

# The Künneth Theorem in equivariant $K$ -theory for actions of a cyclic group of order 2

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The Künneth Theorem for equivariant (complex)  $K$ -theory  $K_G^*$ , in the form developed by Hodgkin and others, fails dramatically when  $G$  is a finite group, and even when  $G$  is cyclic of order 2. We remedy this situation in this very simplest case  $G = \mathbb{Z}/2$  by using the power of  $\mathrm{RO}(G)$ -graded equivariant  $K$ -theory.

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## 1 Introduction

Equivariant  $K$ -theory, invented by Atiyah and Segal (for the original exposition, see Segal [19]), is the simplest equivariant cohomology theory to define. It is enormously useful in equivariant topology, in index theory (where it is needed for the equivariant index theorem), and in the theory of operator algebras. (If  $X$  is a locally compact  $G$ -space, then  $C_0(X)$ , the algebra of continuous functions on  $X$  vanishing at infinity, is a  $G$ - $C^*$ -algebra, and  $K_G^{-*}(X) \cong K_*^G(C_0(X))$ . Note that on  $G$ -algebras, equivariant  $K$ -theory becomes a homology theory instead of a cohomology theory. For the theory of equivariant  $K$ -theory for operator algebras, see Blackadar [2, Chapter V, Section 11].)

Despite its apparent simplicity, equivariant  $K$ -theory is still quite puzzling in many respects. This is already evident when one studies the Künneth Theorem, or in other words, when one attempts to compute  $K_G^*(X \times Y)$  given knowledge of  $K_G^*(X)$  and  $K_G^*(Y)$  (or dually, to compute  $K_*^G(A \otimes B)$  in terms of  $K_*^G(A)$  and  $K_*^G(B)$ ). The first, and still the most important, work on this problem was done by Hodgkin [7]. Hodgkin observed that since the coefficient ring for  $G$ -equivariant  $K$ -theory is the (complex) representation ring  $R(G)$  of  $G$ , a Künneth Theorem for  $K_G^*$  should take the form of a spectral sequence

$$(1) \quad \mathrm{Tor}_p^{R(G)}(K_G^*(X), K_G^{-*+q}(Y)) \Rightarrow K_G^{p+q}(X \times Y),$$

which he constructed. However, Hodgkin noticed that there are two big problems with this:

- (1) If  $G$  is a disconnected compact Lie group, then  $R(G)$  never has finite homological dimension, so the best one can hope for is that the spectral sequence is associated to a filtration of infinite length. Sometimes the limit does not seem to resemble  $K_G^*(X \times Y)$  at all (see [7, Example 1, p. 68]).
- (2) Even if  $G$  is connected, but if  $\pi_1(G)$  is not torsion-free, then the spectral sequence may converge to the wrong limit (see [7, Example 2, p. 68]).

In particular, Hodgkin's theorem, which was improved a bit by Snaith [21] and McLeod [14], is of no help at all if  $G$  is finite or if  $\pi_1(G)$  has torsion. In joint work of the author with Schochet [17], we extended Hodgkin's theorem to the  $C^*$ -algebraic case of  $K_*^G(A \otimes B)$ , with  $A$  and  $B$  nuclear  $G$ -algebras in a suitable "bootstrap" class (containing all countable inductive limits of separable commutative  $G$ - $C^*$ -algebras), but again only for  $G$  connected compact Lie with  $\pi_1(G)$  torsion-free. (We did, however, manage to elucidate the meaning of the condition that  $\pi_1(G)$  be torsion-free. For connected compact Lie groups  $G$ , this is equivalent to the condition that every action of  $G$  on the compact operators  $\mathcal{K}$  be exterior equivalent to a trivial action.) Thus the "puzzle" of what should replace the Künneth Theorem when  $G$  is finite remained open.

The other major piece of work on this problem was done by Chris Phillips [15]. He did address the Künneth Theorem for equivariant  $K$ -theory for  $G$  finite, but only obtained a partial result, since he was relying on the Localization Theorem of Segal [19, Proposition 4.1].

While in this paper we sometimes work in the generality of group actions on  $C^*$ -algebras, the reader should realize that the case where the  $C^*$ -algebras are commutative is highly non-trivial and already new, and those not interested in operator algebras can restrict themselves to this case without missing very much. However, generalizing to the noncommutative case makes the proofs easier, since as first pointed out in Schochet [18], geometric resolutions are actually easier to construct in the noncommutative world.

## Acknowledgements

This work was supported by NSF Grant DMS-1206159. It grew out of the author's puzzlement about certain aspects of equivariant  $K$ -theory that came up in joint work with Mathai Varghese [12] begun in March, 2012, with partial support from the University of Adelaide and the Australian Mathematical Sciences Institute. Some of the ideas in this paper go back to the author's joint work with Claude Schochet in the 1980's.

The author would like to thank the referee for the observation that there is a close formal analogy between the main theorem in this paper and Künneth Theorem for KO proved by Bousfield [4]. Bousfield's "unified  $K$ -theory"  $K_{CRT}^*$  plays a role similar to our  $\mathbb{K}_G^*$ . While at first sight, real  $K$ -theory KO does not appear to be an equivariant theory, Boersema [3] has extended Bousfield's work to the case of real  $C^*$ -algebras. These include in particular algebras of the form  $C_0^\tau(X) = \{f \in C_0(X) \mid f(\tau(x)) = \overline{f(x)}\}$ , where  $\tau$  is an involution on the locally compact space  $X$ . The KO-theory of such algebras recovers the Real  $K$ -theory KR of Atiyah [1], which can be viewed as another form of  $\mathbb{Z}/2$ -equivariant  $K$ -theory for spaces. So the analogy is actually rather close.

The author would also like to thank Peter May for pointing out to him the work of Lewis and Mandell [11] on a more general class of equivariant Künneth Theorems, from which the main theorem of this paper (in a slightly different category) could be obtained. However, translating the Lewis–Mandell machinery into the explicit results here still requires a fair amount of work, and does not apply (without considerable modification) to the noncommutative world of  $G$ - $C^*$ -algebras. A remaining project would be to use this machinery to extend the results of this paper to the case of general finite groups. As the referee has noted, the results might look rather different at odd primes and at the prime 2. Some work along these lines, leading to a Künneth Theorem for equivariant  $K$ -theory with a somewhat different appearance from the one in this paper, has appeared in [5].

## 2 Background, notation, and previous results

We begin by recalling some previous results and establishing notation. In particular, we restate the results of Phillips [15] and Izumi [8] in terms a topologist would appreciate since it is likely that their work is not known to most topologists interested in equivariant  $K$ -theory.

Throughout this paper,  $K$ -theory or equivariant  $K$ -theory for spaces always means complex topological  $K$ -theory *with compact supports* for locally compact Hausdorff spaces. In most cases these spaces will be second countable and thus paracompact. Because of Bott periodicity, we will sometimes regard this theory as being  $\mathbb{Z}/2$ -graded. This theory satisfies a very strong form of excision—if  $X$  is a closed  $G$ -subspace of  $Y$ , then  $K_G^*(Y, X) \cong K_G^*(Y \setminus X)$ . (Note that  $Y \setminus X$  is indeed locally compact.)

From now on, let  $G$  be a cyclic group of prime order  $q$  and let  $R = R(G)$  be its representation ring, which we identify with  $\mathbb{Z}[t]/(t^q - 1)$ . Here  $t$  represents the standard representation of  $G$  on  $\mathbb{C}$  in which a fixed generator  $g$  of  $G$  is sent to  $\zeta = \exp(2\pi i/q)$ . This ring is the coefficient ring for equivariant  $K$ -theory. Its ideal structure was studied

in Segal [20]. Let  $I = (t-1)$  be the augmentation ideal and let  $J = (1+t+\dots+t^{q-1})$ . Since  $(t-1)(1+t+\dots+t^{q-1}) = 0$  in  $R$ , each prime ideal  $\mathfrak{p}$  of  $R$  contains either  $I$  or  $J$ , and these are the unique minimal prime ideals of  $R$  by [20, Proposition 3.7]. In the language of Segal, the prime ideal  $I$  has support  $\{1\}$ , while the prime ideal  $J$  has support  $G$ . (Since  $\{1\}$  and  $G$  are the only subgroups of  $G$ , these are the only two possibilities.) Note that  $R/I \cong \mathbb{Z}$ , while  $R/J \cong \mathbb{Z}[\zeta]$  is the ring of integers in the cyclotomic field  $\mathbb{Q}[\zeta]$ . Similarly, the localizations of  $R$  at these two prime ideals are  $R_I \cong \mathbb{Q}$  and  $R_J \cong \mathbb{Q}[\zeta]$ . Since  $R/I$  and  $R/J$  are both Dedekind domains, the other prime ideals of  $R$  are all maximal ideals. If such a maximal ideal  $\mathfrak{p}$  contains  $I$ , then it is of the form  $(I, p)$  for  $p$  a prime of  $\mathbb{Z}$  generating  $(\mathfrak{p} \cap \mathbb{Z}) \triangleleft \mathbb{Z}$ , while if it contains  $J$ , then it contains  $(J, p)$  for some prime  $p$ . The arithmetic of the cyclotomic field (the splitting of primes  $p$  in  $\mathbb{Z}[\zeta]$ ) will come in at this point when one classifies the prime (and necessarily maximal) ideals over  $(J, p)$ . There are now two cases: if  $p = q$ , then  $R/(p) \cong \mathbb{F}_q[t]/(t^q - 1) \cong \mathbb{F}_q[u]/(u^q)$ , where  $u = t - 1$ . So,  $p$  ramifies in the cyclotomic field and  $(J, p) \subseteq (I, p)$ , with  $(I, p)$  the unique maximal ideal of  $R$  containing  $p$ . This ideal has support  $\{1\}$  in the sense of Segal. Otherwise, if  $q \neq p$ , the primes over  $(J, p)$  are distinct from the primes over  $I$ , and have support  $G$  (cf [15, Proposition 6.2.2]). In any event, the localization  $R_{\mathfrak{p}}$  of  $R$  at a maximal ideal will be isomorphic to  $\mathbb{Z}_{(p)}$  if  $\mathfrak{p} = (I, p)$  and to a localization of  $\mathbb{Z}[\zeta]$  if  $\mathfrak{p} \supseteq (J, p)$ . So  $R_{\mathfrak{p}}$  is a discrete valuation ring and thus has global dimension 1. For purposes of this paper we will eventually further restrict to the case  $q = 2$ , in which case  $J = (t + 1)$  and  $R/J$  is also isomorphic to  $\mathbb{Z}$ , and the maximal ideals of  $R$  are precisely  $(I, p)$  or  $(J, p)$  for  $p$  a prime. In this case  $(I, 2) = (J, 2)$  since  $(t - 1) + 2 = t + 1$ ; otherwise the ideals  $(I, p)$  and  $(J, p)$  are all distinct. The picture of  $\text{Spec } R$ , showing the inclusion relations among prime ideals, is shown in Figure 1. Note the left–right reflection symmetry of the diagram, which can be explained by the existence of an automorphism  $t \mapsto -t$  of  $R$  (quite special to the case  $q = 2$ ), which interchanges  $I$  and  $J$ .

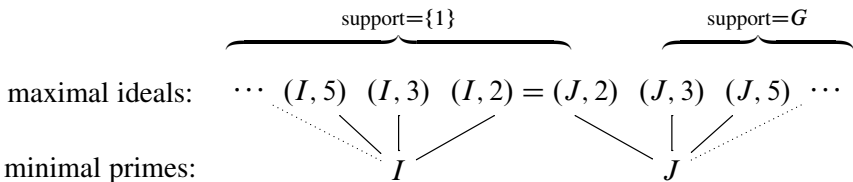


Figure 1: A picture of  $\text{Spec } R$  for  $G = \mathbb{Z}/2$

Before proceeding, it is convenient to recall the following calculation of equivariant  $K$ -theory for free actions, which almost certainly is known to experts but is not explicit in [19].

**Proposition 2.1** *Let  $G$  be a cyclic group of order  $q$  and let  $X$  be a compact free  $G$ -space. Then, the  $R$ -module structure on  $K_G^*(X) \cong K^*(X/G)$  is defined by letting  $t$  act by tensoring with the line bundle  $V$  with  $c_1(V) = c$ , where  $c$  is the image in  $H^2(X/G, \mathbb{Z})$  under the Bockstein homomorphism of the class in  $H^1(X/G, \mathbb{F}_q)$  classifying the  $q$ -to-1 covering map  $X \rightarrow X/G$ . One can also realize  $V$  more explicitly as the fiber product  $X \times_G \mathbb{C}$ , where  $G$  acts on  $\mathbb{C}$  by the nontrivial character  $t$ .*

*If  $A$  is a closed  $G$ -invariant subspace of  $X$ , then the  $R(G)$ -module structure on*

$$K_G^*(X, A) \cong K^*(Y, B), \quad Y = X/G, \quad B = A/G,$$

*is again defined by letting  $t$  act by cup-product with  $[V] \in K^0(X/G)$ . (Recall that for any pair  $(Y, B)$ , we have the cup-product  $K^0(Y) \otimes K^*(Y, B) \rightarrow K^*(Y, B)$ .)*

**Proof** The definition of the  $R(G)$ -action on  $K_G^*(X)$  or on  $K_G^*(X, A)$  implies that the result of applying the module action of  $t$  corresponds to tensoring with  $(\mathbb{C}, t)$ , which is the same after applying the isomorphism  $K_G^*(X) \cong K^*(Y)$  or  $K_G^*(X, A) \cong K^*(Y, B)$  as the vector bundle tensor product with  $V$ . The rest is immediate.  $\square$

The following result of Izumi constrains  $K_*^G(A)$  if  $A$  is a  $G$ - $C^*$ -algebra, or  $K_G^*(X)$  if  $X$  is a locally compact  $G$ -space, and if  $K_*(A) = 0$  or  $K^*(X) = 0$  (in all degrees).

**Theorem 2.2** (Izumi [8, Lemma 4.4]) *Let  $A$  be a  $G$ - $C^*$ -algebra, where  $G$  is a cyclic group of prime order  $q$ . Assume that  $A$  is  $K$ -contractible, i.e., that  $K_*(A) = 0$  (non-equivariantly, in both odd and even degrees). Then, as a  $\mathbb{Z}$ -module (i.e., forgetting the  $R$ -module structure),  $K_*^G(A)$  is uniquely  $q$ -divisible. Similarly, if  $X$  is a locally compact  $G$ -space and  $K^*(X) = 0$ , then  $K_G^*(X)$  is uniquely  $q$ -divisible as a  $\mathbb{Z}$ -module.*

Note that Izumi phrased [Theorem 2.2](#) in terms of the  $K$ -theory of the crossed product  $A \rtimes \mathbb{Z}/q$ , but this is the same as  $K_*^G(A)$  because of the Green–Julg Theorem (Julg [10], Blackadar [2, Theorem 11.7.1]).

The theorem cannot be improved, even in the abelian case, because if  $X$  is a locally compact  $G$ -space and  $K^*(X) = 0$ , then  $K_G^*(X)$  is *not necessarily zero*. It was pointed out in [16, Lemma 5.7] that Lowell Jones’ converse [9] to P A Smith’s Theorem provides a counterexample. However, since the proof there was slightly garbled (as pointed out by Thomas Schick in the review in MathSciNet), we restate it again.

**Proposition 2.3** *Let  $G$  be a cyclic group of prime order  $q$ . Then there is a contractible finite  $G$ -CW-complex  $Y$  for which  $L = Y^G$  has torsion in its homology of order prime to  $q$ . We can choose a basepoint  $x_0 \in L$  so that if  $X = Y \setminus \{x_0\}$ ,  $K^*(X) = 0$  while  $K_G^*(X) \neq 0$ .*

**Proof** By [9], if  $L$  is a finite CW-complex with  $\tilde{H}^*(L, \mathbb{Z}/q) = 0$  (in all degrees), then we can choose a contractible finite  $G$ -CW-complex  $Y$  with  $L = Y^G$ . Clearly  $L$  can be chosen with a basepoint  $x_0$  so that  $K^*(L \setminus \{x_0\})$  contains torsion of order a prime  $\ell \neq q$  (though  $K^*(L \setminus \{x_0\}; \mathbb{Z}/q) = \tilde{K}^*(L; \mathbb{Z}/q) = 0$ ). Since  $Y$  is contractible, if  $X = Y \setminus \{x_0\}$ , then  $K^*(X) = \tilde{K}^*(Y) = 0$ . Choose a maximal ideal  $\mathfrak{p}$  of  $R$  containing  $(J, \ell)$ . Then  $\mathfrak{p}$  has support  $G$  in the sense of Segal, so by the Localization Theorem [19, Proposition 4.1],  $K_G^*(X)_{\mathfrak{p}} \cong K_G^*(L \setminus \{x_0\})_{\mathfrak{p}} \cong K^*(L \setminus \{x_0\}) \otimes_{\mathbb{Z}} R_{\mathfrak{p}}$ . This is non-zero since  $K^*(L \setminus \{x_0\}) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell \neq 0$  and  $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$  is a finite field of characteristic  $\ell$ . Thus  $K_G^*(X)_{\mathfrak{p}} \neq 0$  and  $K_G^*(X) \neq 0$ . □

Applying Takai Duality [22] to Theorem 2.2, we deduce the following.

**Corollary 2.4** *Let  $A$  be a  $G$ - $C^*$ -algebra, where  $G$  is a cyclic group of prime order  $q$ . If  $K_*^G(A) = 0$  (in all degrees), then  $K_*(A)$  is uniquely  $q$ -divisible as a  $\mathbb{Z}$ -module. Similarly, if  $X$  is a locally compact  $G$ -space and  $K_G^*(X) = 0$ , then  $K^*(X)$  is uniquely  $q$ -divisible.*

Now suppose one has a  $G$ - $C^*$ -algebra  $B$  with  $K_*^G(B) = 0$  but  $K_*(B) \neq 0$ . One can get such an example by starting with a  $K$ -contractible  $G$ - $C^*$ -algebra for which  $K_*^G(A)$  is non-zero, as provided by Proposition 2.3 or by [8, Lemma 4.7]. Then let  $B = A \rtimes \mathbb{Z}/q$  and consider the dual action of  $\mathbb{Z}/q$  on  $B$ ; then  $B$  is not  $K$ -contractible since  $K_*(B) \cong K_*^G(A) \neq 0$ , but by Takai duality, one finds that  $K_*^G(B) \cong K_*(A) = 0$ . However,  $K_*(B) \cong K_*^G(C(G) \otimes B)$ , so we find that the Künneth Theorem in equivariant  $K$ -theory fails for  $B$ , in the sense that  $K_*^G(B)$  is identically 0 but there is a  $G$ -algebra (namely  $C(G)$ ) for which the tensor product has non-vanishing (but uniquely  $q$ -divisible) equivariant  $K$ -theory. (See [12] for more details.) To sum up, knowing just  $K_*^G(C)$  and  $K_*^G(D)$ , we cannot hope for a spectral sequence computing  $K_*^G(C \otimes D)$ . The fact that the (naïve) Künneth Theorem in equivariant  $K$ -theory fails for actions of finite groups was already pointed out in Phillips [15, Example 6.6.9].

However, we can now state the Künneth Theorem of Phillips.

**Theorem 2.5** (Phillips, [15, Theorem 6.4.6]) *Let  $G$  be a cyclic group of prime order  $q$ , and let  $\mathfrak{p}$  be a prime ideal of  $R = R(G)$  with support  $G$ . (Thus either  $\mathfrak{p} = J$*

or  $\mathfrak{p}$  contains  $(J, p)$  for  $p$  a prime not equal to  $q$ .) Let  $A$  and  $B$  be separable  $G$ - $C^*$ -algebras with  $B$  nuclear and with  $A$  in a suitable bootstrap category containing all equivariant inductive limits of separable type I  $G$ - $C^*$ -algebras and closed under various conditions (see [15, Theorem 6.4.7] for more details). Then there is a functorial short exact sequence

$$0 \rightarrow K_*^G(A)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} K_*^G(B)_{\mathfrak{p}} \xrightarrow{\omega_{\mathfrak{p}}} K_*^G(A \otimes B)_{\mathfrak{p}} \rightarrow \text{Tor}_1^{R_{\mathfrak{p}}}(K_*^G(A)_{\mathfrak{p}}, K_{*+1}^G(B)_{\mathfrak{p}}) \rightarrow 0,$$

which splits, though not naturally. The theorem holds in particular if  $A = C_0(X)$  and  $B = C_0(Y)$  with  $X$  and  $Y$  second countable locally compact  $G$ -spaces.

**Remark 2.6** Note that Theorem 2.5 is exactly what would expect from a Hodgkin-type spectral sequence (1), since localization is an exact functor, and thus one would get a spectral sequence

$$(2) \quad \text{Tor}_p^{R(G)_{\mathfrak{p}}}(K_G^*(X)_{\mathfrak{p}}, K_G^{-*+q}(Y)_{\mathfrak{p}}) \Rightarrow K_G^{p+q}(X \times Y)_{\mathfrak{p}},$$

which would collapse at  $E_2$ , giving a short exact sequence, since  $R_{\mathfrak{p}}$  is a PID in this case, and thus all higher Tor's (beyond  $\text{Tor}_1$ ) vanish. Phillips' insight is that (2) holds for the prime ideals mentioned in the theorem, even though the Hodgkin-type spectral sequence (1) fails.

**Remark 2.7** Note that Theorem 2.5 fails, even in the commutative case, if  $\mathfrak{p} = I$  or  $\mathfrak{p} = (I, p)$ , even though  $R_{\mathfrak{p}}$  has global dimension 1, so that homological algebra alone is not the explanation. Indeed, note that  $K_G^*(G) \cong R/I$ , which when localized at  $\mathfrak{p}$  gives just  $R_{\mathfrak{p}}$ , which is free of rank 1 as an  $R_{\mathfrak{p}}$ -module. Thus the theorem, if true, would say that  $K_G^*(G \times X)_{\mathfrak{p}} \cong K^*(X)_{(p)}$  is always isomorphic to  $K_G^*(X)_{\mathfrak{p}}$  at least as a  $\mathbb{Z}_{(p)}$ -module. But this is false even in the rather trivial case of  $X = G$  with the simply transitive  $G$ -action, since  $K_G^*(G \times G)_{\mathfrak{p}} \cong R_{\mathfrak{p}}^q$  while  $K_G^*(G)_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ .

### 3 A finer invariant

To deal with the failure of the Künneth Theorem, we need a finer invariant than just equivariant  $K$ -theory alone. An important fact about equivariant  $K$ -theory for a compact group  $G$  is that it is naturally  $\text{RO}(G)$ -graded (see for example May [13, Chapters IX, X, XIII, and XIV]). Given a compact group  $G$ , a locally compact  $G$ -space, and a finite-dimensional real orthogonal representation  $V$  of  $G$ , we can form  $K_{G,V}^*(X) = K^*(X \times V)$ . Similarly, given a  $G$ - $C^*$ -algebra  $A$ , we can define

$$K_*^{G,V}(A) = K_*^G(A \otimes C_0(V)),$$

where  $G$  acts on the second factor via the linear representation and acts on the tensor product by the tensor product action. Note that if  $V$  happens to be a complex vector space and the action of  $G$  is complex linear, then equivariant Bott periodicity [19, Proposition 3.2] gives an isomorphism  $K_{G,V}^* \cong K_G^*$  or  $K_{*,V}^{G,V} \cong K_*^G$ . (This is also true more generally if  $V$  is even-dimensional over  $\mathbb{R}$  and if the action of  $G$  factors through  $\text{Spin}^c(V)$ .) And if  $G$  acts trivially on  $V$ ,  $K_{*,V}^{G,V} \cong K_{*+\dim V}^G$ . But in general the groups  $K_{*,V}^{G,V}$  are not the same as  $K_*^G$ , even modulo a grading shift. In the noncommutative world, another approach to the groups  $K_{*,V}^{G,V}$  is possible via graded Clifford algebras, since  $C_0(V)$  is  $KK^G$ -equivalent to  $\text{Cl}(V)$ , the complex Clifford algebra of  $V$  viewed as a *graded*  $G$ -algebra [2, Theorem 20.3.2]. But this requires introducing graded  $C^*$ -algebras, which we would prefer to avoid.

For the rest of the paper we will deal only with the case where  $G = \{1, g\}$  is cyclic of order 2.<sup>1</sup> This group  $G$  has exactly two real characters, the trivial character 1 and the non-trivial character  $t$ , the sign representation  $-$  (where the generator  $g$  of the group acts by  $-1$  on  $\mathbb{R}$ ). From the sign representation we get the twisted equivariant  $K$ -groups  $K_{G,-}^*$  (on spaces) or  $K_{*, -}^{G,-}$  (on algebras). These are modules over the representation ring  $R$ . The coefficient groups for  $K_{G,-}^*$  are computed in Echterhoff and Pfante [6], for example. It turns  $K_{G,-}^*(\text{pt}) \cong R/J$ , concentrated in even degree. Twisting twice brings us back to conventional equivariant  $K$ -theory since a direct sum of two copies of the sign character is a complex representation, where equivariant Bott periodicity applies.

We can now define an invariant of a  $G$ -space (or  $G$ - $C^*$ -algebra) finer than just the equivariant  $K$ -theory alone. Namely, note that if  $V$  is  $\mathbb{R}$  with the sign representation of  $G$ , then we have a  $G$ -map  $\{0\} \hookrightarrow V$  inducing (for any  $G$ -space) a natural  $R$ -module homomorphism  $\varphi: K_{G,-}^*(X) \rightarrow K_G^*(X)$ , which when  $X$  is a point can be identified with the composite  $R/J \cong I \hookrightarrow R$ . Via the composite

$$K_G^*(X) \xrightarrow{\text{equivariant Bott}} K_G^*(X \times V \times V) \rightarrow K_G^*(X \times V \times \{0\}),$$

we also have a natural  $R$ -module homomorphism  $\psi: K_G^*(X) \rightarrow K_{G,-}^*(X)$ . The composite  $\varphi \circ \psi$  is the product with the element of  $R$  associated to

$$R \cong K_G^0(\text{pt}) \xrightarrow{\text{Bott}} K_G^0(V_{\mathbb{C}}) \rightarrow K_G^0(\{0\}) = R$$

coming from the inclusion  $\{0\} \hookrightarrow \mathbb{C}$ , where  $V_{\mathbb{C}}$  is the complexification of  $V$  ( $\mathbb{C}$  with the action of  $G$  by multiplication by  $-1$ ). This composite is  $1 - t$  (see [19, Section

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<sup>1</sup>Unfortunately,  $\text{RO}(G)$ -graded  $K$ -theory does not give anything new when  $G$  is cyclic of odd prime order, since then all non-trivial real irreducible representations of  $G$  are actually complex.



3]), and since  $\psi \circ \varphi$  is the same thing (except applied to  $X \times V$  instead of to  $X$ ), we have proved the following.

**Proposition 3.1** *Let  $G$  be the cyclic group of two elements. To any  $G$ -space there  $X$  is naturally associated a diagram*

$$\mathbb{K}_G^*(X): \quad K_G^*(X) \begin{array}{c} \xrightarrow{\psi} \\ \xleftarrow{\varphi} \end{array} K_{G,-}^*(X) \quad b$$

where the maps  $\varphi$  and  $\psi$  preserve the  $\mathbb{Z}/2$ -grading and the composite in either order is multiplication by  $1 - t$ .

**Examples 3.2**

- (1) If  $X = \text{pt}$ , then  $K_G^*(X) = R$  (concentrated in degree 0) and  $K_{G,-}^*(X) \cong R/J \cong I$  (concentrated in degree 0). The map  $\varphi$  is the inclusion  $I \hookrightarrow R$ . The map  $\psi$  is the projection  $R \twoheadrightarrow R/J$ .
- (2) If  $X = V$  ( $\mathbb{R}$  with the sign representation of  $G$ ), then  $\mathbb{K}_G^*(V)$  is the same as  $\mathbb{K}_G^*(\text{pt})$ , but with the two  $R$ -modules interchanged.
- (3) If  $X = G$  with the simply transitive action of  $G$ , then  $K_G^*(X) \cong R/I$  (concentrated in degree 0) and  $K_{G,-}^*(X) = K_G^*(G \times V) \cong K^*(V) \cong R/I$  (concentrated in degree 1). The connecting maps are both necessarily 0.

Note of course that  $\mathbb{K}_*^G(A)$  can be defined for a  $G$ - $C^*$ -algebra  $A$  in exactly the same way.

The following proposition is a precursor of the main theorem in the next section.

**Proposition 3.3** *Let  $G$  be a cyclic group of two elements and let  $X$  be a locally compact  $G$ -space. Then there is a natural 6-term exact sequence*

$$\begin{array}{ccccc} K^1(X) & \longrightarrow & K_{G,-}^0(X) & \xrightarrow{\varphi} & K_G^0(X) \\ f \uparrow & & & & \downarrow f \\ K_G^1(X) & \xleftarrow{\varphi} & K_{G,-}^1(X) & \longleftarrow & K^0(X) \end{array}$$

where the vertical arrows marked  $f$  on the left and right are the forgetful maps from equivariant to non-equivariant  $K$ -theory. The same (with the indices lowered) holds similarly for  $G$ - $C^*$ -algebras.

**Proof** We use the fact that if  $V$  is the sign representation of  $G$  as above, then  $V \setminus \{0\} \cong \mathbb{R} \times G$  (equivariantly). Here  $\mathbb{R}$  carries the trivial  $G$ -action but  $G$  acts simply transitively on itself. Thus we get an induced long exact sequence:

$$\cdots \rightarrow K_G^*(X \times \mathbb{R} \times G) \rightarrow K_G^*(X \times V) \rightarrow K_G^*(X \times \{0\}) \rightarrow \cdots$$

Here the middle group is  $K_{G,-}^*(X)$  and the map to  $K_G^*(X \times \{0\})$  is what we defined to be  $\varphi$ . The group on the left is isomorphic to the non-equivariant  $K$ -group  $K^*(X \times \mathbb{R}) \cong K^{*+1}(X)$ . It remains to show that the connecting map  $K_G^*(X \times \{0\}) \rightarrow K^{*+1}(X \times \mathbb{R}) \cong K^*(X)$  is the forgetful map  $f$ . This follows by naturality of products from the fact that it is true for  $X = \text{pt}$  (which one can check from the exact sequence and the identification of  $K_G^0(X)$  with  $R$  and of  $K^0(X)$  with  $R/I$ ).  $\square$

**Corollary 3.4** *If  $G$  is a cyclic group of two elements and  $X$  is a locally compact  $G$ -space, then there is a short exact sequence*

$$0 \rightarrow \text{coker } \varphi \rightarrow K^*(X) \rightarrow \ker \varphi \rightarrow 0,$$

where  $\varphi$  is as in [Proposition 3.1](#).

**Proof** This is just a restatement of the exactness in [Proposition 3.1](#).  $\square$

## 4 The main theorem and its proof

In this section  $G$  will always denote a cyclic group of order 2.

Because of [Theorem 2.5](#), as well as the fact that an  $R$ -module is completely determined by its localizations at maximal ideals  $\mathfrak{p}$  of  $R$ , to complete the problem of getting a Künneth Theorem for  $K_G^*$  (for  $G$ -spaces) or  $K_*^G$  (for  $G$ -algebras) we just need to compute  $\mathbb{K}_G^*(X \times Y)_{\mathfrak{p}}$  in terms of  $\mathbb{K}_G^*(X)_{\mathfrak{p}}$  and  $\mathbb{K}_G^*(Y)_{\mathfrak{p}}$  (and similarly for algebras in a suitable bootstrap category) for maximal ideals  $\mathfrak{p} = (I, p)$ ,  $p$  a prime. (Once again, see [Figure 1](#).)

After localization at  $\mathfrak{p} = (I, p)$ , an interesting thing happens: since  $1 - t$  lies in the kernel of the localization map  $R \rightarrow R_{\mathfrak{p}}$ ,  $\psi \circ \varphi$  and  $\varphi \circ \psi$  are both 0. But something much stronger is true.

**Proposition 4.1** *Let  $G$  be a cyclic group of order 2 and let  $\mathfrak{p} = (I, p) \triangleleft R = R(G)$ ,  $p$  a prime. Then for any  $G$ - $C^*$ -algebra  $A$ ,  $\varphi: K_*^{G,-}(A) \rightarrow K_*^G(A)$  and  $\psi: K_*^G(A) \rightarrow K_*^{G,-}(A)$  vanish after localization at  $\mathfrak{p}$ .*

**Proof** It is enough to treat one of  $\varphi$  and  $\psi$  since each can be obtained from the other by replacing  $A$  by  $A \otimes C_0(V)$ . (As usual,  $V$  here denotes the sign representation of  $G$  on  $\mathbb{R}$ .) Furthermore, by the usual tricks with suspensions and unitalizations, we can restrict attention to  $K_0$  and assume  $A$  is unital. By [2, Section 11.3], any class in  $K_0(A)$  comes from a  $G$ -invariant projection  $p$  in  $\text{End}(W) \otimes A$ ,  $W$  a finite-dimensional  $G$ -module (and thus of the form  $\mathbb{C}^n \oplus V_{\mathbb{C}}^m$ ). Such a projection  $p$  defines a  $G$ -homomorphism  $\mathbb{C} \rightarrow \text{End}(W) \otimes A$ . By functoriality of  $\psi$ , we get a commutative diagram

$$\begin{CD} K_0^G(\mathbb{C})_{\mathfrak{p}} = R_{\mathfrak{p}} @>\psi>> K_0^{G,-}(\mathbb{C})_{\mathfrak{p}} = 0 \\ @VpVV @VVp \otimes 1V \\ K_0^G(\text{End}(W) \otimes A)_{\mathfrak{p}} @>\psi>> K_0^{G,-}(\text{End}(W) \otimes A)_{\mathfrak{p}} \end{CD}$$

which shows that  $\psi([p]) = 0$  after localization at  $\mathfrak{p}$ . □

Because of Proposition 4.1, we can ignore  $\varphi$  and  $\psi$  after localization at  $\mathfrak{p}$  and treat  $\mathbb{K}_G^*(X)_{\mathfrak{p}}$  as a  $\mathbb{Z}/2$ -graded  $R_{\mathfrak{p}}$ -module, by putting  $K_{G,-}^0(X)_{\mathfrak{p}} \oplus K_G^{-1}(X)_{\mathfrak{p}}$  in odd degree and  $K_G^0(X)_{\mathfrak{p}} \oplus K_{G,-}^{-1}(X)_{\mathfrak{p}}$  in even degree. Our main result is suggested by the following reformulation of Corollary 3.4:

**Proposition 4.2** *Let  $G$  be a cyclic group of order 2 and let  $X$  be a locally compact  $G$ -space. Let  $\mathfrak{p} = (I, p) \triangleleft R$ ,  $p$  a prime. Then there is a filtration on a direct sum of two copies of  $K^*(X)_{\mathfrak{p}} \cong K_G^*(X \times G)_{\mathfrak{p}}$  for which the associated graded module is  $\mathbb{K}_G^*(X)_{\mathfrak{p}} \otimes \mathbb{K}_G^*(G)_{\mathfrak{p}}$ .*

**Proof** By Examples 3.2(3),  $\mathbb{K}_G^*(G)_{\mathfrak{p}}$  contains two copies of  $R_{\mathfrak{p}}$  concentrated in degree 0. Thus tensoring with  $\mathbb{K}_G^*(X)_{\mathfrak{p}}$  is simply tensoring over  $R_{\mathfrak{p}}$  with  $R_{\mathfrak{p}} \oplus R_{\mathfrak{p}}$ , i.e., the operation of “doubling.” But by Proposition 3.3 together with Proposition 4.1,  $K^*(X)_{\mathfrak{p}}$  is an extension of  $K_{G,-}^{*+1}(X)_{\mathfrak{p}}$  by  $K_G^*(X)_{\mathfrak{p}}$ . So the result follows. □

**Remark 4.3** It might seem strange that the formulation of Proposition 4.2 is limited to the case of maximal ideals  $\mathfrak{p} \supset I$ . If we localize instead at a maximal ideal  $\mathfrak{p}$  with support  $G$  (and thus of the form  $(J, p)$ ,  $p \neq 2$ ), then by Examples 3.2(3),  $\mathbb{K}_G^*(G)_{\mathfrak{p}} = 0$ . At the same time,  $K^*(X) \cong K_G^*(X \times G)$ , and  $X \times G$  is a free  $G$ -space, so by the Localization Theorem,  $K_G^*(X \times G)_{\mathfrak{p}} = 0$ . Thus Proposition 4.2 is still true in this case, but it just says  $0 = 0$ , which obviously is not very interesting. By way of further explanation, Proposition 2.1 shows that for a free compact  $G$ -space, the augmentation ideal  $I$  acts nilpotently on the equivariant  $K$ -theory, so only localization at prime ideals containing  $I$  gives anything useful.

Now we are ready to state and prove the main theorem.

**Theorem 4.4** *Let  $G$  be a cyclic group of order 2 and let  $X$  and  $Y$  be second countable locally compact  $G$ -spaces. Let  $\mathfrak{p} = (I, p) \triangleleft R$ ,  $p$  a prime. Then there is a natural short exact sequence of  $\mathbb{Z}/2$ -graded  $R_{\mathfrak{p}}$ -modules:*

$$(3) \quad 0 \rightarrow \mathbb{K}_G^p(X)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \mathbb{K}_G^q(Y)_{\mathfrak{p}} \xrightarrow{\omega_{\mathfrak{p}}} \mathbb{K}_G^{p+q}(X \times Y)_{\mathfrak{p}} \rightarrow \text{Tor}_1^{R_{\mathfrak{p}}}(\mathbb{K}_G^p(X)_{\mathfrak{p}}, \mathbb{K}_G^{q+1}(Y)_{\mathfrak{p}}) \rightarrow 0$$

Note the similarity to (2). Similarly, for separable  $G$ - $C^*$ -algebras  $A$  and  $B$  with  $B$  in a suitable “bootstrap” category, containing all inductive limits of separable type I  $G$ - $C^*$ -algebras and closed under exterior equivalence, equivariant Morita equivalence, and the “2 out of 3 property for short exact sequences,” we have a natural short exact sequence:

$$0 \rightarrow \mathbb{K}_p^G(A)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \mathbb{K}_q^G(B)_{\mathfrak{p}} \xrightarrow{\omega_{\mathfrak{p}}} \mathbb{K}_{p+q}^G(A \otimes B)_{\mathfrak{p}} \rightarrow \text{Tor}_1^{R_{\mathfrak{p}}}(\mathbb{K}_p^G(A)_{\mathfrak{p}}, \mathbb{K}_{q+1}^G(B)_{\mathfrak{p}}) \rightarrow 0$$

The method of proof of this theorem is similar to the one used in [17] and [15], or in other words, is based on the method of geometric resolutions. First we need to see that the problem that occurred in our previous counterexample to the Künneth Theorem, having  $K_G^*(X) = 0$  but  $K_G^*(X \times Y) \neq 0$  for some  $Y$ , cannot recur.

**Lemma 4.5** *Let  $\mathfrak{p} = (I, p) \triangleleft R$ ,  $p$  a prime. Let  $A$  be a  $G$ - $C^*$ -algebra with  $\mathbb{K}_*^G(A)_{\mathfrak{p}} = 0$ . Then for any finite  $G$ -CW-complex  $Y$ ,  $\mathbb{K}_*^G(A \otimes C(Y))_{\mathfrak{p}} = 0$ .*

**Proof** We are assuming  $K_*^G(A) = 0$  and  $K_*^{G,-}(A) = 0$ , and we need to show  $K_*^G(A \otimes C(Y)) = 0$ . (A similar conclusion for  $K_*^{G,-}(A \otimes C(Y))$  follows upon replacing  $A \otimes C(Y)$  by  $A \otimes C(Y) \otimes C_0(V)$ , where  $V$  as usual is the sign representation of  $G$  on  $\mathbb{R}$ .) First we show the result holds when  $Y$  is a single (open)  $G$ -cell, i.e., either  $\mathbb{R}^n$  or  $G \times \mathbb{R}^n$  with trivial action on  $\mathbb{R}^n$ . (Since we are using open cells here,  $C(Y)$  should be replaced by  $C_0(Y)$ .) But  $K_*^G(A \otimes C_0(\mathbb{R}^n)) \cong K_{*+n}^G(A)$  and  $K_*^G(A \otimes C_0(G \times \mathbb{R}^n)) \cong K_{*+n}^G(A)$ , to which we can apply Proposition 3.3. Now assume  $Z$  is a  $G$ -space for which we know  $K_*^G(A \times C_0(Z)) = 0$  and  $K_*^{G,-}(A \times C_0(Z)) = 0$ , and assume  $Y$  is obtained from  $Z$  by adding a single equivariant cell, so that  $Y \setminus Z$  is either  $\mathbb{R}^n$  or  $G \times \mathbb{R}^n$  with trivial action on  $\mathbb{R}^n$ . Applying the 5-Lemma to the  $K$ -theory sequences associated to the equivariant short exact sequence

$$0 \rightarrow A \otimes C_0(Z) \rightarrow A \otimes C_0(Y) \rightarrow A \otimes C_0(Y \setminus Z) \rightarrow 0,$$

we get the result for  $Y$ . Finally, the lemma in full generality follows by an induction on the  $G$ -cells of  $Y$ . □

**Corollary 4.6** *Let  $\mathcal{A}_G$  denote the category of separable abelian  $G$ - $C^*$ -algebras (contravariantly equivalent to the category of second countable locally compact Hausdorff  $G$ -spaces) and let  $\mathcal{C}_G$  denote the smallest category of separable  $G$ - $C^*$ -algebras containing the separable type I  $G$ - $C^*$ -algebras and closed under  $G$ -kernels,  $G$ -quotients,  $G$ -extensions, equivariant inductive limits, crossed products by actions of  $\mathbb{R}$  or  $\mathbb{Z}$  commuting with the  $G$ -action, exterior equivalence, and  $G$ -Morita equivalence. Let  $\mathfrak{p} = (I, p) \triangleleft R$ ,  $p$  a prime. Let  $A$  be a  $G$ - $C^*$ -algebra with  $\mathbb{K}_*^G(A)_{\mathfrak{p}} = 0$ . Then for any  $G$ - $C^*$ -algebra  $B$  in  $\mathcal{A}_G$  or  $\mathcal{C}_G$ ,  $\mathbb{K}_*^G(A \otimes B)_{\mathfrak{p}} = 0$ .*

**Proof** This follows from Lemma 4.5 by an application of [17, Theorem 2.8] or [15, Theorem 6.4.7]. We just recall the essence of the argument for the abelian case  $B = C_0(Y)$ . Because of the exact sequence for the pair  $(Y, Y^G)$ , it is enough to treat the cases of free and trivial  $G$ -spaces. There is also an easy reduction to the case where the space is compact. But any compact metrizable space is a countable inverse limit of finite CW-complexes. This plus Lemma 4.5 immediately gives the case of a trivial  $G$ -space. If  $Y$  is a free compact metrizable  $G$ -space, write  $Y/G$  as a countable inverse limit of finite CW-complexes and pull back to write  $Y$  as a countable inverse limit of free finite  $G$ -CW-complexes. Since equivariant  $K$ -theory commutes with equivariant countable  $C^*$ -inductive limits, the result follows.  $\square$

The next step is to prove Theorem 4.4 with a projectivity restriction on  $\mathbb{K}_*^G(A)$ . This first requires a lemma which will also be needed to do the general case.

**Lemma 4.7** *Let  $G$  be a cyclic group of order 2,  $\mathfrak{p} = (I, p) \triangleleft R$ ,  $p$  a prime. Let  $A$  be a separable  $G$ - $C^*$ -algebra. Let  $\mathcal{H}$  denote an infinite-dimensional separable complex Hilbert space equipped with a unitary representation of  $G$  that contains both irreducible representations with infinite multiplicity. Then there is a commutative  $G$ - $C^*$ -algebra  $C = C_0(X \amalg (Y \times V))$ , where  $X$  and  $Y$  are disjoint unions of finite-dimensional real vector spaces with trivial  $G$ -action and  $V$  is the sign representation of  $G$ , and there is a  $G$ -homomorphism  $\alpha: C \rightarrow A \otimes C_0(V_{\mathbb{C}} \oplus \mathbb{C}) \otimes \mathcal{K}(\mathcal{H})$ , such that  $\alpha$  induces a surjection*

$$\mathbb{K}_*^G(C)_{\mathfrak{p}} \xrightarrow{\alpha_*} \mathbb{K}_*^G(A \otimes C_0(V_{\mathbb{C}} \oplus \mathbb{C}) \otimes \mathcal{K}(\mathcal{H}))_{\mathfrak{p}} \xleftarrow{\cong} \mathbb{K}_*^G(A)_{\mathfrak{p}}$$

where the isomorphism  $\mathbb{K}_*^G(A)_{\mathfrak{p}} \cong \mathbb{K}_*^G(A \otimes C_0(V_{\mathbb{C}} \oplus \mathbb{C}) \otimes \mathcal{K}(\mathcal{H}))_{\mathfrak{p}}$  is the canonical one coming from equivariant Bott periodicity.

If  $\mathbb{K}_*^G(A)_{\mathfrak{p}}$  is free over  $R_{\mathfrak{p}}$ , then  $C$  and  $\alpha$  can be chosen so that  $\alpha_*$  is an isomorphism.

**Proof** By [17, Proposition 4.1 and Remark 4.2], which are proved using the same trick that appeared in the proof of Proposition 4.1, there is a commutative  $C^*$ -algebra,

which we may take to be of the form  $C_0(X')$  with  $X'$  a disjoint union of countably many points and lines on which  $G$  acts trivially, and there is a  $G$ -map  $C_0(X' \times \mathbb{R}) \rightarrow A \otimes C_0(V_{\mathbb{C}} \times \mathbb{R} \times \mathbb{R}) \otimes \mathcal{K}(\mathcal{H})$  inducing a surjection on  $K_*^G$ . If  $K_*^G(A)_{\mathfrak{p}}$  is free over  $R_{\mathfrak{p}}$ , we can choose the induced map on  $K_*^G$  localized at  $\mathfrak{p}$  to be an isomorphism. We take  $X = X' \times \mathbb{R}$ . Note by [Examples 3.2\(1\)](#) (and its suspension) that  $K_{G,-}^*(X)_{\mathfrak{p}} = 0$ .

Similarly, we can apply the same argument to  $A \otimes C_0(V)$ , and by definition,  $K_*^G(A \otimes C_0(V))_{\mathfrak{p}} = K_*^{G,-}(A)_{\mathfrak{p}}$ . We get a trivial  $G$ -space  $Y'$ , again a disjoint union of countably many points and lines, and a  $G$ -map  $C_0(Y' \times \mathbb{R}) \rightarrow A \otimes C_0(V \times \mathbb{R} \times \mathbb{R}) \otimes \mathcal{K}(\mathcal{H})$  inducing a surjection on  $K_*^G$ . Again, if  $K_*^G(A \otimes C_0(V \times \mathbb{R}^2))_{\mathfrak{p}} \cong K_*^{G,-}(A)_{\mathfrak{p}}$  is free over  $R_{\mathfrak{p}}$ , we can choose the induced map on  $K_*^G$  localized at  $\mathfrak{p}$  to be an isomorphism. Take  $Y = Y' \times \mathbb{R}$  and tensor everything with  $C_0(V)$ . Recall that  $C_0(V) \otimes C_0(V) = C_0(V_{\mathbb{C}})$ . We now have a  $G$ -map  $C_0(Y \times V) \rightarrow A \otimes C_0(V_{\mathbb{C}} \oplus \mathbb{C}) \otimes \mathcal{K}(\mathcal{H})$  inducing a surjection on  $K_*^{G,-}$  after localization at  $\mathfrak{p}$ , and inducing an isomorphism if  $K_*^{G,-}(A)_{\mathfrak{p}}$  is free over  $R_{\mathfrak{p}}$ . Since  $K_{G,-}^*(Y)_{\mathfrak{p}} = 0$ ,  $K_G^*(Y \times V)_{\mathfrak{p}} = 0$ . The lemma follows upon assembling everything together.  $\square$

**Theorem 4.8** *Let  $G$  be a cyclic group of order 2,  $\mathfrak{p} = (I, p) \triangleleft R$ ,  $p$  a prime. Let  $A$  be a separable  $G$ - $C^*$ -algebra with  $\mathbb{K}_*^G(A)_{\mathfrak{p}}$  free over  $R_{\mathfrak{p}}$ . Let  $B$  be any  $G$ - $C^*$ -algebra in  $\mathcal{A}_G$  or  $\mathcal{C}_G$  (in the notation of [Corollary 4.6](#)). Then there is a natural isomorphism  $\mathbb{K}_G(A)_{\mathfrak{p}} \otimes_{\mathfrak{p}} \mathbb{K}_G(B)_{\mathfrak{p}} \rightarrow \mathbb{K}_G(A \otimes B)_{\mathfrak{p}}$  coming from the pairing  $\omega_{\mathfrak{p}}$  discussed in [\[15, Section 6.1\]](#).*

**Proof** The theorem is obviously true for  $A = \mathbb{C}$  with trivial action, and it follows that it is true for  $A = C_0(V)$  also, since this has essentially the same  $K$ -theory (except for interchange of  $K_0^G$  with  $K_0^{G,-}$ , by [Examples 3.2\(2\)](#)). Note that since we are localizing at a maximal ideal containing  $I$ ,  $K_*^{G,-}(\mathbb{C})_{\mathfrak{p}} = 0$ , whereas  $K_*^G(C_0(V))_{\mathfrak{p}} = 0$ . The theorem is also true for  $A = C(G)$  by [Proposition 4.2](#), and then follows for  $A = C_0(G \times V)$  or  $A = C_0(G \times \mathbb{R})$  as well.

Now let us do the general case. Assume  $C$  and  $\alpha$  are chosen as in [Lemma 4.7](#) to induce isomorphisms on  $K_*^G$  and on  $K_*^{G,-}$  after localization at  $\mathfrak{p}$ . Let  $W$  denote the mapping cone of  $\alpha$ . Then we obtain a short exact sequence of  $G$ -algebras

$$(4) \quad 0 \rightarrow A \otimes C_0(V_{\mathbb{C}} \oplus \mathbb{C}) \otimes \mathcal{K}(\mathcal{H}) \rightarrow W \rightarrow C \rightarrow 0,$$

for which the induced long exact sequence in  $K_*^G$  localized at  $\mathfrak{p}$  gives an isomorphism  $K_*^G(C)_{\mathfrak{p}} \xrightarrow{\cong} K_*^G(A)_{\mathfrak{p}}$  and an isomorphism  $K_*^{G,-}(C)_{\mathfrak{p}} \xrightarrow{\cong} K_*^{G,-}(A)_{\mathfrak{p}}$ . It follows that  $\mathbb{K}_*^G(W)_{\mathfrak{p}} = 0$ . By [Corollary 4.6](#),  $\mathbb{K}_*^G(W \otimes B) = 0$ . But tensoring with  $B$  is exact, since  $B$  is nuclear, so we get a short exact sequence

$$0 \rightarrow A \otimes C_0(V_{\mathbb{C}} \oplus \mathbb{C}) \otimes \mathcal{K}(\mathcal{H}) \otimes B \rightarrow W \otimes B \rightarrow C \otimes B \rightarrow 0$$

from (4). The long exact sequence now gives an isomorphism  $(\alpha \otimes 1)_* : \mathbb{K}_*^G(C \otimes B)_p \rightarrow \mathbb{K}_*^G(A \otimes B)_p$ . But by the case already considered (because of the special structure of the algebra  $C$ ),

$$\omega_p : \mathbb{K}_*^G(C)_p \otimes_{R_p} \mathbb{K}_*^G(B)_p \xrightarrow{\cong} \mathbb{K}_*^G(C \otimes B)_p.$$

Combining the isomorphisms and a little diagram chase now shows that:

$$\omega_p : \mathbb{K}_*^G(A)_p \otimes_{R_p} \mathbb{K}_*^G(B)_p \xrightarrow{\cong} \mathbb{K}_*^G(A \otimes B)_p \quad \square$$

Finally we can prove the main theorem.

**Proof of Theorem 4.4** We apply Lemma 4.7 and take the mapping cone sequence (4). Since

$$\mathbb{K}_*^G(C)_p \xrightarrow{\alpha_*} \mathbb{K}_*^G(A \otimes C_0(V_{\mathbb{C}} \oplus \mathbb{C}) \otimes \mathcal{K}(\mathcal{H}))_p \cong \mathbb{K}_*^G(A)_p$$

is surjective,  $R_p$  is a PID, and  $\mathbb{K}_*^G(C)_p$  is free over  $R_p$ ,  $\mathbb{K}_*^G(W)_p$  is also free over  $R_p$ ; in fact

$$(5) \quad \mathbb{K}_*^G(W)_p \rightarrow \mathbb{K}_*^G(C)_p \rightarrow \mathbb{K}_*^G(A)_p$$

is a free  $R_p$ -resolution of  $\mathbb{K}_*^G(A)_p$ . To simplify the notation, let us replace  $A$  by  $A \otimes C_0(V_{\mathbb{C}} \oplus \mathbb{C}) \otimes \mathcal{K}(\mathcal{H})$ ; this is harmless since the theorem will be true for  $A$  if it is true for the latter. As in the proof of Theorem 4.8, we tensor (4) with  $B$  and apply  $\mathbb{K}_*^G$ . We now get a commutative diagram with exact columns, where the first column comes from tensoring (5) with  $\mathbb{K}_*^G(B)_p$  and the second column is the long exact sequence in  $\mathbb{K}_*^G$ . The result is as follows:

$$\begin{array}{ccc} \text{Tor}_1^{R_p}(\mathbb{K}_*^G(A)_p, \mathbb{K}_*^G(B)_p) & & \mathbb{K}_{*+1}^G(A \otimes B)_p \\ \downarrow & & \downarrow \\ \mathbb{K}_*^G(W)_p \otimes_{R_p} \mathbb{K}_*^G(B)_p & \xrightarrow[\cong]{\omega_p} & \mathbb{K}_*^G(W \otimes B)_p \\ \downarrow & & \downarrow \\ \mathbb{K}_*^G(C)_p \otimes_{R_p} \mathbb{K}_*^G(B)_p & \xrightarrow[\cong]{\omega_p} & \mathbb{K}_*^G(C \otimes B)_p \\ \downarrow \alpha_* \otimes 1 & & \downarrow (\alpha \otimes 1_B)_* \\ \mathbb{K}_*^G(A)_p \otimes_{R_p} \mathbb{K}_*^G(B)_p & \xrightarrow{\omega_p} & \mathbb{K}_*^G(A \otimes B)_p \end{array}$$

The fact that the middle horizontal arrows are isomorphisms follows from Theorem 4.8.

We finish the proof with a diagram chase. If a class in  $\mathbb{K}_*^G(A \otimes B)_p$  maps to 0 in  $\mathbb{K}_{*-1}^G(W \otimes B)_p$ , then it comes from  $\mathbb{K}_*^G(C \otimes B)_p$  and thus lies in the image of

$\mathbb{K}_*^G(A)_p \otimes_{R_p} \mathbb{K}_*^G(B)_p$  under  $\omega_p$ . Furthermore, if a class  $c \in \mathbb{K}_*^G(A)_p \otimes_{R_p} \mathbb{K}_*^G(B)_p$  maps to 0 under  $\omega_p$ , then  $c = (\alpha_* \otimes 1)(d)$  for some  $d \in \mathbb{K}_*^G(C)_p \otimes_{R_p} \mathbb{K}_*^G(B)_p$ , and chasing the diagram shows that  $d$  came from  $\mathbb{K}_*^G(W)_p \otimes_{R_p} \mathbb{K}_*^G(B)_p$ , which shows that  $c = 0$ . Thus  $\omega_p$  is injective, and its image is the same as the image of  $\mathbb{K}_*^G(C \otimes B)_p$  in  $\mathbb{K}_*^G(A \otimes B)_p$ , which is precisely the kernel of the boundary map to  $\mathbb{K}_{*-1}^G(W \otimes B)_p$ .

Now consider the cokernel of  $\omega_p$ . By the above discussion, this is identified with the image of  $\mathbb{K}_*^G(A \otimes B)_p$  in  $\mathbb{K}_{*-1}^G(W \otimes B)_p$ , which is the kernel of the map  $\mathbb{K}_{*-1}^G(W \otimes B)_p \rightarrow \mathbb{K}_{*-1}^G(C \otimes B)_p$ , which by the diagram again is identified with

$$\mathrm{Tor}_1^{R_p}(\mathbb{K}_*^G(A)_p, \mathbb{K}_{*-1}^G(B)_p).$$

That completes the proof of the main theorem.  $\square$

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