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Proof of a stronger version of the AJ Conjecture for torus knots

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For a knot K in S^3, the sl_2–colored Jones function $J_K(n)$ is a sequence of Laurent polynomials in the variable t that is known to satisfy non-trivial linear recurrence relations. The operator corresponding to the minimal linear recurrence relation is called the recurrence polynomial of K. The AJ Conjecture (see Garoufalidis [4]) states that when reducing $t=-1$, the recurrence polynomial is essentially equal to the A–polynomial of K. In this paper we consider a stronger version of the AJ Conjecture, proposed by Sikora [14], and confirm it for all torus knots.

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1 Introduction

1.1 The AJ Conjecture

For a knot K in S^3, let $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$ be the colored Jones polynomial of K colored by the n–dimensional simple sl_2–representation (Jones [8], and Reshetikhin and Turaev [13]), normalized so that for the unknot U,

$$J_U(n) = [n] := \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$

The color n can be assumed to take negative integer values by setting $J_K(-n) = -J_K(n)$. In particular, $J_K(0) = 0$. It is known that $J_K(1) = 1$, and $J_K(2)$ is the ordinary Jones polynomial.

Define two operators L, M acting on the set of discrete functions $f\colon \mathbb{Z} \to \mathcal{R} := \mathbb{C}[t^{\pm 1}]$ by

$$(Lf)(n) := f(n+1), \quad (Mf)(n) := t^{2n} f(n).$$

It is easy to see that $LM = t^2 ML$. Besides, the inverse operators L^{-1}, M^{-1} are well-defined. One can consider L, M as elements of the quantum torus

$$\mathcal{T} := \mathcal{R}\langle L^{\pm 1}, M^{\pm 1} \rangle / (LM - t^2 ML),$$

which is not commutative, but almost commutative.

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Let

$$\mathcal{A}_K = \{P \in \mathcal{T} \mid PJ_K = 0\},$$

which is a left-ideal of $\mathcal{T}$, called the *recurrence ideal* of K. It was proved by Garoufalidis and Lê in [5] that for every knot K, the recurrence ideal $\mathcal{A}_K$ is non-zero. An element in $\mathcal{A}_K$ is called a recurrence relation for the colored Jones polynomials of K.

The ring $\mathcal{T}$ is not a principal left-ideal domain, ie, not every left-ideal of $\mathcal{T}$ is generated by one element. By adding all inverses of polynomials in t, M to $\mathcal{T}$, one gets a principal left-ideal domain $\widetilde{\mathcal{T}}$; cf [4]. The ring $\widetilde{\mathcal{T}}$ can be formally defined as follows. Let $\mathcal{R}(M)$ be the fractional field of the polynomial ring $\mathcal{R}[M]$. Let $\widetilde{\mathcal{T}}$ be the set of all Laurent polynomials in the variable L with coefficients in $\mathcal{R}(M)$,

$$\widetilde{\mathcal{T}} = \left\{ \sum_{j \in \mathbb{Z}} f_j(M) L^j \;\middle|\; f_j(M) \in \mathcal{R}(M),\ f_j = 0 \text{ almost everywhere} \right\},$$

and define the product in $\widetilde{\mathcal{T}}$ by $f(M)L^k \cdot g(M)L^l = f(M)g(t^{2k}M)L^{k+l}$.

The left-ideal extension $\widetilde{\mathcal{A}}_K := \widetilde{\mathcal{T}}\mathcal{A}_K$ of $\mathcal{A}_K$ in $\widetilde{\mathcal{T}}$ is then generated by a polynomial

$$\alpha_K(t; M, L) = \sum_{j=0}^{d} \alpha_{K,j}(t, M) L^j,$$

where d is assumed to be minimal and all the coefficients $\alpha_{K,j}(t, M) \in \mathbb{Z}[t^{\pm 1}, M]$ are assumed to be co-prime. That α_K can be chosen to have integer coefficients follows from the fact that $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$. The polynomial α_K is defined up to a polynomial in $\mathbb{Z}[t^{\pm 1}, M]$. Moreover, one can choose $\alpha_K \in \mathcal{A}_K$, ie, it is a recurrence relation for the colored Jones polynomials. We will call α_K the *recurrence polynomial* of K.

Let ε be the map reducing $t = -1$. Garoufalidis [4] formulated the following conjecture (see also Frohman, Gelca and Lofaro [3], and Gelca [6]).

Conjecture 1 (**AJ Conjecture**) *For every knot K, $\varepsilon(\alpha_K)$ is equal to the A–polynomial, up to a polynomial depending on M only.*

The A–polynomial of a knot was introduced by Cooper, Culler, Gillet, Long and Shalen [1]; it describes the $\mathrm{SL}_2(\mathbb{C})$–character variety of the knot complement as viewed from the boundary torus. Here in the definition of the A–polynomial, we also allow the factor $L-1$ coming from the abelian component of the character variety of the knot group. Hence the A–polynomial in this paper is equal to $L-1$ times the A–polynomial defined in [1].

The AJ Conjecture was verified for the trefoil and figure 8 knots by Garoufalidis [4], and was partially checked for all torus knots by Hikami [7]. It was established for some classes of two–bridge knots and pretzel knots, including all twist knots and $(-2, 3, 6n \pm 1)$–pretzel knots, by Lê and the author [9; 10]. Here we provide a full proof of the AJ Conjecture for all torus knots. Moreover, we show that a stronger version of the conjecture, due to Sikora, holds true for all torus knots.

1.2 Main results

For a finitely generated group G, let $\chi(G)$ denote the $SL_2(\mathbb{C})$–character variety of G; see eg Culler and Shalen [2], and Lubotzky and Magid [11]. For a manifold Y we use $\chi(Y)$ also to denote $\chi(\pi_1(Y))$. Suppose $G = \mathbb{Z}^2$, the free abelian group with 2 generators. Every pair of generators μ, λ will define an isomorphism between $\chi(G)$ and $(\mathbb{C}^*)^2/\tau$, where $(\mathbb{C}^*)^2$ is the set of non-zero complex pairs (M, L) and τ is the involution $\tau(M, L) := (M^{-1}, L^{-1})$, as follows: Every representation is conjugate to an upper diagonal one, with M and L being the upper left entries of μ and λ, respectively. The isomorphism does not change if one replaces (μ, λ) by (μ^{-1}, λ^{-1}).

For an algebraic set V (over $\mathbb{C}$), let $\mathbb{C}[V]$ denote the ring of regular functions on V. For example, $\mathbb{C}[(\mathbb{C}^*)^2/\tau] = \mathfrak{t}^\sigma$, the σ–invariant subspace of $\mathfrak{t} := \mathbb{C}[M^{\pm 1}, L^{\pm 1}]$, where $\sigma(M^k L^l) := M^{-k} L^{-l}$.

Let K be a knot in S^3 and $X = S^3 \setminus K$ its complement. The boundary of X is a torus whose fundamental group is free abelian of rank two. An orientation of K will define a unique pair of an oriented meridian μ and an oriented longitude λ such that the linking number between the longitude and the knot is zero. The pair provides an identification of $\chi(\partial X)$ and $(\mathbb{C}^*)^2/\tau$ that actually does not depend on the orientation of K.

The inclusion $\partial X \hookrightarrow X$ induces an algebra homomorphism

$$\theta \colon \mathbb{C}[\chi(\partial X)] \equiv \mathfrak{t}^\sigma \longrightarrow \mathbb{C}[\chi(X)].$$

We will call the kernel $\mathfrak{p}$ of θ the *A–ideal* of K; it is an ideal of $\mathfrak{t}^\sigma$. The A–ideal was first introduced in [3]; it determines the A–polynomial of K. In fact $\mathfrak{p} = (A_K \cdot \mathfrak{t})^\sigma$, the σ–invariant part of the ideal $A_K \cdot \mathfrak{t} \subset \mathfrak{t}$ generated by the A–polynomial A_K.

The involution σ acts on the quantum torus $\mathcal{T}$ also by $\sigma(M^k L^l) = M^{-k} L^{-l}$. Let $\mathcal{A}_K^\sigma$ be the σ–invariant part of the recurrence ideal $\mathcal{A}_K$; it is an ideal of $\mathcal{T}^\sigma$. Sikora [14] proposed the following conjecture.

Conjecture 2 *Suppose K is a knot. Then* $\sqrt{\varepsilon(\mathcal{A}_K^\sigma)} = \mathfrak{p}$.

Here $\sqrt{\varepsilon(\mathcal{A}_K^\sigma)}$ denotes the radical of the ideal $\varepsilon(\mathcal{A}_K^\sigma)$ in the ring $\mathfrak{t}^\sigma = \varepsilon(\mathcal{T}^\sigma)$.

It is easy to see that Conjecture 2 implies the AJ Conjecture. Conjecture 2 was verified for the unknot and the trefoil knot by Sikora [14]. In the present paper we confirm it for all torus knots.

Theorem 1 *Conjecture 2 holds true for all torus knots.*

1.3 Plan of the paper

We provide a full proof of the AJ Conjecture for all torus knots in Section 2 and prove Theorem 1 in Section 2.

1.4 Acknowledgements

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2 Proof of the AJ Conjecture for torus knots

We will always assume that knots have framings 0.

Let $T(a,b)$ denote the (a,b)–torus knot. We consider the two cases, $a,b>2$ and $a=2$, separately. Lemmas 2.1 and 2.5 below were first proved in [7] using formulas for the colored Jones polynomials and the Alexander polynomial of torus knots given in Morton [12]. We present direct proofs here.

2.1 The case $a,b>2$

Lemma 2.1 *One has*

$$J_{T(a,b)}(n+2)=t^{-4ab(n+1)}J_{T(a,b)}(n)+t^{-2ab(n+1)}\,\frac{t^2\lambda_{(a+b)(n+1)}-t^{-2}\lambda_{(a-b)(n+1)}}{t^2-t^{-2}},$$

where $\lambda_k := t^{2k}+t^{-2k}$.

Proof By [12], we have

$$J_{T(a,b)}(n) = t^{-ab(n^2-1)}\sum_{j=-(n-1)/2}^{(n-1)/2} t^{4bj(aj+1)}[2aj+1], \tag{1}$$

where $[k] := (t^{2k} - t^{-2k})/(t^2 - t^{-2})$. Hence:

$$\begin{aligned}
&J_{T(a,b)}(n+2)\\
&\qquad = t^{-ab((n+2)^2-1)} \sum_{j=-(n+1)/2}^{(n+1)/2} t^{4bj(aj+1)}[2aj+1]\\
&\qquad = t^{-ab((n+2)^2-1)} \sum_{j=-(n-1)/2}^{(n-1)/2} t^{4bj(aj+1)}[2aj+1] + t^{-ab((n+2)^2-1)}\\
&\qquad\quad \times \big(t^{b(n+1)(a(n+1)+2)}[a(n+1)+1] - t^{b(n+1)(a(n+1)-2)}[a(n+1)-1]\big)\\
&\qquad = t^{-4ab(n+1)} J_{T(a,b)}(n) + t^{-2ab(n+1)} \frac{t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}}{t^2 - t^{-2}} \qquad \square
\end{aligned}$$

Lemma 2.2 *The colored Jones function of* $T(a,b)$ *is annihilated by the operator* $F_{a,b} = c_3 L^3 + c_2 L^2 + c_1 L + c_0$ *where:*

$$\begin{aligned}
c_3 &:= t^2\big(t^{2(a+b)} M^{a+b} + t^{-2(a+b)} M^{-(a+b)}\big)\\
&\qquad\qquad - t^{-2}\big(t^{2(a-b)} M^{a-b} + t^{-2(a-b)} M^{-(a-b)}\big)\\
c_2 &:= -t^{-2ab}\big(t^2\big(t^{4(a+b)} M^{a+b} + t^{-4(a+b)} M^{-(a+b)}\big)\\
&\qquad\qquad + t^{-2}\big(t^{4(a-b)} M^{a-b} + t^{-4(a-b)} M^{-(a-b)}\big)\big)\\
c_1 &:= -t^{-8ab} M^{-2ab} c_3\\
c_0 &:= -t^{-4ab} M^{-2ab} c_2
\end{aligned}$$

Proof It is easy to check that $c_3 t^{-4ab(n+2)} + c_1 = c_2 t^{-4ab(n+1)} + c_0 = 0$ and

$$c_3\big(t^2\lambda_{(a+b)(n+2)} - t^{-2}\lambda_{(a-b)(n+2)}\big) + c_2 t^{2ab}\big(t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}\big) = 0.$$

Hence, by Lemma 2.1, $F_{a,b} J_{T(a,b)}(n)$ is equal to:

$$
\begin{aligned}
&c_3 J_{T(a,b)}(n+3) + c_2 J_{T(a,b)}(n+2) + c_1 J_{T(a,b)}(n+1) + c_0 J_{T(a,b)}(n) \\
&\quad = c_3\Big(t^{-4ab(n+2)} J_{T(a,b)}(n+1) + t^{-2ab(n+2)} \frac{t^2\lambda_{(a+b)(n+2)} - t^{-2}\lambda_{(a-b)(n+2)}}{t^2 - t^{-2}}\Big) \\
&\qquad + c_2\Big(t^{-4ab(n+1)} J_{T(a,b)}(n) + t^{-2ab(n+1)} \frac{t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}}{t^2 - t^{-2}}\Big) \\
&\qquad\qquad + c_1 J_{T(a,b)}(n+1) + c_0 J_{T(a,b)}(n) \\
&\quad = (c_3 t^{-4ab(n+2)} + c_1) J_{T(a,b)}(n+1) + (c_2 t^{-4ab(n+1)} + c_0) J_{T(a,b)}(n) \\
&\qquad + t^{-2ab(n+1)}\Big(c_3 \frac{t^2\lambda_{(a+b)(n+2)} - t^{-2}\lambda_{(a-b)(n+2)}}{t^2 - t^{-2}} \\
&\qquad\qquad + c_2 t^{2ab} \frac{t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}}{t^2 - t^{-2}}\Big) \\
&\quad = 0
\end{aligned}
$$

This proves Lemma 2.2. □

Recall that $\alpha_{T(a,b)}$ is the recurrence polynomial of $T(a,b)$.

Proposition 2.3 *For $a, b > 2$, one has $\alpha_{T(a,b)} = F_{a,b}$.*

Proof By Lemma 2.2 it suffices to show that if an operator $P = P_2 L^2 + P_1 L + P_0$, where the P_j are polynomials in $\mathbb{C}[t^{\pm 1}, M]$, annihilates the colored Jones polynomials of $T(a,b)$ then $P = 0$.

Indeed, suppose $P\, J_{T(a,b)}(n) = 0$. Then, by Lemma 2.1:

$$
\begin{aligned}
0 &= P_2 J_{T(a,b)}(n+2) + P_1 J_{T(a,b)}(n+1) + P_0 J_{T(a,b)}(n) \\
&= P_2\Big(t^{-4ab(n+1)} J_{T(a,b)}(n) + t^{-2ab(n+1)} \frac{t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}}{t^2 - t^{-2}}\Big) \\
&\qquad + P_1 J_{T(a,b)}(n+1) + P_0 J_{T(a,b)}(n) \\
&= (t^{-4ab(n+1)} P_2 + P_0) J_{T(a,b)}(n) + P_1 J_{T(a,b)}(n+1) \\
&\qquad + P_2 t^{-2ab(n+1)} \frac{t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}}{t^2 - t^{-2}}
\end{aligned}
$$

Let $P_2' = t^{-4ab(n+1)} P_2 + P_0$ and

$$P_0' = P_2 t^{-2ab(n+1)} \frac{t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}}{t^2 - t^{-2}}.$$

Then,

$$P_2' J_{T(a,b)}(n) + P_1 J_{T(a,b)}(n+1) + P_0' = 0. \tag{2}$$

Note that P_2' and P_0' are polynomials in $\mathbb{C}[t^{\pm 1}, M]$. We need the following lemma.

Lemma 2.4 *The lowest degree in t of $J_{T(a,b)}(n)$ is*

$$l_n = -abn^2 + ab + \tfrac{1}{2}(1-(-1)^{n-1})(a-2)(b-2).$$

Proof From (1), it follows easily that $l_n = -abn^2 + ab$ if n is odd, and $l_n = (-abn^2 + ab) + (ab - 2b - 2a + 4)$ if n is even. □

Let us complete the proof of Proposition 2.3. Suppose $P_2', P_1 \neq 0$. Let r_n and s_n be the lowest degrees (in t) of P_2' and P_1 respectively. Note that, when n is large enough, r_n and s_n are polynomials in n of degrees at most 1. Equation (2) then implies that $r_n + l_n = s_n + l_{n+1}$, ie:

$$r_n - s_n = l_{n+1} - l_n = -ab(2n+1) - (-1)^n(a-2)(b-2)$$

This cannot happen since the LHS is a polynomial in n, when n is large enough, while the RHS is not (since $(a-2)(b-2) > 0$). Hence $P_2' = P_1 = P_0' = 0$, which means $P = 0$. □

It is easy to see that $\varepsilon(\alpha_{T(a,b)}) = M^{-2ab}(M^a - M^{-a})(M^b - M^{-b})A_{T(a,b)}$ where $A_{T(a,b)} = (L-1)(L^2M^{2ab} - 1)$ is the A–polynomial of $T(a,b)$ when $a, b > 2$. This means the AJ Conjecture holds true for $T(a,b)$ when $a, b > 2$.

2.2 The case $a = 2$

Lemma 2.5 *One has*

$$J_{T(2,b)}(n+1) = -t^{-(4n+2)b} J_{T(2,b)}(n) + t^{-2nb}[2n+1].$$

Proof By (1), we have

$$J_{T(2,b)}(n) = t^{-2b(n^2-1)} \sum_{j=-(n-1)/2}^{(n-1)/2} t^{4bj(2j+1)}[4j+1].$$

Hence

$$J_{T(2,b)}(n+1) = t^{-2b((n+1)^2-1)} \sum_{k=-n/2}^{n/2} t^{4bk(2k+1)}[4k+1].$$

Set $k=-(j+\frac{1}{2})$. Then:

$$\begin{aligned}
J_{T(2,b)}&(n+1)\\
&=t^{-2b((n+1)^2-1)}\sum_{j=(n-1)/2}^{-(n+1)/2} t^{4bj(2j+1)}[-(4j+1)]\\
&=t^{-2b((n+1)^2-1)}\Big(-\sum_{j=-(n-1)/2}^{(n-1)/2} t^{4bj(2j+1)}[4j+1]+t^{2bn(n+1)}[2n+1]\Big)\\
&=-t^{-(4n+2)b}J_{T(2,b)}(n)+t^{-2nb}[2n+1]
\end{aligned}$$

This proves Lemma 2.5. □

Lemma 2.6 *The colored Jones function of $T(2,b)$ is annihilated by the operator $G_{2,b}=d_2L^2+d_1L+d_0$ where*

$$\begin{aligned}
d_2 &:= t^2M^2-t^{-2}M^{-2},\\
d_1 &:= t^{-2b}\big(t^{-4b}M^{-2b}(t^2M^2-t^{-2}M^{-2})-(t^6M^2-t^{-6}M^{-2})\big),\\
d_0 &:= -t^{-4b}M^{-2b}(t^6M^2-t^{-6}M^{-2}).
\end{aligned}$$

Proof From Lemma 2.5 we have

$$\begin{aligned}
J_{T(2,b)}(n+1) &= -t^{-(4n+2)b}J_{T(2,b)}(n)+t^{-2nb}[2n+1],\\
J_{T(2,b)}(n+2) &= t^{-8(n+1)b}J_{T(2,b)}(n)-t^{-6(n+1)b}[2n+1]+t^{-2(n+1)b}[2n+3].
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
&t^{-8(n+1)b}d_2-t^{-(4n+2)b}d_1+d_0=0,\\
&d_2\big(-t^{-6(n+1)b}[2n+1]+t^{-2(n+1)b}[2n+3]\big)+d_1t^{-2nb}[2n+1]=0.
\end{aligned}$$

Hence:

$$\begin{aligned}
G_{2,b}&J_{T(2,b)}(n)\\
&=d_2J_{T(2,b)}(n+2)+d_1J_{T(2,b)}(n+1)+d_0J_{T(2,b)}(n)\\
&=d_2\big(t^{-8(n+1)b}J_{T(2,b)}(n)-t^{-6(n+1)b}[2n+1]+t^{-2(n+1)b}[2n+3]\big)\\
&\qquad+d_1\big(-t^{-(4n+2)b}J_{T(2,b)}(n)+t^{-2nb}[2n+1]\big)+d_0J_{T(2,b)}(n)\\
&=\big(t^{-8(n+1)b}d_2-t^{-(4n+2)b}d_1+d_0\big)J_{T(2,b)}(n)\\
&\quad+d_2\big(-t^{-6(n+1)b}[2n+1]+t^{-2(n+1)b}[2n+3]\big)+d_1t^{-2nb}[2n+1]\\
&=0
\end{aligned}$$

This proves Lemma 2.6. □

Proposition 2.7 *One has $\alpha_{T(2,b)} = G_{2,b}$.*

Proof By Lemma 2.6, it suffices to show that if an operator $P = P_1 L + P_0$, where the P_j are polynomials in $\mathbb{C}[t^{\pm 1}, M]$, annihilates the colored Jones polynomials of $T(2,b)$ then $P = 0$.

Indeed, suppose $PJ_{T(2,b)}(n) = 0$. Then:

$$\begin{aligned} 0 &= P_1 J_{T(2,b)}(n+1) + P_0 J_{T(2,b)}(n) \\ &= P_1\big(-t^{-(4n+2)b} J_{T(2,b)}(n) + t^{-2nb}[2n+1]\big) + P_0 J_{T(2,b)}(n) \\ &= \big(-t^{-(4n+2)b} P_1 + P_0\big) J_{T(2,b)}(n) + t^{-2nb}[2n+1] P_1 \end{aligned}$$

Let $P_1' = -t^{-(4n+2)b} P_1 + P_0$ and $P_0' = t^{-2nb}[2n+1]P_1$. Then P_1', P_0' are polynomials in $\mathbb{C}[t^{\pm 1}, M]$ and $P_1' J(n) + P_0' = 0$. This implies that $P_1' = P_0' = 0$ since the lowest degree in t of $J_{T(2,b)}(n)$ is $-2bn^2 + 2b$, which is quadratic in n, by Lemma 2.4. Hence $P = 0$. □

It is easy to see that $\varepsilon(\alpha_{T(2,b)}) = M^{-2b}(M^2 - M^{-2}) A_{T(2,b)}$ where $A_{T(2,b)} = (L-1)(LM^{2b}+1)$ is the A–polynomial of $T(2,b)$. This means the AJ Conjecture holds true for $T(2,b)$.

3 Proof of Theorem 1

As in the previous section, we consider the two cases, $a, b > 2$ and $a = 2$, separately.

3.1 The case $a, b > 2$

We claim that:

Proposition 3.1 *The colored Jones function of $T(a,b)$ is annihilated by the operator PQ where:*

$$\begin{aligned} P := {}& t^{-10ab}(L^3 M^{2ab} + L^{-3} M^{-2ab}) \\ & -(t^{2(a-b)} + t^{2(b-a)}) t^{-4ab}(L^2 M^{2ab} + L^{-2} M^{-2ab}) + t^{2ab}(LM^{2ab} + L^{-1} M^{-2ab}) \\ & \qquad -(t^{2ab} + t^{-2ab})(L + L^{-1}) + (t^{2(a-b)} + t^{2(b-a)})(t^{4ab} + t^{-4ab}) \\ Q := {}& t^{-6ab}(L^3 M^{2ab} + L^{-3} M^{-2ab}) \\ & -(t^{2(a+b)} + t^{-2(a+b)}) q^{-ab}(L^2 M^{2ab} + L^{-2} M^{-2ab}) + t^{-2ab}(LM^{2ab} + L^{-1} M^{-2ab}) \\ & \qquad -(t^{2ab} + t^{-2ab})(L + L^{-1}) + 2(t^{2(a+b)} + t^{-2(a+b)}) \end{aligned}$$

Proof We first prove the following two lemmas.

Lemma 3.2 *One has*

$$QJ_{T(a,b)}(n) = t^{4ab-2}(\lambda_{a+b} - \lambda_{a-b})\,\frac{t^{2abn}\lambda_{(a-b)(n+1)} - t^{-2abn}\lambda_{(a-b)(n-1)}}{t^2 - t^{-2}}.$$

Proof Let

$$g(n) := t^{-2abn}\,\frac{t^2\lambda_{(a+b)n} - t^{-2}\lambda_{(a-b)n}}{t^2 - t^{-2}}.$$

Then, by Lemma 2.1, $J_{T(a,b)}(n+2) = t^{-4ab(n+1)}J_{T(a,b)}(n) + g(n+1)$. Hence:

$$\begin{aligned}
&QJ_{T(a,b)}(n)\\
&= t^{-6ab}\big(t^{4ab(n+3)}J_{T(a,b)}(n+3) + t^{-4ab(n-3)}J_{T(a,b)}(n-3)\big)\\
&\quad - (t^{2(a+b)} + t^{-2(a+b)})t^{-4ab}\big(t^{4ab(n+2)}J_{T(a,b)}(n+2) + t^{-4ab(n-2)}J_{T(a,b)}(n-2)\big)\\
&\qquad + t^{-2ab}\big(t^{4ab(n+1)}J_{T(a,b)}(n+1) + t^{-4ab(n-1)}J_{T(a,b)}(n-1)\big)\\
&\qquad - (t^{2ab} + t^{-2ab})\big(J_{T(a,b)}(n+1) + J_{T(a,b)}(n-1)\big)\\
&\qquad + 2(t^{2(a+b)} + t^{-2(a+b)})J_{T(a,b)}(n)\\
&= t^{-6ab}\big(t^{4ab}\big(J_{T(a,b)}(n+1) + J_{T(a,b)}(n-1)\big)\\
&\qquad + t^{2ab(n+5)}g(n+2) - t^{-2ab(n-5)}g(n-2)\big)\\
&\quad - (t^{2(a+b)} + t^{-2(a+b)})t^{-4ab}\big(2t^{4ab}J_{T(a,b)}(n)\\
&\qquad + t^{2ab(n+4)}g(n+1) - t^{-2ab(n-4)}g(n-1)\big)\\
&\quad + t^{-2ab}\big(t^{4ab}\big(J_{T(a,b)}(n-1) + J_{T(a,b)}(n+1)\big) + \big(t^{2ab(n+3)} - t^{-2ab(n-3)}\big)g(n)\big)\\
&\qquad - (t^{2ab} + t^{-2ab})\big(J_{T(a,b)}(n+1) + J_{T(a,b)}(n-1)\big)\\
&\qquad + 2(t^{2(a+b)} + t^{-2(a+b)})J_{T(a,b)}(n)\\
&= t^{-6ab}\big(t^{2ab(n+5)}g(n+2) - t^{-2ab(n-5)}g(n-2)\big)\\
&\quad - \big(t^{2(a+b)} + t^{-2(a+b)}\big)t^{-4ab}\big(t^{2ab(n+4)}g(n+1) - t^{-2ab(n-4)}g(n-1)\big)\\
&\qquad + t^{-2ab}\big(t^{2ab(n+3)} - t^{-2ab(n-3)}\big)g(n)
\end{aligned}$$

Using the definition of $g(n)$, we get:

$$\begin{aligned}
QJ_{T(a,b)}(n) = t^{4ab}\Big(&t^{2abn}\frac{t^2\lambda_{(a+b)(n+2)} - t^{-2}\lambda_{(a-b)(n+2)}}{t^2 - t^{-2}}\\
&-t^{-2abn}\,\frac{t^2\lambda_{(a+b)(n-2)} - t^{-2}\lambda_{(a-b)(n-2)}}{t^2 - t^{-2}}\Big)
\end{aligned}$$

$$
\begin{aligned}
&-(t^{2(a+b)}+t^{-2(a+b)})t^{4ab}\times\\
&\quad\Big(t^{2abn}\,\frac{t^2\lambda_{(a+b)(n+1)}-t^{-2}\lambda_{(a-b)(n+1)}}{t^2-t^{-2}}\\
&\qquad\qquad -t^{-2abn}\,\frac{t^2\lambda_{(a+b)(n-1)}-t^{-2}\lambda_{(a-b)(n-1)}}{t^2-t^{-2}}\Big)\\
&+t^{4ab}(t^{2abn}-t^{-2abn})\,\frac{t^2\lambda_{(a+b)n}-t^{-2}\lambda_{(a-b)n}}{t^2-t^{-2}}
\end{aligned}
$$

Now applying the equality $\lambda_{k+l}+\lambda_{k-l}=\lambda_k\lambda_l$, we then obtain

$$
Q\,J_{T(a,b)}(n)=t^{4ab-2}(\lambda_{a+b}-\lambda_{a-b})\,\frac{t^{2abn}\lambda_{(a-b)(n+1)}-t^{-2abn}\lambda_{(a-b)(n-1)}}{t^2-t^{-2}}.
$$

This proves Lemma 3.2. □

Let $h(n):=t^{2abn}\lambda_{(a-b)(n+1)}-t^{-2abn}\lambda_{(a-b)(n-1)}$.

Lemma 3.3 *The function $h(n)$ is annihilated by the operator P, ie, $Ph(n)=0$.*

Proof Let $c=a-b$. Then:

$$
\begin{aligned}
&Ph(n)\\
&\quad=t^{-10ab}\big(t^{4ab(n+3)}h(n+3)+t^{-4ab(n-3)}h(n-3)\big)\\
&\quad-(t^{2(a-b)}+t^{2(b-a)})t^{-4ab}\big(t^{4ab(n+2)}h(n+2)+t^{-4ab(n-2)}h(n-2)\big)\\
&\quad+t^{2ab}\big(t^{4ab(n+1)}h(n+1)+t^{-4ab(n-1)}h(n-1)\big)\\
&\quad-(t^{2ab}+t^{-2ab})\big(h(n+1)+h(n-1)\big)+(t^{2(a-b)}+t^{2(b-a)})(t^{4ab}+t^{-4ab})h(n)\\
&\quad=\big(t^{2ab(3n+4)}\lambda_{c(n+4)}-t^{2ab(n-2)}\lambda_{c(n+2)}\\
&\qquad\qquad+t^{-2ab(n+2)}\lambda_{c(n-2)}-t^{-2ab(3n-4)}\lambda_{c(n-4)}\big)\\
&\quad-\lambda_c\big(t^{2ab(3n+4)}\lambda_{c(n+3)}-t^{2abn}\lambda_{c(n+1)}+t^{-2abn}\lambda_{c(n-1)}-t^{-2ab(3n-4)}\lambda_{c(n-3)}\big)\\
&\quad+\big(t^{2ab(3n+4)}\lambda_{c(n+2)}-t^{2ab(n+2)}\lambda_{cn}+t^{-2ab(n-2)}\lambda_{cn}-t^{-2ab(3n-4)}\lambda_{c(n-2)}\big)\\
&\quad-(t^{2ab}+t^{-2ab})\big(t^{2ab(n+1)}\lambda_{c(n+2)}-t^{-2ab(n+1)}\lambda_{cn}\\
&\quad+t^{2ab(n-1)}\lambda_{cn}-t^{-2ab(n-1)}\lambda_{c(n-2)}\big)\\
&\quad+\lambda_c(t^{4ab}+t^{-4ab})\big(t^{2abn}\lambda_{c(n+1)}-t^{-2abn}\lambda_{c(n-1)}\big)
\end{aligned}
$$

Note that $\lambda_{k+l}+\lambda_{k-l}=\lambda_k\lambda_l$. Hence:

$$\begin{aligned}
Ph(n) &= \big(-t^{2ab(n-2)}\lambda_{c(n+2)}+t^{-2ab(n+2)}\lambda_{c(n-2)}\big) \\
&\qquad\qquad -\lambda_c\big(-t^{2abn}\lambda_{c(n+1)}+t^{-2abn}\lambda_{c(n-1)}\big) \\
&\quad +\big(-t^{2ab(n+2)}\lambda_{cn}+t^{-2ab(n-2)}\lambda_{cn}\big) \\
&\quad -(t^{2ab}+t^{-2ab})\big(t^{2ab(n+1)}\lambda_{c(n+2)}-t^{-2ab(n+1)}\lambda_{cn} \\
&\qquad\qquad +t^{2ab(n-1)}\lambda_{cn}-t^{-2ab(n-1)}\lambda_{c(n-2)}\big) \\
&\quad +\lambda_c(t^{4ab}+t^{-4ab})\big(t^{2abn}\lambda_{c(n+1)}-t^{-2abn}\lambda_{c(n-1)}\big) \\
&= -(t^{4ab}+t^{-4ab}+1)t^{2abn}\lambda_{c(n+2)}+(t^{4ab}+t^{-4ab}+1)t^{-2abn}\lambda_{c(n-2)} \\
&\quad -(t^{4ab}+t^{-4ab}+1)(t^{2abn}-t^{-2abn})\lambda_{cn} \\
&\qquad\qquad +\lambda_c(t^{4ab}+t^{-4ab}+1)\big(t^{2abn}\lambda_{c(n+1)}-t^{-2abn}\lambda_{c(n-1)}\big) \\
&= -(t^{4ab}+t^{-4ab}+1)t^{2abn}\big(\lambda_{c(n+2)}+\lambda_{cn}-\lambda_c\lambda_{c(n+1)}\big) \\
&\quad +(t^{4ab}+t^{-4ab}+1)t^{-2abn}\big(\lambda_{c(n-2)}+\lambda_{cn}-\lambda_c\lambda_{c(n-1)}\big) \\
&= 0
\end{aligned}$$

This proves Lemma 3.3. □

Proposition 3.1 follows directly from Lemmas 3.2 and 3.3. □

3.2 The case $a=2$

We claim that:

Proposition 3.4 *The colored Jones function of $T(2,b)$ is annihilated by the operator*

$$\begin{aligned}
R &= t^{-4b}(L^2M^{2b}+L^{-2}M^{-2b})+(t^{2b}+t^{-2b})(L+L^{-1}) \\
&\qquad -(t^4+t^{-4})t^{-2b}(LM^{2b}+L^{-1}M^{-2b})+(M^{2b}+M^{-2b})-2(t^4+t^{-4}).
\end{aligned}$$

Proof From Lemma 2.5 we have

$$\begin{aligned}
J_{T(2,b)}(n+1) &= -t^{-(4n+2)b}J_{T(2,b)}(n)+t^{-2nb}[2n+1], \\
J_{T(2,b)}(n+2) &= t^{-8(n+1)b}J_{T(2,b)}(n)-t^{-6(n+1)b}[2n+1]+t^{-2(n+1)b}[2n+3], \\
J_{T(2,b)}(n-1) &= -t^{(4n-2)b}J_{T(2,b)}(n)+t^{2nb}[2n-1], \\
J_{T(2,b)}(n-2) &= t^{8(n-1)b}J_{T(2,b)}(n)-t^{6(n-1)b}[2n-1]+t^{2(n-1)b}[2n-3].
\end{aligned}$$

Hence

$$
\begin{aligned}
RJ_{T(2,b)}(n) &= t^{-4b}\big(t^{4(n+2)b}J_{T(2,b)}(n+2)+t^{-4(n-2)b}J_{T(2,b)}(n-2)\big)\\
&\qquad + (t^{2b}+t^{-2b})\big(J_{T(2,b)}(n+1)+J_{T(2,b)}(n-1)\big)\\
&\quad -(t^4+t^{-4})t^{-2b}\big(t^{4(n+1)b}J_{T(2,b)}(n+1)+t^{-4(n-1)b}J_{T(2,b)}(n-1)\big)\\
&\qquad + \big((t^{4nb}+t^{-4nb})-2(t+t^{-4})\big)J_{T(2,b)}(n)\\
&= t^{-4b}\big(-t^{-2(n-1)b}[2n+1]+t^{2(n+3)b}[2n+3]\\
&\qquad -t^{2(n+1)b}[2n-1]+t^{-2(n-3)b}[2n-3]\big)\\
&\qquad +(t^{2b}+t^{-2b})\big(t^{-2nb}[2n+1]+t^{2nb}[2n-1]\big)\\
&\qquad -(t^4+t^{-4})t^{-2b}\big(t^{(2n+4)b}[2n+1]+t^{(-2n+4)b}[2n-1]\big)\\
&\qquad -(t+t^{-4})t^{2b}\big(t^{2nb}[2n+1]+t^{-2nb}[2n-1]\big)\\
&= t^{2b}t^{2nb}\big([2n+3]+[2n-1]-(t^4+t^{-4})[2n+1]\big)\\
&\qquad + t^{2b}t^{-2nb}\big([2n-3]+[2n+1]-(t^4+t^{-4})[2n-1]\big)\\
&= 0,
\end{aligned}
$$

since $[k+l]+[k-l]=(t^{2l}+t^{-2l})[k]$. □

3.3 Proof of Theorem 1

We first note that the A–ideal $\mathfrak{p}$, the kernel of $\theta\colon \mathfrak{t}^\sigma \longrightarrow \mathbb{C}[\chi(X)]$, is radical, ie, $\sqrt{\mathfrak{p}}=\mathfrak{p}$, since the character ring $\mathbb{C}[\chi(X)]$ is reduced, ie, has nil-radical 0, by definition.

Lemma 3.5 *Suppose $\delta(t,M,L)\in\mathcal{A}_K$. Then there are polynomials $g(t,M)\in\mathbb{C}[t^{\pm1},M]$ and $\gamma(t,M,L)\in\mathcal{T}$ such that*

$$
\delta(t,M,L)=\frac{1}{g(t,M)}\gamma(t,M,L)\alpha_K(t,M,L). \tag{3}
$$

Moreover, $g(t,M)$ and $\gamma(t,M,L)$ can be chosen so that $\varepsilon(g)\neq 0$.

Proof By definition α_K is a generator of $\widetilde{\mathcal{A}}_K$, the extension of $\mathcal{A}_K$ in the principal left-ideal domain $\widetilde{\mathcal{T}}$. Since $\delta\in\mathcal{A}_K$, it is divisible by α_K in $\widetilde{\mathcal{T}}$. Hence (3) follows.

We can assume that $t+1$ does not divide both $g(t,M)$ and $\gamma(t,M,L)$ simultaneously. If $\varepsilon(g)=0$ then g is divisible by $t+1$, and hence γ is not. But then from the equality $g\delta=\gamma\alpha_K$, it follows that α_K is divisible by $t+1$, which is impossible, since all the coefficients of powers of L in α_K are supposed to be co-prime. □

Showing $\sqrt{\varepsilon(\mathcal{A}_K^\sigma)} \subset \mathfrak{p}$ For torus knots, by Section 1, we have $\varepsilon(\alpha_K) = f(M)A_K$, where $f(M) \in \mathbb{C}[M^{\pm 1}]$. For every $\delta \in \mathcal{A}_K$, by Lemma 3.5, there exist $g(t,M) \in \mathbb{C}[t^{\pm 1}, M]$ and $\gamma \in \mathcal{T}$ such that $\delta = \frac{1}{g(t,M)}\gamma\,\alpha_K$ and $\varepsilon(g) \neq 0$. It implies that

$$\varepsilon(\gamma) = \frac{1}{\varepsilon(g(M))}\varepsilon(\gamma)\varepsilon(\alpha_K) = \frac{1}{\varepsilon(g(M))}\varepsilon(\gamma) f(M) A_K. \tag{4}$$

The A–polynomial of a torus knot does not contain any non-trivial factor depending on M only. Since $\varepsilon(\gamma) \in \mathfrak{t} = \mathbb{C}[L^{\pm 1}, M^{\pm 1}]$, equation (4) implies that

$$h := \frac{1}{\varepsilon(g(M))}\varepsilon(\gamma) f(M)$$

is an element of $\mathfrak{t}$. Hence $\varepsilon(\gamma) \in A_K \cdot \mathfrak{t}$, the ideal of $\mathfrak{t}$ generated by A_K. It follows that $\varepsilon(\mathcal{A}_K) \subset A_K \cdot \mathfrak{t}$ and thus $\varepsilon(\mathcal{A}_K^\sigma) \subset (A_K \cdot \mathfrak{t})^\sigma = \mathfrak{p}$. Hence $\sqrt{\varepsilon(\mathcal{A}_K^\sigma)} \subset \sqrt{\mathfrak{p}} = \mathfrak{p}$.

Showing $\mathfrak{p} \subset \sqrt{\varepsilon(\mathcal{A}_K^\sigma)}$ For $a, b > 2$, by Proposition 3.1 the colored Jones function of $T(a,b)$ is annihilated by the operator PQ. Note that

$$\begin{aligned}\varepsilon(PQ) &= (L + L^{-1} - 2)^2 (L^2 M^{2ab} + L^{-2} M^{-2ab} - 2)^2 \\ &= L^{-2}\big(L^{-1} M^{-ab}(L-1)(L^2 M^{2ab} - 1)\big)^4.\end{aligned}$$

If $u \in \mathfrak{p}$ then $u = vA'_{T(a,b)}$, where

$$A'_{T(a,b)} := L^{-1} M^{-ab}(L-1)(L^2 M^{2ab} - 1) = L^{-1} M^{-ab} A_{T(a,b)}$$

and $v \in \mathbb{C}[M^{\pm 1}, L^{\pm 1}]$. It is easy to see that $\sigma(v) = Lv$, since $\sigma(u) = u$ and $\sigma(A'_{T(a,b)}) = L^{-1} A'_{T(a,b)}$. This implies that $\sigma(v^2 L) = \sigma(v)^2 L^{-1} = v^2 L$. We then have

$$u^4 = v^4 A'^4_{T(a,b)} = \varepsilon(v^4 L^2 PQ) \in \varepsilon(\mathcal{A}_K^\sigma),$$

hence $u \in \sqrt{\varepsilon(\mathcal{A}_K^\sigma)}$.

For $a = 2$, by Proposition 3.4 the colored Jones function of $T(2,b)$ is annihilated by the operator R. Note that $\sigma(R) = R$ and

$$\varepsilon(R) = (L + L^{-1} - 2)(LM^{2b} + L^{-1}M^{-2b} + 2) = \big(L^{-1}M^{-b}(L-1)(LM^{2b}+1)\big)^2.$$

If $u \in \mathfrak{p}$ then $u = vA'_{T(2,b)}$, where

$$A'_{T(2,b)} := L^{-1} M^{-b}(L-1)(LM^{2b} + 1) = L^{-1} M^{-b} A_{T(2,b)}$$

and $v \in \mathbb{C}[M^{\pm 1}, L^{\pm 1}]$. It is easy to see that $\sigma(v) = -v$ and hence $\sigma(v^2) = \sigma(v)\sigma(v) = v^2$. We then have

$$u^2 = v^2 A'^2_{T(2,b)} = \varepsilon(v^2 R) \in \varepsilon(\mathcal{A}_K^\sigma),$$

hence $u \in \sqrt{\varepsilon(\mathcal{A}_K^\sigma)}$.

In both cases $\mathfrak{p} \subset \sqrt{\varepsilon(\mathcal{A}_K^\sigma)}$. Hence $\sqrt{\varepsilon(\mathcal{A}_K^\sigma)} = \mathfrak{p}$ for all torus knots.

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