta(K) \sqcup w_2^\beta(K) \sqcup (w'_1)^\beta(K) \sqcup (w'_2)^\beta(K)$ and the set $\bar{K}$ are pairwise disjoint.

In addition, the β–trajectories of w_1 and w_2 from the points in K do not intersect with the shadows of the points of $w_1^\beta(K) \sqcup w_2^\beta(K)$ before their last points, since as soon as the β–trajectories leave K, they are geodesic lines by Claim 3.5. This implies that, for any action $\alpha \in \mathcal{A}$ which differs from β only inside the shadows of the points of $w_1^\beta(K) \sqcup w_2^\beta(K)$, one has $w_j^\alpha(x) = w_j^\beta(x)$ for all $j=1,2$ and $x \in K$ (here we use the fact that the normal form of w_j ($j=1,2$) starts with an element of G).

Let us produce such an action α by modifying β as follows (see Figure 4). For all $i=1,\dots,k$, we consider the permutation ξ_i which exchanges the points $(hw_1)^\beta(x_i)$ and $w_2^\beta(y_i)$, and we define $\xi = \xi_1 \cdots \xi_k$ (note that the ξ_i's have disjoint supports). Then, we set $\alpha_t = \xi^{-1}\beta_t\xi$ (that is, $t^\alpha = \xi^{-1}t^\beta\xi$) for all $t \in H$. It is clear that α differs from β only inside the shadows of the points of $w_1^\beta(K) \sqcup w_2^\beta(K)$. Let us now prove that α is admissible.

- If the connected component of x in X_β contains a point x_i (with $1 \le i \le k$), then its connected component in X_α contains either $w_1^\beta(x_i)$, or $(hw_1)^\beta(x_i)$. In both cases, the latter one is infinite, since it contains (the intersection of $G*H$ with) a shadow: the shadow of $(w'_1)^\beta(x_i)$ in the first case and the shadow of $(hw_1)^\beta(x_i)$ in the second one.
- Similarly, if the connected component of x in X_β contains a point y_i (with $1 \le i \le k$), then its connected component in X_α is infinite.
- Finally, if the connected component of x in X_β does not contain any point in $\{x_1, y_1, \dots, x_k, y_k\}$, then its connected component in X_α coincides with the one in X_β and is thus infinite since β is admissible (see Claim 3.4).

Hence, all orbits of the $\langle G, \alpha(H)\rangle$–action are infinite, which means that α is in $\mathcal{A}$.

Moreover, one has $(w_2^{-1}hw_1)^\alpha(\bar{x}) = \bar{y}$, so that α is in $\mathcal{U}_{k,\bar{x},\bar{y}}$, and σ, β and α coincide on F. We have thus proven that $\mathcal{U}_{k,\bar{x},\bar{y}}$ is dense in $\mathcal{A}$.

Finally, if σ is in $\mathcal{A}$ and F is a finite subset of X as before, then consider the action $\beta\colon H \to \mathrm{Sym}(X)$ constructed as above. It is clear that the associated $G*H$–action on X is faithful and that $\sigma|_F = \beta|_F$. This proves that $\mathcal{F}$ is dense in $\mathcal{A}$. Therefore, all subsets $\mathcal{U}'_w$, for $w \in (G*H) \setminus \{1\}$, are dense in $\mathcal{A}$. This achieves the proof of Theorem 3.3. □

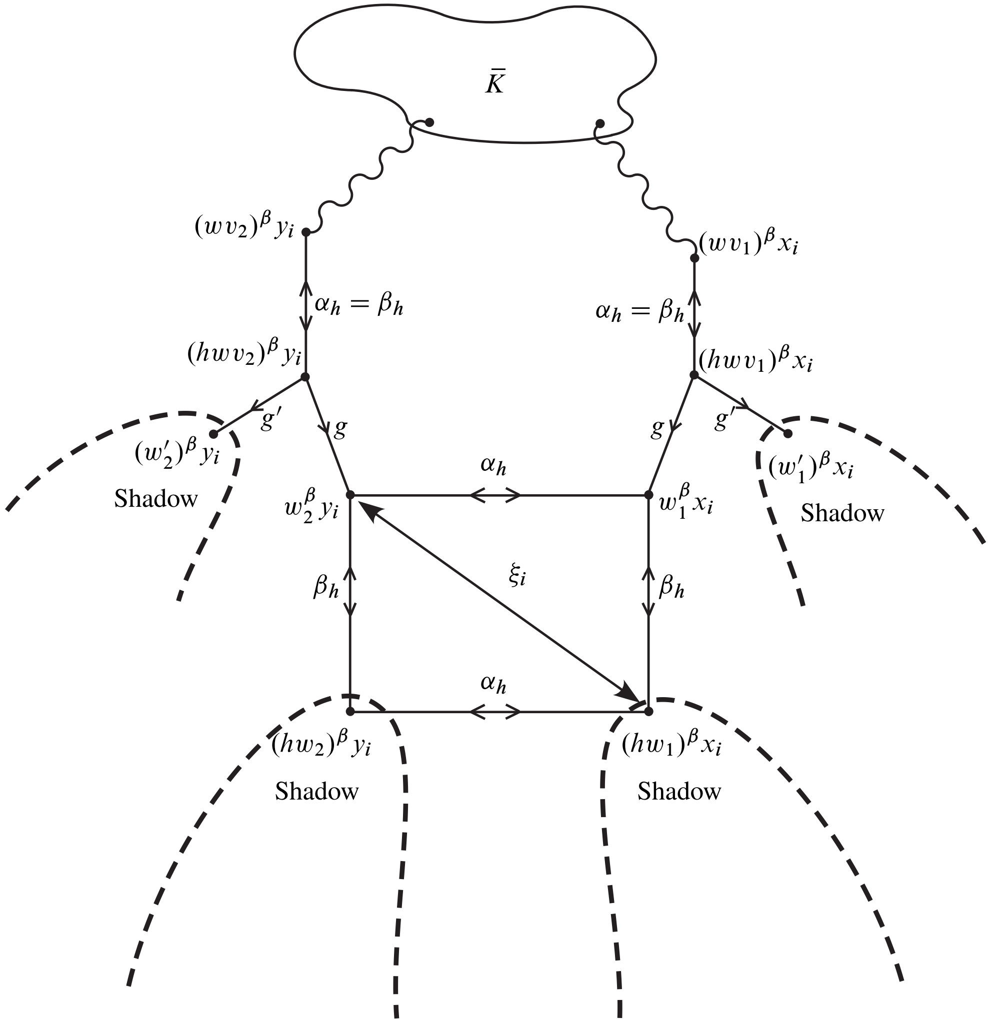

Figure 4: Schreier graph of the action associated to α with $H = \mathbb{Z}/2\mathbb{Z}$

Proof of Theorem 1 In case $G \simeq \mathbb{Z}/2\mathbb{Z} \simeq H$, the group $G * H$ is isomorphic to the infinite dihedral group, which has trivial center but it contains a cyclic subgroup of index 2. Hence $G * H$ does not admit any faithful and highly transitive action by Corollary 1.5.

If at least one of the factors G, H is not isomorphic to the cyclic group $\mathbb{Z}/2\mathbb{Z}$, then by Theorems 2.1 and 3.3, we have that it admits a faithful and highly transitive action. □

References

[1] **M Bestvina**, $\mathbb{R}$*–trees in topology, geometry, and group theory*, from: "Handbook of geometric topology", (R J Daverman, R B Sher, editors), North-Holland, Amsterdam (2002) 55–91 MR1886668

[2] **J D Dixon**, *Most finitely generated permutation groups are free*, Bull. London Math. Soc. 22 (1990) 222–226 MR1041134

[3] **S Garion**, **Y Glasner**, *Highly transitive actions of* $\mathrm{Out}(\mathbb{F}_n)$, to appear in Groups, Geometry and Dynamics (2011) arXiv:1008.0563

[4] **A M W Glass**, **S H McCleary**, *Highly transitive representations of free groups and free products*, Bull. Austral. Math. Soc. 43 (1991) 19–36 MR1086715

[5] **S V Gunhouse**, *Highly transitive representations of free products on the natural numbers*, Arch. Math. (Basel) 58 (1992) 435–443 MR1156575

[6] **K K Hickin**, *Highly transitive Jordan representations of free products*, J. London Math. Soc. (2) 46 (1992) 81–91 MR1180884

[7] **D Kitroser**, *Highly-transitive actions of surface groups*, Proc. Amer. Math. Soc. 140 (2012) 3365–3375 MR2929006

[8] **S Moon**, *Permanence properties of amenable, transitive and faithful actions*, Bull. Belg. Math. Soc. Simon Stevin 18 (2011) 287–296 MR2848804

[9] **B H Neumann**, *Groups covered by permutable subsets*, J. London Math. Soc. 29 (1954) 236–248 MR0062122

[10] **P M Neumann**, *The structure of finitary permutation groups*, Arch. Math. (Basel) 27 (1976) 3–17 MR0401928

[11] **J-P Serre**, *Arbres, amalgames,* SL_2, Astérisque 46, Société Mathématique de France, Paris (1977) MR0476875

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