msp

*Algebraic & Geometric Topology* 13 (2013) 1049–1051

# A streamlined proof of Goodwillie's $n$–excisive approximation

CHARLES REZK

We give a shorter proof of Goodwillie's [1, Lemma 1.9], which is the key step in proving that the construction $P_n F$ gives an $n$–excisive functor.

55P99

## 1 Introduction

For a homotopy functor $F$ from spaces to spaces, Goodwillie has defined the notion of an "$n$–excisive approximation," which is a homotopy functor $P_n F$ together with a natural transformation $p_n F\colon F \to P_n F$. In [1, Theorem 1.8] it is shown that the functor $P_n F$ is in fact an $n$–excisive functor, and therefore that $p_n F$ is the universal example of a map from $F$ to an $n$–excisive functor.

Earlier work of Goodwillie had shown that $P_n F$ is $n$–excisive under additional hypotheses involving connectivity. The notable feature of the proof given in [1] is that no hypotheses involving connectivity are needed. In fact, the argument is entirely general, and will work in any homotopy theory in which directed homotopy colimits commute with finite homotopy limits.

Goodwillie's proof relies the following "lemma" [1, Lemma 1.9]. (The notions of "cartesian" and "strongly cocartesian cube" are defined in [1, Section 1]. The definitions of $T_n F$ and $t_n F$ are given below.)

**1.1 Lemma** *Let $\mathcal{X}$ be any strongly cocartesian $n$–cube in $\mathcal{U}$, and let $F$ be any homotopy functor. The map of cubes $(t_n F)(\mathcal{X})\colon F(\mathcal{X}) \to (T_n F)(\mathcal{X})$ factors through some cartesian cube.*

Goodwillie's proof of this lemma is, as he notes, "a little opaque". In fact, though the proof gives an explicit factorization of $(t_n F)(\mathcal{X})$ through a cartesian cube, the cube in question is difficult to describe, and does not seem to play any natural role.

*Published: 17 April 2013* DOI: 10.2140/agt.2013.13.1049

The purpose of this note is to give a much simplified proof of Goodwillie's lemma (though in the same spirit as Goodwillie's), and thus a simplified proof of the construction of the $n$–excisive approximation. We will assume that the reader is familiar with [1], and we assume the context and notation of Section 1 of that paper.

The author was supported under NSF grants number DMS-0505056 and DMS-1006054.

## 2 Proof of the lemma

Let $\mathcal{P}(n)$ denote the poset of subsets of $\{1,\dots,n\}$, and let $\mathcal{P}_0(n) \subset \mathcal{P}(n)$ be the poset of nonempty subsets.

If $F\colon \mathcal{C} \to \mathcal{D}$ is a homotopy functor, Goodwillie defines a functor $T_n F\colon \mathcal{C} \to \mathcal{D}$ and natural map $t_n F\colon F \to T_{n-1} F$ by

$$F(X) \xrightarrow{t_n F} \operatorname*{holim}_{U \in \mathcal{P}_0(n+1)} F(X * U).$$

**Proof of Lemma 1.1** We write $n$ instead of $n+1$. Given any cube $\mathcal{X}$ and a set $U \in \mathcal{P}(n)$, define a cube $\mathcal{X}_U$ by

$$\mathcal{X}_U(T) = \operatorname{hocolim}\Big(\mathcal{X}(T) \longleftarrow \coprod_{s \in U} \mathcal{X}(T) \longrightarrow \coprod_{s \in U} \mathcal{X}(T \cup \{s\})\Big).$$

We have $\mathcal{X}_\varnothing(T) \approx \mathcal{X}(T)$, and there is an evident map $\alpha\colon \mathcal{X}_U(T) \to \mathcal{X}(T) * U$, which is natural in both $T$ and $U$.

The map $(t_{n-1}F)(\mathcal{X})$ factors as follows:

$$F(\mathcal{X}(T)) \longrightarrow \operatorname*{holim}_{U \in \mathcal{P}_0(n)} F(\mathcal{X}_U(T)) \longrightarrow \operatorname*{holim}_{U \in \mathcal{P}_0(n)} F(\mathcal{X}(T) * U) \approx (T_{n-1}F)(\mathcal{X}(T)).$$

Now suppose that $\mathcal{X}$ is strongly cocartesian. Then there are natural weak equivalences $\mathcal{X}_U(T) \approx \mathcal{X}(T \cup U)$. The maps $\mathcal{X}(T \cup U) \to \mathcal{X}(T \cup \{s\} \cup U)$ are isomorphisms for $s \in U$, and thus if $U$ is nonempty the cube $T \mapsto F(\mathcal{X}_U(T))$ is cartesian. Therefore $\operatorname*{holim}_{U \in \mathcal{P}_0(n)} F(\mathcal{X}_U(T))$ is a homotopy limit of cartesian cubes, and thus is cartesian. □

Note that this shows that if $T$ is nonempty, then $U \mapsto F(\mathcal{X}_U(T))$ is cartesian, so that $F(\mathcal{X}(T)) \to \operatorname{holim}_{U \in \mathcal{P}_0(n)} F(\mathcal{X}_U(T))$ is a weak equivalence for $T \neq \varnothing$. For $T = \varnothing$, we see that $\operatorname{holim}_{U \in \mathcal{P}_0(n)} F(\mathcal{X}_U(\varnothing)) \approx \operatorname{holim}_{U \in \mathcal{P}_0(n)} F(\mathcal{X}(U))$.

# References

[1] **T G Goodwillie**, *Calculus. III. Taylor series*, Geom. Topol. 7 (2003) 645–711 MR2026544

*Department of Mathematics, University of Illinois at Urbana-Champaign*
*273 Altgeld Hall, MC-382, 1409 W. Green Street, Urbana, IL 61801, USA*

rezk@math.uiuc.edu

http://www.math.uiuc.edu/~rezk/

Received: 21 March 2011 Revised: 8 December 2012